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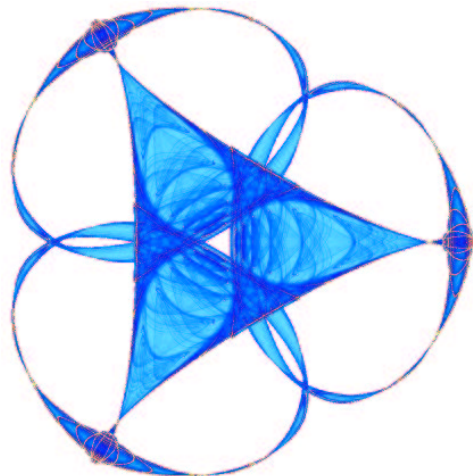
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IMA Preprint Series # 1989

(September 2004)



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ON A RESTRICTED WEAK LOWER SEMICONTINUITY FOR SMOOTH FUNCTIONALS ON SOBOLEV SPACES

DANIEL VASILIU AND BAISHENG YAN

ABSTRACT. We study a restricted weak lower semicontinuity property, which we call the *(PS)-weak lower semicontinuity*, for a smooth integral functional on the Sobolev space along all weakly convergent Palais-Smale sequences of the functional. By the Ekeland variational principle, the (PS)-weak lower semicontinuity is sufficient for the existence of minimizers under the usual coercivity assumption. In general, this condition is not equivalent to the usual (unrestricted) weak lower semicontinuity condition, but we show that, in certain cases, these two conditions are equivalent and both reduce to the usual convexity or quasiconvexity condition in the calculus of variations.

1. INTRODUCTION

In this paper, we study the integral functionals of the type

$$I(u) = \int_{\Omega} f(x, u(x), Du(x)) dx,$$

where Ω is a domain in \mathbf{R}^n , $u: \Omega \rightarrow \mathbf{R}^m$ is a vector field with Jacobi matrix $Du(x) = (\partial u^i / \partial x_j)$ and $f(x, s, \xi)$ is a given function of point $x \in \Omega$, vector $s \in \mathbf{R}^n$ and matrix $\xi \in M^{m \times n}$, the set of all $m \times n$ matrices. Such integral functionals have been extensively studied with important applications in partial differential equations, nonlinear elasticity and the materials science; see, e.g., [3, 4, 5, 9, 11] and the references therein.

Under some structure and growth conditions, one can define functional $I(u)$ on the Sobolev space $W^{1,p}(\Omega; \mathbf{R}^m)$ of mappings from Ω to \mathbf{R}^m and study the minimization problem of the functional I on a given Dirichlet class. Such a problem can be studied by the direct method of calculus of variations [5]. An important property often linked to the direct method of calculus of variations is the so-called *weak lower semicontinuity* of the

2000 *Mathematics Subject Classification.* 49J45, 49J20, 35G30.

Key words and phrases. (PS)-weak lower semicontinuity, Ekeland variational principle, Morrey's quasiconvexity.

This paper has been completed during the second author's stay at the Institute for Mathematics and its Applications (IMA), University of Minnesota. He is grateful for the IMA's hospitality and partial support with funds provided by the National Science Foundation.

functional I . Recall that I is (sequentially) weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbf{R}^m)$ if

$$(1.1) \quad I(u) \leq \liminf_{k \rightarrow \infty} I(u_k) \quad \text{whenever } u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbf{R}^m),$$

where $u_k \rightharpoonup u$ means u_k weakly converges to u .

It has been well-known that, under certain mild conditions on f , the weak lower semicontinuity of integral functional I defined above is equivalent to the important *quasiconvexity* condition introduced by Morrey; see, e.g., [1, 3, 5, 10, 11]. Recall that $f(x, s, \xi)$ is *quasiconvex* in ξ in the Morrey sense provided that the inequality

$$f(x, s, \xi) \leq \frac{1}{|\Omega|} \int_{\Omega} f(x, s, \xi + D\varphi(y)) dy$$

holds for a.e. $x \in \Omega$, all $s \in \mathbf{R}^n$, $\xi \in M^{m \times n}$ and all $\varphi \in C_0^\infty(\Omega; \mathbf{R}^m)$.

We assume the functional I defined above is C^1 on $W^{1,p}(\Omega; \mathbf{R}^m)$. This requires that f be C^1 in (s, ξ) and satisfy certain growth conditions. As in many application problems, I is often also bounded below. When minimizing bounded-below C^1 functionals over a Banach space, an important variational principle discovered by Ekeland [6] (see also [2]) can provide more special minimizing sequences. For our functional I minimized over a Dirichlet class \mathcal{A}_g in $W^{1,p}(\Omega; \mathbf{R}^m)$, we can always obtain a minimizing sequence $\{u_k\}$ in \mathcal{A}_g which satisfies $I'(u_k) \rightarrow 0$ in $W^{-1,p'}(\Omega; \mathbf{R}^m)$. Here, we assume $p > 1$ and $p' = \frac{p}{p-1}$, and $W^{-1,p'}(\Omega; \mathbf{R}^m)$ denotes the dual space of $W_0^{1,p}(\Omega; \mathbf{R}^m)$. Consequently, the weak limit (if exists) of any such minimizing sequence will be an energy minimizer provided that $I(u)$ only satisfies the condition:

$$(1.2) \quad I(u) \leq \liminf_{k \rightarrow \infty} I(u_k) \quad \text{whenever } \begin{cases} u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbf{R}^m) \text{ and} \\ I'(u_k) \rightarrow 0 \text{ in } W^{-1,p'}(\Omega; \mathbf{R}^m). \end{cases}$$

Such a condition has been also mentioned in [11] as a point of view to replace Morrey's quasiconvexity condition.

Note that in nonlinear analysis [2] the sequence $\{u_k\}$ with bounded $I(u_k)$ satisfying $I'(u_k) \rightarrow 0$ is usually called a *Palais-Smale sequence* or a *(PS) sequence* of the functional $I(u)$. Therefore, in this paper, we shall say that a sequence $\{u_k\}$ *(PS)-weakly converges to u* (with respect to I) and denote by $u_k \xrightarrow{\text{PS}} u$ in $W^{1,p}$ if it satisfies $u_k \rightharpoonup u$ in $W^{1,p}$ and $I'(u_k) \rightarrow 0$ in $W^{-1,p'}$. Certainly there may be functionals I for which no (PS)-weakly convergent sequences exist (see Example 3.1 below); such a case is not interesting since it rules out the existence of even critical points of the functional. In any case, we shall say the functional $I(u)$ is *(PS)-weakly lower semicontinuous* on $W^{1,p}(\Omega; \mathbf{R}^m)$ if it satisfies condition (1.2); we automatically assume the

(PS)-weak lower semicontinuity if no (PS)-weak convergent sequences exist. If the condition holds only for all u_k, u in the Dirichlet class \mathcal{A}_g , we then say I is (PS)-weakly lower semicontinuous on \mathcal{A}_g .

As we shall see later, the (PS)-weak lower semicontinuity imposes some intrinsic property on the function $f(x, s, \xi)$, which may be nonlocal even in x . In general, as shown in the paper (see Proposition 4.2), the (PS)-weak lower semicontinuity is not equivalent to the usual weak lower semicontinuity even for one dimensional scalar problems. However, the main results of the paper deal with certain cases where the (PS)-weak lower semicontinuity is actually equivalent to the usual weak lower semicontinuity of the functional (hence the convexity or quasiconvexity of f). In general cases, we do not know the necessary and sufficient condition for the (PS)-weak lower semicontinuity. We point out that the major difficulty in handling this type of restricted weak lower semicontinuity lies in that the test sequences $\{u_k\}$ in the usual techniques [1, 5, 10] do not satisfy the condition $I'(u_k) \rightarrow 0$ in $W^{-1,p'}(\Omega; \mathbf{R}^m)$.

A closely related problem to the (PS)-weak lower semicontinuity of functional I is to characterize all the gradient Young measures [8] generated by weakly convergent (PS) sequences in $W^{1,p}(\Omega; \mathbf{R}^m)$. This problem is associated with the theory of compensated compactness [12]. The difficulty lies in that in this case the strong convergence $I'(u_k) \rightarrow 0$ in $W^{-1,p'}(\Omega; \mathbf{R}^m)$ can not be realized by the Young measure of $\{Du_k\}$ in the dimension $n \geq 2$. Recently the weak lower semicontinuity of functionals under certain linear differential constraints has been studied using the Young measure theory [7]. These linear constraints $\mathcal{A}(u)$ are independent of the functional and usually have large kernel. Then the constrained lower semicontinuity of functionals may be characterized through the Jensen's inequality of the integrant with the associated Young measures supported on the kernel of \mathcal{A} ; this is the so-called \mathcal{A} -quasiconvexity [7]. In this paper, we do not pursue the Young measure method for our restricted weak lower semicontinuity studied here because the Young measures do not realize the strong convergence $I'(u_k) \rightarrow 0$ in dimensions $n \geq 2$.

We now outline the plan of the paper. In Section 2, we present some notation and preliminaries that are needed in this paper and quote a suitable version of Ekeland's variational principle; we refer to [2] for a proof and more on this principle. The definition of (PS)-weak lower semicontinuity is given in Section 3, along with some basic properties about this condition. We also present an existence result based on the (PS)-weak lower semicontinuity using the Ekeland variational principle. Section 4 deals with one dimensional scalar cases, where we show without a coercivity condition the (PS)-weak lower semicontinuity may not be equivalent to the usual weak

lower semicontinuity, but with a coercivity and some technical smoothness assumption the (PS)-weak lower semicontinuity is equivalent to the usual one at least in the case when $f = f(x, \xi)$. Section 5 is devoted to the special cases for functions $f = f(\xi)$, where $f: M^{m \times n} \rightarrow \mathbf{R}$ is a C^1 function satisfying certain growth conditions. In the scalar case ($m = 1$), using certain one-dimensional constructions, we show that the (PS)-weak lower semicontinuity is equivalent to the convexity of f and hence equivalent to the usual weak lower semicontinuity. We also study the general case with function $f(\xi)$ satisfying a coercivity condition. In this case, we again show the (PS)-weak lower semicontinuity is equivalent to the usual one using a Vitali type construction [5]. The result under the coercivity assumption has been announced in [13] and we include it here for the completeness of results and for the convenience of the reader.

2. NOTATION AND PRELIMINARIES

Let Ω be a bounded domain in \mathbf{R}^n . Let $M^{m \times n}$ be the set of $m \times n$ matrices. For vectors $a, b \in \mathbf{R}^n$ and matrices $\xi, \eta \in M^{m \times n}$, we define the inner products by

$$a \cdot b = \sum_{j=1}^n a_j b_j, \quad \xi : \eta = \sum_{i=1}^m \sum_{j=1}^n \xi_{ij} \eta_{ij}$$

with the corresponding Euclidean norms denoted both by $|\cdot|$. For vectors $q \in \mathbf{R}^m$, $a \in \mathbf{R}^n$, we denote by $q \otimes a$ the rank-one $m \times n$ matrix $(q_i a_j)$.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space of scalar functions on Ω , and define $W^{1,p}(\Omega; \mathbf{R}^m)$ to be the space of vector functions $u: \Omega \rightarrow \mathbf{R}^m$ with each component $u^i \in W^{1,p}(\Omega)$ and we denote by Du the Jacobi matrix of u defined by

$$Du(x) = (\partial u^i / \partial x_j)_{i=1, \dots, m}^{j=1, \dots, n}.$$

Let $1 \leq p < \infty$. We make $W^{1,p}(\Omega; \mathbf{R}^m)$ a Banach space with the norm

$$\|u\|_{W^{1,p}(\Omega; \mathbf{R}^m)} = \left(\int_{\Omega} (|u|^p + |Du|^p) dx \right)^{\frac{1}{p}}.$$

Let $C_0^\infty(\Omega; \mathbf{R}^m)$ be the set of infinitely differentiable vector functions with compact support in Ω , and let $W_0^{1,p}(\Omega; \mathbf{R}^m)$ be the closure of $C_0^\infty(\Omega; \mathbf{R}^m)$ in $W^{1,p}(\Omega; \mathbf{R}^m)$. Then $W_0^{1,p}(\Omega; \mathbf{R}^m)$ is itself a Banach space and has an equivalent norm defined by $\|Du\|_{L^p(\Omega)}$.

Let $g \in W^{1,p}(\Omega; \mathbf{R}^m)$. Define the Dirichlet class $\mathcal{A}_g = W_g^{1,p}(\Omega; \mathbf{R}^m)$ to be $g + W_0^{1,p}(\Omega; \mathbf{R}^m)$; that is,

$$\mathcal{A}_g = W_g^{1,p}(\Omega; \mathbf{R}^m) = \{u \in W^{1,p}(\Omega; \mathbf{R}^m) \mid u - g \in W_0^{1,p}(\Omega; \mathbf{R}^m)\}.$$

For $A \in M^{m \times n}$, define $g_A(x) = Ax$ and let $W_A^{1,p} = W_{g_A}^{1,p}(\Omega; \mathbf{R}^m)$ be the Dirichlet class for the linear function g_A .

We use $u_k \rightharpoonup u$ to denote the weak convergence in $W^{1,p}(\Omega; \mathbf{R}^m)$. Note that if $1 < p < \infty$ then any bounded sequence in $W^{1,p}(\Omega; \mathbf{R}^m)$ has a weakly convergent subsequence and if, furthermore, $\partial\Omega$ is smooth or the sequence is in $W_g^{1,p}(\Omega; \mathbf{R}^m)$ one can assume the subsequence is also convergent strongly in $L^p(\Omega; \mathbf{R}^m)$.

Definition 2.1. A functional I on $W^{1,p}(\Omega; \mathbf{R}^m)$ is said to be (sequentially) *weakly lower semicontinuous* on $W^{1,p}(\Omega; \mathbf{R}^m)$ provided

$$(2.1) \quad I(u) \leq \liminf_{k \rightarrow \infty} I(u_k) \quad \text{whenever } u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbf{R}^m).$$

Definition 2.2. Let $h: M^{m \times n} \rightarrow \mathbf{R}$.

(i) We say that h is *convex* on $M^{m \times n}$ if the inequality

$$(2.2) \quad h(\lambda\xi + (1-\lambda)\eta) \leq \lambda h(\xi) + (1-\lambda)h(\eta)$$

holds for all $0 < \lambda < 1$ and $\xi, \eta \in M^{m \times n}$.

(ii) We say that h is *quasiconvex* in the Morrey sense if the inequality

$$(2.3) \quad h(A) \leq \frac{1}{|\Omega|} \int_{\Omega} h(A + D\varphi(x)) \, dx$$

holds for all $A \in M^{m \times n}$ and $\varphi \in C_0^\infty(\Omega; \mathbf{R}^m)$.

For more on Morrey's quasiconvexity condition, we refer to [1, 3, 5, 10, 11]. Note that in the case of $n = 1$ (one-dimensional) or $m = 1$ (scalar) the convexity and quasiconvexity for the function $h: M^{m \times n} \rightarrow \mathbf{R}$ are equivalent.

Note also that h is convex if and only if $g(t) = h(\xi + t\eta)$ is a convex function of t on \mathbf{R} for all $\xi, \eta \in M^{m \times n}$. For C^1 functions h , the convexity condition is equivalent to the condition

$$(2.4) \quad h(\eta) \geq h(\xi) + D_\xi h(\xi) : (\eta - \xi), \quad \forall \eta, \xi \in M^{m \times n}.$$

Furthermore, a C^1 function h on \mathbf{R} is convex if and only if h' is nondecreasing, or equivalently, the following condition holds:

$$(2.5) \quad (h'(a) - h'(b))(a - b) \geq 0, \quad \forall a, b \in \mathbf{R}.$$

Let $f: \Omega \times \mathbf{R}^n \times M^{m \times n} \rightarrow \mathbf{R}$. We say f is *Carathéodory* if $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbf{R}^n \times M^{m \times n}$ and continuous in $(s, \xi) \in \mathbf{R}^n \times M^{m \times n}$ for almost every $x \in \Omega$. If $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbf{R}^n \times M^{m \times n}$ and is C^1 in $(s, \xi) \in \mathbf{R}^n \times M^{m \times n}$ for almost every $x \in \Omega$, we shall use the following notation to denote the derivatives of f on s and ξ :

$$D_s f(x, s, \xi) = \left(\frac{\partial f}{\partial s_1}, \dots, \frac{\partial f}{\partial s_n} \right), \quad D_\xi f(x, s, \xi) = (\partial f / \partial \xi_{ij})_{i=1, \dots, m}^{j=1, \dots, n}.$$

Given a function $f(x, s, \xi)$, define the integral functional I on $W^{1,p}(\Omega; \mathbf{R}^m)$ by

$$I(u) = \int_{\Omega} f(x, u(x), Du(x)) dx, \quad u \in W^{1,p}(\Omega; \mathbf{R}^m).$$

The following important result has been proved by Acerbi and Fusco [1].

Theorem 2.1. *Assume f is Carathéodory and satisfies*

$$0 \leq f(x, s, \xi) \leq c_1(|\xi|^p + |s|^p) + A(x),$$

where $c_1 > 0$ and $A \in L^1(\Omega)$. Then the functional I defined above is weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbf{R}^m)$ if and only if $f(x, s, \cdot)$ is quasiconvex for almost every $x \in \Omega$ and all $s \in \mathbf{R}^n$; that is, the inequality

$$f(x, s, \xi) \leq \frac{1}{|\Omega|} \int_{\Omega} f(x, s, \xi + D\varphi(y)) dy$$

holds for a.e. $x \in \Omega$, all $s \in \mathbf{R}^n$, $\xi \in M^{m \times n}$ and all $\varphi \in C_0^\infty(\Omega; \mathbf{R}^m)$.

For smooth and bounded-below functionals on Banach space, we have the following result from the Ekeland variational principle [2, 6]. We refer to [2, 6] for the proof and more on the applications of the Ekeland variational principle.

Theorem 2.2. *Let X be a Banach space and X^* its dual space, and let $\Phi: X \rightarrow \mathbf{R}$ be a C^1 functional which is bounded below. Then, for each $\epsilon > 0$, there exists $u_\epsilon \in X$ such that*

$$(2.6) \quad \Phi(u_\epsilon) \leq \inf_X \Phi + \epsilon$$

$$(2.7) \quad \|\Phi'(u_\epsilon)\|_{X^*} \leq \epsilon.$$

Therefore, there exists a minimizing sequence $\{u_k\}$ in X such that

$$\lim_{k \rightarrow \infty} \Phi(u_k) = \inf_X \Phi, \quad \lim_{k \rightarrow \infty} \|\Phi'(u_k)\|_{X^*} = 0.$$

3. THE (PS)-WEAK LOWER SEMICONTINUITY

In this section, we assume $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbf{R}^n \times M^{m \times n}$ and is C^1 in $(s, \xi) \in \mathbf{R}^n \times M^{m \times n}$ for almost every $x \in \Omega$. We also assume $1 < p < \infty$ and f satisfies the growth conditions

$$(3.1) \quad |f(x, s, \xi)| \leq c_1(|s|^p + |\xi|^p) + A(x),$$

$$(3.2) \quad |D_s f(x, s, \xi)| + |D_\xi f(x, s, \xi)| \leq c_2(|s|^{p-1} + |\xi|^{p-1}) + B(x)$$

for almost every $x \in \Omega$ and for all $s \in \mathbf{R}^n$, $\xi \in M^{m \times n}$, where c_1, c_2 are positive constants and A, B are positive functions with $A \in L^1(\Omega)$, $B \in L^{\frac{p}{p-1}}(\Omega)$.

From these assumptions, we easily obtain the following result, whose proof is left to the interested reader.

Proposition 3.1. *Under the above conditions, the functional I defined above is a C^1 functional on $W^{1,p}(\Omega; \mathbf{R}^m)$ and for each u the Fréchet derivative $I'(u)$ is given by*

$$\langle I'(u), v \rangle = \int_{\Omega} [D_s f(x, u, Du) \cdot v + D_{\xi} f(x, u, Du) : Dv] dx$$

for all $v \in W^{1,p}(\Omega; \mathbf{R}^m)$.

When minimizing the functional I on a Dirichlet class \mathcal{A}_g , one can shift the class to the Banach space $X = W_0^{1,p}(\Omega; \mathbf{R}^m)$ since

$$(3.3) \quad \inf_{u \in \mathcal{A}_g} I(u) = \inf_{w \in X} \Phi(w),$$

where $\Phi(w) = I(w + g)$. We easily have the following result.

Proposition 3.2. *Let $X = W_0^{1,p}(\Omega; \mathbf{R}^m)$. For any $g \in W^{1,p}(\Omega; \mathbf{R}^m)$, the functional $\Phi: X \rightarrow \mathbf{R}$ defined by $\Phi(w) = I(w + g)$ is C^1 and $\Phi'(w) = I'(w + g)$ as elements in X^* , the dual space of X .*

In the following we write $X^* = W^{-1,p'}(\Omega; \mathbf{R}^m)$, where $p' = \frac{p}{p-1}$. As usual, we define

$$(3.4) \quad \|I'(u)\|_{W^{-1,p'}} = \sup\{\langle I'(u), v \rangle \mid v \in W_0^{1,p}(\Omega; \mathbf{R}^m), \|v\|_{W_0^{1,p}} \leq 1\}.$$

Note that, given a smooth functional I on $X = W_0^{1,p}(\Omega; \mathbf{R}^m)$, the sequences $\{u_k\}$ in X satisfying

$$|I(u_k)| \leq M, \quad I'(u_k) \rightarrow 0 \quad \text{in } W^{-1,p'}(\Omega; \mathbf{R}^m)$$

are usually called the *Palais-Smale sequences* or *(PS) sequences* for the functional I . Therefore, for simplicity, we use the following definition.

Definition 3.1. A sequence $\{u_k\}$ is said to *(PS)-weakly converges to u* (with respect to I) in $W^{1,p}(\Omega; \mathbf{R}^m)$ and denoted by $u_k \xrightarrow{\text{ps}} u$ provided that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbf{R}^m)$ and $I'(u_k) \rightarrow 0$ in $W^{-1,p'}(\Omega; \mathbf{R}^m)$. Define the set of all (PS)-weak limits to be

$$(3.5) \quad \mathcal{S} = \{u \in W^{1,p}(\Omega; \mathbf{R}^m) \mid \exists u_k \in W^{1,p}(\Omega; \mathbf{R}^m) \text{ such that } u_k \xrightarrow{\text{ps}} u\}.$$

Let $\mathcal{C} = \{u \in W^{1,p}(\Omega; \mathbf{R}^m) \mid \|I'(u)\|_{W^{-1,p'}} = 0\}$. Then clearly $\mathcal{C} \subseteq \mathcal{S}$, and hence \mathcal{S} can be viewed as a relaxation of \mathcal{C} under the (PS)-weak convergence. However, for certain functionals I the set \mathcal{S} may be empty.

Example 3.1. Let $f(x, \xi) = \chi_E(x)h(\xi)$, where χ_E is the characteristic function of a measurable set E in $(0, 1)$ with $0 < |E| < 1$ and $h(\xi) = \frac{\pi}{2} + \arctan \xi$. Define

$$I(u) = \int_0^1 f(x, u'(x)) dx, \quad u \in W^{1,2}(0, 1).$$

We claim that for the functional I the (PS)-weak limit set $\mathcal{S} = \emptyset$. Suppose to the contrary $u_k \xrightarrow{\text{PS}} u$ in $W^{1,2}(0,1)$. Let $g_k(x) = \chi_E(x)h'(u'_k(x))$. Then, by Proposition 4.1 below, there exists a subsequence $g_{k_j} \rightarrow L$ strongly in $L^2(0,1)$ for some constant L . We also assume $g_{k_j}(x) \rightarrow L$ for almost every $x \in (0,1)$. Hence we must have $L = 0$ and $g_{k_j}(x) = h'(u'_{k_j}(x)) \rightarrow 0$ for almost every $x \in E$. By Egoroff's theorem, it follows that $|u'_{k_j}(x)| \rightarrow \infty$ almost uniformly on E , which implies $\|u'_{k_j}\|_{L^2(E)} \rightarrow \infty$, a contradiction.

Definition 3.2. Given any nonempty family $\mathcal{A} \subseteq W^{1,p}(\Omega; \mathbf{R}^m)$, we say that I is (PS)-weakly lower semicontinuous on \mathcal{A} provided that

$$(3.6) \quad I(u) \leq \liminf_{k \rightarrow \infty} I(u_k) \quad \text{whenever} \quad u_k, u \in \mathcal{A}, \quad u_k \xrightarrow{\text{PS}} u.$$

We shall technically assume this property if $\mathcal{A} \cap \mathcal{S} = \emptyset$.

The following result shows that if $f = f(x, \xi)$ is convex in ξ then the functional I is in fact (PS)-weakly continuous on all Dirichlet classes.

Proposition 3.3. *Assume $f = f(x, \xi)$ satisfies the corresponding growth conditions as (3.1) and (3.2) above. Suppose $f(x, \xi)$ is convex in ξ for almost every $x \in \Omega$. Then both I and $-I$ are (PS)-weakly lower semicontinuous on all Dirichlet classes \mathcal{A}_g with $g \in W^{1,p}(\Omega; \mathbf{R}^m)$. Therefore the functional I is (PS)-weakly continuous on \mathcal{A}_g in the sense that*

$$(3.7) \quad I(u) = \lim_{k \rightarrow \infty} I(u_k) \quad \forall u_k, u \in \mathcal{A}_g, \quad u_k \xrightarrow{\text{PS}} u.$$

Proof. For any $u_k, u \in W^{1,p}(\Omega; \mathbf{R}^m)$, by the convexity of f , it follows from (2.4) that

$$(3.8) \quad f(x, Du_k) \geq f(x, Du) + D_\xi f(x, Du) : (Du_k - Du),$$

$$(3.9) \quad f(x, Du) \geq f(x, Du_k) + D_\xi f(x, Du_k) : (Du - Du_k)$$

for almost every $x \in \Omega$. If $u_k \xrightarrow{\text{PS}} u$, and $u - u_k \in W_0^{1,p}(\Omega; \mathbf{R}^m)$, then integrating the above inequalities, we have

$$\liminf_{k \rightarrow \infty} I(u_k) \geq I(u) \geq \limsup_{k \rightarrow \infty} I(u_k),$$

and hence (3.7) follows. \square

We show that in general the (PS)-weak lower semicontinuity on all Dirichlet classes does not imply the (PS)-weak lower semicontinuity on the whole space $W^{1,p}(\Omega; \mathbf{R}^m)$ (without the fixed boundary conditions).

Proposition 3.4. *Let Ω be the unit disc in \mathbf{R}^2 and $I(u) = -\int_\Omega |Du|^2 dx$ for $u: \Omega \rightarrow \mathbf{R}$. Then I is (PS)-weakly lower semicontinuous on all Dirichlet classes of $W^{1,2}(\Omega)$ but not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$.*

Proof. By the preceding proposition, I is (PS)-weakly lower semicontinuous on all Dirichlet classes of $W^{1,2}(\Omega)$. We now show it is not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$ (without the fixed boundary conditions). We identify $\mathbf{R}^2 \cong \mathbf{C}^1$. For $z = x_1 + ix_2 \in \Omega$ and $k = 1, 2, \dots$, we define $u_k(x_1, x_2) = \frac{1}{\sqrt{\pi k}} \operatorname{Re}(z^k)$. Then u_k is harmonic in Ω and $\partial_{x_1} u_k - i \partial_{x_2} u_k = \sqrt{\frac{k}{\pi}} z^{k-1}$. Hence $|Du_k(x)| = \sqrt{\frac{k}{\pi}} |z|^{k-1}$ and thus we have $\|Du_k\|_{L^2(\Omega)} = 1$. So u_k is bounded in $W^{1,2}(\Omega)$. It is easy to see $u_k \rightarrow 0$ uniformly on $\bar{\Omega}$ and hence $u_k \rightarrow 0$ in $W^{1,2}(\Omega)$. Since u_k is harmonic in Ω , it also follows that $Du_k \rightarrow 0$ in $W^{-1,2}(\Omega)$. Therefore, for functional $I(u) = -\int_{\Omega} |Du|^2 dx$, we have $u_k \xrightarrow{\text{PS}} 0$, but $I(0) = 0$ and $\liminf_k I(u_k) = -1$. Hence I is not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$. \square

As we mentioned in the introduction, the (PS)-weak lower semicontinuity has been motivated by using the Ekeland variational principle in the direct method for the minimization problem. We have the following existence result.

Theorem 3.5. *Assume f satisfies, in addition to (3.1) and (3.2), the following coercivity condition*

$$(3.10) \quad c_0 |\xi|^p - a(x) \leq f(x, s, \xi) \leq c_1 (|\xi|^p + |s|^p) + A(x),$$

where $c_0 > 0$ is a positive constant, $a \in L^1(\Omega)$ is a function. Given $g \in W^{1,p}(\Omega; \mathbf{R}^m)$, assume the functional I defined above is (PS)-weakly lower semicontinuous on \mathcal{A}_g . Then the minimization problem $\inf_{u \in \mathcal{A}_g} I(u)$ has at least one solution $u \in \mathcal{A}_g$.

Proof. The proof uses a standard direct method of the calculus of variations. Let $X = W_0^{1,p}(\Omega; \mathbf{R}^m)$. Define $\Phi: X \rightarrow \mathbf{R}$ by

$$\Phi(u) = I(u + g) = \int_{\Omega} f(x, u(x) + g(x), Du(x) + Dg(x)) dx.$$

Then Φ is C^1 and bounded below on X , and $\Phi'(u) = I'(u + g)$ in X^* . By Theorem 2.2, there exists a sequence $\{u_k\}$ in X such that

$$\Phi(u_k) \rightarrow \inf_X \Phi, \quad \|\Phi'(u_k)\|_{X^*} \rightarrow 0.$$

Let $w_k = u_k + g \in \mathcal{A}_g$. Then

$$(3.11) \quad I(w_k) \rightarrow \inf_{w \in \mathcal{A}_g} I(w), \quad \|I'(w_k)\|_{W^{-1,p'}} \rightarrow 0.$$

Under the condition $c_0 > 0$ the sequence $\{w_k\}$ determined by (3.11) above is bounded in $W^{1,p}(\Omega; \mathbf{R}^m)$ and, since $1 < p < \infty$, has a weakly convergence

subsequence, relabeled $\{w_k\}$ again. Let u be the weak limit. Then $u \in \mathcal{A}_g$ and $w_k \xrightarrow{\text{ps}} u$; hence the (PS)-weak lower semicontinuity on \mathcal{A}_g implies

$$I(u) \leq \lim_{k \rightarrow \infty} I(w_k) = \inf_{w \in \mathcal{A}_g} I(w).$$

Hence $I(u) = \inf_{w \in \mathcal{A}_g} I(w)$. \square

Remark 3.1. Under the growth assumptions (3.1) and (3.2), any minimizer u of I over \mathcal{A}_g is a weak solution to the Dirichlet problem of the Euler-Lagrange equation of functional I ; that is,

$$(3.12) \quad \begin{cases} -\operatorname{div} D_\xi f(x, u, Du) + D_s f(x, u, Du) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

4. ONE DIMENSIONAL SCALAR CASES

In this section we study the (PS)-weak lower semicontinuity in some special one dimensional scalar cases.

We first consider the Sobolev space $H^1(0, 1) = W^{1,2}(0, 1)$ and functions $f(x, \xi)$ satisfying

$$(4.1) \quad 0 \leq f(x, \xi) \leq C|\xi|^2 + A(x), \quad |f_\xi(x, \xi)| \leq C|\xi| + B(x),$$

with $A \in L^1(0, 1)$, $B \in L^2(0, 1)$. Define

$$I(u) = \int_0^1 f(x, u'(x)) dx, \quad \forall u \in H^1(0, 1).$$

Proposition 4.1. *If $u_k \xrightarrow{\text{ps}} u$ in $H^1(0, 1)$, then there exists a subsequence $\{u_{k_j}\}$ such that $f_\xi(x, u'_{k_j}(x)) \rightarrow L$ strongly in $L^2(0, 1)$ as $j \rightarrow \infty$, where L is a constant.*

Proof. Let $g_k(x) = f_\xi(x, u'_k(x))$ and $L_k = \int_0^1 g_k(x) dx$. Since $\{g_k\}$ is bounded in $L^2(0, 1)$, we assume for a subsequence $g_{k_j} \rightarrow g$ in $L^2(0, 1)$ as $j \rightarrow \infty$, where $g \in L^2(0, 1)$. We define v_k on $[0, 1]$ by

$$v_k(x) = \int_0^x (g_k(t) - L_k) dt, \quad x \in [0, 1].$$

Then it is easily seen that $v_k \in H_0^1(0, 1) = W_0^{1,2}(0, 1)$ and $v'_k = g_k - L_k$. Moreover, $\{v_k\}$ is bounded in $H_0^1(0, 1)$ and hence

$$\langle I'(u_k), v_k \rangle = \int_0^1 f_\xi(x, u'_k(x)) v'_k(x) dx = \int_0^1 g_k^2(x) dx - L_k^2 \rightarrow 0$$

as $k \rightarrow \infty$. Since $g_{k_j} \rightarrow g$ in $L^2(0, 1)$, we have $L_{k_j} \rightarrow L = \int_0^1 g dx$ and

$$\int_0^1 g^2(x) dx \leq \liminf_{j \rightarrow \infty} \int_0^1 g_{k_j}^2 dx = \liminf_{j \rightarrow \infty} L_{k_j}^2 = \left(\int_0^1 g(x) dx \right)^2.$$

This implies $g(x) = L$ a.e. on $[0, 1]$ and $g_{k_j} \rightarrow L$ strongly in $L^2(0, 1)$. \square

In contrast to the theorem of Acerbi and Fusco (Theorem 2.1), we show below by an example that the (PS)-weak lower semicontinuity of I may not imply f being quasiconvex in ξ even for smooth functions $f(x, \xi)$ in the scalar case.

Proposition 4.2. *There exists a C^1 function $f(x, \xi)$ satisfying condition (4.1) above for which the corresponding functional I is (PS)-weakly, but not (unrestricted) weakly, lower semicontinuous on $H^1(0, 1)$.*

Proof. Assume $f(x, \xi) = a(x)h(\xi)$ with $a, h \geq 0$, both C^1 and satisfying the following conditions:

$$(4.2) \quad a(x) = 0 \quad x \in [0, \theta], \quad a(x) > 0 \quad x \in (\theta, 1],$$

$$(4.3) \quad h \geq 0, \quad (h')^{-1}(0) = \{0\}, \quad \liminf_{|\xi| \rightarrow \infty} |h'(\xi)| > 0.,$$

where $\theta \in (0, 1)$ is a constant. Note that the condition (4.3) implies $h(0) < h(\xi)$ for all $\xi \in \mathbf{R}$. Given any $u_k \xrightarrow{\text{ps}} u$ in $H^1(0, 1)$, using subsequence if necessary, we assume $\lim_{k \rightarrow \infty} I(u_k)$ exists. By Proposition 4.1 above, there exists a subsequence $\{u_{k_j}\}$ such that $f_\xi(x, u'_{k_j}) = a(x)h'(u'_{k_j}) \rightarrow L$ strongly in $L^2(0, 1)$ for some constant L . Since $a = 0$ on $[0, \theta]$, one must have the limit $L = 0$; this also implies the whole sequence $a(x)h'(u'_k) \rightarrow 0$ strongly in $L^2(0, 1)$. Therefore $h'(u'_k) \rightarrow 0$ strongly in $L^2(\theta', 1)$ for any $\theta' \in (\theta, 1)$. Hence, for a subsequence it follows that $h'(u'_{k_j}(x)) \rightarrow 0$ for almost every $x \in (\theta', 1)$. By (4.3), we have that $u'_{k_j}(x) \rightarrow 0$ for almost every $x \in (\theta', 1)$. Therefore the weak limit $u' = 0$ on $(\theta', 1)$ for all $\theta' \in (\theta, 1)$. This implies $u' = 0$ in $(\theta, 1)$. Since $h(\xi) \geq h(0)$ for all ξ , we have

$$\lim_{k \rightarrow \infty} I(u_k) = \lim_{k \rightarrow \infty} \int_\theta^1 a(x)h(u'_k(x)) dx \geq \int_\theta^1 a(x)h(0)dx = I(u).$$

Hence I satisfies the (PS)-weak lower semicontinuity on $H^1(0, 1)$. Note that the condition (4.3) does not imply that h is convex. (See, e.g., condition (2.5).) Hence I may not be weakly lower semicontinuous on $H^1(0, 1)$ by Theorem 2.1 above. \square

Remark 4.1. For the functional I defined by a function $f(x, \xi) = a(x)h(\xi)$ as above, the minimization problem

$$\inf_{\substack{u \in H^1(0,1) \\ u(0)=a, u(1)=b}} I(u)$$

has as only minimizers any functions $u \in H^1(0, 1)$ with $u(0) = a$ and $u \equiv b$ on $[\theta, 1]$ and the minimum equals $h(0) \int_\theta^1 a(x)dx$, for any constants $a, b \in \mathbf{R}$. These minimizers are exactly those functions u in the Dirichlet class for which there exists a sequence $\{u_k\}$ in the class such that $u_k \xrightarrow{\text{ps}} u$.

Despite of the result above, we shall show that the (PS)-weak lower semi-continuity is equivalent to the usual weak lower semicontinuity if $f(x, \xi)$ satisfies certain coercivity condition.

In the following, for $\beta \in \mathbf{R}$, let $W_\beta^{1,p}(0, 1)$ be the Dirichlet class of functions u in $W^{1,p}(0, 1)$ with $u(0) = 0$, $u(1) = \beta$. Then we have the following result.

Theorem 4.3. *Assume $f(x, \xi)$ and $f_\xi(x, \xi)$ are both C^1 on $[0, 1] \times \mathbf{R}$ and satisfy, for some $p > 1$ and positive constants $c_0, c_1, c_2 > 0$,*

$$(4.4) \quad c_0|\xi|^p \leq f(x, \xi) \leq c_1(|\xi|^p + 1), \quad |f_\xi(x, \xi)| \leq c_2(|\xi|^{p-1} + 1)$$

for all x and ξ . If the functional I defined by f is (PS)-weakly lower semi-continuous on $W_\beta^{1,p}(0, 1)$ for all $\beta \in \mathbf{R}$, then $f(x, \xi)$ is convex in ξ for all $x \in (0, 1)$.

For the technical reason of using the following Sard's theorem [9], we have assumed that f is sufficiently smooth in both x and ξ in the theorem.

Lemma 4.4. *Let $h: \mathbf{R} \rightarrow \mathbf{R}$ be C^1 and $S = \{y \in \mathbf{R} \mid \exists x \in \mathbf{R}, y = h(x), h'(x) = 0\}$. Then the Lebesgue measure $|S| = 0$ and, in particular, the set of regular values of h , $\mathbf{R} \setminus S$, is dense in \mathbf{R} .*

We proceed with several lemmas to prove Theorem 4.3. First of all, for $\beta \in \mathbf{R}$, we define $m(\beta) = \inf\{I(u) \mid u \in W_\beta^{1,p}(0, 1)\}$. It follows easily from the growth condition (4.4) that

$$(4.5) \quad c_0|\beta|^p \leq m(\beta) \leq c_1(|\beta|^p + 1).$$

From Theorem 3.5 above, it follows that, if I is (PS)-weakly lower semi-continuous on $W_\beta^{1,p}(0, 1)$, then there exists at least one minimizer $u_\beta \in W_\beta^{1,p}(0, 1)$ such that $I(u_\beta) = m(\beta)$. Hence $I'(u_\beta) = 0$ in $W^{-1,p'}(0, 1)$. This implies $f_\xi(x, u'_\beta(x))$ is constant in $(0, 1)$ and let $\mu(\beta)$ denote this constant. Note that $\mu(\beta)$ depends also on the minimizer u_β .

Lemma 4.5. *It follows that*

$$(4.6) \quad \limsup_{\beta \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0^+} \frac{m(\beta + \epsilon) - m(\beta)}{\epsilon} = +\infty,$$

$$(4.7) \quad \liminf_{\beta \rightarrow -\infty} \liminf_{\epsilon \rightarrow 0^-} \frac{m(\beta + \epsilon) - m(\beta)}{\epsilon} = -\infty.$$

Proof. We only prove the equation (4.6); the other one follows similarly. By contradiction, suppose the limit is not $+\infty$. Then there exist positive constants β_0, ϵ_0 and M such that

$$\frac{m(\beta + \epsilon) - m(\beta)}{\epsilon} \leq M, \quad \forall \beta \geq \beta_0, \quad \epsilon \in (0, \epsilon_0],$$

which, in particular, implies that

$$(4.8) \quad m(\beta_0 + k\epsilon_0) - m(\beta_0) \leq Mk\epsilon_0 \quad \forall k = 1, 2, \dots$$

Using (4.5), we have $m(\beta_0 + k\epsilon_0) \geq c_0|\beta_0 + k\epsilon_0|^p \geq \gamma k^p - C_0$ for some positive constants γ and C_0 , and for all $k = 1, 2, \dots$. This combined with (4.8) yields a desired contradiction since $p > 1$. This proves (4.6). \square

Lemma 4.6. *For any $\beta \in \mathbf{R}$, it follows that*

$$(4.9) \quad \limsup_{\epsilon \rightarrow 0^+} \frac{m(\beta + \epsilon) - m(\beta)}{\epsilon} \leq \mu(\beta) \leq \liminf_{\epsilon \rightarrow 0^-} \frac{m(\beta + \epsilon) - m(\beta)}{\epsilon}.$$

Proof. For $0 < \delta < 1$ we define w to be the linear function with $w(1 - \delta) = 0$, $w(1) = \epsilon$. Hence $w'(x) = \epsilon/\delta$. Let u_β be a minimizer for $m(\beta)$ and let $v(x) = u_\beta(x)$ on $[0, 1 - \delta]$ and $v(x) = u_\beta(x) + w(x)$ on $[1 - \delta, 1]$. Then $v \in W^{1,p}(0, 1)$ satisfies $v(0) = 0$, $v(1) = \beta + \epsilon$. Hence

$$m(\beta + \epsilon) \leq I(v) = I(u_\beta) + \int_{1-\delta}^1 [f(x, v') - f(x, u'_\beta)] dx.$$

Since $f(x, v') - f(x, u'_\beta) = f_\xi(x, u'_\beta)\epsilon/\delta + o(\epsilon/\delta)$ for $\epsilon/\delta \rightarrow 0$, we have

$$m(\beta + \epsilon) \leq m(\beta) + \mu(\beta)\epsilon + o\left(\frac{\epsilon}{\delta}\right)\delta \leq m(\beta) + \mu(\beta)\epsilon + o(\epsilon),$$

as $\epsilon \rightarrow 0$. From this the lemma follows. \square

The lemmas above imply

$$(4.10) \quad \limsup_{\beta \rightarrow +\infty} \mu(\beta) = +\infty, \quad \liminf_{\beta \rightarrow -\infty} \mu(\beta) = -\infty.$$

Lemma 4.7. *For any constant $\theta \in \mathbf{R}$, there exists a function $q_\theta \in L^p(0, 1)$ such that $f_\xi(x, q_\theta(x)) = \theta$ for almost every $x \in (0, 1)$.*

Proof. In view of (4.10) above, there exist $\beta_1 < \beta_2$ such that $\mu(\beta_1) < \theta < \mu(\beta_2)$. Hence for almost every $x \in (0, 1)$ we have $f_\xi(x, u'_{\beta_1}(x)) < \theta < f_\xi(x, u'_{\beta_2}(x))$. Let

$$q^-(x) = \min\{u'_{\beta_1}(x), u'_{\beta_2}(x)\}, \quad q^+(x) = \max\{u'_{\beta_1}(x), u'_{\beta_2}(x)\}.$$

Then $q^\pm \in L^p(0, 1)$. By the intermediate value property of $f_\xi(x, \cdot)$, there exists $q \in (q^-(x), q^+(x))$ such that $f_\xi(x, q) = \theta$. Let $q_\theta(x)$ be the infimum of all such q 's. Then $f_\xi(x, q_\theta(x)) = \theta$, $q_\theta(x)$ is lower semicontinuous and $q^-(x) \leq q_\theta(x) \leq q^+(x)$ at almost every $x \in (0, 1)$ and hence $q_\theta \in L^p(0, 1)$. \square

The following result turns out to be quite useful in the proof of the theorem and also later; the proof is elementary and included here for the convenience of the reader.

Lemma 4.8. *Let $h: \mathbf{R} \rightarrow \mathbf{R}$ be C^1 and let*

$$S = \{(a, b) \in \mathbf{R}^2 \mid h'(a) = h'(b)\},$$

$$S_1 = \{(\alpha, \beta) \in S \mid \alpha < \beta, h'(t) \neq h'(\alpha) \forall t \in (\alpha, \beta)\}.$$

Assume $h(t) \geq 0$ for all $t \in \mathbf{R}$. Then the following statements are equivalent:

- (i) *h is convex.*
- (ii) $S_1 = \emptyset$.
- (iii) $h(\lambda a + (1 - \lambda)b) \leq \lambda h(a) + (1 - \lambda)h(b) \quad \forall (a, b) \in S, \lambda \in [0, 1]$.

Proof. We prove the result by showing that (i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).

1. (i) \Rightarrow (iii). This is easy.

2. (iii) \Rightarrow (ii). Suppose to the contrary $S_1 \neq \emptyset$ and $(\alpha, \beta) \in S_1$. Using inequality (iii) with $(\alpha, \beta) \in S$ yields that $\forall 0 < \lambda < 1$ with $t_\lambda = \lambda\alpha + (1 - \lambda)\beta$,

$$(4.11) \quad \frac{h(t_\lambda) - h(\alpha)}{t_\lambda - \alpha} \leq \frac{h(\beta) - h(\alpha)}{\beta - \alpha} \leq \frac{h(t_\lambda) - h(\beta)}{t_\lambda - \beta}.$$

Letting $\lambda \rightarrow 1^-$ and 0^+ in (4.11) respectively yields that $h'(\alpha) \leq \frac{h(\beta) - h(\alpha)}{\beta - \alpha} \leq h'(\beta)$; hence $h'(\alpha) = h'(\beta) = \frac{h(\beta) - h(\alpha)}{\beta - \alpha}$. However, by the mean value property, $h'(\alpha) = \frac{h(\beta) - h(\alpha)}{\beta - \alpha} = h'(t)$ for some $t \in (\alpha, \beta)$, which shows $(\alpha, \beta) \notin S_1$, a desired contradiction.

3. (ii) \Rightarrow (i). Again, to the contrary, suppose h is not convex. Then there exist $a < b$ such that $h'(a) > h'(b)$. We consider only the case when $h'(a) > 0$; otherwise, consider $\bar{h}(t) = h(-t)$, $\bar{a} = -b$ and $\bar{b} = -a$. We claim there exist $c < d \leq a$ such that $h'(c) < h'(d)$. If not, h' would be nonincreasing on $(-\infty, a]$ and hence h would be concave on $(-\infty, a]$. Therefore we would have $h(t) \leq h(a) + h'(a)(t - a)$ for all $t < a$. Since $h'(a) > 0$, letting $t \rightarrow -\infty$, we would have $h(t) \rightarrow -\infty$, a contradiction with $h \geq 0$. Let $c < d \leq a$ be any points as above. Let $m = \max_{[c, b]} h'$. Define $S = \{t \in [c, b] \mid h'(t) = m\}$, $s^- = \min S$, and $s^+ = \max S$. Then $s^-, s^+ \in S$ and $c < s^- \leq s^+ < b$. Hence $h'(c) < m$, $h'(b) < m$. We define $\alpha' < \beta'$ as follows: If $h'(c) = h'(b)$, define $\alpha' = c$, $\beta' = b$. If $h'(c) > h'(b)$, then $h'(c) \in (h'(s^+), h'(b))$ and hence by the intermediate value property of h' , define $\beta' \in (s^+, b)$ so that $h'(\beta') = h'(c)$, and define $\alpha' = c$. If $h'(c) < h'(b)$, then $h'(b) \in (h'(c), h'(s^-))$ and hence again by the intermediate value property of h' , we define $\alpha' \in (c, s^-)$ so that $h'(\alpha') = h'(b)$, and define $\beta' = b$. The points $\alpha' < \beta'$ defined this way will satisfy $\alpha' < s^- \leq s^+ < \beta'$ and $h'(\alpha') = h'(\beta') < h'(s^-)$. Let $G = \{t \in (\alpha', \beta') \mid h'(t) > h'(\alpha')\}$. Then G is an open set and $s^- \in G$. Let (α, β) be the component of G containing s^- . Then it follows that $(\alpha, \beta) \in S_1$, a contradiction with (ii); hence h is convex. \square

Remark 4.2. It can be shown that conditions (i) and (iii) in the lemma above are not equivalent for functions $h: \mathbf{R}^m \rightarrow \mathbf{R}$ if $m \geq 2$ (see Remark 5.2 later).

Proof of Theorem 4.3. Given any $x_0 \in (0, 1)$, we prove $f(x_0, \cdot)$ is convex. By Lemma 4.8, it suffices to show that there exist no numbers $\xi_1 < \xi_2$ such that

$$(4.12) \quad f_\xi(x_0, \xi_1) = f_\xi(x_0, \xi_2), \quad f_\xi(x_0, t) \neq f_\xi(x_0, \xi_1) \quad \forall t \in (\xi_1, \xi_2).$$

We prove this by contradiction. Suppose $\xi_1 < \xi_2$ satisfy (4.12). We will derive a contradiction by showing such ξ_i 's must satisfy

$$(4.13) \quad f(x_0, \lambda\xi_1 + (1 - \lambda)\xi_2) \leq \lambda f(x_0, \xi_1) + (1 - \lambda)f(x_0, \xi_2)$$

for all $\lambda \in (0, 1)$, which gives a desired contradiction as in the step 2 of the proof of Lemma 4.8.

To this end, assume $f_\xi(x_0, \xi_1) = f_\xi(x_0, \xi_2) = \theta_0$. Without loss of generality, assume $f_\xi(x_0, t) > f_\xi(x_0, \xi_1)$ for all $t \in (\xi_1, \xi_2)$. Let

$$\max_{[\xi_1, \xi_2]} f_\xi(x_0, \cdot) = f_\xi(x_0, \bar{\xi}) = \bar{\theta} > \theta_0.$$

To proceed, we need the following lemma, which is the only place we use the smooth assumption of $f_\xi(x, \xi)$ on (x, ξ) .

Lemma 4.9. *There exist a sequence $\theta_n \in (\theta_0, \bar{\theta})$ with $\theta_n \rightarrow \theta_0$ as $n \rightarrow \infty$, a closed interval $J_n = [a_n, b_n] \subset (0, 1)$ containing x_0 , and two continuous functions $q_n^\pm: J_n \rightarrow (\xi_1, \xi_2)$ such that $q_n^-(x) < q_n^+(x)$ and $f_\xi(x, q_n^\pm(x)) = \theta_n$ for all $x \in J_n$. Moreover, $q_n^{-,+}(x_0) \rightarrow \xi_{1,2}$ as $n \rightarrow \infty$.*

Proof. The proof is based on a use of Sard's theorem. By Lemma 4.4 above with $h(\xi) = f_\xi(x_0, \xi)$, the set of regular values of $f_\xi(x_0, \cdot)$ is dense. Hence there exists a sequence of regular values θ_n of $f_\xi(x_0, \cdot)$ in $(\theta_0, \bar{\theta})$ such that $\theta_n \rightarrow \theta_0$ as $n \rightarrow \infty$. Since $f_\xi(x_0, \xi_{1,2}) = \theta_0 < \theta_n < \bar{\theta} = f_\xi(x_0, \bar{\xi})$, by intermediate value property, there exist $\xi_n^- \in (\xi_1, \bar{\xi})$ and $\xi_n^+ \in (\bar{\xi}, \xi_2)$ such that $f_\xi(x_0, \xi_n^\pm) = \theta_n$. The assumption (4.12) implies $\xi_n^- \rightarrow \xi_1$ and $\xi_n^+ \rightarrow \xi_2$ as $n \rightarrow \infty$. Since θ_n is a regular value of $f_\xi(x_0, \cdot)$, it follows that $f_{\xi\xi}(x_0, \xi_n^\pm) \neq 0$. By the implicit function theorem, we have interval $J_n = [a_n, b_n] \subset (0, 1)$ containing x_0 and two differentiable functions $q_n^\pm: J_n \rightarrow (\xi_1, \xi_2)$ such that

$$(4.14) \quad q_n^\pm(x_0) = \xi_n^\pm, \quad f_\xi(x, q_n^\pm(x)) = \theta_n \quad \forall x \in J_n.$$

Then the functions $q_n^\pm(x)$ satisfy the requirements of the lemma. \square

We continue the proof of the theorem. Let $\theta_n \in (\theta_0, \bar{\theta})$, $J_n = [a_n, b_n]$ and $q_n^\pm: J_n \rightarrow (\xi_1, \xi_2)$ be given as in the lemma above. Let $J = [a, b] \subset J_n$ be any interval containing x_0 . Let $q_n \in L^p(0, 1)$ be the function q_θ determined by Lemma 4.7 with $\theta = \theta_n$. In what follows, we fix n . For each $k = 1, 2, \dots$,

we define function $u_k(x)$ by $u_k(x) = \int_0^x w_k(t)dt$, where $w_k(t)$ is defined as follows:

$$(4.15) \quad w_k(t) = \begin{cases} q_n(t) & t \in [0, 1] \setminus [a, b], \\ q_n^-(t) & t \in \cup_{j=1}^k (a + \frac{(j-1)}{k}(b-a), a + \frac{(j-1+\lambda)}{k}(b-a)), \\ q_n^+(t) & t \in \cup_{j=1}^k (a + \frac{(j-1+\lambda)}{k}(b-a), a + \frac{j}{k}(b-a)). \end{cases}$$

It is easily seen that $u_k \in W^{1,p}(0, 1)$ and $\{u_k\}$ is bounded in $W^{1,p}(0, 1)$.

Lemma 4.10. *For all continuous functions $\Phi(x, \xi)$, it follows that*

$$\lim_{k \rightarrow \infty} \int_a^b \Phi(x, u'_k(x))dx = \int_a^b [\lambda \Phi(x, q_n^-(x)) + (1 - \lambda) \Phi(x, q_n^+(x))]dx.$$

Proof. It is easy to see

$$(4.16) \quad \int_a^b \Phi(x, u'_k(x))dx = \sum_{j=1}^k \int_{a + \frac{(j-1)}{k}(b-a)}^{a + \frac{(j-1+\lambda)}{k}(b-a)} \Phi(x, q_n^-(x))dx$$

$$(4.17) \quad + \sum_{j=1}^k \int_{a + \frac{(j-1+\lambda)}{k}(b-a)}^{a + \frac{j}{k}(b-a)} \Phi(x, q_n^+(x))dx$$

$$(4.18) \quad = \lambda \sum_{j=1}^k \Phi(c_j, q_n^-(c_j)) \frac{(b-a)}{k}$$

$$(4.19) \quad + (1 - \lambda) \sum_{j=1}^k \Phi(d_j, q_n^+(d_j)) \frac{(b-a)}{k},$$

where $a + \frac{(j-1)}{k}(b-a) \leq c_j \leq a + \frac{(j-1+\lambda)}{k}(b-a) \leq d_j \leq a + \frac{j}{k}(b-a)$ are some points. Hence the sums in (4.18) and (4.19) are Riemann sums; therefore, as $k \rightarrow \infty$, the lemma follows. \square

Let $\bar{u} \in W^{1,p}(0, 1)$ be defined by $\bar{u}(x) = \int_0^x \bar{w}(t)dt$, where

$$\bar{w}(t) = \begin{cases} q_n(t) & t \in [0, 1] \setminus [a, b], \\ \lambda q_n^-(t) + (1 - \lambda) q_n^+(t) & t \in [a, b]. \end{cases}$$

From the lemma above, it easily follows that $u_k \rightharpoonup \bar{u}$ in $W^{1,p}(0, 1)$. In particular, $\epsilon_k = \bar{u}(1) - u_k(1) \rightarrow 0$ as $k \rightarrow \infty$. By the definition of u_k it follows easily that $f_\xi(x, u'_k(x)) = \theta_n$ for almost every $x \in (0, 1)$; hence $I'(u_k) = 0$ in $W^{-1,p'}(0, 1)$. We now modify u_k to a function $\tilde{u}_k \in W_\beta^{1,p}(0, 1)$ with $\beta = \bar{u}(1)$. For $0 < \delta < 1 - b$ to be selected later, we define $\tilde{u}_k(x) = u_k(x)$ for $x \in [0, 1 - \delta]$, and $\tilde{u}_k(x) = u_k(x) + v_k(x)$ for $x \in [1 - \delta, 1]$, where v_k is a linear function on $[1 - \delta, 1]$ with $v_k(1 - \delta) = 0$, $v_k(1) = \epsilon_k$. Hence $\tilde{u}_k \in W^{1,p}(0, 1)$ with $\tilde{u}_k(0) = 0$, $\tilde{u}_k(1) = \bar{u}(1) = \beta$. Note that $v'_k(x) = \frac{\epsilon_k}{\delta}$. Hence we select $\delta = \delta_k = |\epsilon_k|^{1/2}$ for all sufficiently large k . For this choice of δ , it

is easily shown that the function $\tilde{u}_k \in W_\beta^{1,p}(0,1)$ satisfies $u_k - \tilde{u}_k \rightarrow 0$ in $W^{1,p}(0,1)$, and hence it follows that $I'(\tilde{u}_k) \rightarrow 0$ in $W^{-1,p'}(0,1)$ and $I(u_k) - I(\tilde{u}_k) \rightarrow 0$ as $k \rightarrow \infty$. In particular, $\tilde{u}_k \xrightarrow{\text{PS}} \bar{u}$ in $W_\beta^{1,p}(0,1)$. Therefore, by the (PS)-weak lower semicontinuity of I on $W_\beta^{1,p}(0,1)$, we have $I(\bar{u}) \leq \liminf_k I(\tilde{u}_k) = \liminf_k I(u_k)$. Using Lemma 4.10, after easy computations, this implies

$$\begin{aligned} & \int_a^b f(x, \lambda q_n^-(x) + (1-\lambda)q_n^+(x)) dx \\ & \leq \int_a^b [\lambda f(x, q_n^-(x)) + (1-\lambda)f(x, q_n^+(x))] dx. \end{aligned}$$

This holds for all intervals $[a, b] \subset J_n$ containing x_0 and hence, letting $[a, b]$ shrink to $\{x_0\}$, we have

$$f(x_0, \lambda q_n^-(x_0) + (1-\lambda)q_n^+(x_0)) \leq \lambda f(x_0, q_n^-(x_0)) + (1-\lambda)f(x_0, q_n^+(x_0)).$$

Finally letting $n \rightarrow \infty$, by Lemma 4.9, we have

$$f(x_0, \lambda \xi_1 + (1-\lambda)\xi_2) \leq \lambda f(x_0, \xi_1) + (1-\lambda)f(x_0, \xi_2),$$

as desired by (4.13).

The proof of the theorem is now completed. \square

5. SPECIAL CASES WITH $f = f(\xi)$

In this section, we study some special cases with function $f = f(\xi)$, where $f: M^{m \times n} \rightarrow \mathbf{R}$ is a C^1 function satisfying the following growth conditions:

$$(5.1) \quad c_0 |\xi|^p \leq f(\xi) \leq c_1 (|\xi|^p + 1),$$

$$(5.2) \quad |D_\xi f(\xi)| \leq c_2 (|\xi|^{p-1} + 1),$$

where $1 < p < \infty$ and $c_0 \geq 0, c_1 > 0, c_2 > 0$ are constants. In this case, we shall also use the simplified notation $D_\xi f(\xi) = Df(\xi) = f'(\xi)$. As before, let I be the functional associated with f :

$$I(u) = \int_\Omega f(Du(x)) dx \quad u \in W^{1,p}(\Omega; \mathbf{R}^m).$$

We first have the following result when $m = 1$ (the scalar case) with $c_0 = 0$ in (5.1), which is in contrast to Proposition 4.2 above

Theorem 5.1. *Let $m = 1$ and let $f: \mathbf{R}^n \rightarrow \mathbf{R}$ satisfy the conditions (5.1) and (5.2) above with $c_0 = 0$. Then the functional I is (PS)-weakly lower semicontinuous on the Dirichlet classes $W_A^{1,p}$ for all $A \in \mathbf{R}^n$ if and only if f is convex on \mathbf{R}^n .*

Proof. By Theorem 2.1, we only need to show the necessary part of the theorem. Thus assume I is (PS)-weakly lower semicontinuous on the Dirichlet classes $W_A^{1,p}$ for all $A \in \mathbf{R}^n$. We prove that f is convex on \mathbf{R}^n . To this end, let $\xi, \eta \in \mathbf{R}^n$ and $|\eta| = 1$ be given and let $h(t) = f(\xi + t\eta)$. We show that h is a convex function of $t \in \mathbf{R}$; this implies f is convex on \mathbf{R}^n . By virtue of Lemma 4.8 above, to show h is convex, it suffices to establish the inequality (iii) in that lemma for all $a, b \in \mathbf{R}$ with $a < b$ and $h'(a) = h'(b)$. Note that $h'(t) = f'(\xi + t\eta) \cdot \eta$. Given such a, b , let $\alpha = \xi + a\eta$, $\beta = \xi + b\eta$. Then $h(a) = f(\alpha)$, $h(b) = f(\beta)$ and hence

$$(5.3) \quad h'(a) - h'(b) = (f'(\alpha) - f'(\beta)) \cdot \eta = 0.$$

Given any $\lambda \in (0, 1)$, let $\theta(t)$ be the periodic function on \mathbf{R} of period 1 satisfying $\theta = 0$ on $[0, \lambda)$ and $\theta = 1$ on $[\lambda, 1)$. Let $\rho(t)$ be the Lipschitz function on \mathbf{R} with $\rho(0) = 0$ and $\rho'(t) = \theta(t)$ for almost every $t \in \mathbf{R}$. For $k = 1, 2, \dots$, we define functions

$$(5.4) \quad u_k(x) = \alpha x + \frac{b-a}{k} \rho(kx \cdot \eta) \quad x \in \mathbf{R}^n.$$

Then $Du_k(x) = \alpha + (b-a)\theta(kx \cdot \eta)\eta$ and hence

$$(5.5) \quad Du_k(x) = \begin{cases} \alpha & \text{if } x \cdot \eta \in \cup_{j=-\infty}^{\infty} \left(\frac{j}{k}, \frac{j+\lambda}{k}\right) \\ \beta & \text{if } x \cdot \eta \in \cup_{j=-\infty}^{\infty} \left(\frac{j+\lambda}{k}, \frac{j+1}{k}\right). \end{cases}$$

Let $\{\eta_1, \eta_2, \dots, \eta_n\}$ be an orthonormal basis of \mathbf{R}^n with $\eta_1 = \eta$. For each $x \in \mathbf{R}^n$, we write $x = \sum_{i=1}^n t_i \eta_i$ and define

$$A_k^j = \left\{ x \in \mathbf{R}^n \mid t_1 \in \left(\frac{j}{k}, \frac{j+\lambda}{k}\right) \right\}, \quad B_k^j = \left\{ x \in \mathbf{R}^n \mid t_1 \in \left(\frac{j+\lambda}{k}, \frac{j+1}{k}\right) \right\}.$$

Let $\Omega_\alpha^k = \Omega \cap (\cup_j A_k^j)$, $\Omega_\beta^k = \Omega \cap (\cup_j B_k^j)$. Then one can easily show that

$$(5.6) \quad \lim_{k \rightarrow \infty} |\Omega_\alpha^k| = \lambda |\Omega|, \quad \lim_{k \rightarrow \infty} |\Omega_\beta^k| = (1-\lambda) |\Omega|.$$

For any $1 \leq p < \infty$, the sequence $\{u_k\}$ defined by (5.4) above satisfies $u_k \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$ as $k \rightarrow \infty$, where $\bar{u}(x) = [\lambda\alpha + (1-\lambda)\beta]x$. In fact, one can show that $u_k \rightarrow \bar{u}$ uniformly on $\bar{\Omega}$. We leave the proof of these facts to the interested reader.

Lemma 5.2. $I'(u_k) = 0$ in $W^{-1,p'}(\Omega)$.

Proof. Given any $v \in W_0^{1,p}(\Omega)$, we extend v to be zero outside Ω . Let Q_N be the cube

$$Q_N = \left\{ x \in \mathbf{R}^n \mid x = \sum_{i=1}^n t_i \eta_i, \quad |t_i| < N, \quad \forall i = 1, 2, \dots, n \right\}.$$

Assume N is large enough so that $\bar{\Omega} \subset Q_N$. Then

$$\int_{\Omega} f'(Du_k) \cdot Dv = \int_{Q_N} f'(Du_k) \cdot Dv = \sum_{j=-kN}^{kN-1} \int_{Q_k^j} f'(Du_k) \cdot Dv,$$

where $Q_k^j = \{x \in Q_N \mid x \cdot \eta \in (\frac{j}{k}, \frac{j+1}{k})\}$. We write $Q_k^j = A^j \cup B^j \cup \Gamma^j$, where $A^j = Q_k^j \cap A_k^j$, $B^j = Q_k^j \cap B_k^j$ and $\Gamma^j = \{x \in Q_k^j \mid x \cdot \eta = \frac{j+\lambda}{k}\}$. We also define $F^j = \{x \in Q_N \mid x \cdot \eta = \frac{j}{k}\}$. Note that $Du_k = \alpha$ on A^j and $Du_k = \beta$ on B^j and hence, by the divergence theorem and (5.3) above as well, we have

$$\begin{aligned} \int_{Q_k^j} f'(Du_k) \cdot Dv &= \int_{A^j} f'(Du_k) \cdot Dv + \int_{B^j} f'(Du_k) \cdot Dv \\ &= f'(\alpha) \cdot \int_{A^j} Dv + f'(\beta) \cdot \int_{B^j} Dv \\ &= f'(\alpha) \cdot \left(\int_{F^j} v dS \right) (-\eta) + f'(\alpha) \cdot \left(\int_{\Gamma^j} v dS \right) \eta \\ &\quad + f'(\beta) \cdot \left(\int_{\Gamma^j} v dS \right) (-\eta) + f'(\beta) \cdot \left(\int_{F^{j+1}} v dS \right) \eta \\ &= f'(\alpha) \cdot \eta \left(\int_{F^{j+1}} v dS - \int_{F^j} v dS \right). \end{aligned}$$

Hence, since $F^{\pm kN}$ lies in $\mathbf{R}^n \setminus \bar{\Omega}$, where $v = 0$, it follows that

$$\int_{\Omega} f'(Du_k(x)) \cdot Dv(x) dx = f'(\alpha) \cdot \eta \left(\int_{F^{kN}} v dS - \int_{F^{-kN}} v dS \right) = 0.$$

This proves $I'(u_k) = 0$ in $W^{-1,p'}(\Omega)$. \square

To continue the proof of the theorem, we now modify the sequence $\{u_k\}$ above into a sequence in $W_A^{1,p}(\Omega)$, where $A = \lambda\alpha + (1-\lambda)\beta$. For all sufficiently large j , say $j \geq j_0$, consider nonempty open sets

$$\Omega_j = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > 1/j\}.$$

Note that the measure $|\Omega \setminus \Omega_j| \rightarrow 0$ as $j \rightarrow \infty$. Let $\varphi_j \in C_0^\infty(\Omega)$ be the cut-off functions such that $\varphi_j = 1$ on Ω_j and $0 \leq \varphi_j \leq 1$ in Ω . Since $u_k \rightarrow \bar{u}$ uniformly on $\bar{\Omega}$, we have that, for each $j \geq j_0$, there exists $k_j > j$ satisfying

$$(5.7) \quad \|(u_{k_j} - \bar{u})D\varphi_j\|_{L^p(\Omega)} < \frac{1}{j}.$$

Let $\tilde{u}_j = \varphi_j u_{k_j} + (1 - \varphi_j)\bar{u}$. Then $\tilde{u}_j \in W_{\bar{u}}^{1,p}(\Omega) = W_A^{1,p}(\Omega)$ and $D\tilde{u}_j = \varphi_j Du_{k_j} + (1 - \varphi_j)D\bar{u} + (u_{k_j} - \bar{u})D\varphi_j$. Hence, by (5.7) and also since $Du_{k_j}, D\bar{u}$ are bounded, it follows that

$$(5.8) \quad \lim_{j \rightarrow \infty} \|D\tilde{u}_j\|_{L^p(\Omega \setminus \Omega_j)} = 0.$$

Therefore $\tilde{u}_j \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$ as $j \rightarrow \infty$. Since $\tilde{u}_j = u_{k_j}$ on Ω_j , by (5.8) and the growth conditions (5.1)-(5.2), it easily follows that

$$\lim_{j \rightarrow \infty} \|I'(\tilde{u}_j) - I'(u_{k_j})\|_{W^{-1,p'}(\Omega)} = 0, \quad \lim_{j \rightarrow \infty} |I(\tilde{u}_j) - I(u_{k_j})| = 0.$$

Hence $\tilde{u}_j, \bar{u} \in W_A^{1,p}(\Omega)$, and $\tilde{u}_j \xrightarrow{\text{ps}} \bar{u}$ since $I'(u_{k_j}) = 0$. By the (PS)-weak lower semicontinuity of I on $W_A^{1,p}(\Omega)$, we have $I(\bar{u}) \leq \liminf_j I(\tilde{u}_j) = \liminf_j I(u_{k_j})$. Using (5.6), we easily see that this implies

$$h(\lambda a + (1 - \lambda)b) \leq \lambda h(a) + (1 - \lambda)h(b).$$

Hence by Lemma 4.8 above, $h(t) = f(\xi + t\eta)$ is convex for all ξ, η with $|\eta| = 1$. This proves f is convex on \mathbf{R}^n . \square

We now study the general case with $m \geq 2$. Under the coercivity condition that $c_0 > 0$ in (5.1), we have the following result, which was announced a few years ago in [13], but has never been published; we include it here for the convenience of the reader.

Theorem 5.3. *Let $n, m \geq 1$ and let $f: M^{m \times n} \rightarrow \mathbf{R}$ satisfy the conditions (5.1) and (5.2) above with $c_0 > 0$. Then the following statements are equivalent:*

- (i) I is weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbf{R}^m)$.
- (ii) I is (PS)-weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbf{R}^m)$.
- (iii) I is (PS)-weakly lower semicontinuous on all $W_A^{1,p}(\Omega; \mathbf{R}^m)$.
- (iv) f is quasiconvex.
- (v) $I(g_A) \leq \liminf_{k \rightarrow \infty} I(u_k)$ whenever $A \in M^{m \times n}$, $u_k \in W_A^{1,p}$ and $I'(u_k) \rightarrow 0$ in $W^{-1,p'}(\Omega; \mathbf{R}^m)$.

Proof. By the theorem of Acerbi-Fusco (Theorem 2.1), (i) \Leftrightarrow (iv) even when $c_0 = 0$. Moreover, by the definition of quasiconvexity and using approximation, if f is quasiconvex and only satisfies (5.1) with $c_0 \in \mathbf{R}$, then it readily follows that $I(g_A) \leq I(u)$ for all $u \in W_A^{1,p}(\Omega; \mathbf{R}^m)$; hence (iv) \Rightarrow (v). It is also obvious that (i) \Rightarrow (ii) \Rightarrow (iii) in general cases.

Therefore, to prove the theorem, it suffices to show that (iii) \Rightarrow (iv) and that (v) \Rightarrow (iv) (in fact if $c_0 \geq 0$). We prove them as separate results in two lemmas below. \square

Lemma 5.4. *Under the assumptions (5.1) with $c_0 \geq 0$ and (5.2), (v) \Rightarrow (iv).*

Proof. The proof uses the Ekeland variational principle as given by Theorem 2.2. Given $A \in M^{m \times n}$, define a functional Φ on $X = W_0^{1,p}(\Omega; \mathbf{R}^m)$ by

$$\Phi(v) = \int_{\Omega} f(A + Dv(x)) dx = I(g_A + v).$$

Then $\Phi: X \rightarrow \mathbf{R}$ is C^1 and bounded below, and $\Phi'(v) = I'(g_A + v)$. By Theorem 2.2, there exists a sequence $\{v_k\}$ in X such that

$$\Phi(v_k) \rightarrow \inf_{v \in X} \Phi(v); \quad \|\Phi'(v_k)\|_{X^*} \rightarrow 0.$$

Let $u_k = g_A + v_k \in W_A^{1,p}(\Omega; \mathbf{R}^m)$. Then $I(u_k) = \Phi(v_k) \rightarrow \inf_X \Phi$ and $\|I'(u_k)\|_{X^*} = \|\Phi'(v_k)\|_{X^*} \rightarrow 0$. Therefore, by (v), it follows that

$$f(A)|\Omega| = I(g_A) \leq \liminf_{k \rightarrow \infty} I(u_k) = \inf_{v \in X} \Phi(v) = \inf_{v \in X} I(g_A + v).$$

This implies

$$f(A)|\Omega| \leq \int_{\Omega} f(A + Dv(x)) \, dx, \quad \forall v \in W_0^{1,p}(\Omega; \mathbf{R}^m)$$

and thus f is quasiconvex. \square

Lemma 5.5. *Under the assumptions (5.1) with $c_0 > 0$ and (5.2), (iii) \Rightarrow (iv).*

Proof. Given $A \in M^{m \times n}$, by Theorem 3.5 (note that $c_0 > 0$ is needed here), there exists $\bar{u} \in W_A^{1,p}(\Omega; \mathbf{R}^m)$ which is a minimizer of $I(u)$ on $W_A^{1,p}(\Omega; \mathbf{R}^m)$. We now apply the standard technique of Vitali covering [5] to construct a sequence $\{u_k\}$ in $W_A^{1,p}(\Omega; \mathbf{R}^m)$ satisfying

$$(5.9) \quad I(u_k) = I(\bar{u}) = \inf_{u \in W_A^{1,p}} I(u);$$

$$(5.10) \quad u_k \rightharpoonup g_A \quad \text{in } W^{1,p}(\Omega; \mathbf{R}^m) \text{ as } k \rightarrow \infty;$$

$$(5.11) \quad I'(u_k) = 0 \quad \text{in } W^{-1,p'}(\Omega; \mathbf{R}^m).$$

Note that (5.11) will follow from (5.9) since $u_k \in W_A^{1,p}(\Omega; \mathbf{R}^m)$ is also a minimizer of $I(u)$ on $W_A^{1,p}(\Omega; \mathbf{R}^m)$. Once we have constructed such a sequence $\{u_k\}$, which certainly satisfies $u_k \xrightarrow{\text{ps}} g_A$, the (PS)-weak lower semicontinuity condition (iii) will imply

$$I(g_A) \leq \liminf_{k \rightarrow \infty} I(u_k) = I(\bar{u}) = \inf_{u \in W_A^{1,p}} I(u),$$

for all $A \in M^{m \times n}$, which is exactly the quasiconvexity condition of f , and hence the result follows. Assume, without loss of generality, $0 \in \Omega$ and then we use the Vitali covering theorem to decompose Ω as follows:

$$\Omega = \cup_{j=1}^{\infty} \bar{\Omega}_j \cup N; \quad \bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset \quad (i \neq j),$$

where $\Omega_j = a_j + \epsilon_j \Omega \subset \subset \Omega$ with $a_j \in \Omega$, $0 < \epsilon_j < 1/k$, and $|N| = 0$. Let $\bar{u} = g_A + \bar{v}$, where $\bar{v} \in W_0^{1,p}(\Omega; \mathbf{R}^m)$. We define

$$(5.12) \quad u_k(x) = \begin{cases} Ax + \epsilon_j \bar{v}(\frac{x-a_j}{\epsilon_j}) & \text{if } x \in \Omega_j, \\ Ax & \text{otherwise.} \end{cases}$$

Then one can easily check that u_k belongs to $W_A^{1,p}(\Omega; \mathbf{R}^m)$ and satisfies

$$\int_{\Omega} \psi(Du_k(x)) dx = \int_{\Omega} \psi(D\bar{u}(x)) dx$$

for all continuous functions $\psi: M^{m \times n} \rightarrow \mathbf{R}$ satisfying $|\psi(\xi)| \leq C(|\xi|^p + 1)$. Certainly this implies (5.9). Furthermore, it is easy to see

$$\|u_k - g_A\|_{L^p(\Omega)} \leq \frac{1}{k} \|\bar{u} - g_A\|_{L^p(\Omega)}.$$

Hence $u_k \rightarrow g_A$ as $k \rightarrow \infty$. As mentioned above, condition (5.11) follows from (5.9). This completes the construction of $\{u_k\}$ and thus the proof of the lemma. \square

Remark 5.1. For any $u \in W_A^{1,p}(\Omega; \mathbf{R}^m)$, we write $u = g_A + v$ with $v \in W_0^{1,p}(\Omega; \mathbf{R}^m)$ and define sequence $u_k \in W_A^{1,p}(\Omega; \mathbf{R}^m)$ as in (5.12) above with $\bar{v} = v$. Then, if u is not a minimizer of I over $W_A^{1,p}(\Omega; \mathbf{R}^m)$, one only has $I'(u_k) \rightarrow 0$, but not strongly, in $W^{-1,p'}(\Omega; \mathbf{R}^m)$, as $k \rightarrow \infty$, even when $I'(u) = 0$; hence the (PS)-weak lower semicontinuity can not be applied to this sequence.

Finally we show that, without the coercivity assumption $c_0 > 0$ in (5.1), the results of Theorem 5.3 may fail in general; we give an example in the case $n = 1, m = 2$.

Theorem 5.6. *Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ be a C^1 function satisfying the conditions (5.1) and (5.2) with $c_0 = 0$. Suppose the derivative map $Df = f': \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is one-to-one. Then the functional $I(u) = \int_0^1 f(u'(x)) dx$ is (PS)-weakly lower semicontinuous on the Sobolev space $X = W^{1,p}((0, 1); \mathbf{R}^2)$.*

Proof. Let $u \in X = W^{1,p}((0, 1); \mathbf{R}^2)$. Then

$$\langle I'(u), v \rangle = \int_0^1 [f_{\xi_1}(u'(x))(v^1)' + f_{\xi_2}(u'(x))(v^2)'] dx, \quad \forall v = (v^1, v^2) \in X,$$

and hence it can be shown that

$$\|I'(u)\|_{W^{-1,p'}} \cong \|f_{\xi_1}(u') - C_1(u)\|_{L^{p'}(0,1)} + \|f_{\xi_2}(u') - C_2(u)\|_{L^{p'}(0,1)},$$

where $C_1(u), C_2(u)$ are two constants depending *boundedly* on $u \in X$. Assume $u_k \xrightarrow{\text{ps}} u$ in X . We also assume that $\lim_{k \rightarrow \infty} I(u_k)$ exists. Then there exists a subsequence $\{u_{k_j}\}$ such that

$$\|f_{\xi_1}(u'_{k_j}) - C_1\|_{L^{p'}(0,1)} + \|f_{\xi_2}(u'_{k_j}) - C_2\|_{L^{p'}(0,1)} \rightarrow 0$$

as $j \rightarrow \infty$, where C_1, C_2 are some constants; from this, we also assume there exists a measurable set $E \subset (0, 1)$ such that

$$(5.13) \quad |E| = 1, \quad \lim_{j \rightarrow \infty} f_{\xi_\nu}(u'_{k_j}(x)) = C_\nu \quad \forall x \in E \quad (\nu = 1, 2).$$

Note that for all $M > 0$ the measure $|\{x \in E \mid |u'_k(x)| > M\}| \leq \frac{C}{M^p}$ for all k , where C is a constant; hence there exists a sufficiently large $M > 0$ such that the measure $|E_j| > \frac{1}{2}$, where $E_j = \{x \in E \mid |u'_{k_j}(x)| \leq M\}$. It is then an easy exercise that there exists a subsequence $\{E_{j_s}\}$, with $j_s \rightarrow \infty$ as $s \rightarrow \infty$, of sets $\{E_j\}$ such that the set $E_\infty = \bigcap_{s=1}^\infty E_{j_s}$ is non-empty; we leave the proof to the interested reader. Therefore there exists at least one $x_0 \in E_\infty$, for which $|u'_{k_{j_s}}(x_0)| \leq M$ for all $s = 1, 2, \dots$. By taking a further subsequence, we have $u'_{k_{j_s}}(x_0) \rightarrow \alpha \in \mathbf{R}^2$ along a subsequence of $s \rightarrow \infty$. Therefore, by (5.13), it follows that $f'(u_{k_j}) = (C_1, C_2)$. Since the map f' is one-to-one, from (5.13) it follows that $u'_{k_j}(x) \rightarrow \alpha$ as $j \rightarrow \infty$ for all $x \in E$. This implies that $u'(x) = \alpha$ and $f(u'_{k_j}(x)) \rightarrow f(\alpha)$ as $j \rightarrow \infty$ for all $x \in E$. Hence, by Fatou's lemma,

$$I(u) = f(\alpha) \leq \liminf_{j \rightarrow \infty} I(u_{k_j}) = \lim_{k \rightarrow \infty} I(u_k),$$

which proves the (PS)-weak lower semicontinuity of I on X . \square

Remark 5.2. Note that if $f(\xi_1, \xi_2) = \varphi(\xi_1 - \xi_2^2)$ on \mathbf{R}^2 , where $\varphi \geq 0$ is any C^1 function with a strictly increasing derivative $\varphi' > 0$ on \mathbf{R} , then $f': \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is one-to-one, but f is not convex on \mathbf{R}^2 . An example of such a φ is given by

$$\varphi(t) = \begin{cases} e^t & t \leq 0; \\ t^2 + t + 1 & t > 0. \end{cases}$$

Note that the corresponding function $f(\xi) = f(\xi_1, \xi_2) = \varphi(\xi_1 - \xi_2^2)$ then also satisfies the conditions (5.1) and (5.2) with $c_0 = 0$ and $p = 4$. For such a function $h = f$, the condition (iii) in Lemma 4.8 above holds automatically, but h is not convex; this shows that conditions (i) and (iii) in Lemma 4.8 are not equivalent in general for functions $h: \mathbf{R}^2 \rightarrow \mathbf{R}$.

REFERENCES

- [1] E. Acerbi & N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal., **86** (1984), 125–145.
- [2] J.-P. Aubin & I. Ekeland, “Applied Nonlinear Analysis,” John Wiley & Sons, New York, 1984.
- [3] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal., **63** (1977), 337–403.
- [4] J. M. Ball & R. D. James, *Proposed experimental tests of a theory of fine microstructures and the two well problem*, Phil. Trans. Roy. Soc. London, A **338** (1992), 389–450.
- [5] B. Dacorogna, “Direct Methods in the Calculus of Variations,” Springer-Verlag, New York, 1989.
- [6] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., **47** (1974), 324–353.
- [7] I. Fonseca & S. Müller, *A-quasiconvexity, lower semicontinuity and Young measures*, SIAM J. Math. Anal., **30** (1999), 1355–1390.

- [8] D. Kinderlehrer & P. Pedregal, *Gradient Young measures generated by sequences in Sobolev spaces*, J. Geom. Anal., **4**(1) (1994), 59–90.
- [9] J. Milnor, “Topology from the Differentiable Viewpoint,” University of Virginia Press, 1965.
- [10] C. B. Morrey, Jr., *Quasiconvexity and the lower semicontinuity of multiple integrals*, Pacific J. Math., **2** (1952), 25–53.
- [11] S. Müller, *Variational models for microstructure and phase transitions*, in “Calculus of Variations and Geometric Evolution Problems,” (Cetraro, 1996), pp. 85–210, Lecture Notes in Math. **1713**, Springer, 1999.
- [12] L. Tartar, *Compensated compactness and applications to partial differential equations*, in “Nonlinear Analysis and Mechanics: Heriot-Watt Symposium,” Vol. IV, pp. 136–212, Res. Notes in Math., 39. Pitman, Boston, Mass.-London, 1979.
- [13] B. Yan, “Quasiconvexity and Restricted Lower Semicontinuity in the Calculus of Variations,” AMS Meeting, Ann Arbor, Michigan, (March 2, 2002).

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