NULL CONTROLLABILITY OF THE VON KÁRMÁN THERMOELASTIC PLATES UNDER THE CLAMPED OR FREE MECHANICAL BOUNDARY CONDITIONS

By

George Avalos

IMA Preprint Series # 2008

(November 2004)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370
URL: http://www.ima.umn.edu

Null Controllability of the von Kármán Thermoelastic Plates Under the Clamped or Free Mechanical Boundary Conditions

George Avalos
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588

November 26, 2004

Abstract

In this paper, we prove the local exact null controllability of the thermoelastic plate model, in the absence of rotational inertia, and under the influence of the (non-Lipschitz) von Kármán nonlinearity. The plate component may be taken to satisfy either the clamped or higher order (and physically relevant) free boundary conditions. In the accompanying analysis, critical use is made of sharp observability estimates which obtain for the linearization of the thermoelastic plate (these being derived in [3] and [4]). Moreover, another key ingredient in our work to steer the given nonlinear dynamics is the recent result in [10] concerning the sharp regularity of the von Karman nonlinearity.

1 Statement of Problem and Main Results

Throughout, $\Omega \subset \mathbb{R}^2$ will be a bounded, open set with smooth boundary Γ. Given terminal time T, $0 < T < 1$[1], we consider the following nonlinear thermoelastic system on $\Omega \times (0,T)$:

$$\begin{cases} \begin{cases} \omega_{tt} + \Delta^2 \omega + \alpha \Delta \theta = [\mathcal{F}(\omega), \omega] \\ \theta_t - \Delta \theta - \alpha \Delta \omega_t = u \end{cases} \text{ on } (0,T) \times \Omega \\ \\ \omega(t=0) = \omega_0;\ \omega_t(t=0) = \omega_1;\ \theta(t=0) = \theta_0 \text{ on } \Omega. \end{cases} \tag{1}$$

In this model the parameter α, which couples the hyperbolic-like (plate) and parabolic (heat) dynamics, is nonzero with, say, $M \geq \alpha > 0$. Concerning the nonlinearity which appears in the plate component of this system: the so-called *von Kármán* bracket $[\cdot, \cdot]$ is defined by having for all $v, \tilde{v} \in H^2(\Omega)$,

$$[v, \tilde{v}] = v_{xx}\tilde{v}_{yy} + v_{yy}\tilde{v}_{xx} - 2v_{xy}\tilde{v}_{xy}.$$

Moreover, the *Airy Stress* function $\mathcal{F}(\cdot)$ which appears within the bracket in (1) is defined by the solution of the following elliptic problem:

$$\Delta^2 \mathcal{F}(v) = -[v, v] \text{ in } \Omega;\ \ \mathcal{F}(v) = \left.\frac{\partial \mathcal{F}(v)}{\partial \nu}\right|_{\Gamma} = 0 \text{ on } \Gamma.$$

(see; e.g., [9], [15], [22]).

[1] Our taking $0 < T < 1$ is merely for convenience; since in the context of null controllability, terminal time T should be arbitrarily "small", this range of T will not at all influence the course of events.

In addition, the thermoelastic variables will throughout satisfy the one or other of the following boundary conditions:

(i) The "clamped" mechanical boundary conditions:

$$\omega = \frac{\partial \omega}{\partial \nu} = \theta = 0 \text{ on } (0,T)\times\Gamma; \tag{2}$$

(ii) The "free" mechanical boundary conditions:

$$\begin{cases} \begin{cases} \Delta\omega + (1-\mu)B_1\omega + \alpha\theta = 0 \\ \dfrac{\partial \Delta\omega}{\partial \nu} + (1-\mu)\dfrac{\partial B_2 \omega}{\partial \tau} - \omega + \alpha\dfrac{\partial \theta}{\partial \nu} = 0 \end{cases} \text{on } (0,T)\times\Gamma \\ \\ \dfrac{\partial \theta}{\partial \nu} + \lambda\theta = 0 \ \text{ on } (0,T)\times\Gamma, \quad \text{where } \lambda > 0. \end{cases} \tag{3}$$

Here, the parameter $\mu \in (0,1)$ is Poisson's ratio. Also, the second order boundary operators B_i appearing above are given by

$$\begin{aligned} B_1 w &\equiv 2\nu_1\nu_2\frac{\partial^2 w}{\partial x \partial y} - \nu_1^2\frac{\partial^2 w}{\partial y^2} - \nu_2^2\frac{\partial^2 w}{\partial x^2}; \\ B_2 w &\equiv (\nu_1^2 - \nu_2^2)\frac{\partial^2 w}{\partial x \partial y} + \nu_1\nu_2\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right) \end{aligned} \tag{4}$$

(as usual $\nu(x) = [\nu_1(x), \nu_2(x)]$ denotes the unit normal vector exterior to Ω at x).

The space of wellposedness for this PDE will be,

$$\mathbf{H} \equiv \begin{cases} H_0^2(\Omega)\times L^2(\Omega)\times L^2(\Omega), \text{ in the case of the boundary conditions (2)} \\ \\ H^2(\Omega)\times L^2(\Omega)\times L^2(\Omega), \text{ in the case of the boundary conditions (3)} \end{cases}.$$

In particular, with $\{[\omega_0, \omega_1, \theta_0], f\} \in \mathbf{H}\times L^2(Q)$, where $Q \equiv \Omega\times(0,T)$, the solution of the *linearization* of the PDE (10)–subject to either (2) or (3)–will satisfy $[\omega(t), \omega_t(t), \theta(t)] \in C([0,T];\mathbf{H})$. (This assertion can be verified readily in each case by an application of the Lumer Phillips Theorem.) Given then data $\{[\omega_0, \omega_1, \theta_0], f\} \in \mathbf{H}\times L^2(Q)$, the problem is as follows: find a thermal control $u \in L^2(Q)$ such that the corresponding solution to (6), (2) (resp., (6), (3)) satisfies $[\omega(T), \omega_t(T), \theta(T)] = [0,0,0]$.

Concerning the null controllability of the fully nonlinear model, our main result is as follows:

Theorem 1 *The system (1) is locally null controllable within the class of $L^2(Q)$-controls. That is to say: there is a positive number $\rho(T) > 0$ such that if $\|[\omega_0, \omega_1, \theta_0]\|_{\mathbf{H}} < \rho$, then there exists a control $u \in L^2(Q)$ such that the corresponding solution of (1) satisfies $[\omega(T), \omega_t(T), \theta(T)] = [0,0,0]$. In regards to the size of the initial data, we have furthermore*

$$\rho = \begin{cases} \mathcal{O}\left(T^{\frac{5}{4}}\right), \ \textit{in the case of the boundary conditions (2)} \\ \mathcal{O}\left(T^{\frac{13}{8}+\frac{\delta}{2}}\right), \ \textit{in the case of the boundary conditions (3).} \end{cases} \tag{5}$$

This present work is a continuation of that in [5], which deals with the null controllability of von Kármán thermoelastic plates in the relatively simple case–at least from a PDE estimate point of view–that the plate variables satisfy the canonical *hinged* mechanical boundary conditions. On the other hand, the analysis of the nonlinear model (1)–be it in the clamped case (2) or free case (3)–draws on, and builds upon, the techniques which were employed in [3] and [4] to analyze the rate of blowup for the minimal energy function which corresponds to null controllability. The heart of the matter in these papers was the derivation of associated *sharp* observability estimates for the associated homogenous adjoint problem (see (10) below). In [3] and [4], critical use is made of delicate trace estimates for solutions of the linearization of (1), estimates which do not come about by the standard Sobolev Trace Theorem; rather, these estimates are a direct consequence of the underlying analyticity of the thermoelastic plate in the absence of rotation inertia (see; e.g., Lemma 9 of [3], or Lemma 5 of [4]). With the sharp observability inequalities in hand, [3] and [4] subsequently provides an algorithmic argument so as to compute the precise rate of singularity for the minimal norm null controller.

In the present work, we use the sharp observability estimates–relevant for the purely linear null controllability problem–to obtain the local null controllability of the thermoelastic plate under the influence of the well-known (and nonLipschitz) *von Kármán* nonlinearity (see [22]). In fact, the estimates previously derived in [3] and [4] for the "energy" $\mathcal{E}(T)$ of the adjoint system (6) will be used in this paper to help generate the observability estimate which is associated with the *affine* thermoelastic problem (6) (i.e., the thermoelastic system (1) with nonlinear term $[\mathcal{F}(\omega), \omega]$ replaced by forcing term $f(t)$). In addition to the said reverse estimates for $\mathcal{E}(T)$ (posted below in (14) for the clamped case, and in (15) for the free), another necessary ingredient for the proof of the null controllability of the affine problem is the appropriate use of the underlying analyticity for the associated thermoelastic C_0-semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$. In particular, we will have need to use the classic regularity result (but quite important in the present context),

$$e^{\mathcal{A}(\cdot)} \in \mathcal{L}\left(\mathbf{H}, L^2(0,T; D(\mathcal{A}^{\frac{1}{2}})\right)$$

(since the generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ is analytic, its fractional powers are well-defined). It is the use of this result which allows for a relatively short proof of null controllability for the affine problem (given of course, that we already have in hand the fundamental observability estimates for $\mathcal{E}(T)$ from [3] and [4]).

Having ascertained the null controllability of the affine problem, we subsequently employ a fixed point argument to obtain a null controller for fixed initial data $[\omega_0, \omega_1, \theta_0\ ]$ which has the size prescribed in (5). Our precise specification in Theorem 1 on the size of radius ρ is ultimately the result of our keeping close track of the observability constant C_T which corresponds with null controllability of the affine thermoelastic plate. An optimization argument will yield that the affine problem's minimal norm null controller has its L^2-measurement to be of order C_T; in the end, it is this fact which will dictate the choice of radius ρ in our Contraction Mapping Principle argument.

Another important component, in our work to show the null controllability of the von Kármán thermoelastic plate, is our usage of the relatively recent result in [10] that the mapping $\phi \to [\mathcal{F}(\phi), \phi]$ is locally Lipschitz from $H^2(\Omega)$ into $L^2(\Omega)$. This subtle topological improvement over previously known boundedness results (see e.g., [8], [16]) is a *sine qua non* for our present work: since our fixed point mapping argument critically hinges on our having the null controllability of the affine thermoelastic plate, with initial data $[\omega_0, \omega_1, \theta_0]$ in $\mathbf{H}$ and (more to the point here) forcing term $f(t) \in L^2(Q)$, we must have $[\mathcal{F}(\phi), \phi] \in L^2(\Omega)$ for $\phi \in H^2(\Omega)$. (The previously known $[\mathcal{F}(\phi), \phi] \in H^{-\epsilon}(\Omega)$, for $\phi \in H^2(\Omega)$, will not do.)

In regard to the issue of deriving sharp observability inequalities for null controllability of thermoelastic plates, which is the really the arch theme of the present paper: we should mention the work of R. Triggiani in [28], which derives the results in [3] and [4] (but by an altogether different, spectral, methodology) for the canonical case of *hinged* mechanical boundary conditions. In addition, there are the papers [7] and [12] which deal with the null controllability of the thermoelastic variables by means of locally distributed control; in this case of locally distributed control, the observability constants will be necessarily obey an *exponential* rate of blowup, vis-à-vis the *rational* rates of blowup which are seen in [3], [4] and in the present paper (see Lemma 4 below).

2 The Affine Problem

By way of proving the Theorem 1, we will consider the 2-D thermoelastic plate equation, under the influence of initial data $[\omega_0, \omega_1, \theta_0]$, forcing term $f(t)$ and control function $u(t)$:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \omega_{tt} + \Delta^2\omega + \alpha\Delta\theta = f(t) \\ \theta_t - \Delta\theta - \alpha\Delta\omega_t = u \end{array} \right. \text{ on } (0,T)\times\Omega \\ \\ \omega(t=0) = \omega_0;\ \omega_t(t=0) = \omega_1;\ \theta(t=0) = \theta_0 \ \text{ on } \Omega. \end{array} \right. \tag{6}$$

The variables $[\omega, \theta]$ for this *affine* problem satisfy either the (clamped) boundary conditions (2) or the free boundary conditions (3). Given then data $\{[\omega_0, \omega_1, \theta_0], f\} \in \mathbf{H} \times L^2(Q)$, the problem is as follows: find a thermal control $u \in L^2(Q)$ such that the corresponding solution to (6), (2) (resp., (6), (3)) satisfies $[\omega(T), \omega_t(T), \theta(T)] = [0,0,0]$. In other words, we wish to establish that the affine system (6) is *null controllable* within the class of L^2-thermal controls.

To this end, let $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ denote the spatial realization which models the linear thermoelastic dynamics of (1). Under the clamped boundary conditions (2) $\mathcal{A}$ is given explicitly in (21) below; in the free case (3) it admits the representation (32). By the Lumer-Phillips Theorem, one can straightforwardly show the existence of a contraction semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ on $\mathbf{H}$, under either set of mechanical boundary conditions. Accordingly, the solution of (6), for any data $\{[\omega_0, \omega_1, \vartheta_0], f\} \in \mathbf{H} \times L^2(Q)$, may be written as

$$\begin{bmatrix} \omega(t) \\ \omega_t(t) \\ \theta(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix} + \int_0^t e^{\mathcal{A}(t-\tau)} \begin{bmatrix} 0 \\ f(\tau) \\ 0 \end{bmatrix} d\tau. \tag{7}$$

With this representation in mind, we define $\mathcal{L}_T \in \mathcal{L}(L^2(Q), \mathbf{H})$ by

$$\mathcal{L}_T u = \int_0^T e^{\mathcal{A}(T-\tau)} \begin{bmatrix} 0 \\ 0 \\ u(\tau) \end{bmatrix} d\tau \text{ for all } u \in L^2(Q). \tag{8}$$

With the affine term in (6) in mind, let us also define $\mathcal{N}_T \in \mathcal{L}(\mathbf{H} \times L^2(Q), \mathbf{H})$ by having

$$\mathcal{N}_T\left([\omega_0, \omega_1, \theta_0], f\right) = e^{\mathcal{A}T} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix} + \int_0^T e^{\mathcal{A}(T-\tau)} \begin{bmatrix} 0 \\ f(\tau) \\ 0 \end{bmatrix} d\tau \text{ for all } f \in L^2(Q). \tag{9}$$

In regard to these operators, thermal null controllability of the equation (6),(2) (resp., (6),(3)), within the class of controls $u \in L^2(Q)$, will be established if we show the following:

Lemma 2 *For all $T > 0$ (and under either mechanical boundary conditions (2) or (3)), we have $Range(\mathcal{N}_T) \subset Range(\mathcal{L}_T)$.*

With Lemma 2 in hand, one can readily combine this null controllability statement with the Closed Graph Theorem so as to have the following:

Proposition 3 *Let $\mathcal{L}_0$ denote the restriction of $\mathcal{L}_T$ to $[Null(\mathcal{L}_T)]^\perp$; that is, $\mathcal{L}_0 = \mathcal{L}_T|_{[Null(\mathcal{L}_T)]^\perp}$. Then under the validity of Lemma 2, one has that $(\mathcal{L}_0)^{-1}\mathcal{N}_T \in \mathcal{L}(\mathbf{H} \times L^2(Q), L^2(Q))$ (where $(\mathcal{L}_0)^{-1}$ denotes the algebraic inverse from $Range(\mathcal{L}_T)$ into $[Null(\mathcal{L}_T)]^\perp$). Moreover, $\mathcal{L}_T(\mathcal{L}_0)^{-1}\mathcal{N}_T = \mathcal{N}_T$.*

3 The Proof of Lemma 2

Consider the homogeneous problem, in variables $[\phi(t), \phi_t(t), \vartheta(t)]$, which is dual with respect to (6):

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \phi_{tt} + \Delta^2\phi + \alpha\Delta\vartheta = 0 \;\text{ on } (0,T)\times\Omega \\ \vartheta_t - \Delta\vartheta - \alpha\Delta\phi_t = 0 \;\text{ on } (0,T)\times\Omega \end{array} \right. \\ \\ [\phi(0), -\phi_t(0), \vartheta(0)] = [\phi_0, \phi_1, \vartheta_0] \in \mathbf{H}. \end{array} \right. \tag{10}$$

For this adjoint, problem we have the following homogeneous boundary conditions:

$$\left\{ \begin{array}{l} \phi = \dfrac{\partial\phi}{\partial\nu} = \vartheta = 0 \text{ on } (0,T)\times\Gamma, \;\text{ if boundary conditions (2) are in play;} \\ \\ \left\{ \begin{array}{l} \left\{ \begin{array}{l} \Delta\phi + (1-\mu)B_1\phi + \alpha\vartheta = 0 \\ \dfrac{\partial\Delta\phi}{\partial\nu} + (1-\mu)\dfrac{\partial B_2\phi}{\partial\tau} - \phi + \alpha\dfrac{\partial\vartheta}{\partial\nu} = 0 \end{array} \right. \text{ on } (0,T)\times\Gamma \\ \\ \dfrac{\partial\vartheta}{\partial\nu} + \lambda\vartheta = 0 \;\text{ on } (0,T)\times\Gamma, \quad \text{ where } \lambda > 0, \end{array} \right. \text{ if (3) are active.} \end{array} \right. \tag{11}$$

For initial data $[\phi_0, \phi_1, \vartheta_0] \in \mathbf{H}$, one has by semigroup theory that

$$\begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \vartheta(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ -\phi_1 \\ \vartheta_0 \end{bmatrix} \in C([0,T];\mathbf{H}).$$

Now, to show the containment $Range(\mathcal{N}_T) \subset Range(\mathcal{L}_T)$ in either case of clamped or free mechanical boundary conditions, where $\mathcal{N}_T$ (resp. $\mathcal{L}_T$) is given by (9) (resp. (8)), it suffices to show the following observability inequality (see e.g., Theorem 2.6 of [29]):

$$\left\| \mathcal{N}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix} \right\|_{\mathbf{H}\times L^2(Q)} \le C_T \left\| \mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix} \right\|_{L^2(Q)}. \tag{12}$$

In terms of the variables for the adjoint PDE (10), this abstract inequality becomes

$$\mathcal{E}(T) + \int_0^T \mathcal{E}(t)dt \le C_T^2 \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt, \tag{13}$$

where the "energy" of the homogeneous system (10), for all $0 \le t \le T$, is given by

$$\mathcal{E}(t) = \left\{ \begin{array}{l} \frac{1}{2}\left(\|\phi(t)\|^2_{H_0^2(\Omega)} + \|\phi_t(t)\|^2_{L^2(\Omega)} + \|\vartheta(t)\|^2_{L^2(\Omega)} \right), \;\text{ in the clamped case (2);} \\ \\ \frac{1}{2}\left(\|\phi(t)\|^2_{H^2(\Omega)} + \|\phi_t(t)\|^2_{L^2(\Omega)} + \|\vartheta(t)\|^2_{L^2(\Omega)} \right), \text{ in the free case (3).} \end{array} \right.$$

This necessary inequality, be it for the clamped or free case, will essentially follow from the principal estimate for the terminal energy $\mathcal{E}(T)$. For the clamped case (2) this was established in [3]:

$$\mathcal{E}(T) \le \frac{C}{T^5} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt, \tag{14}$$

where positive constant C above is independent of time $T>0$.

On the other hand, for the free case (3), the following estimate for the energy at terminal time $t=T$ was established in [4]:

$$\mathcal{E}(T) \leq \frac{C}{T^{\frac{13}{2}+2\delta}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt, \text{ for arbitrary } \delta > 0, \tag{15}$$

where again the positive constant C is independent of time T.

In this connection, our intent is to show that the integral term in (13) retains the observability constant posted in (14) (for the clamped case) or in (15) (for the free case).

Lemma 4 *The energy of the adjoint homogeneous system (11) obeys the estimate (13), with*

$$\mathcal{C}_T = \begin{cases} \mathcal{O}\left(T^{-\frac{5}{2}}\right), & \text{if the boundary conditions (2) are in play} \\ \mathcal{O}\left(T^{-\frac{13}{4}-\delta}\right), & \text{if the boundary conditions (3) are in play.} \end{cases}$$

The Proof of Theorem 4 will be undertaken in the next two subsections.

3.1 The Derivation of the Estimate (13) for the Clamped Case

Here, the solution $[\phi, \phi_t, \vartheta]$ of the adjoint problem (10) satisfies the clamped boundary conditions posted in (11).

Step 1: (A preliminary estimate for lower order terms): Let $\mathring{\mathbf{A}}$: $D(\mathring{\mathbf{A}}) \subset L^2(\Omega) \to L^2(\Omega)$ denote the biharmonic under homogeneous clamped boundary conditions. That is,

$$\mathring{\mathbf{A}} f = \Delta^2 f; \;\; D(\mathring{\mathbf{A}}) = H^4(\Omega) \cap H_0^2(\Omega). \tag{16}$$

Moreover, let $A_D : D(A_D) : L^2(\Omega) \to L^2(\Omega)$ denote the Laplace operator under homogeneous Dirichlet boundary conditions; i.e.,

$$A_D f = -\Delta f, \; D(A_D) = H^2(\Omega) \cap H_0^1(\Omega) \tag{17}$$

As defined, each of these operators, $\mathring{\mathbf{A}}$ and A_D are positive definite, and self-adjoint.

With these definitons in mind, we multiply the plate equation in (10) by the multiplier $\mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t$; we multiply the heat equation by $A_D^{-1}\vartheta$ (by the spectral calculus for self-adjoint operators, $\mathring{\mathbf{A}}^s$ (resp. A_D^s) is well-defined for all real s). Integrating both resulting relations in time and space, we arrive, for any $0 \leq t < T$.

$$\begin{aligned} &\left\|\mathring{\mathbf{A}}^{\frac{1}{4}}\phi(t)\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{-\frac{1}{4}}\phi_t(t)\right\|^2_{L^2(\Omega)} + \left\|A_D^{-\frac{1}{2}}\vartheta(t)\right\|^2_{L^2(\Omega)} \\ = \;& \left\|\mathring{\mathbf{A}}^{\frac{1}{4}}\phi(T)\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{-\frac{1}{4}}\phi_t(T)\right\|^2_{L^2(\Omega)} + \left\|A_D^{-\frac{1}{2}}\vartheta(T)\right\|^2_{L^2(\Omega)} + \int_t^T \left(\vartheta, \alpha\left(\Delta\mathring{\mathbf{A}}^{-\frac{1}{2}}+1\right)\phi_t + \vartheta\right)_\Omega d\tau \end{aligned} \tag{18}$$

Using the observability estimate (14), we then obtain the following pointwise estimate (in norms below that of finite energy):

$$\begin{aligned} &\|\phi(t)\|^2_{H_0^1(\Omega)} + \|\phi_t(t)\|^2_{H^{-1}(\Omega)} + \|\vartheta(t)\|^2_{H^{-1}(\Omega)} \\ \leq \;& \frac{C}{T^5}\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, d\tau + \epsilon\int_0^T \mathcal{E}(\tau)d\tau, \text{ for all } 0 \leq t < T \end{aligned} \tag{19}$$

(note that in obtaining this estimate, we are using the fact that $\Delta\mathring{\mathbf{A}}^{-\frac{1}{2}} \in \mathcal{L}(L^2(\Omega))$. We are also using the characterization of the fractional powers of $\mathring{\mathbf{A}}$, A_D in [13]).

Step 2 In this step, we use the key fact that the thermoelastic generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$–in the present clamped case, and indeed for all boundary conditions–gives rise to an *analytic* semigroup of contractions (see [25] and [21]). As such, and as $\mathcal{A}$ is moreover similar to a normal operator (see [21]), we then have the following regularity for its semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$:

$$e^{\mathcal{A}(\cdot)} \in \mathcal{L}(\mathbf{H}, L^2(0,T; D(\mathcal{A}^{\frac{1}{2}})), \tag{20}$$

with a norm bound which is independent of T; see Lemma 4.1 and Remark 4.1 of [18]. (Note also that as $\mathcal{A}$ is analytic, its fractional powers are well-defined).

Moreover, the generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, in the present clamped case, can be written explicitly as

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\mathring{\mathbf{A}} & 0 & \alpha A_D \\ 0 & -\alpha A_D & -A_D \end{bmatrix}. \tag{21}$$

From this expression, we can in turn write out the inverse $\mathcal{A}^{-1} \in \mathcal{L}\left(\mathbf{H}, D(\mathcal{A})\right)$ as

$$\mathcal{A}^{-1} = \begin{bmatrix} -\alpha^2\mathring{\mathbf{A}}^{-1}A_D & -\mathring{\mathbf{A}}^{-1} & -\alpha\mathring{\mathbf{A}}^{-1} \\ I & 0 & 0 \\ -\alpha I & 0 & -A_D^{-1} \end{bmatrix}.$$

Given then this representation for $\mathcal{A}^{-1}$ and the definition of the elliptic operators in (16)-(17) whose inverses make up the components of this matrix, we have that $\mathcal{A}^{-1} \in \mathcal{L}\left(L^2(\Omega) \times [D(\mathring{\mathbf{A}}^{\frac{1}{2}})]' \times [D(A_D)]', \mathbf{H}\right)$. Interpolation between this inverse and the identity $\mathcal{A}^0 \in \mathcal{L}\left(\mathbf{H}\right)$ gives then that

$$\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}\left(H_0^1(\Omega) \times [D(\mathring{\mathbf{A}}^{\frac{1}{4}})]' \times [D(A_D^{\frac{1}{2}})]', \mathbf{H}\right). \tag{22}$$

Therewith, we can now estimate the term $\int_0^T \mathcal{E}(t)dt$: Let $\vec{x}_0 \equiv [\phi_0, -\phi_1, \vartheta_0] \in \mathbf{H}$ (i.e., the initial data of the problem (10)). Then with (20) and (22), we have

$$\begin{aligned} \int_0^T \mathcal{E}(t)dt &= \int_0^T \left\|\mathcal{A}^{\frac{1}{2}}e^{\mathcal{A}t}\mathcal{A}^{-\frac{1}{2}}\vec{x}_0\right\|_{\mathbf{H}}^2 dt \\ &\leq C\left\|\mathcal{A}^{-\frac{1}{2}}\vec{x}_0\right\|_{\mathbf{H}}^2 \\ &\leq C\left(\left\|\mathring{\mathbf{A}}^{\frac{1}{4}}\phi_0\right\|_{L^2(\Omega)}^2 + \left\|\mathring{\mathbf{A}}^{-\frac{1}{4}}\phi_1\right\|_{L^2(\Omega)}^2 + \left\|A_D^{-\frac{1}{2}}\vartheta_0\right\|_{L^2(\Omega)}^2\right). \end{aligned}$$

Invoking the pointwise estimate (19)–which is valid for the lower topology $H_0^1(\Omega) \times [D(\mathring{\mathbf{A}}^{\frac{1}{4}})]' \times [D(A_D^{\frac{1}{2}})]'$–gives now

$$\int_0^T \mathcal{E}(t)dt \leq \frac{C}{T^5}\int_0^T \|\vartheta\|_{L^2(\Omega)}^2 d\tau + \epsilon\int_0^T \mathcal{E}(\tau)d\tau,$$

or,

$$(1-\epsilon)\int_0^T \mathcal{E}(t)dt \leq \frac{C}{T^5}\int_0^T \|\vartheta\|_{L^2(\Omega)}^2 d\tau. \tag{23}$$

Combining (14) and (23) gives now the sharp observability estimate (13), which is requisite to null controllability. The observability constant in (12) (as it is in [3] for dual of the purely linear problem) is thus,

$$\mathcal{C}_T = \mathcal{O}\left(T^{-\frac{5}{2}}\right) \text{ in the clamped case.} \tag{24}$$

This completes the proof of Lemma 4 (and so too of Lemma 2) for the clamped case (2).

3.2 The Derivation of the Estimate (13) for the Free Case

Here, we have the solution $[\phi, \phi_t, \vartheta]$ of the adjoint problem (10) satisfying the free mechanical/Robin thermal boundary conditions posted in (11). In this case, the relatively simple argument availed of in Section 3.1 will not directly apply.

By way of justifying this last assertion, let us define the following elliptic operators:

- We set the linear operator (cf. (16) $\mathring{\mathbf{A}} : D(\mathring{\mathbf{A}}) \subset L^2(\Omega) \to L^2(\Omega)$ to be $\mathring{\mathbf{A}}\varpi = \Delta^2\varpi$, for $\varpi \in D(\mathring{\mathbf{A}})$, where

  $$D(\mathring{\mathbf{A}}) = \left\{\varpi \in H^4(\Omega) : [\Delta\varpi + (1-\mu)B_1\varpi]_\Gamma = 0 \quad \text{and} \quad \left[\frac{\partial\Delta\varpi}{\partial\nu} + (1-\mu)\frac{\partial B_2\varpi}{\partial\tau} - \varpi\right]_\Gamma = 0\right\},$$

  where the boundary operators B_i are as defined in (4).

  This operator is densely defined, positive definite and self-adjoint. Consequently by [13], one has the characterization

  $$D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \approx H^2(\Omega); \text{ with moreover } \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} = a(\phi,\phi) + \int_\Gamma \phi^2 d\Gamma, \tag{25}$$

  where the bilinear form $a(\cdot,\cdot)$ on $H^2(\Omega)$ is given by

  $$\begin{aligned} &a(w,\tilde{w}) \\ \equiv\ &\int_\Omega [w_{xx}\tilde{w}_{xx} + w_{yy}\tilde{w}_{yy} + \mu(w_{xx}\tilde{w}_{yy} + w_{yy}\tilde{w}_{xx}) + 2(1-\mu)w_{xy}\tilde{w}_{xy}]\, d\Omega. \end{aligned} \tag{26}$$

  This characterization involving the bilinear form $a(\cdot,\cdot)$ comes from the "Green's formula" in [17]: Namely, for functions $w, \hat{w}$ smooth enough, there is the relation

  $$\int_\Omega \left(\Delta^2 w\right)\hat{w}d\Omega = a(w,\hat{w}) + \int_\Gamma \left(\frac{\partial\Delta w}{\partial\nu} + (1-\mu)\frac{\partial B_2 w}{\partial\tau}\right)\hat{w}d\Gamma - \int_\Gamma (\Delta w + (1-\mu)B_1 w)\frac{\partial\hat{w}}{\partial\nu}d\Gamma \tag{27}$$

  (here as usual, $\tau = [-\nu_2, \nu_1]$ denotes the unit tangent vector).

- Moreover, we define the elliptic operators G_i by

  $$G_1 h = v \Leftrightarrow \begin{cases} \Delta^2 v = 0 \quad \text{on } \Omega \\ \left\{\begin{array}{l} \Delta v + (1-\mu)B_1 v = h \\ \dfrac{\partial\Delta v}{\partial\nu} + (1-\mu)\dfrac{\partial B_2 v}{\partial\tau} - v = 0 \end{array}\right. \quad \text{on } \Gamma \end{cases};$$

  $$G_2 h = v \Leftrightarrow \begin{cases} \Delta^2 v = 0 \quad \text{on } \Omega \\ \left\{\begin{array}{l} \Delta v + (1-\mu)B_1 v = 0 \\ \dfrac{\partial\Delta v}{\partial\nu} + (1-\mu)\dfrac{\partial B_2 v}{\partial\tau} - v = h \end{array}\right. \quad \text{on } \Gamma \end{cases}. \tag{28}$$

By elliptic regularity-see e.g., [23]-one has that for all real s,

$$G_1 \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{5}{2}}(\Omega)); \quad G_2 \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{7}{2}}(\Omega)). \tag{29}$$

A fortiori then, one has $\mathring{\mathbf{A}} G_i \in \mathcal{L}\left(L^2(\Gamma), [D(\mathring{\mathbf{A}}^{\frac{1}{2}})]'\right)$, with the adjoints $G_i^* \mathring{\mathbf{A}} \in \mathcal{L}\left(D(\mathring{\mathbf{A}}^{\frac{1}{2}}), L^2(\Gamma)\right)$ being readily computed–by means of the formula (27)–as

$$G_1^* \mathring{\mathbf{A}} f = \frac{\partial f}{\partial \nu} \quad \text{and} \quad G_2^* \mathring{\mathbf{A}} f = - f|_\Gamma \tag{30}$$

- In addition, we set the operator $A_R : D(A_R) \subset L^2(\Omega) \to L^2(\Omega)$ to be the elliptic operator defined by

$$A_R f = -\Delta f; \quad D(A_R) = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu} + \lambda f = 0, \ \lambda > 0 \right\}. \tag{31}$$

With these quantities and with $\gamma_0 \in \mathcal{L}\left(H^1(\Omega), H^{\frac{1}{2}}(\Gamma)\right)$ denoting (as usual) the Sobolov trace map–i.e., $\gamma_0(f) = f|_\Gamma$ for $f \in C^\infty(\bar{\Omega})$–then the generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ may be written in the free case as

$$\begin{pmatrix} 0 & \mathbf{I} & 0 \\ -\mathring{\mathbf{A}} & 0 & \alpha\left(A_R - \mathring{\mathbf{A}} G_1 \gamma_0 + \lambda \mathring{\mathbf{A}} G_2 \gamma_0\right) \\ 0 & \alpha\Delta & -A_R \end{pmatrix};$$

$$D(\mathcal{A}) \quad = \quad \left\{ [\omega_0, \omega_1, \theta_0] \in H^2(\Omega) \times H^2(\Omega) \times D(A_R) : \mathring{\mathbf{A}} \left[\omega_0 + \alpha \left(G_1 \gamma_0 - \lambda G_2 \gamma_0\right) \theta_0\right] \in L^2(\Omega) \right\} \tag{32}$$

Subsequently, one write out explicitly the inverse $\mathcal{A}^{-1} \in \mathcal{L}(\mathbf{H})$ as

$$\mathcal{A}^{-1} = \begin{pmatrix} \alpha^2 \mathring{\mathbf{A}}^{-1}\left(A_R - \mathring{\mathbf{A}} G_1 \gamma_0 + \lambda \mathring{\mathbf{A}} G_2 \gamma_0\right) A_R^{-1} \Delta & -\mathring{\mathbf{A}}^{-1} & -\alpha \mathring{\mathbf{A}}^{-1}\left(A_R - \mathring{\mathbf{A}} G_1 \gamma_0 + \lambda \mathring{\mathbf{A}} G_2 \gamma_0\right) A_R^{-1} \\ I & 0 & 0 \\ \alpha A_R^{-1} \Delta & 0 & -A_R^{-1} \end{pmatrix}. \tag{33}$$

From this representation, we see that $\mathcal{A}^{-1}$ will *not* map continuously $L^2(\Omega) \times [D(\mathring{\mathbf{A}}^{\frac{1}{2}})]' \times [D(A_R)]'$ into $\mathbf{H}$ (in particular, $A_R^{-1}\Delta$, as a mapping on $L^2(\Omega)$ is not well-defined). Thus, the interpolation argument employed for the clamped case in Section (3.1) is not directly applicable.

However, we can modify the argument, as follows:

Step 1 (a priori estimates). To begin, we derive a pointwise estimate for the solution, which is attainable given the estimate (23) which we have in hand. This estimate is essentially analogous to that in (19) for the clamped case; however it accounts for the incompatibility between the plate and heat dynamics. Multiplying the beam equation in (10) by $\mathring{\mathbf{A}}^{-\frac{1}{2}} \phi_t$ and the heat equation by $A_R^{-1} \vartheta$, and subsequently integrating in time and space, we have the relation, for all $0 \le t \le T$

$$\begin{aligned} & \left[\left\| \mathring{\mathbf{A}}^{-\frac{1}{4}} \phi_t(s) \right\|^2_{L^2(\Omega)} + \left\| \mathring{\mathbf{A}}^{\frac{1}{4}} \phi(s) \right\|^2_{L^2(\Omega)} + \left\| A_R^{-\frac{1}{2}} \vartheta(s) \right\|^2_{L^2(\Omega)} \right]_{s=t}^{s=T} \\ = \quad & \alpha \int_t^T \left[\left\langle \mathring{\mathbf{A}} \left(\lambda G_2 - G_1\right) \gamma_0 \vartheta, \mathring{\mathbf{A}}^{-\frac{1}{2}} \phi_t \right\rangle_{[D(\mathring{\mathbf{A}}^{\frac{1}{2}})]' \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})} - \left(\Delta \vartheta, \mathring{\mathbf{A}}^{-\frac{1}{2}} \phi_t\right)_{L^2(\Omega)} + \left(\Delta \phi_t, A_R^{-1} \vartheta\right)_{L^2(\Omega)} \right] dt \\ & - \int_t^T \|\vartheta\|^2_{L^2(\Omega)} \, dt. \end{aligned} \tag{34}$$

With regard to the right hand side: Using the representation for the adjoint $\mathring{\mathbf{A}}G_2$ in (30), as well as two applications of Green's Theorem, we have for all

$$
\begin{aligned}
&\lambda \int_t^T \left\langle \mathring{\mathbf{A}}G_2\gamma_0\vartheta, \mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t \right\rangle_{[D(\mathring{\mathbf{A}}^{\frac{1}{2}})]'\times D(\mathring{\mathbf{A}}^{\frac{1}{2}})} = -\lambda \int_t^T \left(\gamma_0\vartheta, \mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t\Big|_\Gamma\right)_{L^2(\Gamma)} dt \\
= \;& \int_t^T \left(\frac{\partial\vartheta}{\partial\nu}, \mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t\Big|_\Gamma\right)_{L^2(\Gamma)} dt \\
= \;& \int_t^T \left(\gamma_0\vartheta, \frac{\partial}{\partial\nu}\mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t\right)_{L^2(\Gamma)} dt + \int_t^T \left(\Delta\vartheta, \mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t\right)_{L^2(\Omega)} dt - \int_t^T \left(\vartheta, \Delta\mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t\right)_{L^2(\Omega)} dt
\end{aligned} \tag{35}
$$

Conbining this relation with (34) and the representation of $\mathring{\mathbf{A}}G_1$ in (30) gives now the f0llowing relation for all $0 \le t \le T$:

$$
\begin{aligned}
&\frac{1}{2}\left[\left\|\mathring{\mathbf{A}}^{-\frac{1}{4}}\phi_t(s)\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{4}}\phi(s)\right\|^2_{L^2(\Omega)} + \left\|A_R^{-\frac{1}{2}}\vartheta(s)\right\|^2_{L^2(\Omega)}\right]_{s=t}^{s=T} \\
= \;& \alpha\int_t^T \left(\Delta\phi_t, A_R^{-1}\vartheta\right)_{L^2(\Omega)} dt - \alpha\int_t^T \left(\vartheta, \Delta\mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t\right)_{L^2(\Omega)} dt - \int_t^T \|\vartheta\|^2_{L^2(\Omega)}\, dt.
\end{aligned} \tag{36}
$$

Note that because of the incompatibility between the domains $D(A_R)$ and $D(\mathring{\mathbf{A}})$, one is left with high order terms on the right hand side of this relation (cf. the right hand of (18) for the clamped case). To deal with this first term on the right hand side, we integrate by parts again, and reinvoke the heat equation in (10):

$$
\begin{aligned}
&\int_t^T \left(\Delta\phi_t, A_R^{-1}\vartheta\right)_{L^2(\Omega)} dt = \left[\left(\Delta\phi(s), A_R^{-1}\vartheta(s)\right)_{L^2(\Omega)}\right]_{s=t}^{s=T} - \int_t^T \left(\Delta\phi, A_R^{-1}\vartheta_t\right)_{L^2(\Omega)} dt \\
= \;& \left[\left(\Delta\phi(s), A_R^{-1}\vartheta(s)\right)_{L^2(\Omega)}\right]_{s=t}^{s=T} - \int_t^T \alpha\left(\Delta\phi, A_R^{-1}\Delta\phi_t\right)_{L^2(\Omega)} dt + \int_t^T \left(\Delta\phi, \vartheta\right)_{L^2(\Omega)} dt \\
= \;& \left[\left(A_R^{-\frac{1}{2}}\Delta\phi(s), A_R^{-\frac{1}{2}}\vartheta(s)\right)_{L^2(\Omega)} - \frac{\alpha}{2}\left\|A_R^{-\frac{1}{2}}\Delta\phi(s)\right\|^2\right]_{s=t}^{s=T} + \int_t^T \left(\Delta\phi, \vartheta\right)_{L^2(\Omega)} dt.
\end{aligned} \tag{37}
$$

After now applying (37) to (36) and rearranging terms, we obtain the following relation for all $0 \le t \le T$:

$$
\begin{aligned}
&\frac{1}{2}\left[\left\|\mathring{\mathbf{A}}^{-\frac{1}{4}}\phi_t(t)\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{4}}\phi(t)\right\|^2_{L^2(\Omega)} + \left\|A_R^{-\frac{1}{2}}\vartheta(t) - \alpha A_R^{-\frac{1}{2}}\Delta\phi(t)\right\|^2_{L^2(\Omega)}\right] \\
= \;& \frac{1}{2}\left[\left\|\mathring{\mathbf{A}}^{-\frac{1}{4}}\phi_t(T)\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{4}}\phi(T)\right\|^2_{L^2(\Omega)} + \left\|A_R^{-\frac{1}{2}}\vartheta(T) - \alpha A_R^{-\frac{1}{2}}\Delta\phi(T)\right\|^2_{L^2(\Omega)}\right] \\
&+\alpha\int_t^T \left(\Delta\mathring{\mathbf{A}}^{-\frac{1}{2}}\phi_t - \Delta\phi, \vartheta\right)_{L^2(\Omega)} dt + \int_t^T \|\vartheta\|^2_{L^2(\Omega)}\, dt.
\end{aligned}
$$

Estimating this by means of the observability inequality (15), we have then the following *a priori* bounds, for all $0 \le t \le T$:

$$
\left\|\mathring{\mathbf{A}}^{-\frac{1}{4}}\phi_t(t)\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{4}}\phi(t)\right\|^2_{L^2(\Omega)} + \left\|A_R^{-\frac{1}{2}}\vartheta(t) - \alpha A_R^{-\frac{1}{2}}\Delta\phi(t)\right\|^2_{L^2(\Omega)} \le \frac{C}{T^{\frac{13}{2}+2\delta}}\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon\int_0^T \mathcal{E}(t)dt, \tag{38}
$$

where $\delta, \epsilon > 0$ are arbitarily small.

Step 2. Define the operator $\mathcal{P} \in \mathcal{L}(H^2(\Omega), \mathbf{H})$ by

$$
\mathcal{P}\phi_0 = \begin{bmatrix} \phi_0 \\ 0 \\ \alpha\Delta\phi_0 \end{bmatrix}. \tag{39}
$$

Note that $\mathcal{A}^{-1}\mathcal{P} \in \mathcal{L}(L^2(\Omega), \mathbf{H})$. In fact, if $\phi_0 \in D(\mathring{\mathbf{A}}^{\frac{1}{2}})$, then using the representation in (33) we have

$$\mathcal{A}^{-1}\mathcal{P}\phi_0 = \begin{bmatrix} 0 \\ \phi_0 \\ 0 \end{bmatrix}.$$

Extension by continuity gives then the asserted boundedness, with $\mathcal{A}^{-1}\mathcal{P}\phi_0 = [0, \phi_0, 0]$ for all $\phi_0 \in L^2(\Omega)$. Interpolation will subsequently yield the boundedess

$$\mathcal{A}^{-\frac{1}{2}}\mathcal{P} \in \mathcal{L}(H^1(\Omega), \mathbf{H}). \tag{40}$$

Moreover, we define the injection Π^* by setting

$$\Pi^* \begin{bmatrix} \phi_1 \\ \vartheta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix}.$$

As such, then for $[\phi_1, \vartheta_0] \in [D(\mathring{\mathbf{A}}^{\frac{1}{2}})]' \times [D(A_R)]'$, we have from (33) that

$$\mathcal{A}^{-1}\Pi^* \begin{bmatrix} \phi_1 \\ \vartheta_0 \end{bmatrix} = \begin{bmatrix} -\mathring{\mathbf{A}}^{-1}\phi_1 - \alpha\mathring{\mathbf{A}}^{-1}\left(A_R - \mathring{\mathbf{A}}G_1\gamma_0 + \lambda\mathring{\mathbf{A}}G_2\gamma_0\right)A_R^{-1}\vartheta_0 \\ 0 \\ -A_R^{-1}\vartheta_0 \end{bmatrix} \in \mathbf{H}$$

(note that implicitly we are using the fact that $\left(A_R - \mathring{\mathbf{A}}G_1\gamma_0 + \lambda\mathring{\mathbf{A}}G_2\gamma_0\right) \in \mathcal{L}(L^2(\Omega), [D(\mathring{\mathbf{A}}^{\frac{1}{2}})]')$; see e.g., Proposition 4.1, p. 372 of [2]). In short, $\mathcal{A}^{-1}\Pi^* \in \mathcal{L}([D(\mathring{\mathbf{A}}^{\frac{1}{2}})]' \times [D(A_R)]', \mathbf{H})$. Since also $\Pi^* \in \mathcal{L}(L^2(\Omega) \times L^2(\Omega), \mathbf{H})$, interpolation then gives

$$\mathcal{A}^{-\frac{1}{2}}\Pi^* \in \mathcal{L}([D(\mathring{\mathbf{A}}^{\frac{1}{4}})]' \times [D(A_R^{\frac{1}{2}})]', \mathbf{H}). \tag{41}$$

Step 3. The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, which models the present free mechanical boundary conditions, generates an *analytic* semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ and is moreover similar to a normal operator (see [21]). Consequently, we have the regularity

$$e^{\mathcal{A}(\cdot)} \in \mathcal{L}(\mathbf{H}, L^2(0, T; D(\mathcal{A}^{\frac{1}{2}})) \tag{42}$$

(see Lemma 4.1 and Remsrk 4.1 of [18]). With this boundedness in mind, as well as those in (40) and (41), we can now estimate the energy integral in (13):

$$\begin{aligned} \int_0^T \mathcal{E}(t)dt = \int_0^T \left\| e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 dt \\ = \int_0^T \left\| e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \alpha\Delta\phi_0 + \vartheta_0 - \alpha\Delta\phi_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 dt \\ \leq 2\int_0^T \left\| e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ 0 \\ \alpha\Delta\phi_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 dt + 2\int_0^T \left\| e^{\mathcal{A}t} \begin{bmatrix} 0 \\ \phi_1 \\ \vartheta_0 - \alpha\Delta\phi_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 dt. \end{aligned}$$

Applying now the analyticity result in (42), followed by a consideration of (40) and (41), we obtain

$$
\begin{aligned}
&\int_0^T \mathcal{E}(t)dt \\
\leq \;& 2\int_0^T \left\| \mathcal{A}^{\frac{1}{2}} e^{\mathcal{A}t} \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} \phi_0 \\ 0 \\ \alpha\Delta\phi_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 dt + 2\int_0^T \left\| \mathcal{A}^{\frac{1}{2}} e^{\mathcal{A}t} \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \phi_1 \\ \vartheta_0 - \alpha\Delta\phi_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 dt \\
= \;& 2\int_0^T \left\| \mathcal{A}^{\frac{1}{2}} e^{\mathcal{A}t} \mathcal{A}^{-\frac{1}{2}} \mathcal{P}\phi_0 \right\|_{\mathbf{H}}^2 dt + 2\int_0^T \left\| \mathcal{A}^{\frac{1}{2}} e^{\mathcal{A}t} \mathcal{A}^{-\frac{1}{2}} \Pi^* \begin{bmatrix} \phi_1 \\ \vartheta_0 - \alpha\Delta\phi_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 dt \\
\leq \;& C\left(\left\| \mathcal{A}^{-\frac{1}{2}} \mathcal{P}\phi_0 \right\|_{\mathbf{H}}^2 + \left\| \mathcal{A}^{-\frac{1}{2}} \Pi^* \begin{bmatrix} \phi_1 \\ \vartheta_0 - \alpha\Delta\phi_0 \end{bmatrix} \right\|_{\mathbf{H}}^2 \right) \\
\leq \;& C\left(\|\phi_0\|_{H^1(\Omega)}^2 + \|\phi_1\|_{[D(\mathring{\mathbf{A}}^{\frac{1}{4}})]'}^2 + \|\vartheta_0 - \alpha\Delta\phi_0\|_{[D(A_R^{\frac{1}{2}})]'}^2 \right).
\end{aligned}
$$

Invoking finally the *a priori* estimate in (38), we obtain

$$
(1-\epsilon)\int_0^T \mathcal{E}(t)dt \leq \frac{C}{T^{\frac{13}{2}+2\delta}} \int_0^T \|\vartheta\|_{L^2(\Omega)}^2 dt.
$$

Combining this estimate with (15) gives now the inequality (13), with observability constant in (12) having the following order, for arbitrary $\delta > 0$,

$$
\mathcal{C}_T = \mathcal{O}(T^{-\frac{13}{4}-\delta}), \text{ in the free case.} \tag{43}
$$

This completes the proof of Lemma 4 (and hence of Lemma 2) for the free case.

4 Characterization of the Optimal Control

Having established the null controllability of the affine linear thermoelastic plate (6), one can subsequently consider the following minimization problem for given $\{[\omega_0, \omega_1\theta_0], f\} \in \mathbf{H} \times L^2(Q)$:

Problem 5

$$
\begin{aligned}
&\textit{Minimize} \;\; \tfrac{1}{2}\textstyle\int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt \\
&\textit{Subject to} \;\; \mathcal{L}_T u + \mathcal{N}_T([\vec{x}_0, f]) = 0,
\end{aligned} \tag{44}
$$

where initial data $\vec{x}_0 = [\omega_0, \omega_1, \theta_0] \in \mathbf{H}$. The operator theoretic quantities $\{\mathcal{L}_T, \mathcal{N}_T\}$ are again as given in (8) and (9). Because of the Lemma 2, this problem is well-defined. In fact, from the classical convex optimization (see e.g., [6]), there exists a unique control $u^* \in [Null(\mathcal{L}_T)]^\perp$ which solves (44). We now proceed to qualitatively describe this minimizer.

Lemma 6 *Let $u^* = u^*(\vec{x}_0, f) \in L^2(Q)$ be the solution of the optimization problem (44). Then the following hold true:*

(i) With $\mathcal{L}_0 = \mathcal{L}_T|_{[Null(\mathcal{L}_T)]^\perp}$,

$$
u^* = -\mathcal{L}_0^{-1}\mathcal{N}_T(\vec{x}_0, f);
$$

(ii)

$$
\|u^*(\vec{x}_0, f)\|_{L^2(Q)} \leq \mathcal{C}_T \|(\vec{x}_0, f)\|_{\mathbf{H}\times L^2(Q)}, \tag{45}
$$

where the positive constant $\mathcal{C}_T$ is the observability constant from Lemma 4;

(iii)

$$\left\| (\mathcal{L}_0)^{-1} \mathcal{N}_T \right\|_{\mathcal{L}(\mathbf{H}\times L^2(Q), L^2(Q))} \leq \mathcal{C}_T, \tag{46}$$

where again $\mathcal{C}_T$ is the observability constant from Lemma 4.

Proof of Lemma 6: Since $\{e^{\mathcal{A}^*t}\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ is an *analytic* semigroup for either mechanical boundary conditions (2) or (3) (see [21]), then the "backwards uniqueness" property obtains for these dynamics; that is to say,

$$e^{\mathcal{A}^*T}\vec{x}_0 = \vec{0} \Longrightarrow e^{\mathcal{A}^*t}\vec{x}_0 = \vec{0}, \text{ for } 0 \leq t < T. \tag{47}$$

In short, $e^{\mathcal{A}^*T}$ is injective. Consequently, the pre-Hilbert space $\mathbf{W}_0 \equiv Range(e^{\mathcal{A}^*T})$, with inner product

$$\left(e^{\mathcal{A}^*T}\vec{x}, e^{\mathcal{A}^*T}\vec{y}\right)_{\mathbf{W}_0}, \text{for } \vec{x} \text{ and } \vec{y} \in \mathbf{H},$$

has a completion $\mathbf{W} \subset \mathbf{H}$ (see e.g., Proposition 1, p. 132 of [1]). Since $Null(e^{\mathcal{A}T})$ is empty, we then have the continuous and dense inclusions

$$\mathbf{W} \subset \mathbf{H} \subset \mathbf{W}'. \tag{48}$$

From these dense inclusions and the inequality (12) (equivalent to Lemma 2), we infer that $\mathcal{L}_T\mathcal{L}_T^* \in \mathcal{L}(\mathbf{W}, \mathbf{W}')$ is $\mathbf{W}$-elliptic: in fact,

$$\left\langle \mathcal{L}_T\mathcal{L}_T^*\vec{\phi}, \vec{\phi} \right\rangle_{\mathbf{W}'\times\mathbf{W}} = \left(\mathcal{L}_T\mathcal{L}_T^*\vec{\phi}, \vec{\phi}\right)_{\mathbf{H}} \geq \frac{1}{\mathcal{C}_T} \left\| e^{\mathcal{A}^*T}\vec{\phi} \right\|_{\mathbf{H}}^2 = \frac{1}{\mathcal{C}_T} \left\| \vec{\phi} \right\|_{\mathbf{W}}^2 \text{ for every } \vec{\phi} \in Range(e^{\mathcal{A}^*T}).$$

(In obtaining this inequality, we are using the fact that $Range(e^{\mathcal{A}T}) \subset Range(\mathcal{N}_T)$; alternatively, we could appeal directly to the observability inequalities obtained in the earlier [3] and [4]). The Lax-Milgram Theorem thus gives that

$$\lambda^* = -\left(\mathcal{L}_T\mathcal{L}_T^*\right)^{-1} \mathcal{N}_T(\vec{x}_0, f) \in \mathbf{W}. \tag{49}$$

Comparing this relation with the constraint satisfied by the minimizer $u^*(\vec{x}_0, f)$ of (44), we have that $\lambda^* \in \mathbf{H}$ and the minimizer $u^* \in [Null(\mathcal{L}_T)]^\perp$ satisfy the following *optimality conditions*:

$$u^* + \mathcal{L}_T^*\lambda^* \in Null\left(\mathcal{L}_T\right); \mathcal{L}_T u^* = -\mathcal{N}_T(\vec{x}_0, f). \tag{50}$$

Applying the inverse of $\mathcal{L}_0 = \mathcal{L}_T|_{Null[\mathcal{L}_T]^\perp}$ to both sides of the second relation of (50) now gives (i). To show (ii): With the multiplier λ^* and the optimality conditions (50), we have for given data $(\vec{x}_0, f) \in L^2(Q) \times \mathbf{H}$,

$$\begin{aligned}
\left(\left(\vec{x}_0, f\right), \mathcal{N}_T^*\lambda^*\right)_{\mathbf{H}\times L^2(Q)} &= \left(\mathcal{N}_T\left(\vec{x}_0, f\right), \lambda^*\right)_{\mathbf{H}} \\
&= \left(-\mathcal{L}_T u^*, \lambda^*\right)_{\mathbf{H}} \\
&= \left(\mathcal{L}_T\mathcal{L}_T^*\lambda^*, \lambda^*\right)_{\mathbf{H}} = \left\|\mathcal{L}_T^*\lambda^*\right\|_{L^2(Q)}^2 \geq \frac{1}{\mathcal{C}_T^2} \left\|\mathcal{N}_T^*\lambda^*\right\|_{\mathbf{H}\times L^2(Q)}^2,
\end{aligned}$$

where in the last inequality we have used the observability estimate (12), with the value of $\mathcal{C}_T$ as prescribed in Lemma 4. From this we have then the estimate

$$\left\|\mathcal{N}_T^*\lambda^*\right\|_{\mathbf{H}\times L^2(Q)} \leq \mathcal{C}_T^2 \left\|(\vec{x}_0, f)\right\|_{\mathbf{H}\times L^2(Q)}. \tag{51}$$

Combining this estimate with (50), we have further

$$\begin{aligned}
\left\| u^* \left(\vec{x}_0, f \right) \right\|^2_{L^2(Q)} &= \left(u^*, -\mathcal{L}_T^* \lambda^* \right)_{L^2(Q)} \\
&= \left(-\mathcal{L}_T u^*, \lambda^* \right)_{\mathbf{H}} \\
&= \left(\mathcal{N}_T \left(\vec{x}_0, f \right), \lambda^* \right)_{\mathbf{H}} \\
&= \left(\left(\vec{x}_0, f \right), \mathcal{N}_T^* \lambda^* \right)_{\mathbf{H} \times L^2(Q)} \\
&\leq \left\| \left(\vec{x}_0, f \right) \right\|_{\mathbf{H} \times L^2(Q)} \left\| \mathcal{N}_T^* \lambda^* \right\|_{\mathbf{H} \times L^2(Q)} \leq \mathcal{C}_T^2 \left\| \left(\vec{x}_0, f \right) \right\|^2_{\mathbf{H} \times L^2(Q)},
\end{aligned} \tag{52}$$

where in deriving the last inequality, we have used (51). This gives the estimate (45). In turn, the estimate (46) comes from combining the second relation in (50) with that in (52) (So this optimization argument provides a estimate on the size of $\left\| \left(\mathcal{L}_0 \right)^{-1} \mathcal{N}_T \right\|_{\mathcal{L}(\mathbf{H} \times L^2(Q), L^2(Q))}$, which is not provided by the Proposition 3). This concludes the proof of Lemma 6.

5 Proof of Lemma 1

To begin, we define the (Green's) map G by

$$Gf = g \Leftrightarrow \begin{cases} \Delta^2 g = f & \text{in } \Omega \\ g = 0, \ \frac{\partial}{\partial \nu} g = 0 & \text{on } \Gamma. \end{cases}$$

(So in particular, $\mathcal{F}(w) = -G\left[w, w\right]$.) In regard to G, a rather important fact in the present context is the following:

Theorem 7 *(See Remark 0.2 of [11]; also [10]). The mapping*

$$\{w, v, z\} \rightarrow \left[w, G\left[z, v\right]\right]$$

is bounded from $H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$ into $L^2(\Omega)$.

Because of this boundedness of the von Kármán nonlinearity and the exact null controllability for the affine problem (6), the proof of Theorem 1 will be a ready consequence of the Contraction Mapping Principle. In fact, we can define the map $\mathcal{T} : C([0,T]; \mathbf{H}) \rightarrow C([0,T]; \mathbf{H})$, by having for all $\left[\phi, \tilde{\phi}, \vartheta\right] \in C([0,T]; \mathbf{H})$,

$$\begin{aligned}
\mathcal{T}\left(\begin{bmatrix} \phi \\ \tilde{\phi} \\ \vartheta \end{bmatrix} \right) &= e^{\mathcal{A}(\cdot)} \vec{x}_0 + \int_0^{(\cdot)} e^{\mathcal{A}(\cdot - s)} \begin{bmatrix} 0 \\ \left[\mathcal{F}(\phi), \phi\right] \\ 0 \end{bmatrix} ds \\
&- \int_0^{(\cdot)} e^{\mathcal{A}(\cdot - s)} \begin{bmatrix} 0 \\ 0 \\ \left(\mathcal{L}_0 \right)^{-1} \mathcal{N}_T \left(\vec{x}_0, \left[\mathcal{F}(\phi), \phi\right] \right) \end{bmatrix} ds,
\end{aligned} \tag{53}$$

where $\vec{x}_0 = \left[\omega_0, \omega_1, \theta_0\right] \in \mathbf{H}$. By a Corollary to the Contraction Mapping Principle (see e.g., p. 117 of [14]), there will be a unique fixed point of $\mathcal{T}$ in $\overline{B(0;r)} \subset C([0,T]; \mathbf{H})$ if we can find a radius r so that: (i) $\mathcal{T}$ is a contraction on $\overline{B(0;r)}$ with Lipschitz constant q; (ii) we have the inequality

$$r \geq (1-q)^{-1} \left\| e^{\mathcal{A}(\cdot)} \vec{x}_0 \right\|_{C([0,T]; \mathbf{H})}. \tag{54}$$

In fact, for $\vec{f} = [f_1, f_2, f_3]$ and $\vec{g} = [g_1, g_2, g_3]$ in $\overline{B(0;r)}$, we have, by the contraction of the semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$, that for $0 \leq t \leq T$, $T < 1$ (say),

$$\begin{aligned}
&\left\| \mathcal{T}\left(\vec{f}\right)(t) - \mathcal{T}\left(\vec{g}\right)(t) \right\|_{\mathbf{H}} \\
= &\left\| \int_0^t e^{\mathcal{A}(t-s)} \begin{bmatrix} 0 \\ [\mathcal{F}(f_2), f_2] - [\mathcal{F}(g_2), g_2] \\ (\mathcal{L}_0)^{-1} \mathcal{N}_T\left(0, [\mathcal{F}(f_2), f_2] - [\mathcal{F}(g_2), g_2]\right) \end{bmatrix} ds \right\|_{\mathbf{H}} \\
\leq &\int_0^T \left(\left\| [\mathcal{F}(f_2), f_2] - [\mathcal{F}(g_2), g_2] \right\|_{L^2(\Omega)} + \left\| (\mathcal{L}_0)^{-1} \mathcal{N}_T\left(0, [\mathcal{F}(f_2), f_2] - [\mathcal{F}(g_2), g_2]\right) \right\|_{L^2(\Omega)} \right) dt.
\end{aligned}$$

Using the estimate on the norm $(\mathcal{L}_0)^{-1} \mathcal{N}_T$ given in Lemma 6(iii), and the locally Lipschitz continuity implied by Theorem 7, we subsequently obtain

$$\begin{aligned}
&\left\| \mathcal{T}\left(\vec{f}\right) - \mathcal{T}\left(\vec{g}\right) \right\|_{L^\infty(0,T;\mathbf{H})} \\
\leq\ &\mathcal{C}_T\sqrt{T} \left\| [\mathcal{F}(f_2), f_2] - [\mathcal{F}(g_2), g_2] \right\|_{L^2(Q)} \\
\leq\ &\mathcal{C}_T\sqrt{T} \left(\left\| [\mathcal{F}(f_2 - g_2), f_2] \right\|_{L^2(Q)} + \left\| [\mathcal{F}(g_2), f_2 - g_2] \right\|_{L^2(Q)} \right) \\
\leq\ &\mathcal{C}_T\sqrt{T} \left(\int_0^T \left(\|f_2 - g_2\|^2_{H^2(\Omega)} \|f_2\|_{H^2(\Omega)} + \|g_2\|^2_{H^2(\Omega)} \|f_2 - g_2\|_{H^2(\Omega)} \right)^2 dt \right)^{\frac{1}{2}} \\
\leq\ &\mathcal{C}_T T r^2 \|f_2 - g_2\|_{C([0,T;H^2(\Omega))} \leq r^2 \mathcal{C}_T T \left\| \vec{f} - \vec{g} \right\|_{C([0,T];\mathbf{H})}
\end{aligned}$$

(here, the positive constant $\mathcal{C}_T$ has the same order prescribed in Lemma 4). For $T < 1$, if we take

$$r = \frac{1}{\sqrt{C_T}}, \tag{55}$$

then $\mathcal{T} : \overline{B(0;r)} \to \overline{B(0;r)}$ is a contraction with Lipschitz constant $q = T$. In addition, if we take initial data

$$\|\vec{x}_0\|_{\mathbf{H}} < \rho \equiv r(1 - T), \tag{56}$$

where r is as in (55), then for all $0 \leq t \leq T$,

$$(1 - T)^{-1} \left\| e^{\mathcal{A}t} \vec{x}_0 \right\|_{\mathbf{H}} \leq (1 - T)^{-1} \|\vec{x}_0\|_{\mathbf{H}} < r.$$

Thus by the Contraction Mapping Principle, for $\|\vec{x}_0\|_{\mathbf{H}} < \rho(T)$ as given in (56), there exists a fixed point $\left[\phi, \widetilde{\phi}, \vartheta\right] \in \overline{B(0;r)}$ to the mapping $\mathcal{T}$. The solution to (1) may subsequently be given by

$$\begin{aligned}
\omega &= \phi;\ \omega_t = \widetilde{\phi};\ \theta = \vartheta; \\
u &= (\mathcal{L}_0)^{-1} \mathcal{N}_T\left([\omega_0, \omega_1, \theta_0], [\mathcal{F}(\omega), \omega]\right).
\end{aligned}$$

Moreover, by construction this control u will steer the dynamics to zero. This completes the proof of Theorem 1.

References

[1] J. P. Aubin, "Analyse Fonctionnelle Appliquée, Tome 2, Presses Universitaires de France (1979).

[2] G. Avalos and I. Lasiecka, *Boundary controllability of thermoelastic plates via the free boundary conditions*, SIAM J. Control Optim., Vol. 38, No. 2 (2000), pp. 337-383.

[3] G. Avalos and I. Lasiecka, *The null controllability of thermoelastic plates and singularity of the associated minimal energy function*, Journal of Mathematical Analysis and its Applications, 294 (2004), pp. 34-61.

[4] G. Avalos and I. Lasiecka, *Asymptotic Rates of Blowup for the Minimal Energy Function for the Null Controllability of Thermoelastic Plates: The Free Case*, to appear in the Proceedings of the Conference for the Control of Partial Differential Equations, Georgetown University, Marcel Dekker, in press.

[5] G. Avalos, *Concerning the exact null controllability of a nonlinear thermoelastic plate*, to appear in a special issue of Nonlinear Analysis which contains the Proceeedings of the Fourth World Congress of Nonlinear Analysts (WCNA 2004), Orlando, Florida.

[6] J. Cea, "Lectures in Optimization", Tata Institute of Fundamental Research.

[7] A. Benabdallah and M. G. Naso, *Null controllability of a thermoelastic plate*, Abstr. Appl. Anal. **7**, (2002), pp. 1024-1065.

[8] I. D. Chueshov, *Strong solutions and the attractor of the von Kármán equations*, Math. USSR Sbornik, 69 (1991), pp. 25-36.

[9] G. Duvaut and J. L. Lions, "Les Inéquations en Méchanique et en Physique", Dunod, Paris (1972).

[10] A. Favini, M. A. Horn, I. Lasiecka and D. Tataru, **ADDENDUM** to the paper, *Global existence, uniqueness and regularity of solution to a von Karman system with nonlinear boundary dissipation*, Differential and Integral Equations, 10 (January 1997), pp. 197-200.

[11] A. Favini, M. A. Horn, I. Lasiecka and D. Tataru, *Global existence, uniqueness and regularity of solution to a von Karman system with nonlinear boundary dissipation*, Differential and Integral Equations, 9 (1996), pp. 267-294.

[12] P. Gervasio and M. G. Naso, *Numerical approximation of controllability of trajectories for Euler-Bernoulli thermoelastic plates*, Mathematical Models and Methods in Applied Sciences, Vol. 14, No. 5 (2004), pp. 701-733.

[13] P. Grisvard, *Caracterization de quelques espaces d'interpolation*, Arch. Rational Mech. Anal. **25** (1967), pp. 40-63.

[14] V. Hutson and J. S. Pym, "Applications of Functional Analysis and Operator Theory", Academic Press, New York (1980).

[15] T. von Kármán, *Festigkeitsprobleme in Maschinenbau*, Encyklopedie der Mathematischen Wissenschaften, Vol. 4 (1910), pp. 314-385.

[16] H. Koch and A. Stahel, *Global existence of classical solutions to the dynamical von Kármán equations*, Mathematical Methods in the Applied Sciences, 161 (1993), pp. 581-586.

[17] J. E. Lagnese, "Boundary Stabilization of Thin Plates", SIAM Stud. Appl. Math. 10, SIAM, Philadelphia (1989).

[18] I. Lasiecka, *Unified theory for abstract parabolic boundary value problems*, *A semigroup approach*, Appl. Math. Optim. 6 (1980), pp. 287-233.

[19] I. Lasiecka and R. Triggiani, *Exact controllability of semilinear abstract systems*, Applied Mathematics and Optimization, Vol. 23 (1991), pp. 109-154.

[20] I. Lasiecka and R. Triggiani, *Analyticity of thermo-elastic semigroups with coupled BC. Part II: The case of free BC,* Annali Scuola Normale di Pisa, Classes Scienze (Serie IV, Fasicolo 3-4), Vol. XXVII (1998(c)), pp. 457-497.

[21] I. Lasiecka and R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories. I: Abstract Parabolic Systems, Cambridge University Press, New York (2000).

[22] J. L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéares", Dunod, Paris (1969).

[23] J. L. Lions and E. Magenes, "Non-Homogeneous Boundary Value Problems and Applications", Vol. 1, Springer-Verlag, New York (1972).

[24] D. G. Luenberger, "Optimization by Vector Space Methods", John Wiley & Sons, Inc. (1969).

[25] Z. Liu and M. Renardy, *A note on the equations of a thermoelastic plate*, Appl. Math. Lett., Vol. **8**, No. 3 (1995), pp. 1-6.

[26] A. Pazy, "Semigroups of Linear Operators", Springer-Verlag, New York (1983).

[27] J. T. Schwartz, "Nonlinear functional analysis", Gordon and Breach Publishers, New York (1969).

[28] R. Triggiani, "Optimal estimates of norms of fast controls in exact null controllability of two non-classical abstract parabolic systems", Advances in Differential Equations, Vol. 8, No. 2 (February 2003), pp. 189-229.

[29] J. Zabcyk, "Mathematical Control Theory: An Introduction", Birkhäuser Boston (1992).