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ENERGY NORM A POSTERIORI ERROR ESTIMATION OF HP-ADAPTIVE DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC PROBLEMS

PAUL HOUSTON *, DOMINIK SCHÖTZAU †, AND THOMAS P. WIHLER ‡

Abstract. In this paper, we develop the a posteriori error estimation of hp-version interior penalty discontinuous Galerkin discretizations of elliptic boundary-value problems. Computable upper and lower bounds on the error measured in terms of a natural (mesh-dependent) energy norm are derived. The bounds are explicit in the local mesh sizes and approximation orders. A series of numerical experiments illustrate the performance of the proposed estimators within an automatic hp-adaptive refinement procedure.

Key words. Discontinuous Galerkin methods, a posteriori error estimation, hp-adaptivity, elliptic problems.

AMS subject classifications. 65N30, 65N35, 65N50

1. Introduction. In this article, we derive hp-version energy norm a posteriori error bounds for discontinuous Galerkin discretizations of the following elliptic model problem: find u such that

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = g \qquad \text{on } \Gamma. \tag{1.2}$$

Here, we take Ω to be a polygonal Lipschitz domain in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$, f is a given source term in $L^2(\Omega)$ and g is a Dirichlet boundary datum in $H^{1/2}(\Gamma)$. The standard weak formulation of (1.1)–(1.2) is to find $u \in H^1(\Omega)$ such that $u|_\Gamma = g$ and

$$A(u, v) := \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} = \int_\Omega f v \, d\mathbf{x} \tag{1.3}$$

for all $v \in H_0^1(\Omega)$.

It is well-known that solutions of elliptic problems in polygons may exhibit corner singularities. This lack of smoothness typically results in degraded convergence rates of h- and also p-version finite element methods used for the discretization of such problems. However, it has been shown that hp-version finite element methods, which are based on locally refined meshes and variable approximation orders, can achieve exponential rates of convergence despite the presence of singularities; for details, we refer the reader to [5, 16, 17, 37] and the references cited therein.

Discontinuous Galerkin methods (DG, for short) are particularly well-suited for the application within hp-version algorithms. Indeed, within a DG approach, non-matching grids containing hanging nodes and non-uniform polynomial approximation degrees can easily be handled. For a recent survey on DG methods, we refer the reader to the articles [11, 12, 13]. The a priori error analysis of DG methods for elliptic problems is relatively well understood by now and a large body of literature is available; in the context of the hp-version of the DG method, we mention here the articles [10, 15, 23, 31, 32, 34, 35, 36, 39] and the references cited therein. However, there are considerably fewer papers that are concerned with

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the a posteriori error estimation for DG methods applied to elliptic boundary-value problems. Regarding L^2-norm or functional error estimation for the h-version of the DG method, we refer the reader to [7, 27, 33] and the references therein. An extension to hp-adaptive DG methods was considered in [18]; see, also, [25] for a recent review of goal-oriented a posteriori error estimation for both conforming and DG finite element methods. Energy norm error estimation for h-version DG methods for diffusion problems has been studied in [6, 7, 8, 28]. Finally, mixed DG approximations to the time-harmonic Maxwell operator and the Stokes equations are considered in [20, 21], respectively.

While the proof of the energy norm a posteriori error bounds in [7] hinges on a saturation assumption on the finite element spaces, the analyses in [6, 8, 20] are based on exploiting suitable Helmholtz decompositions of the error, together with the local conservation properties of the underlying DG method (see [9] and the references therein for closely related a posteriori error estimation techniques for nonconforming methods). Yet another approach was developed in [28]; it crucially relies on the approximation of discontinuous finite element functions by conforming ones. A similar idea was then used in the recent article [21], where energy norm a posteriori error bounds are derived for the mixed h-version DG approximation of the Stokes problem. Indeed, the approach in [21] employs a norm equivalence property for discontinuous finite element spaces (see [19, Theorem 5.3]), which is a consequence of the approximation results in [28, Section 2.1]. However, in contrast to the analysis in [28], the proof of the bound in [21] is based on rewriting the method in a non-consistent manner using lifting operators in the spirit of [1], see also [32]. Thereby, it is possible to derive DG a posteriori error bounds under minimal regularity assumptions on the underlying analytical solution. Here, we emphasize that all the work above is concerned with the h-version of DG methods only.

In this paper, we extend the technique proposed in [21] to the hp-version of the DG method and derive reliable upper bounds on the error measured in terms of a natural (mesh-dependent) energy norm for the DG approximation of the elliptic boundary-value problem (1.1)–(1.2); we refer to our recent conference paper [22] for the extension of this analysis to the hp-DG approximation of the Stokes equations. In particular, we generalize the crucial norm equivalence result in [21] to hp-DG finite element spaces consisting of either triangular or quadrilateral elements, under the assumption that the computational mesh is conforming. With this result, the upper (reliability) bounds are then obtained using the arguments from [21] and the hp-version Scott-Zhang interpolant from [29]. Additionally, lower (efficiency) bounds will also be derived; they follow with only minor modifications from the techniques presented in [30]. As in the case of the conforming hp-finite element methods considered there, reliability and efficiency of our error bounds cannot be established uniformly with respect to the polynomial degree, since the proof of efficiency relies on employing inverse estimates which are suboptimal in the spectral order. Finally, numerical experiments highlighting the performance of the proposed estimator within an hp-adaptive mesh refinement algorithm will also be undertaken.

The outline of this article is as follows. In Section 2, we introduce the hp-DG method for the numerical approximation of the boundary-value problem (1.1)–(1.2). In Section 3, our a posteriori error bounds are presented and discussed; here both upper and lower energy norm bounds will be derived. The proofs of these results will be presented in Section 4; Section 5 will be devoted to the proof of the norm equivalence property for hp-DG spaces. In Section 6, we present a series of numerical experiments to illustrate the performance of the proposed error estimators within an automatic hp-mesh refinement algorithm. Finally, in Section 7 we summarize the work presented in this paper and draw some conclusions.

Throughout the paper, we use the following standard function spaces. For a polygonal

Lipschitz domain $D \subset \mathbb{R}^2$ or an interval $D = (a, b)$ we denote by $H^k(D)$ the Sobolev space of order $k \in \mathbb{N}_0$, endowed with the semi-norm $|\cdot|_{k,D}$ and norm $\|\cdot\|_{k,D}$. For $k \geq 0$ non-integer, we define $H^k(D)$ and $\|\cdot\|_{k,D}$ by the K-method of interpolation; see [38]. We denote by $H^k(D)^2$ the space of vector fields with components in $H^k(D)$; for simplicity, the standard product norm in $H^k(D)^2$ is also denoted by $\|\cdot\|_{k,D}$. Furthermore, we set $L^2(D) = H^0(D)$ and write $H^1_0(D)$ for the subspace of $H^1(D)$ of functions with vanishing trace on D. The space $H^{1/2}(\partial D)$ is the space of traces of functions in $H^1(\Omega)$; we denote its usual norm by $\|\cdot\|_{1/2,\partial D}$. Finally, for an interval $D = (a, b)$, let $H^{1/2}_{00}(D)$ be the interpolation space $(L^2(D), H^1_0(D))_{1/2}$ defined by the K-method of interpolation. We endow it with the interpolation norm $\|\cdot\|_{1/2,00,D}$. It is well-known that this norm is equivalent to the following expression:

$$\|v\|^2_{1/2,00,D} \approx \|v\|^2_{1/2,D} + \int_a^b \frac{v(x)^2}{(x-a)(x-b)}\, dx, \qquad v \in H^{1/2}_{00}(D).$$

If D is a straight bounded line segment in $\mathbb{R}^2$, the space $H^{1/2}_{00}(D)$ can be defined straightforwardly by mapping D onto an interval.

2. The hp-version interior penalty method. In this section, we introduce the hp-version interior penalty discontinuous Galerkin method for the approximation of (1.1)–(1.2).

2.1. Finite element spaces. We consider shape-regular conforming meshes $\mathcal{T}_h$ that partition $\Omega \subset \mathbb{R}^2$ into open triangles and parallelograms $\{K\}_{K\in\mathcal{T}_h}$. Each element $K \in \mathcal{T}_h$ can then be affinely mapped onto the generic reference element $\widehat{K}$ which is either the triangle $\widehat{T} = \{\, (x, y) \,:\, -1 < x < 1,\ 0 < y < \sqrt{3}\min(1 + x, 1 - x) \,\}$ or the square $\widehat{S} = (-1, 1)^2$, respectively.

The diameter of an element $K \in \mathcal{T}_h$ is denoted by h_K. Due to our assumptions on the meshes, these diameters are of bounded variation, that is, there is a constant $\rho_1 \geq 1$ such that

$$\rho_1^{-1} \leq h_K/h_{K'} \leq \rho_1, \tag{2.1}$$

whenever K and K' share a common edge. We store the elemental diameters in the mesh size vector $\mathbf{h}$ given by $\mathbf{h} = \{h_K \,:\, K \in \mathcal{T}_h\}$. Similarly, we associate with each element $K \in \mathcal{T}_h$ a polynomial degree $k_K \geq 1$ and define the degree vector $\mathbf{k} = \{k_K \,:\, K \in \mathcal{T}_h\}$. We assume that $\mathbf{k}$ is of bounded variation as well, that is, there is a constant $\rho_2 \geq 1$ such that

$$\rho_2^{-1} \leq k_K/k_{K'} \leq \rho_2, \tag{2.2}$$

whenever K and K' share a common edge. Additionally, we suppose that there exists a constant $\rho_3 > 0$ such that

$$|k_K - k_{K'}| \leq \rho_3, \tag{2.3}$$

for all pairs of boundary elements $K, K' \in \mathcal{T}_h$ with $\partial K \cap \partial K' \cap \Gamma \neq \emptyset$. The technical assumption (2.3) is required in Lemma 4.6 where we recall the hp-Scott-Zhang interpolation result from [29, Theorem 2.4] for the approximation of H^1-functions with inhomogeneous boundary conditions.

For a partition $\mathcal{T}_h$ of Ω and a degree distribution $\mathbf{k}$, we define the hp-version discontinuous Galerkin finite element space V_h by

$$V_h = \{\, v \in L^2(\Omega) \,:\, v|_K \in \mathcal{S}_{k_K}(K),\ K \in \mathcal{T}_h \,\}. \tag{2.4}$$

Here, $\mathcal{S}_{k_K}(K)$ is the space $\mathcal{P}_{k_K}(K)$ of polynomials of total degree $\leq k_K$, if K is a triangle, or the space $\mathcal{Q}_{k_K}(K)$ of polynomials of degree $\leq k_K$ in each variable, if K is a parallelogram.

2.2. Trace operators. Next, we define the trace operators that are required for the interior penalty method. To this end, we denote by $\mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)$ the set of all interior edges of the partition $\mathcal{T}_h$ of Ω, and by $\mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)$ the set of all boundary edges of $\mathcal{T}_h$. Furthermore, we define $\mathcal{E}(\mathcal{T}_h) = \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h) \cup \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)$. The boundary ∂K of an element K and the sets $\partial K \setminus \Gamma$ and $\partial K \cap \Gamma$ will be identified in a natural way with the corresponding subsets of $\mathcal{E}(\mathcal{T}_h)$.

Let K^+ and K^- be two adjacent elements of $\mathcal{T}_h$, and $\mathbf{x}$ an arbitrary point on the interior edge $\kappa \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)$ given by $\kappa = \partial K^+ \cap \partial K^-$. Furthermore, let v and $\mathbf{q}$ be scalar- and vector-valued functions, respectively, that are smooth inside each element $K^\pm$. By $(v^\pm, \mathbf{q}^\pm)$, we denote the traces of $(v, \mathbf{q})$ on κ taken from within the interior of $K^\pm$, respectively. Then, the averages of v and $\mathbf{q}$ at $\mathbf{x} \in \kappa$ are given by

$$\{\!\{v\}\!\} = \frac{1}{2}(v^+ + v^-), \qquad \{\!\{\mathbf{q}\}\!\} = \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-),$$

respectively. Similarly, the jumps of v and $\mathbf{q}$ at $\mathbf{x} \in \kappa$ are given by

$$[\![v]\!] = v^+ \,\mathbf{n}_{K^+} + v^- \,\mathbf{n}_{K^-}, \qquad [\![\mathbf{q}]\!] = \mathbf{q}^+ \cdot \mathbf{n}_{K^+} + \mathbf{q}^- \cdot \mathbf{n}_{K^-},$$

respectively, where we denote by $\mathbf{n}_{K^\pm}$ the unit outward normal vector of $\partial K^\pm$, respectively.

On a boundary edge $\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)$, we set $\{\!\{v\}\!\} = v$, $\{\!\{\mathbf{q}\}\!\} = \mathbf{q}$ and $[\![v]\!] = v\mathbf{n}$, with $\mathbf{n}$ denoting the unit outward normal vector on the boundary Γ.

2.3. Interior penalty discretization. For a mesh $\mathcal{T}_h$ on Ω and a polynomial degree vector $\mathbf{k}$, let V_h be the hp-version finite element space defined in (2.4). We consider the interior penalty discretization of (1.1)–(1.2): find $u_h \in V_h$ such that

$$A_h(u_h, v) = F_h(v) \tag{2.5}$$

for all $v \in V_h$, where

$$\begin{aligned} A_h(u, v) = &\sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, d\mathbf{x} - \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \left(\{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] + \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \right) ds \\ &+ \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \mathtt{c}\, [\![u]\!] \cdot [\![v]\!] \, ds, \\ F_h(v) = &\int_\Omega f v \, d\mathbf{x} - \sum_{\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)} \int_\kappa g \nabla_h v \cdot \mathbf{n} \, ds + \sum_{\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)} \int_\kappa \mathtt{c} g v \, ds. \end{aligned}$$

Here, ∇_h denotes the elementwise gradient operator. Moreover, the function $\mathtt{c} \in L^\infty(\mathcal{E}(\mathcal{T}_h))$ is the discontinuity stabilization function that is chosen as follows: we define the functions $\mathtt{h} \in L^\infty(\mathcal{E}(\mathcal{T}_h))$ and $\mathtt{k} \in L^\infty(\mathcal{E}(\mathcal{T}_h))$ by

$$\mathtt{h}(\mathbf{x}) := \begin{cases} \min(h_K, h_{K'}), & \mathbf{x} \in \kappa \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h),\ \kappa = \partial K \cap \partial K', \\ h_K, & \mathbf{x} \in \kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h),\ \kappa \in \partial K \cap \Gamma, \end{cases}$$

$$\mathtt{k}(\mathbf{x}) := \begin{cases} \max(k_K, k_{K'}), & \mathbf{x} \in \kappa \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h),\ \kappa = \partial K \cap \partial K', \\ k_K, & \mathbf{x} \in \kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h),\ \kappa \in \partial K \cap \Gamma, \end{cases}$$

and set

$$\mathtt{c} = \gamma \mathtt{k}^2 \mathtt{h}^{-1}, \tag{2.6}$$

with a parameter $\gamma > 0$ that is independent of $\mathbf{h}$ and $\mathbf{k}$.

It can be shown that there is a parameter $\gamma_{\min} > 0$ independent of $\mathbf{h}$ and $\mathbf{k}$ such that for $\gamma \geq \gamma_{\min}$ the DG method in (2.5) is stable and possesses a unique solution; cf. [39, Proposition 3.8], for example.

3. Energy norm a posteriori error estimation. In this section, we present our main results.

3.1. A reliable a posteriori error bound. We first state and discuss an energy norm hp-a posteriori error estimate which provides a reliable upper bound on the approximation error for the DG method (2.5). To this end, we introduce the space

$$V(h) = V_h + H^1(\Omega),$$

and endow it with the norm

$$\|v\|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} \int_K |\nabla v|^2 \, d\mathbf{x} + \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |[\![v]\!]|^2 \, ds; \tag{3.1}$$

we shall refer to $\| \cdot \|_{1,h}$ as the *energy* norm.

Next, in order to define our a posteriori error indicators, we let g_h be a piecewise polynomial approximation in $H^{1/2}(\Gamma)$ of the Dirichlet datum g. That is, g_h is the trace of a function in $H^1(\Omega)$ and satisfies

$$g_h|_\kappa \in \mathcal{P}_{k_K}(\kappa), \qquad \kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h), \ \kappa \in \partial K \cap \Gamma, K \in \mathcal{T}_h. \tag{3.2}$$

For example, if g belongs to $C^0(\Gamma)$ and is smooth on each edge, it is possible to construct an approximation g_h such that $g_h(P) = g(P)$ at the end points of each edge $\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)$ and such that $\|g - g_h\|_{1/2,00,\kappa}$ is optimally convergent in the mesh size and the polynomial degree (up to a logarithmic loss in the polynomial degree). This leads to an optimal approximation of g in $H^{1/2}(\Gamma)$; we refer the reader to [4] and the references cited therein for details.

For each element $K \in \mathcal{T}_h$, we then introduce the local error indicator η_K which is given by the sum of the three terms

$$\eta_K^2 = \eta_{R_K}^2 + \eta_{E_K}^2 + \eta_{J_K}^2. \tag{3.3}$$

The first term η_{R_K} is the interior residual defined by

$$\eta_{R_K}^2 = k_K^{-2} h_K^2 \|\Pi_h f + \Delta u_h\|_{0,K}^2, \tag{3.4}$$

where $\Pi_h f$ denotes the elementwise $L^2(K)$-projection of f onto the space $\mathcal{S}_{k_K - 1}(K)$ of polynomials of degree $k_K - 1$. The second term η_{E_K} is the edge residual given by

$$\eta_{E_K}^2 = \frac{1}{2} \sum_{\kappa \in \partial K \setminus \Gamma} \|\mathtt{k}^{-1/2} \mathtt{h}^{1/2} [\![\nabla_h u_h]\!]\|_{0,\kappa}^2. \tag{3.5}$$

Finally, the third term η_{J_K} measures the jumps of the approximate solution u_h and is defined by

$$\eta_{J_K}^2 = \sum_{\kappa \in \partial K \setminus \Gamma} \gamma \frac{1}{2} \|\mathtt{k}\mathtt{h}^{-1/2} [\![u_h]\!]\|_{0,\kappa}^2 + \sum_{\kappa \in \partial K \cap \Gamma} \gamma \|\mathtt{k}\mathtt{h}^{-1/2} (u_h - g_h)\|_{0,\kappa}^2, \tag{3.6}$$

with g_h denoting the approximation of g in (3.2).

The following theorem states that, up to standard data approximation terms, the error indicators in (3.3)–(3.6) give rise to a reliable energy norm a posteriori error bound.

THEOREM 3.1. *Let u be the analytical solution of* (1.1)–(1.2) *and $u_h \in V_h$ its DG approximation obtained by* (2.5) *with $\gamma \geq \max(1, \gamma_{\min})$. Let the local error indicators be defined by* (3.3)–(3.6)*. Then, the following a posteriori error bound holds*

$$\|u - u_h\|_{1,h} \leq C_{EST} \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2} + C_{APP}\, \mathcal{A}(f - \Pi_h f, g - g_h), \tag{3.7}$$

with positive constants C_{EST} and C_{APP} which are independent of γ, $\mathbf{h}$, and $\mathbf{k}$. Here, $\mathcal{A}(f - \Pi_h f, g - g_h)$ is the data approximation term given by

$$\mathcal{A}(f - \Pi_h f, g - g_h)^2 = \sum_{K \in \mathcal{T}_h} k_K^{-2} h_K^2 \|f - \Pi_h f\|_{0,K}^2 + \|g - g_h\|_{1/2,\Gamma}^2 + \sum_{\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)} \gamma \|\mathtt{kh}^{-1/2}(g - g_h)\|_{0,\kappa}^2.$$

REMARK 3.2. *We emphasize that the assumption $\gamma \geq \max(1, \gamma_{\min})$ implies that C_{EST} and C_{APP} are independent of γ.*

REMARK 3.3. *The weight $\mathtt{kh}^{-1/2}$ in the jump estimators (as well as in the jump contribution in the energy norm $\|\cdot\|_{1,h}$) is slightly suboptimal in $\mathtt{k}$ with respect to standard hp-approximation properties. This is a notorious difficulty for DG methods which is caused by the fact that the interior penalty stabilization function $\mathtt{c}$ in* (2.6) *has to be chosen suboptimally in $\mathtt{k}$ in order to ensure stability. The same problem shows up in the a priori error analysis of DG methods; see, e.g., [23, 32] and the references cited therein.*

REMARK 3.4. *As has been previously pointed out, the proof of Theorem 3.1 is based on the technique developed in [21] for h-version DG approximations of the Stokes equations, together with the hp-interpolation results of [29]. To this end, we extend the norm equivalence result of [21] to hp-version DG finite element spaces. This crucial result will be established in Section 5 and is the main reason why we restrict our analysis to conforming meshes. While the extension to irregular meshes with* no additional *loss in the polynomial approximation order remains an open problem, the numerical results in Section 6 clearly indicate that the proposed estimator works equally well on 1-irregularly refined quadrilateral meshes.*

3.2. Efficiency. The estimator in Theorem 3.1 cannot be shown to be efficient uniformly in the polynomial degree. Indeed, as for conforming hp-methods, estimators that are both reliable and efficient in the polynomial degree are not currently available within the literature; cf. [30] and the references therein. The main reason for this is that deriving efficiency bounds involves the use of inverse estimates which are suboptimal in the polynomial degree.

In order to minimize the dependence on the polynomial degree in the efficiency bounds, the use of weighted versions of the local error indicators η_K for conforming hp-methods was recently proposed in [30]. Following that approach, we generalize our estimators η_K for the DG method to a family of weighted estimators $\eta_{\alpha;K}$, with $\alpha \in [0,1]$; here, $\eta_{0;K}$ coincides with η_K and is reliable, whereas the best efficiency bounds are obtained for $\eta_{\alpha;K}$, with $\alpha = 1$. As for conforming hp-methods, simultaneous reliability and efficiency cannot be achieved for any fixed $\alpha \in [0,1]$.

On the reference element $\widehat{K}$, we define the weight function $\Phi_{\widehat{K}}(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \partial\widehat{K})$. For an arbitrary element $K \in \mathcal{T}_h$, we set $\Phi_K = c_K \Phi_{\widehat{K}} \circ F_K^{-1}$, where $F_K : \widehat{K} \to K$ is the elemental transformation and c_K is a scaling factor chosen such that $\int_K \Phi_K \, d\mathbf{x} = \mathrm{meas}(K)$. Similarly, we define on the reference interval $\widehat{I} = (-1,1)$, the weight function $\Phi_{\widehat{I}}(x) = 1 - x^2$. For an interior edge κ, the weight Φ_κ is then defined by $\Phi_\kappa = c_\kappa \Phi_{\widehat{I}} \circ F_\kappa^{-1}$, where F_κ is the affine transformation that maps $(-1,1)$ onto κ and c_κ is chosen such that $\int_\kappa \Phi_\kappa\,ds = \mathrm{length}(\kappa)$.

As in [30], for each element $K \in \mathcal{T}_h$ and $\alpha \in [0,1]$, we introduce the weighted local error indicator $\eta_{\alpha;K}$, which is given by

$$\eta^2_{\alpha;K} = \eta^2_{\alpha;R_K} + \eta^2_{\alpha;E_K} + \eta^2_{J_K}, \tag{3.8}$$

where the terms $\eta_{\alpha;R_K}$ and $\eta_{\alpha;E_K}$ are defined, respectively, by

$$\eta^2_{\alpha;R_K} = k_K^{-2} h_K^2 \|(\Pi_h f + \Delta u_h)\Phi_K^{\alpha/2}\|^2_{0,K}, \tag{3.9}$$

$$\eta^2_{\alpha;E_K} = \frac{1}{2} \sum_{\kappa \in \partial K \setminus \Gamma} \|\mathtt{k}^{-1/2}\mathtt{h}^{1/2} [\![\nabla_h u_h]\!] \Phi_\kappa^{\alpha/2}\|^2_{0,\kappa}. \tag{3.10}$$

The third term η_{J_K} in (3.8) is left unchanged and is given by (3.6).

Clearly, for $\alpha = 0$, the estimator $\eta_{\alpha;K}$ coincides with our original estimator η_K in (3.3), that is, $\eta_K = \eta_{0;K}$; Theorem 3.1 then shows that $\eta_{0;K}$ is reliable.

From the inverse estimates in [30, Lemma 2.4 and Theorem 2.5], it can be seen that

$$\eta_{0;R_K} \le C k_K^\alpha \eta_{\alpha;R_K}, \qquad \eta_{0;E_K} \le C k_K^\alpha \eta_{\alpha;E_K}, \qquad K \in \mathcal{T}_h,$$

with a constant C that is independent of the local mesh sizes and the polynomial degrees. Thereby, we deduce the following result.

COROLLARY 3.5. *Let $\alpha \in [0,1]$. Under the assumptions of Theorem 3.1, we have*

$$\|u - u_h\|_{1,h} \le C'_{EST} \left(\sum_{K \in \mathcal{T}_h} k_K^{2\alpha} \eta^2_{\alpha;R_K} + k_K^{2\alpha} \eta^2_{\alpha;E_K} + \eta^2_{J_K} \right)^{1/2} + C_{APP}\, \mathcal{A}(f - \Pi_h f, g - g_h),$$

with a constant C'_{EST} that is independent of α, γ, $\mathbf{h}$, and $\mathbf{k}$. The constant C_{APP} is the same as in Theorem 3.1.

In Corollary 3.5, we see that the best reliability bounds are obtained for $\alpha = 0$, that is, for the original estimator η_K in Theorem 3.1.

Next, we discuss the efficiency of the error indicator $\eta_{\alpha;K}$.

THEOREM 3.6. *Let u be the analytical solution of* (1.1)–(1.2) *and $u_h \in V_h$ its DG approximation obtained by* (2.5)*. Writing $\eta_{\alpha;K}$ to denote the weighted error indicators defined in* (3.8)*, we have the following bounds:*

(i) Let $\alpha \in [0,1]$. For any $\varepsilon > 0$, there is a constant C_ε, independent of α, $\mathbf{h}$, $\mathbf{k}$, and $K \in \mathcal{T}_h$, such that

$$\eta^2_{\alpha;R_K} \le C_\varepsilon \Big[k_K^{2(1-\alpha)} \|\nabla(u - u_h)\|^2_{0,K} + k_K^{\max(1+2\varepsilon-2\alpha,0)} k_K^{-2} h_K^2 \|f - \Pi_h f\|^2_{0,K} \Big].$$

(ii) Let $\alpha \in [0,1]$. For any $\varepsilon > 0$, there is a constant C_ε, independent of α, $\mathbf{h}$, $\mathbf{k}$, and $K \in \mathcal{T}_h$, such that

$$\eta^2_{\alpha;E_K} \le C_\varepsilon k_K^{\max(1+2\varepsilon-2\alpha,0)} \Big[k_K \|\nabla_h(u - u_h)\|^2_{0,\delta_K} + k_K^{2\varepsilon} k_K^{-2} h_K^2 \|f - \Pi_h f\|^2_{0,\delta_K} \Big],$$

where $\delta_K = \bigcup \{K' \in \mathcal{T}_h : K' \text{ and } K \text{ share a common edge}\}$.

(iii) There is a constant C, independent of $\mathbf{h}$, $\mathbf{k}$, and $K \in \mathcal{T}_h$, such that

$$\eta^2_{J_K} \le C \Big[\sum_{\kappa \in \partial K} \gamma \frac{1}{2} \|\mathtt{k}\mathtt{h}^{-1/2} [\![u - u_h]\!]\|^2_{0,\kappa} + \sum_{\kappa \in \partial K \cap \Gamma} \gamma \|\mathtt{k}\mathtt{h}^{-1/2}(g - g_h)\|^2_{0,\kappa} \Big].$$

The proof of Theorem 3.6 follows immediately from the results in [30]; for sake of completeness, we present the main steps in Section 4.5.

We emphasize that the constants in Theorem 3.6 are independent of the mesh size for any $\alpha \in [0,1]$. Hence, the estimators $\eta_{\alpha;K}$, and in particular the estimator $\eta_K = \eta_{0;K}$ in (3.3), are both reliable and efficient in the mesh size.

On the other hand, Corollary 3.5 and Theorem 3.6 indicate that simultaneous reliability and efficiency, with respect to the polynomial degree, cannot be achieved for any $\alpha \in [0,1]$. While the best reliability bounds are the ones for the estimator $\eta_K = \eta_{0;K}$ in (3.3), the best efficiency bounds are obtained for $\eta_{\alpha;K}$ with $\alpha = 1$. For the estimators η_K in (3.3) the efficiency bound is suboptimal in the polynomial degree by one order. Nevertheless, our numerical results in Section 6 confirm that the reliable estimator $\eta_{0;K}$ works fairly well and accurately in combination with hp-refinement. In particular, we show that adaptive hp-refinement based on $\eta_K = \eta_{0;K}$ leads to exponential convergence of the approximations, indicating that the poor dependence on the polynomial degree of the efficiency bound is not significant in this situation.

4. Proofs. In Sections 4.1–4.4, we prove the a posteriori error bound stated in Theorem 3.1; the derivation of this result follows the h-version approach developed in [21]. Finally, in Section 4.5, the proof of Theorem 3.6 is presented.

4.1. Extension of the discontinuous Galerkin forms. We begin by suitably extending the forms A_h and F_h to the continuous level using the lifting operators introduced in [1]; see also [32].

To this end, we introduce the auxiliary space

$$\mathbf{\Sigma}_h = \{\, \mathbf{q} \in L^2(\Omega)^2 \ :\ \mathbf{q}|_K \in \mathcal{S}_{k_K}(K)^2,\ K \in \mathcal{T}_h \,\},$$

and define the operator $\mathbf{L} : V(h) \to \mathbf{\Sigma}_h$ by

$$\int_\Omega \mathbf{L}(v) \cdot \mathbf{q}\, d\mathbf{x} = \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa [\![v]\!] \cdot \{\!\{\mathbf{q}\}\!\}\, ds \qquad \forall \mathbf{q} \in \mathbf{\Sigma}_h.$$

For a function $\widetilde{g} \in H^{1/2}(\Gamma)$, we further define the lifting $\mathbf{U}_{\widetilde{g}} \in \mathbf{\Sigma}_h$ by

$$\int_\Omega \mathbf{U}_{\widetilde{g}} \cdot \mathbf{q}\, d\mathbf{x} = \sum_{\kappa \in \mathcal{E}_\mathcal{B}(\mathcal{T}_h)} \int_\kappa \widetilde{g}\, \mathbf{q} \cdot \mathbf{n}\, ds \qquad \forall \mathbf{q} \in \mathbf{\Sigma}_h.$$

Since the analytical solution u of (1.1)–(1.2) belongs to $H^1(\Omega)$, we have that $[\![u]\!] = \mathbf{0}$ on $\mathcal{E}_\mathcal{I}(\mathcal{T}_h)$, and hence there holds

$$\mathbf{L}(u) = \mathbf{U}_g, \tag{4.1}$$

where g is the Dirichlet boundary datum in (1.2).

We are now ready to introduce the following auxiliary forms

$$\begin{aligned} \widetilde{A}_h(u,v) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v\, d\mathbf{x} - \sum_{K \in \mathcal{T}_h} \int_K \big(\mathbf{L}(u) \cdot \nabla v + \mathbf{L}(v) \cdot \nabla u\big)\, d\mathbf{x} \\ &\quad + \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \mathsf{c}\, [\![u]\!] \cdot [\![v]\!]\, ds, \\ \widetilde{F}_h(v) &= \int_\Omega f v\, d\mathbf{x} - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{U}_g \cdot \nabla v\, d\mathbf{x} + \sum_{\kappa \in \mathcal{E}_\mathcal{B}(\mathcal{T}_h)} \int_\kappa \mathsf{c} g v\, ds. \end{aligned}$$

We observe that

$$\widetilde{A}_h = A_h \quad \text{on} \quad V_h \times V_h, \qquad \widetilde{F}_h = F_h \quad \text{on } V_h.$$

Hence, we may rewrite the discrete problem (2.5) in the following equivalent form: find $u_h \in V_h$ such that

$$\widetilde{A}_h(u_h, v) = \widetilde{F}_h(v) \qquad \forall v \in V_h. \tag{4.2}$$

Furthermore, taking into account (4.1), the definition of $\widetilde{A}_h$ and $\widetilde{F}_h$, and the weak formulation in (1.3), it can be readily seen that the analytical solution u of (1.1)–(1.2) satisfies

$$\widetilde{A}_h(u, v) = \widetilde{F}_h(v) \qquad \forall v \in H_0^1(\Omega). \tag{4.3}$$

4.2. Stability. We now discuss some crucial stability properties. We begin by reviewing the following stability results for $\mathbf{L}$ and $\mathbf{U}_{\widetilde{g}}$; see [32] for details.

LEMMA 4.1. *There exists a constant $C_L > 0$ such that*

$$\|\mathbf{L}(v)\|_{0,\Omega}^2 \leq C_L \gamma^{-1} \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |[\![v]\!]|^2 \, ds, \qquad v \in V(h),$$

$$\|\mathbf{U}_{g_1} - \mathbf{U}_{g_2}\|_{0,\Omega}^2 \leq C_L \gamma^{-1} \sum_{\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |g_1 - g_2|^2 \, ds, \qquad g_1, g_2 \in H^{1/2}(\Gamma).$$

The constant C_L is independent of γ, $\mathbf{h}$, and $\mathbf{k}$, but depends solely on the shape-regularity of the mesh.

REMARK 4.2. *The constant C_L in Lemma 4.1 is independent of the constants ρ_1, ρ_2, and ρ_3 in* (2.1), (2.2), *and* (2.3), *respectively.*

In the next two lemmas, we recall the basic stability properties of the form $\widetilde{A}_h$. The proof of the first lemma follows immediately from Lemma 4.1, the definition of the norm $\|\cdot\|_{1,h}$ and the Cauchy-Schwarz inequality; cf. [32, Proposition 3.1].

LEMMA 4.3. *For any $u, v \in V(h)$, we have*

$$|\widetilde{A}_h(u, v)| \leq C_C \|u\|_{1,h} \|v\|_{1,h},$$

where $C_C = \max(2, 1 + C_L\gamma^{-1})$ and C_L is the constant arising in Lemma 4.1.

We remark that C_C can be bounded independently of γ provided that $\gamma \geq 1$.

The second lemma is an immediate consequence of the definition of $\widetilde{A}_h$.

LEMMA 4.4. *For any $u \in H_0^1(\Omega)$, the following identity holds*

$$\widetilde{A}_h(u, u) = \|u\|_{1,h}^2.$$

Proof. For $u \in H_0^1(\Omega)$, we have that $[\![u]\!] = \mathbf{0}$ over all edges. This implies that $\mathbf{L}(u) = \mathbf{0}$ and $\widetilde{A}_h(u, u) = \int_\Omega |\nabla u|^2 \, d\mathbf{x} = \|u\|_{1,h}^2$, as required. □

Next, we state an hp-version decomposition result for discontinuous finite element spaces. To this end, let $V_h^c = V_h \cap H_0^1(\Omega)$. The orthogonal complement in V_h of V_h^c with respect to the norm $\|\cdot\|_{1,h}$ is denoted by $V_h^\perp$, such that

$$V_h = V_h^c \oplus V_h^\perp. \tag{4.4}$$

The following equivalence result holds; it is an extension to hp-version DG spaces of the corresponding h-version decomposition result derived in [19, Theorem 5.3]; see also [28, Section 2.1].

PROPOSITION 4.5. *Assume that* $\gamma \geq \max(1, \gamma_{\min})$*. The expression*

$$v \mapsto \left(\sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |[\![v]\!]|^2 \, ds \right)^{1/2}$$

is a norm on $V_h^\perp$*. This norm is equivalent to the norm* $\|\cdot\|_{1,h}$ *and there is a constant* $C_P > 0$ *such that*

$$\|v\|_{1,h} \leq C_P \left(\sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |[\![v]\!]|^2 \, ds \right)^{1/2} \leq C_P \|v\|_{1,h}$$

for all $v \in V_h^\perp$*. The constant* C_P *is independent of* γ*,* $\mathbf{h}$*, and* $\mathbf{k}$*, but depends solely on the shape-regularity of the mesh and the constants* ρ_1 *and* ρ_2 *in* (2.1) *and* (2.2)*, respectively.*

The proof of Proposition 4.5 will be given in Section 5.

4.3. hp**-Interpolation of Scott-Zhang type.** Next, we recall the (conforming) hp-Scott-Zhang interpolation result obtained in [29, Theorem 2.4]. To this end, we introduce the space

$$H^1_{\mathcal{B}}(\Omega) = \{ \, u \in H^1(\Omega) \, : \, u|_\kappa \in \mathcal{P}_{k_K}(\kappa), \; \kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h), \; \kappa \in \partial K \cap \Gamma, \; K \in \mathcal{T}_h \, \}.$$

With this notation, the following result holds.

LEMMA 4.6. *There exists a linear operator* $I_h : H^1_{\mathcal{B}}(\Omega) \to V_h \cap H^1_{\mathcal{B}}(\Omega)$ *satisfying* $I_h v|_\kappa = v|_\kappa$*, for all* $\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)$*, and*

$$\sum_{K \in \mathcal{T}_h} \left(k_K^2 h_K^{-2} \|v - I_h v\|^2_{0,K} + \|\nabla(v - I_h v)\|^2_{0,K} + \|\mathtt{k}^{1/2} \mathtt{h}^{-1/2}(v - I_h v)\|^2_{0,\partial K} \right) \leq C_I^2 \|\nabla v\|^2_{0,\Omega},$$

with an interpolation constant C_I*, which is independent of* $\mathbf{h}$ *and* $\mathbf{k}$*, and depends solely on the shape-regularity of the mesh and the constants* ρ_1*,* ρ_2*, and* ρ_3 *in* (2.1)*,* (2.2)*, and* (2.3)*, respectively.*

Moreover, for the a posteriori error analysis in Section 4.4, we will need the following auxiliary result.

PROPOSITION 4.7. *Let* $v \in H^1_0(\Omega)$ *and* $v_h = I_h v \in V_h^c$ *its Scott-Zhang interpolant from Lemma 4.6. Moreover, let* u_h *be the DG approximation defined by* (2.5)*. Then,*

$$\left| \widetilde{F}_h(v - v_h) - \widetilde{A}_h(u_h, v - v_h) \right| \leq C_A \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 + \mathcal{A}(f - \Pi_h f, g - g_h)^2 \Big)^{1/2} \|\nabla v\|_{0,\Omega}.$$

Here, $C_A = \sqrt{2} C_I \max(1, C_L \gamma^{-1})^{1/2}$*, with* C_I *and* C_L *being the constants from Lemma 4.6 and Lemma 4.1, respectively.*

Again, we remark that, for $\gamma \geq 1$, the constant C_A in Proposition 4.7 can be bounded independently of γ.

Proof. Let $z_{g_h} \in H^1_{\mathcal{B}}(\Omega)$ such that $z_{g_h} = g_h$ on Γ. It can then be readily seen that $\mathbf{L}(z_{g_h}) = \mathbf{U}_{g_h}$ in $\mathbf{\Sigma}_h$. Furthermore, we set $\xi = v - v_h \in H^1_0(\Omega)$ and $T = \widetilde{F}_h(\xi) - \widetilde{A}_h(u_h, \xi)$. Using the conformity of ξ, integration by parts and the definition of the lifting operator $\mathbf{L}$, we obtain

$$
\begin{aligned}
T &= \sum_{K\in\mathcal{T}_h}\left(\int_K (f+\Delta u_h)\xi d\mathbf{x} - \int_{\partial K}\nabla_h u_h\cdot(\xi\mathbf{n}_K)ds\right)\\
&\quad + \sum_{K\in\mathcal{T}_h}\int_K \mathbf{L}(u_h)\cdot\nabla\xi\,d\mathbf{x} - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{U}_g\cdot\nabla\xi\,d\mathbf{x}\\
&= \sum_{K\in\mathcal{T}_h}\int_K (\Pi_h f+\Delta u_h)\xi\,d\mathbf{x} - \sum_{\kappa\in\mathcal{E}_\mathcal{I}(\mathcal{T}_h)}\int_\kappa [\![\nabla_h u_h]\!]\xi\,ds\\
&\quad + \sum_{K\in\mathcal{T}_h}\int_K \mathbf{L}(u_h-z_{g_h})\cdot\nabla\xi\,d\mathbf{x} - \sum_{K\in\mathcal{T}_h}\int_K(\mathbf{U}_g-\mathbf{U}_{g_h})\cdot\nabla\xi\,d\mathbf{x}\\
&\quad + \sum_{K\in\mathcal{T}_h}\int_K (f-\Pi_h f)\xi\,d\mathbf{x}.
\end{aligned}
$$

Here, $\mathbf{n}_K$ denotes the unit outward normal vector on the boundary ∂K of an element $K\in\mathcal{T}_h$. Employing the first bound from Lemma 4.1, combined with the fact that $[\![z_{g_h}]\!]=\mathbf{0}$ on interior edges and $[\![z_{g_h}]\!]=g_h\mathbf{n}$ on boundary edges, we deduce that

$$
\|\mathbf{L}(u_h-z_{g_h})\|_{0,\Omega}^2 \le C_L\gamma^{-1}\sum_{K\in\mathcal{T}_h}\eta_{J_K}^2.
$$

In addition, using the second estimate from Lemma 4.1 with $g_1=g$ and $g_2=g_h$, results in

$$
\|\mathbf{U}_g-\mathbf{U}_{g_h})\|_{0,\Omega}^2 \le C_L\gamma^{-1}\mathcal{A}(0,g-g_h)^2. \tag{4.5}
$$

Thus, the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
|T| &\le \sqrt{2}\left(\sum_{K\in\mathcal{T}_h}\eta_{R_K}^2+\eta_{E_K}^2+C_L\gamma^{-1}\eta_{J_K}^2+C_L\gamma^{-1}\mathcal{A}(0,g-g_h)^2+\mathcal{A}(f-\Pi_h f,0)^2\right)^{1/2}\\
&\quad\times\left(\sum_{K\in\mathcal{T}_h}\left(k_K^2h_K^{-2}\|\xi\|_{0,K}^2+\|\nabla\xi\|_{0,K}^2\right)+\sum_{\kappa\in\mathcal{E}_\mathcal{I}(\mathcal{T}_h)}\|\mathtt{k}^{1/2}\mathtt{h}^{-1/2}\xi\|_{0,\kappa}^2\right)^{1/2}.
\end{aligned}
$$

Noting that

$$
\sum_{\kappa\in\mathcal{E}_\mathcal{I}(\mathcal{T}_h)}\|\mathtt{k}^{1/2}\mathtt{h}^{-1/2}\xi\|_{0,\kappa}^2 \le \frac{1}{2}\sum_{K\in\mathcal{T}_h}\|\mathtt{k}^{1/2}\mathtt{h}^{-1/2}\xi\|_{0,\partial K}^2,
$$

and exploiting the approximation properties in Lemma 4.6 leads to

$$
|T| \le \sqrt{2}C_I\max(1,C_L\gamma^{-1})^{1/2}\left(\sum_{K\in\mathcal{T}_h}\eta_K^2+\mathcal{A}(f-\Pi_h f,g-g_h)^2\right)^{1/2}\|\nabla v\|_{0,\Omega}.
$$

This completes the proof. □

4.4. The a posteriori bound in Theorem 3.1. In this section we complete the proof of Theorem 3.1.

To this end, let g_h be the approximation of the boundary datum g. We denote by u_{g_h} the solution of the following problem

$$-\Delta u_{g_h} = f \qquad \text{in } \Omega, \tag{4.6}$$
$$u_{g_h} = g_h \qquad \text{on } \Gamma. \tag{4.7}$$

Since solutions of the Laplace equation depend continuously on their data, we readily obtain that

$$\|\nabla(u - u_{g_h})\|_{0,\Omega} \leq C_S \|g - g_h\|_{1/2,\Gamma}, \tag{4.8}$$

for a stability constant $C_S > 0$.

Next, using the decomposition (4.4) of the space V_h into $V_h = V_h^c \oplus V_h^{\perp}$, we write

$$u_h - I_h u_{g_h} = u_h^c + u_h^{\perp},$$

with I_h denoting the Scott-Zhang interpolation operator from Lemma 4.6. By the triangle inequality, the error $e = u - u_h$ of the hp-DG approximation can then be estimated as follows:

$$\begin{aligned} \|u - u_h\|_{1,h} &\leq \|u - u_{g_h}\|_{1,h} + \|u_{g_h} - I_h u_{g_h} - (u_h - I_h u_{g_h})\|_{1,h} \\ &\leq \|u - u_{g_h}\|_{1,h} + \|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h} + \|u_h^{\perp}\|_{1,h}. \end{aligned} \tag{4.9}$$

Employing (4.8), the first term on the right-hand side of (4.9) may be bounded by the error in the approximation of g; thereby,

$$\begin{aligned} \|u - u_{g_h}\|_{1,h}^2 &= \|\nabla(u - u_{g_h})\|_{0,\Omega}^2 + \sum_{\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |g - g_h|^2 \, ds \\ &\leq C_S^2 \|g - g_h\|_{1/2,\Gamma}^2 + \sum_{\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |g - g_h|^2 \, ds \\ &\leq (C_S^2 + 1) \, \mathcal{A}(0, g - g_h)^2. \end{aligned} \tag{4.10}$$

Using the equivalence result in Proposition 4.5 and noting that, in view of the conformity properties of I_h, we have that $[\![u_h^{\perp}]\!] = [\![u_h]\!]$ on interior edges and $[\![u_h^{\perp}]\!] = (u_h - g_h)\mathbf{n}$ on boundary edges, the third term in (4.9) can be bounded as follows

$$\|u_h^{\perp}\|_{1,h} \leq C_P \left(\sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa \gamma \mathtt{k}^2 \mathtt{h}^{-1} |[\![u_h^{\perp}]\!]|^2 \, ds \right)^{1/2} \leq C_P \left(\sum_{K \in \mathcal{T}_h} \eta_{J_K}^2 \right)^{1/2}. \tag{4.11}$$

Hence, we obtain

$$\|u - u_h\|_{1,h} \leq \|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h} + (C_S^2 + 1)^{1/2} \mathcal{A}(0, g - g_h) + C_P \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}.$$

To bound the term $\|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h}$, we first note that $u_{g_h} - I_h u_{g_h} - u_h^c \in H_0^1(\Omega)$. We then set

$$v = \frac{u_{g_h} - I_h u_{g_h} - u_h^c}{\|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h}} \in H_0^1(\Omega).$$

With this definition, employing the coercivity result from Lemma 4.4, results in

$$\|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h} = \widetilde{A}_h(u_{g_h} - I_h u_{g_h} - u_h^c, v), \qquad \|v\|_{1,h} = \|\nabla v\|_{0,\Omega} = 1. \quad (4.12)$$

We note that

$$\begin{aligned}
\widetilde{A}_h(u_{g_h} - I_h u_{g_h} - u_h^c, v) &= \widetilde{A}_h(u_{g_h} - u_h, v) + \widetilde{A}_h(u_h^\perp, v) \\
&= \widetilde{A}_h(u - u_h, v) + \widetilde{A}_h(u_{g_h} - u, v) + \widetilde{A}_h(u_h^\perp, v) \\
&= \widetilde{F}_h(v) - \widetilde{A}_h(u_h, v) + \widetilde{A}_h(u_{g_h} - u, v) + \widetilde{A}_h(u_h^\perp, v);
\end{aligned} \quad (4.13)$$

here, we have used the weak formulation in (4.3). Furthermore, from (4.2), we recall that

$$\widetilde{A}_h(u_h, v_h) - \widetilde{F}_h(v_h) = 0 \qquad \forall v_h \in V_h.$$

Combining this with the estimates from (4.12) and (4.13), we deduce that

$$\|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h} = \widetilde{F}_h(v - v_h) - \widetilde{A}_h(u_h, v - v_h) + \widetilde{A}_h(u_{g_h} - u, v) + \widetilde{A}_h(u_h^\perp, v),$$

for any $v_h \in V_h$. Choosing $v_h = I_h v \in V_h^c$ to be the Scott-Zhang interpolant from Lemma 4.6 yields

$$\begin{aligned}
&\|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h} \\
&\quad \le |\widetilde{F}_h(v - v_h) - \widetilde{A}_h(u_h, v - v_h)| + |\widetilde{A}_h(u_{g_h} - u, v)| + |\widetilde{A}_h(u_h^\perp, v)| \\
&\quad \le C_A \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 + \mathcal{A}(f - \Pi_h f, g - g_h)^2 \right)^{1/2} \|v\|_{1,h} \\
&\qquad + C_C \|u - u_{g_h}\|_{1,h} \|v\|_{1,h} + C_C \|u_h^\perp\|_{1,h} \|v\|_{1,h} \\
&\quad \le C_A \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2} \|v\|_{1,h} + C_A \mathcal{A}(f - \Pi_h f, g - g_h) \|v\|_{1,h} \\
&\qquad + C_C (C_S^2 + 1)^{1/2} \mathcal{A}(0, g - g_h) \|v\|_{1,h} + C_C C_P \left(\sum_{K \in \mathcal{T}_h} \eta_{J_K}^2 \right)^{1/2} \|v\|_{1,h}.
\end{aligned}$$

Here, we have applied the auxiliary result from Proposition 4.7, the continuity of $\widetilde{A}_h$ from Lemma 4.3, and the bounds in (4.10) and (4.11).

Using the fact that $\|v\|_{1,h} = 1$, we conclude that

$$\begin{aligned}
\|u_{g_h} - I_h u_{g_h} - u_h^c\|_{1,h} &\le (C_A + C_C C_P) \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2} \\
&\qquad + (C_A + C_C (C_S^2 + 1)^{1/2}) \mathcal{A}(f - \Pi_h f, g - g_h).
\end{aligned}$$

Combining the above estimates gives

$$\begin{aligned}
\|u - u_h\|_{1,h} &\le (C_A + (C_C + 1) C_P) \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2} \\
&\qquad + (C_A + (C_C + 1)(C_S^2 + 1)^{1/2}) \mathcal{A}(f - \Pi_h f, g - g_h),
\end{aligned}$$

which completes the proof of Theorem 3.1.

Note that, for $\gamma \ge \max(1, \gamma_{\min})$, all the constants can be bounded independently of γ.

4.5. Proof of Theorem 3.6. We present the proofs of the three assertions in Theorem 3.6 separately. The proofs of (i) and (ii) are analogous to the corresponding bounds derived in [30, Lemma 3.4 and Lemma 3.5]; however, for sake of completeness, we present the main steps.

Assertion (i): We first consider the case $\alpha > 1/2$. To this end, we set $v_K = (\Pi_h f + \Delta u_h)\Phi_K^{\alpha}$. Then, using that $-\Delta u = f$ in $L^2(K)$, we obtain by elementary manipulations

$$\begin{aligned}
\|v_K \Phi_K^{-\alpha/2}\|_{0,K}^2 &= \int_K (\Pi_h f + \Delta u_h) v_K \, d\mathbf{x} \\
&= \int_K (-\Delta u + \Delta u_h) v_K \, d\mathbf{x} + \int_K (\Pi_h f - f) v_K \, d\mathbf{x} \\
&= \int_K \nabla(u - u_h) \cdot \nabla v_K \, d\mathbf{x} + \int_K (\Pi_h f - f) v_K \, d\mathbf{x} \\
&\leq \|\nabla(u - u_h)\|_{0,K} \|\nabla v_K\|_{0,K} + \|(f - \Pi_h f)\Phi_K^{\alpha/2}\|_{0,K} \|v_K \Phi_K^{-\alpha/2}\|_{0,K}.
\end{aligned}$$

From the proof of [30, Lemma 3.4], we have

$$\|\nabla v_K\|_{0,K}^2 \leq C k_K^{2(1-\alpha)} k_K^2 h_K^{-2} \|v_K \Phi_K^{-\alpha/2}\|_{0,K}^2.$$

Since $\|v_K \Phi_K^{-\alpha/2}\|_{0,K} = k_K h_K^{-1} \eta_{\alpha;R_K}$, we readily obtain that

$$\eta_{\alpha;R_K} \leq C \Big[k_K^{1-\alpha} \|\nabla(u - u_h)\|_{0,K} + k_K^{-1} h_K \|f - \Pi_h f\|_{0,K} \Big]. \tag{4.14}$$

This shows the assertion for $\alpha > \frac{1}{2}$. For $\alpha \in [0, 1/2]$, we first use that

$$\eta_{\alpha;R_K} \leq C_\varepsilon k_K^{\beta-\alpha} \eta_{\beta;R_K}$$

, for $\beta = 1/2 + \varepsilon$ with $\varepsilon > 0$, and apply the bound in (4.14) to $\eta_{\beta;R_K}$.

Assertion (ii): We again first consider the case $\alpha > 1/2$ and let κ be an edge shared by two elements K_1 and K_2. Set $\delta_\kappa := (\overline{K}_1 \cup \overline{K}_2)^\circ$; Lemma 2.6 of [30] ensures the existence of a function $w_\kappa \in H_0^1(\delta_\kappa)$ with $w_\kappa|_\kappa = [\![\nabla u_h]\!] \Phi_\kappa^\alpha$, $w_\kappa|_{\partial\delta_\kappa} = 0$ and

$$\begin{aligned}
\|\nabla w_\kappa\|_{0,\delta_\kappa}^2 &\leq C h_K^{-1} \big(\sigma k_K^{2(2-\alpha)} + \sigma^{-1} \big) \|[\![\nabla u_h]\!] \Phi_\kappa^{\alpha/2}\|_{0,\kappa}^2, \\
\|w_\kappa\|_{0,\delta_\kappa}^2 &\leq C h_K \sigma \|[\![\nabla u_h]\!] \Phi_\kappa^{\alpha/2}\|_{0,\kappa}^2,
\end{aligned} \tag{4.15}$$

for any $\sigma \in (0, 1]$. Using that $[\![\nabla u]\!] = 0$ on interior edges and that $-\Delta u = f$ on each element K, it can be readily seen that

$$\begin{aligned}
\|[\![\nabla u_h]\!] \Phi_\kappa^{\alpha/2}\|_{0,\kappa}^2 &= \int_\kappa [\![\nabla u_h]\!] w_\kappa \, ds = \int_\kappa [\![\nabla (u_h - u)]\!] w_\kappa \, ds \\
&= \int_{\partial K_1} \nabla(u_h - u) \cdot \mathbf{n}_{K_1} w_\kappa \, ds + \int_{\partial K_2} \nabla(u_h - u) \cdot \mathbf{n}_{K_2} w_\kappa \, ds \\
&= \int_{\delta_\kappa} \nabla_h(u_h - u) \cdot \nabla w_\kappa \, d\mathbf{x} + \int_{\delta_\kappa} (f + \Delta u_h) w_\kappa \, d\mathbf{x} \\
&\leq \|\nabla_h(u - u_h)\|_{0,\delta_\kappa} \|\nabla w_\kappa\|_{0,\delta_\kappa} \\
&\qquad + \big(\|\Pi_h f + \Delta u_h\|_{0,\delta_\kappa} + \|f - \Pi_h f\|_{0,\delta_\kappa} \big) \|w_\kappa\|_{0,\delta_\kappa}.
\end{aligned}$$

Here, $\mathbf{n}_{K_1}$ and $\mathbf{n}_{K_2}$ denote the unit outward normal vectors on the boundaries ∂K_1 and ∂K_2, respectively. By summing up this estimate over all edges of a given element K, invoking the bounds for $\|\nabla w_\kappa\|_{0,\delta_\kappa}$ and $\|w_\kappa\|_{0,\delta_\kappa}$ from (4.15), and using assertion (i), we obtain

$$\eta^2_{\alpha;E_K} \leq C\Big(k_K^{-1}(\sigma k_K^{2(2-\alpha)} + \sigma^{-1}) + k_K^3\sigma\Big)\|\nabla_h(u-u_h)\|^2_{0,\delta_K} + C\sigma k_K^{2(1+\varepsilon)} k_K^{-2} h_K^2 \|f - \Pi_h f\|^2_{0,\delta_K}.$$

Setting $\sigma = k_K^{-2}$ proves the assertion for $\alpha > 1/2$. For $\alpha \in [0,1/2]$, we have that $\eta_{\alpha;E_K} \leq Ck_K^{\beta-\alpha}\eta_{\beta;E_K}$ and use the above argument for $\eta_{\beta;E_K}$ with $\beta = 1/2+\varepsilon$ to obtain the assertion.

Assertion (iii): This is a simple consequence of the fact that the jump of u vanishes over interior edges and the triangle inequality which gives, for any boundary edge $\kappa \in \partial K \cap \Gamma$,

$$\|\mathtt{kh}^{-1/2}(u_h - g_h)\|^2_{0,\kappa} \leq 2\|\mathtt{kh}^{-1/2}(u_h - u)\|^2_{0,\kappa} + 2\|\mathtt{kh}^{-1/2}(g - g_h)\|^2_{0,\kappa}.$$

This completes the proof of Theorem 3.6.

5. Norm equivalence. In this section, we prove the norm equivalence property stated in Proposition 4.5. This result is an hp-extension of the approximation result in [28, Section 2.1]; see also [19, Theorem 5.3] and [21, Proposition 4.1]

5.1. Preliminaries. First, we introduce some additional notation and definitions.

Polynomial spaces. For an interval $I = (a,b)$, we write $\mathcal{P}_k(I)$ to denote the space of all polynomials of degree less than or equal to k. Furthermore, we set $\mathcal{I}^k_{00}(I) = \{\, q \in \mathcal{P}_k(I) \,:\, q(a) = q(b) = 0 \,\}$.

Meshes. A node P of a finite element mesh $\mathcal{T}_h$ is the vertex of an element $K \in \mathcal{T}_h$. P is called an interior node if $P \notin \Gamma$; similarly, it is a boundary node if $P \in \Gamma$. We denote by $\mathcal{N}_\mathcal{I}(\mathcal{T}_h)$, $\mathcal{N}_\mathcal{B}(\mathcal{T}_h)$ the sets of interior and boundary nodes, respectively, and set $\mathcal{N}(\mathcal{T}_h) = \mathcal{N}_\mathcal{I}(\mathcal{T}_h) \cup \mathcal{N}_\mathcal{B}(\mathcal{T}_h)$. For an element $K \in \mathcal{T}_h$, we write $\mathcal{E}(K)$ and $\mathcal{N}(K)$ to denote the sets of its edges and nodes, respectively. Moreover, for an edge $\kappa \in \mathcal{E}(\mathcal{T}_h)$, let $\mathcal{N}(\kappa)$ be the set of the nodes that belong to κ, and $\mathcal{T}_h(\kappa)$ the set of elements in $\mathcal{T}_h$ that share the edge κ. Finally, for a node $P \in \mathcal{N}(\mathcal{T}_h)$, we define $\mathcal{E}(P)$ and $\mathcal{T}_h(P)$ as the sets of edges and elements that share P, respectively.

5.2. Piecewise linear approximation. We begin by considering the conforming approximation of piecewise linear DG functions. To this end, we define the linear discontinuous Galerkin finite element space $\widetilde{V}_h$ by

$$\widetilde{V}_h = \{\, v \in L^2(\Omega) \,:\, v|_K \in \mathcal{S}_1(K),\ K \in \mathcal{T}_h \,\},$$

as well as the conforming subspace $\widetilde{V}_h^c \subset \widetilde{V}_h$ by $\widetilde{V}_h^c = \widetilde{V}_h \cap H^1_0(\Omega)$. Furthermore, we introduce an operator $\widetilde{\mathcal{A}} : \widetilde{V}_h \to \widetilde{V}_h^c$, where, for $v \in \widetilde{V}_h$, $\widetilde{\mathcal{A}}v \in \widetilde{V}_h^c$ is given by prescribing its nodal values as

$$\widetilde{\mathcal{A}}v(P) = \begin{cases} |\mathcal{T}_h(P)|^{-1} \displaystyle\sum_{K \in \mathcal{T}_h(P)} v|_K(P) & \text{if } P \in \mathcal{N}_\mathcal{I}(\mathcal{T}_h), \\ 0 & \text{if } P \in \mathcal{N}_\mathcal{B}(\mathcal{T}_h). \end{cases}$$

Here, $|\mathcal{T}_h(P)|$ denotes the number of elements in the set $\mathcal{T}_h(P)$.

From [28, Theorem 2.2], the following approximation property holds.

LEMMA 5.1. *The approximant $\widetilde{\mathcal{A}} : \widetilde{V}_h \to \widetilde{V}_h^c$ satisfies*

$$\|\nabla_h(v - \widetilde{\mathcal{A}}v)\|^2_{0,\Omega} \leq C \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \|\mathtt{h}^{-1/2}[\![v]\!]\|^2_{0,\kappa} \qquad \forall v \in \widetilde{V}_h,$$

with a constant $C > 0$ that is independent of the mesh size.

5.3. Approximation over edges. Next, we consider the conforming approximation of higher-order DG functions over edges. To this end, we make use of the following extension result:

PROPOSITION 5.2. *Let $\widehat{K}$ be the reference triangle or the reference square, and $k \in \mathbb{N}$. Furthermore, consider an edge $\widehat{\kappa} \in \mathcal{E}(\widehat{K})$. Then, there is a linear extension operator $\widehat{E}_k^{\widehat{\kappa}} : \mathcal{I}_{00}^k(\widehat{\kappa}) \to \mathcal{S}_k(\widehat{K})$ such that for any $\widehat{q} \in \mathcal{I}_{00}^k(\widehat{\kappa})$, there holds $\widehat{E}_k^{\widehat{\kappa}}(\widehat{q}) = \widehat{q}$ on $\widehat{\kappa}$, $\widehat{E}_k^{\widehat{\kappa}}(\widehat{q}) = 0$ on $\partial\widehat{K} \setminus \widehat{\kappa}$, and $\|\widehat{E}_k^{\widehat{\kappa}}(\widehat{q})\|_{1,\widehat{K}} \leq C\|\widehat{q}\|_{1/2,00,\kappa}$. Here, $C > 0$ is a constant independent of the polynomial degree k.*

Proof. See [3, Theorem 7.4 and Theorem 7.5], [2, Theorem 7.4 and Theorem 7.5], or [29, Theorem B.4]. ◻

For an arbitrary element $K \in \mathcal{T}_h$ and an edge $\kappa \in \mathcal{E}(K)$, let F_K be the elemental mapping that maps $\widehat{K}$ onto K, and $\widehat{\kappa}$ the edge of $\widehat{K}$ such that $F_K(\widehat{\kappa}) = \kappa$. The extension operator $E_{k,K}^{\kappa} : \mathcal{I}_{00}^k(\kappa) \to \mathcal{S}_k(K)$ is then defined by

$$E_{k,K}^{\kappa}(q) = \left[\widehat{E}_k^{\widehat{\kappa}}(q \circ F_K)\right] \circ F_K^{-1}, \qquad q \in \mathcal{I}_{00}^k(\kappa). \tag{5.1}$$

We are now ready to prove the following technical result.

LEMMA 5.3. *Let the two elements $K_1, K_2 \in \mathcal{T}_h$ share the interior edge $\kappa \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}_h)$. Set $\delta_\kappa := (\overline{K}_1 \cup \overline{K}_2)^\circ$ and let $v_1 \in \mathcal{P}_{k_1}(\kappa)$ and $v_2 \in \mathcal{P}_{k_2}(\kappa)$ with $k_1, k_2 \geq 1$. By $v_1^{\mathrm{n}} \in \mathcal{P}_1(\kappa)$ and $v_2^{\mathrm{n}} \in \mathcal{P}_1(\kappa)$ we denote the nodal interpolants of v_1 and v_2, respectively. Then there is a conforming approximation $W^c \in H_0^1(\delta_\kappa)$ such that $W_1^c := W^c|_{K_1} \in \mathcal{S}_{k_1}(K_1)$, $W_2^c := W^c|_{K_2} \in \mathcal{S}_{k_2}(K_2)$, and*

$$\sum_{i=1,2} \left|E_{k_i,K_i}^{\kappa}(v_i - v_i^{\mathrm{n}}) - W_i^c\right|_{1,K_i}^2 \leq C\max(k_1,k_2)^2\|\mathtt{h}^{-1/2}(v_1 - v_2)\|_{0,\kappa}^2 + C\|\mathtt{h}^{-1/2}(v_1^{\mathrm{n}} - v_2^{\mathrm{n}})\|_{0,\kappa}^2,$$

with a constant $C > 0$ independent of the polynomial degrees and the element sizes.

Similarly, let K be the element that contains the boundary edge $\kappa \in \mathcal{E}_{\mathcal{B}}(\mathcal{T}_h)$. Then, for $v \in \mathcal{P}_k(\kappa)$ with $k \geq 1$ and its nodal interpolant $v^{\mathrm{n}} \in \mathcal{P}_1(\kappa)$, there is an approximation $W^c \in \mathcal{S}_k(K)$ such that $W^c = 0$ on $\partial K \setminus \kappa$ and

$$\left|E_{k,K}^{\kappa}(v - v^{\mathrm{n}}) - W^c\right|_{1,K}^2 \leq Ck^2\|\mathtt{h}^{-1/2}v\|_{0,\kappa}^2 + C\|\mathtt{h}^{-1/2}v^{\mathrm{n}}\|_{0,\kappa}^2.$$

Again, the constant $C > 0$ is independent of the polynomial degree and the element size.

Proof. Let κ be an interior edge shared by K_1 and K_2. It is sufficient to consider the case where K_1 and K_2 are of reference size; the general case follows by a simple scaling argument. Without loss of generality, we may further assume that $k_1 \geq k_2$.

In order to construct W^c in this case, we first note that $v_1 - v_1^{\mathrm{n}} \in \mathcal{I}_{00}^{k_1}(\kappa)$ and $v_2 - v_2^{\mathrm{n}} \in \mathcal{I}_{00}^{k_2}(\kappa)$. Furthermore, on κ, we introduce

$$\varphi = 1/2\ \left(\Pi_{\kappa,k_2}^{00}(v_1 - v_1^{\mathrm{n}}) + (v_2 - v_2^{\mathrm{n}})\right) \in \mathcal{I}_{00}^{k_2}(\kappa),$$

with Π_{κ,k_2}^{00} denoting the $H_{00}^{1/2}(\kappa)$-projection onto $\mathcal{I}_{00}^{k_2}(\kappa)$. We then define W^c by

$$W^c = \begin{cases} W_1^c := E_{k_1,K_1}^{\kappa}(\varphi) & \text{on } K_1, \\ W_2^c := E_{k_2,K_2}^{\kappa}(\varphi) & \text{on } K_2. \end{cases}$$

By construction and Proposition 5.2, the function W^c vanishes on $\partial\delta_\kappa$. Moreover, $W_1^c \in \mathcal{S}_{k_1}(K_1)$, $W_2^c \in \mathcal{S}_{k_2}(K_2)$ and $W_1^c|_\kappa = \varphi = W_2^c|_\kappa$ on κ. Therefore, $W^c \in H_0^1(\delta_\kappa)$. From the linearity of $E^{\kappa}_{k_1,K_1}$ and the stability in Proposition 5.2, we obtain

$$\begin{aligned}
&\left|E^{\kappa}_{k_1,K_1}(v_1-v_1^{\mathrm{n}})-W_1^c\right|_{1,K_1}\\
&\qquad\le C\|(v_1-v_1^{\mathrm{n}})-1/2\,\Pi^{00}_{\kappa,k_2}(v_1-v_1^{\mathrm{n}})-1/2\,(v_2-v_2^{\mathrm{n}})\|_{1/2,00,\kappa}\\
&\qquad\le C\|1/2(\mathbb{I}-\Pi^{00}_{\kappa,k_2})(v_1-v_1^{\mathrm{n}})+1/2\,((v_1-v_1^{\mathrm{n}})-(v_2-v_2^{\mathrm{n}}))\|_{1/2,00,\kappa}\\
&\qquad\le C\|(\mathbb{I}-\Pi^{00}_{\kappa,k_2})(v_1-v_1^{\mathrm{n}})\|_{1/2,00,\kappa}+C\|(v_1-v_2)-(v_1^{\mathrm{n}}-v_2^{\mathrm{n}})\|_{1/2,00,\kappa}.
\end{aligned}$$

Since $\Pi^{00}_{\kappa,k_2}(v_2-v_2^{\mathrm{n}})=v_2-v_2^{\mathrm{n}}$ and $\mathbb{I}-\Pi^{00}_{\kappa,k_2}$ is stable in $H^{1/2}_{00}(\kappa)$, we can bound the second term in the above inequality as follows:

$$\begin{aligned}
\|(\mathbb{I}-\Pi^{00}_{\kappa,k_2})(v_1-v_1^{\mathrm{n}})\|_{1/2,00,\kappa}&=\|(\mathbb{I}-\Pi^{00}_{\kappa,k_2})((v_1-v_1^{\mathrm{n}})-(v_2-v_2^{\mathrm{n}}))\|_{1/2,00,\kappa}\\
&\le\|(v_1-v_2)-(v_1^{\mathrm{n}}-v_2^{\mathrm{n}})\|_{1/2,00,\kappa}.
\end{aligned}$$

We conclude that

$$\left|E^{\kappa}_{k_1,K_1}(v_1-v_1^{\mathrm{n}})-W_1^c\right|_{1,K_1}\le C(\|v_1-v_2\|_{1/2,00,\kappa}+\|v_1^{\mathrm{n}}-v_2^{\mathrm{n}}\|_{1/2,00,\kappa}). \tag{5.2}$$

A similar argument shows that

$$\left|E^{\kappa}_{k_2,K_2}(v_2-v_2^{\mathrm{n}})-W_2^c\right|_{1,K_2}\le C(\|v_1-v_2\|_{1/2,00,\kappa}+\|v_1^{\mathrm{n}}-v_2^{\mathrm{n}}\|_{1/2,00,\kappa}). \tag{5.3}$$

From the inverse inequalities in [37, Corollary 3.94] and interpolation, we readily obtain that

$$\|v_1-v_2\|_{1/2,00,\kappa}\le Ck_1\|v_1-v_2\|_{0,\kappa}.$$

Furthermore, a simple h-version norm equivalence property on $\mathcal{P}_1(\kappa)$ shows that

$$\|v_1^{\mathrm{n}}-v_2^{\mathrm{n}}\|_{1/2,00,\kappa}\le C\|v_1^{\mathrm{n}}-v_2^{\mathrm{n}}\|_{0,\kappa}.$$

This completes the proof for an interior edge; for a boundary edge, the result is obtained analogously. □

5.4. Approximation of DG functions. We consider the DG space defined in (2.4) and introduce the conforming subspace V_h^c given by

$$V_h^c=V_h\cap H_0^1(\Omega).$$

The following approximation result holds.

PROPOSITION 5.4. *There is an approximant $\mathcal{A}:\ V_h\to V_h^c$ that satisfies*

$$\|\nabla_h(v-\mathcal{A}v)\|^2_{0,\Omega}\le C\sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)}\int_\kappa \mathtt{k}^2\mathtt{h}^{-1}|[\![v]\!]|^2\,ds,\qquad v\in V_h,$$

with a constant $C>0$ that is independent of $\mathbf{h}$ *and* $\mathbf{k}$.

Proof. Let $v\in V_h$. For any $K\in\mathcal{T}_h$, set $v_K:=v|_K\in\mathcal{S}_{k_K}(K)$. We now proceed in the following steps.

Decomposition of v: We begin by decomposing v into a nodal part, an edge part, and an interior part. To this end, let $v_K^{\mathrm{n}}\in\mathcal{S}_1(K)$ be the nodal interpolant of v_K, $K\in\mathcal{T}_h$. Furthermore, for $K\in\mathcal{T}_h$ and an edge κ of $\mathcal{E}(K)$, we denote by $V_K^{\kappa}=E^{\kappa}_{k_K,K}(v_K-v_K^{\mathrm{n}})$ the lifting of $v_K - v_K^{\rm n}$ that was constructed in (5.1). By construction, $(v_K - v_K^{\rm n}) - \sum_{\kappa\in\mathcal{E}(K)} V_K^\kappa$ vanishes on ∂K and hence, we can define the interior part $v_K^{\rm i}$ of v_K by

$$v_K^{\rm i} = (v_K - v_K^{\rm n}) - \sum_{\kappa\in\mathcal{E}(K)} V_K^\kappa, \qquad K \in \mathcal{T}_h.$$

Thus, we may decompose v into

$$v = v^{\rm n} + v^{\rm e} + v^{\rm i}.$$

Here, $v^{\rm n}$ is the piecewise linear function in $\widetilde{V}_h$ given by $v^{\rm n}|_K = v_K^{\rm n}$ for any $K \in \mathcal{T}_h$. Moreover, $v^{\rm e}$ is the function given by

$$v^{\rm e}|_K = \sum_{\kappa\in\mathcal{E}(K)} V_K^\kappa, \qquad K \in \mathcal{T}_h.$$

Finally, $v^{\rm i}$ is given by $v^{\rm i}|_K = v_K^{\rm i}$, $K \in \mathcal{T}_h$; by construction, $v^{\rm i} \in V_h^c$.

Construction of the approximant: Next, we define the approximant $\mathcal{A}v$. To this end, let $w^{\rm n} = \widetilde{\mathcal{A}}v^{\rm n} \in \widetilde{V}_h^c \subset V_h^c$ be the conforming piecewise linear approximation of $v^{\rm n}$ constructed in Lemma 5.1. Thereby,

$$\|\nabla_h(v^{\rm n} - w^{\rm n})\|_{0,\Omega}^2 \le C \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \|\mathtt{h}^{-1/2}[\![v^{\rm n}]\!]\|_{0,\kappa}^2. \tag{5.4}$$

In addition, for each edge $\kappa \in \mathcal{E}_\mathcal{I}(\mathcal{T}_h)$, shared by the two elements K_1 and K_2, we let W^κ be the conforming approximation in $H_0^1(\delta_\kappa)$ constructed in Lemma 5.3. This satisfies the following properties: $W^\kappa|_{K_1} \in \mathcal{S}_{k_{K_1}}(K_1)$, $W^\kappa|_{K_2} \in \mathcal{S}_{k_{K_2}}(K_2)$, and

$$|V_{K_1}^\kappa - W^\kappa|_{1,K_1}^2 + |V_{K_2}^\kappa - W^\kappa|_{1,K_2}^2 \le C\|\mathtt{k}\mathtt{h}^{-1/2}[\![v]\!]\|_{0,\kappa}^2 + C\|\mathtt{h}^{-1/2}[\![v^{\rm n}]\!]\|_{0,\kappa}^2. \tag{5.5}$$

Similarly, for a boundary edge $\kappa \in \mathcal{E}_\mathcal{B}(\mathcal{T}_h)$, contained in the element K, we denote by $W^\kappa \in \mathcal{S}_k(K)$ the conforming approximation in Lemma 5.3; this satisfies $W^\kappa = 0$ on $\partial K \setminus \kappa$ and

$$|V_K^\kappa - W^\kappa|_{1,K}^2 \le C\|\mathtt{k}\mathtt{h}^{-1/2}[\![v]\!]\|_{0,\kappa}^2 + C\|\mathtt{h}^{-1/2}[\![v^{\rm n}]\!]\|_{0,\kappa}^2. \tag{5.6}$$

We now set

$$w^{\rm e} = \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} W^\kappa.$$

By construction, $w^{\rm e} \in V_h^c$ and we conclude from the above estimates in (5.5) and (5.6) that

$$\|\nabla_h(v^{\rm e} - w^{\rm e})\|_{0,\Omega}^2 \le C \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \|\mathtt{k}\mathtt{h}^{-1/2}[\![v]\!]\|_{0,\kappa}^2 + C \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \|\mathtt{h}^{-1/2}[\![v^{\rm n}]\!]\|_{0,\kappa}^2. \tag{5.7}$$

Finally, we set $w^{\rm i} = v^{\rm i} \in V_h^c$ and define

$$\mathcal{A}v = w^{\rm n} + w^{\rm e} + w^{\rm i}.$$

Then, $v - \mathcal{A}v = v^{\rm n} - w^{\rm n} + v^{\rm e} - w^{\rm e}$. The triangle inequality and the bounds in (5.4) and (5.7) give

$$\|\nabla_h(v - \mathcal{A}v)\|_{0,\Omega}^2 \le C \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \|\mathtt{k}\mathtt{h}^{-1/2}[\![v]\!]\|_{0,\kappa}^2 + C \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \|\mathtt{h}^{-1/2}[\![v^{\rm n}]\!]\|_{0,\kappa}^2. \tag{5.8}$$

Bound of the nodal jumps: The desired estimate follows now from (5.8), provided that

$$\|\mathtt{h}^{-1/2}[\![v^{\mathrm{n}}]\!]\|_{0,\kappa}^2 \leq C\|\mathtt{kh}^{-1/2}[\![v]\!]\|_{0,\kappa}^2, \tag{5.9}$$

for any edge $\kappa \in \mathcal{E}(\mathcal{T}_h)$. To show this, we note that

$$\|\mathtt{h}^{-1/2}[\![v^{\mathrm{n}}]\!]\|_{0,\kappa}^2 \leq C \sum_{P\in\mathcal{N}(\kappa)} [\![v^{\mathrm{n}}]\!](P)^2 \leq C \sum_{P\in\mathcal{N}(\kappa)} [\![v]\!](P)^2.$$

Here, we have used the nodal exactness of v^{n}, that is, $v(P) = v^{\mathrm{n}}(P)$ for all $P \in \mathcal{N}(\kappa)$. Hence, from the inverse estimate in [37, Theorem 3.92], we obtain

$$\|\mathtt{h}^{-1/2}[\![v^{\mathrm{n}}]\!]\|_{0,\kappa}^2 \leq C\|[\![v]\!]\|_{L^\infty(\kappa)}^2 \leq C\|\mathtt{kh}^{-1/2}[\![v]\!]\|_{0,\kappa}^2.$$

This shows (5.9) and completes the proof. □

5.5. Proof of Proposition 4.5. The norm equivalence result, Proposition 4.5, follows now directly from Proposition 5.4 and from the fact that $V_h^\perp$ is orthogonal to V_h^c with respect to the norm $\|\cdot\|_{1,h}$.

6. Numerical Experiments. In this section we present a series of numerical examples to illustrate the practical performance of the proposed a posteriori error estimator derived in Theorem 3.1 within an automatic hp-adaptive refinement procedure which is based on 1-irregular quadrilateral elements. The hp-adaptive meshes are constructed by first marking the elements for refinement/derefinement according to the size of the local error indicators η_K; this is done by employing the fixed fraction strategy, with refinement and derefinement fractions set to 25% and 10%, respectively. Once an element $K \in \mathcal{T}_h$ has been flagged for refinement or derefinement, a decision must be made whether the local mesh size h_K or the local degree k_K of the approximating polynomial should be adjusted accordingly. The choice to perform either h-refinement/derefinement or p-refinement/derefinement is based on estimating the local smoothness of the (unknown) analytical solution. To this end, we employ the hp-adaptive strategy developed in [26], where the local regularity of the analytical solution is estimated from truncated local Legendre expansions of the computed numerical solution; see, also, [24].

Here, the emphasis will be to demonstrate that the proposed a posteriori error indicator converges to zero at the same asymptotic rate as the energy norm of the actual error on a sequence of non-uniform hp-adaptively refined meshes. For simplicity, as in [6], we set the constant C_{EST} arising in Theorem 3.1 equal to one and ensure that the corresponding effectivity indices are roughly constant on all of the meshes employed; here, the effectivity index is defined as the ratio of the a posteriori error bound and the energy norm of the actual error. In general, to ensure the reliability of the error estimator, C_{EST} must be determined numerically for the underlying problem at hand, cf. [14], for example. In all of our experiments, the data approximation terms in the a posteriori bound (3.7) from Theorem 3.1 will be neglected.

6.1. Example 1. In this example, we let Ω be the unit square $(0,1)^2$ in $\mathbb{R}^2$; further, we set $g \equiv 0$ on Γ and select f so that the analytical solution to (1.1)–(1.2) is given by

$$u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-\sigma(2x-1)^2},$$

where σ is a positive constant, cf. [30]. Throughout this section we set $\sigma = 25$; we note that a value of $\sigma = 2.5$ was employed in [30].

In Figure 6.1(a) we present a comparison of the actual and estimated energy norm of the error versus the third root of the number of degrees of freedom in the finite element space

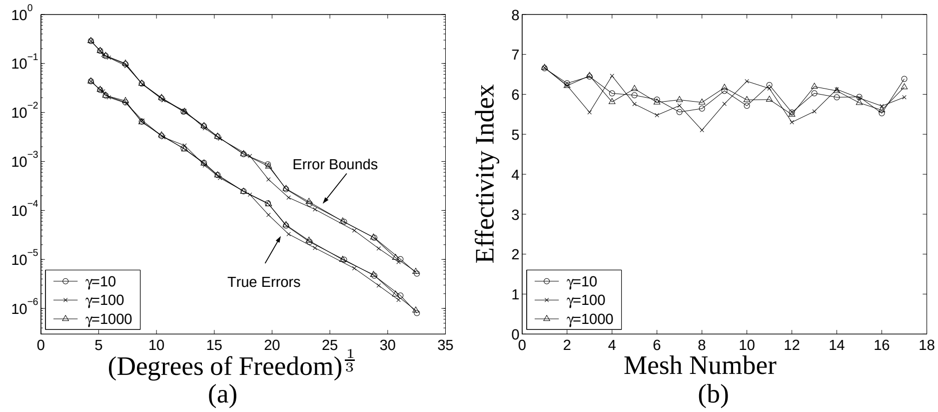


FIG. 6.1. Example 1. *(a) Comparison of the actual and estimated energy norm of the error with respect to the (third root of the) number of degrees of freedom; (b) Effectivity indices.*

V_h on a linear-log scale, for the sequence of meshes generated by our hp-adaptive algorithm. Here, numerical experiments are presented for different values of the parameter γ arising in the definition of the discontinuity stabilization function $\mathtt{c}$, cf. (2.6). We remark that the third root of the number of degrees of freedom is chosen on the basis of the a priori error analysis carried out in [39]; cf., also, [36]. For each value of γ, we observe that the error bound over-estimates the true error by a (reasonably) consistent factor; indeed, from Figure 6.1(b), we see that the computed effectivity indices oscillate around a value of approximately 6. Moreover, we observe that both the actual error in the underlying computed solution and the corresponding a posteriori error bound are relativity insensitive to changes in γ as predicted in Theorem 3.1. Finally, from Figure 6.1(a) we observe that the convergence lines using hp-refinement are (roughly) straight on a linear-log scale, which indicates that exponential convergence is attained for this smooth problem.

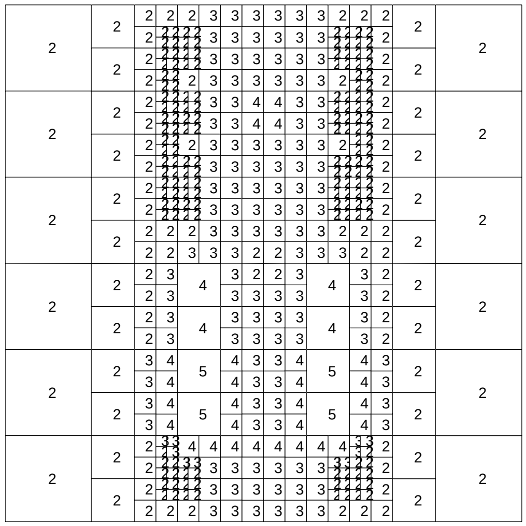
In Figure 6.2 we show the mesh generated using the proposed a posteriori error indicator with $\gamma = 10$ after 9 and 16 hp-adaptive refinement steps. Here, we observe that some h-refinement of the mesh has been performed in the vicinity of the base of the exponential 'hills' situated in the left- and the right-hand sides of the domain, where the gradient/curvature of the analytical solution is relativity large. Once the h-mesh has adequately captured the structure of the solution, the hp-adaptive algorithm increased the degree of the approximating polynomial within the interior part of the domain containing these hills.

6.2. Example 2. In this section we let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$, and select $f = 0$. Then, writing (r, φ) to denote the system of polar coordinates, we impose an appropriate inhomogeneous boundary condition for u so that

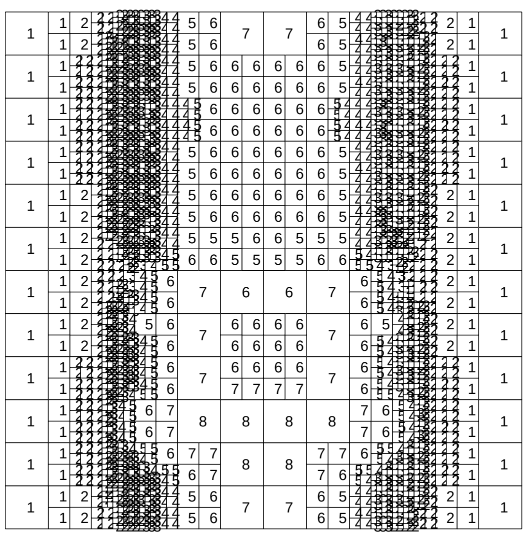
$$u = r^{2/3} \sin(2\varphi/3);$$

cf. [39]. We note that u is analytic in $\overline{\Omega} \setminus \{\mathbf{0}\}$, but ∇u is singular at the origin; indeed, here $u \notin H^2(\Omega)$. This example reflects the typical (singular) behavior that solutions of elliptic boundary value problems exhibit in the vicinity of reentrant corners in the computational domain.

Figure 6.3(a) shows the history of the actual and estimated energy norm of the error on each of the meshes generated by our hp-adaptive algorithm for $\gamma = 10, 100, 1000$. As in the previous example, we observe that, for each γ, the a posteriori bound over-estimates the true error by a consistent factor; for $\gamma = 10$, the effectivity index tends to a value of just under



(a)



(b)

FIG. 6.2. Example 1. *hp-mesh after: (a)* 9 *adaptive refinements, with* 426 *elements and* 5392 *degrees of freedom; (b)* 16 *adaptive refinements, with* 2088 *elements and* 34426 *degrees of freedom.*

3, while for $\gamma = 100, 1000$, this quantity tends to a value just below 4, cf. Figure 6.3(b). Additionally, from Figure 6.3(a) we observe exponential convergence of the energy norm of the error using hp-refinement; indeed, for each γ, on a linear-log scale, the convergence lines are on average straight.

In Figure 6.4 we show the mesh generated using the local error indicators η_K after 13 hp-adaptive refinement steps with $\gamma = 10$. Here, we see that the h-mesh has been largely refined in the vicinity of the re-entrant corner located at the origin; from the zoom, we see that this refinement occurs in the diagonal direction $x = y$. In the other diagonal direction, $x = -y$, p-refinement is employed as the solution is deemed to be smooth here. Additionally, we see

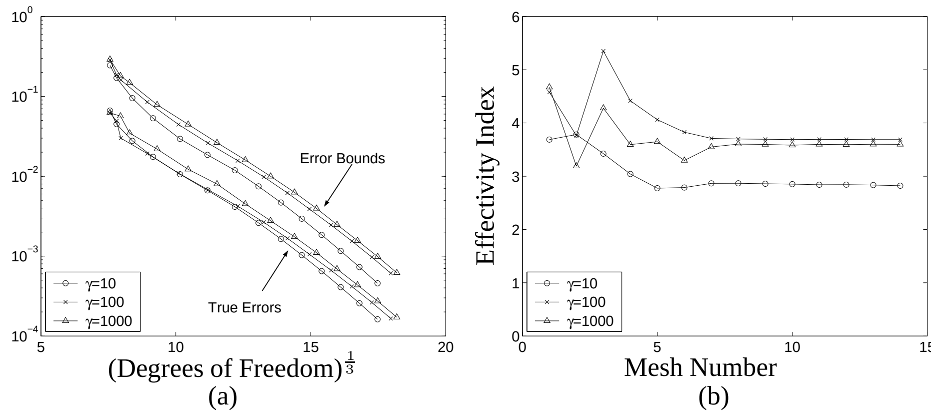


FIG. 6.3. Example 2. *(a) Comparison of the actual and estimated energy norm of the error with respect to the (third root of the) number of degrees of freedom; (b) Effectivity indices.*

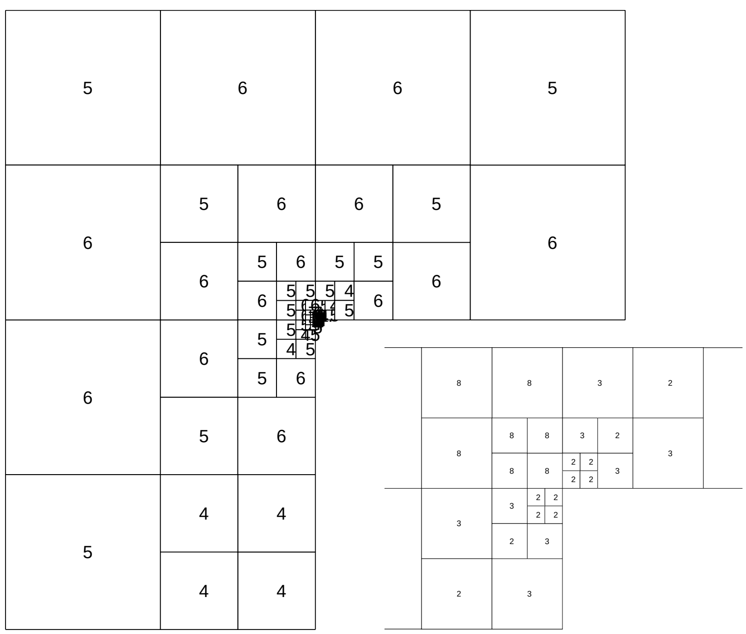


FIG. 6.4. Example 2. hp-mesh after 13 adaptive refinements, with 138 elements and 5335 degrees of freedom.

that the polynomial degrees have been increased away from the origin, since the underlying analytical solution is smooth in this region.

7. Concluding Remarks. In this paper, we have presented the first hp-version energy norm a posteriori error analysis for discontinuous Galerkin discretizations of elliptic boundary-value problems. The analysis is based on employing a non-consistent reformulation of the DG scheme, together with a new hp-version norm equivalence result for the underlying discontinuous finite element space. Although our analysis is restricted to conforming meshes consisting of triangles and quadrilaterals, the numerical tests presented in this article clearly demonstrate that, in practice, the proposed a posteriori estimator works equally well on 1-irregularly refined meshes with hanging nodes. The derivation of a posteriori error bounds on such nonconforming meshes with no additional loss of powers of the approximation degree remains an open problem.

We emphasize that our analysis can be extended with only minor modifications to more general second-order diffusion-dominated problems. Similarly, the proposed estimator can be readily generalized to problems with Neumann and/or mixed boundary conditions. Furthermore, although here we have used the interior penalty approach to discretize the boundary-value problem (1.1)–(1.2), our results remain valid for any other DG formulation whose underlying primal form A_h is both coercive and continuous, such as, e.g., the local discontinuous Galerkin method; for details, we refer the reader to [1]. Finally, we mention that the extension of our hp-version a posteriori error analysis to the Stokes equations of incompressible fluid flow can be found in the recent conference paper [22].

Future work will be devoted to extending our approach to convection-diffusion problems, where the convection dominates, as well as to problems in three space dimensions.

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