TRIVARIATE SPLINE APPROXIMATION OF DIVERGENCE-FREE VECTOR FIELDS

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IMA Preprint Series # 1992

(September 2004)

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Trivariate Spline Approximation of Divergence-Free Vector Fields

Gerard Awanou [1)] and Ming-Jun Lai [2)]

Abstract. We discuss the approximation properties of divergence-free vector fields by using trivariate spline vectors which are also divergence-free. We pay special attention to the approximation constants and show that they depend only on the smallest solid angle in the underlying tetrahedral partition and the nature of the boundary of the domain. The estimates are given in the max-norm and L^p norm.

§1. Introduction

In this paper we show how trivariate spline vectors which are divergence-free can approximate any given divergence-free vector field. More precisely, we give approximation properties of these spline spaces and track the approximation constants. Throughout the paper, we will assume that Ω is a bounded, simply-connected domain of $\mathbb{R}^3$ with a boundary of class $C^{m,1}, m \geq 0$.

Let $\mathbf{f} = (f_1, f_2, f_3)$ be a vector with components $f_i(x, y, z)$ differentiable on Ω, $i = 1, 2, 3$. Recall that $\mathbf{f}$ is a divergence-free vector if

$$\operatorname{div}(\mathbf{f}) = \frac{\partial}{\partial x} f_1(x, y, z) + \frac{\partial}{\partial y} f_2(x, y, z) + \frac{\partial}{\partial z} f_3(x, y, z) \equiv 0, \quad \text{for all } (x, y, z) \in \Omega.$$

Let $\widetilde{\Omega}$ be a polygonal domain approximating Ω with $\widetilde{\Omega} \subseteq \Omega$ and let $\triangle$ be a tetrahedral partition of $\widetilde{\Omega}$. $\widetilde{\Omega}$ may be constructed by picking points on the boundary $\partial\Omega$ of Ω. For T in $\triangle$, T may intersect the exterior of Ω in which case when considering such a T, we mean $T \cap \Omega$. We consider the trivariate spline space of degree d and smoothness r

$$S_d^r(\triangle) := \{s \in C^r(\Omega), s|_T \in \mathbb{P}_d, \forall T \in \triangle\},$$

where $\mathbb{P}_d$ denotes the space of all polynomials of total degree d and T is a tetrahedron in $\triangle$. We are interested in using splines in $S_d^r(\triangle)$ to approximate $\mathbf{f}$. The

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problem is: Given a divergence-free vector $\mathbf{f}$, find a divergence-free spline vector $\mathbf{s} = (s_1, s_2, s_3) \in (S_d^r(\triangle))^3$ which approximates $\mathbf{f}$ reasonably well in the sense that

$$\|\mathbf{f} - \mathbf{s}\|_2 \leq O(|\triangle|^m)$$

for some $m \geq 1$, where $|\triangle|$ denotes the largest diameter of the tetrahedra in $\triangle$, and $\|\mathbf{f}\|_p$ is the Euclidean norm of the vector $\mathbf{f}$ defined by

$$\|\mathbf{f}\|_p = \left(\sum_{i=1}^{3} \int_{\Omega} |f_i(x,y,z)|^p dx\ dy\ dz \right)^{1/p}$$

for $1 \leq p < \infty$ and

$$\|\mathbf{f}\|_\infty = \max_{i=1,2,3} |f_i|_\infty.$$

Our study is motivated by our recent numerical investigation of the Navier-Stokes equations in [Awanou and Lai'02]. There we presented a trivariate spline method to numerically solve the Navier-Stokes equations and concentrated on numerical techniques to solve the discrete problem. Numerical results validated the approach. But there are also many other partial differential equations which solutions are divergence-free, e.g., the magnetohydrodynamics equations and the Maxwell equations.

It turns out that our problem is closely related to the standard spline approximation problem. Given a sufficiently smooth function f defined on Ω, find a continuously differentiable spline $s \in S_d^r(\triangle)$ with $r \geq 1$ such that s approximates f very well. The standard approximation problem requires the spline space to satisfy certain properties and the approximation constant is dependent on the geometry of the tetrahedral partition. For example, in the bivariate setting, the approximation constant is dependent on the smallest angle in the underlying triangulation or dependent on the shape of the triangulation, i.e. the ratio of the diameter and the radius of the inscribed circle in a triangle which is maximal (cf. [Lai and Schumaker'98].

In the 3D setting, our approximation constant will depend on the solid angle of a tetrahedron T at one of its vertices v which definition we now recall. Let B_v be the sphere at v with radius 1. Extending the edges of T starting at v if necessary, the three edges of T sharing the common vertex v intersect B_v at three points. The area of the spherical triangles with vertices at these three intersection points is called the solid angle $\ell(T)$ of T at v. We let $\ell(\triangle)$ be the smallest solid angle of the tetrahedra in $\triangle$. Similar to the bivariate setting, we need the concept of the shape regularity $\theta(\triangle)$ which is defined as follows.

$$\theta(\triangle) = \max_{T \in \triangle} \frac{h_T}{\rho_T},$$

where $h_T = \mathrm{diam}(T)$ and ρ_T is the radius of the inscribed sphere in T.

Before stating the main result of this paper, we need some notation. Let $W_p^m(\Omega)$ be the standard Sobolev space for $m \geq 1$ and $1 \leq p \leq \infty$ with usual norm $\|\ \|_{p,m,\Omega}$ and semi-norm $|\ |_{p,m,\Omega}$ ($|\ |_{m,\Omega} = |\ |_{2,m,\Omega}$). For each tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle \in \triangle$, let

$$\xi_{ijkl}^T = \frac{1}{d}(iv_1 + jv_2 + kv_3 + lv_4), \quad i + j + k + l = d$$

be the domain points of T and $\mathcal{D}_\triangle := \{\xi_{ijkl}^T, i + j + k + l = d, T \in \triangle\}$. For $(x, y, z) \in T$, let b_1, b_2, b_3, b_4 be the barycentric coordinates of (x, y, z) satisfying

$$\begin{aligned}(x, y, z) =& b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 \\ 1 =& b_1 + b_2 + b_3 + b_4.\end{aligned}$$

It is known that any polynomial $P \in \mathcal{P}_d$ can be written in terms of Bernstein polynomials, that is,

$$P = \sum_{i+j+k+l=d} c_{ijkl} B_{ijkl}^d$$

with $B_{ijkl}^d(x, y, z) = \dfrac{d!}{i!j!k!l!}(b_1)^i (b_2)^j (b_3)^k (b_4)^l$. Thus, any spline function s in $S_d^r(\triangle)$ can be written as follows:

$$s_{|T} = \sum_{i+j+k+l=d} c_{ijkl}^T B_{ijkl}^d, \quad \text{for all } T \in \triangle.$$

This is the so-called the B-form of multivariate splines (cf. [de Boor'87]). Since $s \in C^r(\Omega)$, the B-coefficients c_{ijkl}^T must satisfy some smoothness conditions. These conditions are linear relations among these B-coefficients (cf. [de Boor'87]).

Let S be a subspace of $S_d^r(\triangle)$ and let $\mathcal{M}$ be a subset of $\mathcal{D}_\triangle$. We say that $\mathcal{M}$ is a minimal determining set for S if when the B-coefficients c_{ijkl}^T of $s \in S$ are zero for all c_{ijkl}^T whose corresponding domain points $\xi_{ijkl}^T \in \mathcal{M}$, then $s \equiv 0$ and the number of these domain points in $\mathcal{M}$ is minimal. It is natural to associate a minimal determining set $\mathcal{M}$ to a cardinal spline basis $\{\phi_\xi, \xi \in \mathcal{M}\}$ where $\phi_\xi \in S$ is defined by 1) assign the B-coefficient of s_ξ corresponding to the domain point ξ to be 1; 2) assign the B-coefficients of ϕ_ξ corresponding to the domain points in $\mathcal{M}\backslash\xi$ to be 0; 3) use the smoothness conditions to determine the remaining B-coefficients of ϕ_ξ.

Finally let $\text{star}^1(v) = \text{star}(v)$ be the union of all tetrahedra sharing the common vertex v and inductively, let

$$\text{star}^\ell(v) = \{T \in \triangle, T \cap \text{star}^{\ell-1}(v) \neq \emptyset\}$$

for $\ell = 2, \ldots$ We are now ready to state the following result.

Theorem 1.1. *Let $\triangle$ be a tetrahedral partition of Ω and $S \subset S_d^r(\triangle)$ be a subspace which contains the polynomial space $\mathcal{P}_d$. Suppose that we can identify a minimal determining set $\mathcal{M}$ for S and the assocated cardinal basis $\{\phi_\xi, \xi \in \mathcal{M}\}$ satisfies the following property: there exists an integer $\ell > 0$ such that the support of ϕ_ξ is contained in star$^\ell(v_\xi)$ for a vertex $v_\xi \in \triangle$ for all $\xi \in \mathcal{M}$. Then there exists a quasi-linear operator Q which maps any $f \in L_1(\Omega)$ into S such that for any $f \in W_p^{m+1}(\Omega)$,*

$$\|f - Q(f)\|_{p,k,\Omega} \le C|\triangle|^{m+1-k}|f|_{p,m+1,\Omega},$$

for $0 \le k \le r+1$ and $r \le m \le d$, where C is a constant which is dependent on m, d, k and the smallest solid angle $\ell(\triangle)$.

With these results, we are able to prove our main result in the paper. To state the results, we need the following notation:

$$V^m = \{\mathbf{f} \in (H^m(\Omega))^3, \mathrm{div}\ \mathbf{f} = 0\}$$

and

$$\mathcal{S}_d^r = \{\mathbf{s} \in (S_d^r(\triangle))^3,\ \mathrm{div}\ \mathbf{s} = 0\}$$

be spaces of divergence-free vector fields, for $r \ge 1$.

Theorem 1.2. *Suppose that $\mathbf{f} \in V^d$. Fix $r \ge 1$. Then*

$$\inf_{\mathbf{s} \in \mathcal{S}_d^r} \|\mathbf{f} - \mathbf{s}||_{(H^1(\Omega))^3} \le |\triangle|^d||\mathbf{f}||_{(H^d(\Omega))^3}.$$

for an appropriate $d > r$ and a tetrahedral partition $\triangle$.

The paper is organized as follows. In §2, we prove the above approximation result, then in §3, we use the spline approximation method in [Lai and Schumaker'98] to prove Theorem1.1.

§2. The Approximation of Divergence-free Splines

We recall that $|\triangle|$ denote the length of the longest edge in $\triangle$. In this section, we show that divergence-free splines approximate well divergence-free vector fields, i.e. Theorem 1.2. We first recall the definition of the **curl** of a distribution $\mathbf{v} = (v_1, v_2, v_3) \in \mathcal{D}'(\Omega)^3$ which is defined by

$$\mathbf{curl}(\mathbf{v}) = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right).$$

Here, $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions with compact support in Ω and $\mathcal{D}'(\Omega)$ is the dual of $\mathcal{D}(\Omega)$. We also need the following result (cf. [Girault and Raviart'86, p.45]).

Lemma 2.1. *A vector field* $\mathbf{v} \in (L^2(\Omega))^3$ *satisfies div* $\mathbf{v}$=0 *if and only if there exists a vector potential* $\mathbf{V} \in (H^1(\Omega))^3$ *such that* $\mathbf{v} = \mathbf{curl}(\mathbf{V})$.

Proof of Theorem 1.2. Clearly, we have

$$||\mathbf{curl}\ \mathbf{V}||_{(L_2(\Omega))^3} \le ||\mathbf{v}||_{(H^1(\Omega))^3}. \tag{2.1}$$

For a given $\mathbf{f} \in V^d$, let $\mathbf{F} \in (H^{d+1}(\Omega))^3$ be a potential vector satisfying $\mathbf{curl}(\mathbf{F}) = \mathbf{f}$. For each component of $\mathbf{F} = (F_1, F_2, F_3)$, we use Theorem 1.1 to find a spline approximation $S_{F_i} = Q(F_i)$ in $S^{r+1}_{d+1}(\triangle)$ such that

$$|F_i - S_{F_i}|_{1,\Omega} \le C|\triangle|^d |F_i|_{d+1,\Omega},$$

for $i = 1, 2, 3$, where $|\ |_{1,\Omega} = |\ |_{2,1,\Omega}$

Next, we let $\mathbf{S_F} = (S_{F_1}, S_{F_2}, S_{F_3})$ and put $\mathbf{s_f} = \mathbf{curl}(\mathbf{S_f})$. Then $\mathbf{s_f} \in \mathcal{S}^r_d$ and

$$\begin{aligned} ||\mathbf{f} - \mathbf{s_f}||_{((L_2(\Omega))^3} &= ||\mathbf{curl}(\mathbf{F}) - \mathbf{curl}(\mathbf{S_f})||_{(L_2(\Omega))^3} \\ &\le ||\mathbf{F} - \mathbf{S_f}||_{(H^1(\Omega))^3} \\ &\le C|\triangle|^d \big(|F_1|^2_{d+1,\Omega} + |F_2|^2_{d+1,\Omega} + |F_3|^2_{d+1,\Omega}\big)^{1/2}. \end{aligned}$$

This completes the proof. □

§3. Trivariate Spline Approximation

In this section, we prove Theorem 1.1 but need first additional results. We let $\sigma(s)$ denote the support of s and we begin with the following result which is a straightforward generalization of Theorem 4.1 in [Lai and Schumaker'98] from the bivariate setting to the trivariate setting. We keep the same notations, specially the numerotation of the constants for easier reference.

Theorem 3.1. *Fix* $0 \le m \le d$. *Suppose* Γ *is some finite index set, and let* $\{\phi_\xi\}_{\xi\in\Gamma}$ *be a set of splines in* $\mathcal{S}^0_d(\triangle)$ *such that*

H1) *there exists an integer* ℓ *such that for each* ξ, *the support of* ϕ_ξ *is contained in* star$^\ell(v_\xi)$ *for some vertex* $v_\xi \in \triangle$;

H2) $K_8 := \max_\xi \|\phi_\xi\|_{\infty,\Omega} < \infty$;

H3) $K_9 := \max_T \#(\Sigma_T) < \infty$, *where* $\sigma(\phi_\xi)$ *denotes the support of* ϕ_ξ *and*

$$\Sigma_T := \{\xi :\ T \subset \sigma(\phi_\xi)\}. \tag{3.1}$$

Suppose in addition that there exists a set of linear functionals $\{\lambda_{\xi,m}\}_{\xi\in\Gamma}$ *defined on* $L_1(\Omega)$ *with the property that for all* $\xi \in \Gamma$, *there is a tetradedron* T_ξ *contained in the support of* ϕ_ξ *with*

$$|\lambda_{\xi,m} f| \le \frac{K_{10}}{A^{1/p}_{T_\xi}} \|f\|_{p,T_\xi} \quad \text{for all } f \in L_p(\Omega) \text{ when } 1 \le p < \infty \tag{3.2}$$

and

$$|\lambda_{\xi,m} f| \leq K_{10} \|f\|_{\infty, T_\xi} \quad \text{for all } f \in L_\infty(\Omega) \text{ when } p = \infty \tag{3.3}$$

for some constant K_{10}. Finally, suppose that the corresponding quasi-interpolation operator

$$Q_m f = \sum_{\xi \in \Gamma} (\lambda_{\xi,m} f) \phi_\xi \tag{3.4}$$

reproduces polynomials in the sense that

$$Q_m P = P \quad \text{for all } P \in \mathbb{P}_m. \tag{3.5}$$

Then there exists a constant C depending only on the smallest solid angle in the underlying tetrahedral partition, the nature of the boundary of the domain such that if $f \in W_p^{m+1}(\Omega)$, then

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - Q_m f)\|_{p,\Omega} \leq C |\triangle|^{m+1-\alpha-\beta-\gamma} |f|_{m+1,p,\Omega} \tag{3.6}$$

for all $0 \leq \alpha + \beta + \gamma \leq m$ and all $1 \leq p \leq \infty$.

We also have the following result

Lemma 3.2. *There exists a constant K_4 dependent only on d such that for any polynomial $P \in \mathbb{P}_d$,*

$$\frac{\|c\|_p}{K_4} \leq \frac{1}{A_T^{1/p}} \|P\|_{p,T} \leq \|c\|_p \tag{3.7}$$

for all $1 \leq p \leq \infty$. Here c is the vector of coefficients of P in lexicographical order, and

$$\|c\|_p = \left(\sum_{i+j+k=d} |c_{ijk}|^p \right)^{1/p}, \qquad 1 \leq p < \infty, \tag{3.8}$$

$$\|c\|_\infty = \max_{i+j+k=d} |c_{ijk}|, \qquad p = \infty.$$

Next we introduce the so-called averaged Taylor polynomials (cf. [Brenner and Scott'94, p. 91ff]). Let $B(x_0, y_0, z_0, \rho) = \{(x,y,z) \in \mathbb{R}^3 : ((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{1/2} < \rho\}$ be the disk centered about (x_0, y_0, z_0) with radius ρ. For simplicity, we write $B := B(x_0, y_0, z_0, \rho)$. Let

$$g_B(x,y,z) = \begin{cases} c \exp(-1/(1 - ((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)/\rho^2), & \text{if } (x,y,z) \in B(x_0,y_0,z_0,\rho) \\ 0, & \text{otherwise} \end{cases}$$

be a *mollifier* or *cut-off function* such that $\displaystyle\int_{\mathbb{R}^3} g_B(x,y,z)\,dx\,dy = 1$.

For any function $f \in C^m(\mathbb{R}^3)$, let

$$T_{m,(u,v,w)} f(x,y,z) = \sum_{0 \le \alpha+\beta+\gamma \le m} \frac{D_u^\alpha D_v^\beta D_w^\gamma f(u,v,w)}{\alpha!\beta!\gamma!} (x-u)^\alpha (y-v)^\beta (z-w)^\gamma$$

be the Taylor polynomial of degree m of f at (u,v,w). Then the *averaged Taylor polynomial of degree* m *over* $B(x_0,y_0,z_0,\rho)$ is defined as

$$F_{m,B} f(x,y,z) = \int_{B(x_0,y_0,z_0,\rho)} T_{m,(u,v,w)} f(x,y,z)\, g_B(u,v,w)\, du\, dv\, dw. \tag{3.9}$$

Integrating by parts, we have the equivalent formula

$$\begin{aligned}
&F_{m,B} f(x,y,z) \\
&= \sum_{\alpha+\beta+\gamma \le m} \frac{1}{\alpha!\beta!\gamma!} \int_{B(x_0,y_0,z_0,\rho)} D_u^\alpha D_v^\beta D_w^\gamma f(u,v,w) (x-u)^\alpha (y-v)^\beta (z-w)^\gamma g_B(u,v,w)\, du\, dv\, dw \\
&= \sum_{0 \le \alpha+\beta+\gamma \le m} \frac{(-1)^{\alpha+\beta+\gamma}}{\alpha!\beta!\gamma!} \int_{B(x_0,y_0,z_0,\rho)} f(u,v,w) \\
&\qquad \times D_u^\alpha D_v^\beta D_w^\gamma \left[(x-u)^\alpha (y-v)^\beta (z-w)^\gamma g_B(u,v,w) \right] du\, dv\, dw
\end{aligned}$$

which shows that the averaged Taylor polynomial is well-defined for any integrable function $f \in L_1(B(x_0,y_0,z_0,\rho))$. Clearly, $F_{m,B} f$ is a polynomial of degree $\le m$. It is also known (cf. [Brenner and Scott'94]) that

Lemma 3.3. *For any* $0 \le \alpha+\beta+\gamma \le m$ *and* $f \in W_1^{\alpha+\beta+\gamma}(B(x_0,y_0,z_0,\rho))$,

$$D_x^\alpha D_y^\beta D_z^\gamma F_{m,B} f = F_{m-\alpha-\beta,B}(D_x^\alpha D_y^\beta D_z^\gamma f).$$

Lemma 3.4. *For any polynomial* $f \in \mathbb{P}_m$, $f = F_{m,B} f$.

Given a tetradedron $T \in \triangle$, let $B_T := B(x_T, y_T, z_T, \rho_T) \subset T$ be the largest disk contained in T. We now estimate the norm of the operator F_{m,B_T}.

Lemma 3.5. *For any* $f \in L_p(T)$ *with* $1 \le p \le \infty$,

$$||F_{m,B_T} f||_{p,T} \le K_6 ||f||_{p,T}.$$

Here K_6 *is a constant dependent only on* θ_T.

Lemma 3.6. *Fix* $m \ge 0$ *and let* $U_{\mathcal{T}}$ *be a polygonal domain consisting of the union of a set* $\mathcal{T}$ *of tetradedrons lying in* $\mathrm{star}^\ell(v)$ *for some vertex* v. *Let* T *be an arbitrary tetradedron in* $\mathcal{T}$. *Then there exists a positive constant* K_7 *depending only on* m, ℓ, $\theta_{\mathcal{T}}$, *and the Lipschitz constant of* $\partial\Omega$ *such that for all* $f \in W_p^{m+1}(U_{\mathcal{T}})$,

$$\| D_x^\alpha D_y^\beta (f - F_{m,B_T} f) \|_{p,U_{\mathcal{T}}} \le K_7 |U_{\mathcal{T}}|^{m+1-\alpha-\beta} |f|_{m+1,p,U_{\mathcal{T}}}$$

for all $1 \le p \le \infty$.

Proof of Theorem 1.1. We are now in a position to apply Theorem 3.1 to establish Theorem 1.1. Let $\{\phi_\xi\}_{\xi\in\mathcal{M}}$ be the spline basis functions for S satisfying H1). That is, the support of ϕ_ξ is contained in $\mathrm{star}^\ell(v_\xi)$ for a vertex v_ξ inside $\sigma(\phi_\xi)$. By the assumptions, we know that H2) holds. In any case $K_9 \le \#(\Gamma) < \infty$ so H3) is satisfied.

We now define corresponding linear functionals and an associated quasi-interpolation operator. Choose $\xi \in \Gamma$, and suppose T_ξ is a tetradedron in which ξ lies. Let B_{T_ξ} be the largest sphere contained in T_ξ. Then for any function $f \in L_1(\Omega)$, we define

$$\lambda_{\xi,m} f := \gamma_\xi(F_{m,B_{T_\xi}} f),$$

where $F_{m,B_{T_\xi}} f$ is the averaged Taylor polynomial associated with f, and γ_ξ is the functional which when applied to a polynomial written in B-form, picks off the B-coefficient corresponding to the domain point ξ. Note that $\lambda_{\xi,m}$ is a linear functional, and the value of $\lambda_{\xi,m} f$ depends only on values of f on the tetradedron T_ξ.

Using Lemmas 3.2 and 3.5, we have

$$|\lambda_{\xi,m} f| = |\gamma_\xi(F_{m,B_{T_\xi}} f)| \le \frac{K_4}{A_{T_\xi}^{1/p}} \|F_{m,B_{T_\xi}} f\|_{p,T_\xi} \le \frac{K_4 K_6}{A_{T_\xi}^{1/p}} \|f\|_{p,T_\xi}.$$

This shows that condition (3.2) of Theorem 3.1 is satisfied.

We now show that Q_m reproduces polynomials of degree m. Given $f \in \mathbb{P}_m$, let $\sum_{\xi\in\Gamma} a_\xi \phi_\xi$ be its unique expansion in terms of ϕ_ξ. By Lemma 3.4, $F_{m,B_{T_\xi}} f = f$ for each $\xi \in \Gamma$. Thus, $\lambda_{\xi,m} f = \gamma_\xi F_{m,B_{T_\xi}} f = \gamma_\xi f = a_\xi$ for all $\xi \in \Gamma$, which implies that $Q_m f = f$.

We have now verified that Q satisfies all of the hypotheses of Theorem 3.1, and our main result Theorem 1.1 follows immediately.

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