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By

Constantin Bacuta

Victor Nistor

and

Ludmil T. Zikatanov

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: http://www.ima.umn.edu

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CONSTANTIN BACUTA, VICTOR NISTOR, AND LUDMIL T. ZIKATANOV

Abstract. We prove a well-posedness result for second order boundary value problems in weighted Sobolev spaces on curvilinear polyhedral domains in $\mathbb{R}^n$ with Dirichlet boundary conditions. Our typical weight is the distance to the set of singular boundary points.

Contents

Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider the boundary value problem

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = g, \end{cases} \tag{1}$$

where Δ is the Laplace operator. For Ω smooth and bounded, this boundary value problem has a unique solution $u \in H^{s+2}(\Omega)$ depending continuously on $f \in H^s(\Omega)$ and $g \in H^{s-1/2}(\partial\Omega)$, $s > 1/2$. See [15, 28, 41] for a proof of this basic and well known result.

It is also well known that this result does not extend to non-smooth domains Ω. A deep analysis of the difficulties that arise for $\partial\Omega$ Lipschitz is contained in the papers [4, 6, 20, 21, 22, 23, 35, 42], and others (see the references in the aforementioned papers). Results more specific to curvilinear polyhedral domains are contained in [9, 10, 11, 12, 13, 25, 26, 31] and in the monographs [17, 18, 29, 30, 36].

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In this paper, we consider the boundary value problem (1) when Ω to is a *bounded curvilinear polyhedral domain* in $\mathbb{R}^n$, or, more generally, in a manifold M of dimension n and Poisson's equation $\Delta u = f$ is replaced by a strongly elliptic system. We define curvilinear polyhedral domains inductively in Section 6. Let us denote by $\eta_{n-2}(x)$ the distance from a point $x \in \Omega$ to the set $\Omega^{(n-2)} \subset \partial\Omega$ of non-smooth boundary points of Ω (*i.e.,* the set of points $p \in \partial\Omega$ such $\partial\Omega$ is not smooth in a neighborhood of p). We take $\eta_{n-2} = 1$ if there are no such points, that is, if $\partial\Omega$ is smooth. The *distance* is computed within $\overline{\Omega}$, that is, it is the least length of the curves $\gamma : [0,1] \to \overline{\Omega}$ such that $\gamma(0) = x$ and $\gamma(1) = p \in \Omega^{(n-2)}$. We work in the *weighted* Sobolev spaces

$$(2) \qquad \mathcal{K}^l_a(\Omega) = \{u \in L^2_{\text{loc}}(\Omega),\ \eta_{n-2}^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \text{for all } |\alpha| \leq l\}, \quad l \in \mathbb{Z}_+,$$

which we endow with the induced Hilbert space norm. We extend the above definition to the case $l \in \mathbb{R}_+$ by interpolation. A similar definition, Definition 5, yields the weighted Sobolev spaces $\mathcal{K}^l_a(\partial\Omega)$, $l \in \mathbb{R}_+$. These weighted Sobolev spaces are closely related to weighted Sobolev spaces on non-compact manifolds. See [14, 19, 37, 38, 39] for related results on boundary value problems on non-compact manifolds and, more generally, on the analysis on non-compact manifolds.

A slightly simplified version of our main result, Theorem 1.5, when formulated for the Laplace operator Δ on $\mathbb{R}^n$, reads as follows. Before we formulate the result, let us agree that, throughout this paper, Ω will always be an open set, sometimes with additional properties.

Theorem 0.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, curvilinear polyhedral domain. Then there exists $\eta > 0$ such that the boundary value problem* (1) *has a unique solution $u \in \mathcal{K}^{s+1}_{a+1}(\Omega)$ for any $f \in \mathcal{K}^{s-1}_{a-1}(\Omega)$, any $g \in \mathcal{K}^{s+1/2}_{a+1/2}(\partial\Omega)$, any $s > 1/2$, and any $|a| < \eta$. This solution depends continuously on f and g. If $s = -1$, this solutions is the solution of the associated variational problem.*

The weighted Soblev spaces are more general than the ones considered before (see Theorem 1.5) and we also allow the dimension of the ambient Euclidean space $\mathbb{R}^n$ to be arbitrary. However, we consider only second order, strongly elliptic systems and only Dirichlet boundary conditions.

The first four sections of the paper contain a self-contained proof our main result, Theorem 1.5, when $g = 0$ and the dimension is small ($n = 2$ and $n = 3$). The last three sections are devoted to the (formidable) geometric difficulties arising in arbitrary dimensions. They are also devoted to setting up the machinery of [2] and, especially [1], which reduces the case g arbitrary to the case $g = 0$, based on general principle.

We now describe the contents of the sections of the paper in more detail. The first section introduces the weighted Sobolev spaces that we consider and states the main result, Theorem 1.5. The second section contains a proof of the main result based on three intermediate results (a Hardy–Poincaré coercivity result, a regularity result, and a trace result) that would be proved later. A proof of the coercivity result in dimensions $n \leq 3$ is given in Section 4, based on the definition of curvilinear polyhedral domains for the same set of dimensions, contained in Section 3. Section 5 is, to a large extend, a review of Lie manifolds (with or without boundary) from [1], which builds on earlier results from [2] and [33]. Polyhedral domains are defined in 6. They are studied from the point of view of Lie manifolds in 7. In the last section, Section 8, we put together the results of the previous sections to complete the proof of our main result.

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1. Sobolev spaces and statement of the main theorem

We introduce in this section the weighted Sobolev spaces $h\mathcal{K}^s_a(\Omega)$, $h\mathcal{K}^s_a(\partial\Omega)$, and the assumptions on the differential operator P whose Dirichlet boundary value problem we shall study. (Recall that throughout this paper Ω will be an open set.) Then we state the main results of this paper, Theorem 1.5, namely the well posedness of the boundary value problem (or $m \times m$ system)

$$\begin{cases} Pu = f \in h\mathcal{K}^{s-1}_{a-1}(\Omega)^m & \text{on } \Omega \\ u|_{\partial\Omega} = g \in h\mathcal{K}^{s+1/2}_{a+1/2}(\partial\Omega)^m & \text{on } \partial\Omega \end{cases} \tag{3}$$

for $u \in h\mathcal{K}^{s+1}_{a+1}(\Omega)^m$, for suitable a and s and a suitable weight h. The spaces $h\mathcal{K}^s_a(\Omega)$ are defined in terms of the space $\mathcal{K}^s_a(\Omega)$ as follows. If $h > 0$ on Ω, we shall denote by

$$h\mathcal{K}^s_a(\Omega) := \{hu,\, u \in \mathcal{K}^s_a(\Omega)\}, \tag{4}$$

with induced norm, that is $\|hu\|_{h\mathcal{K}^s_a(\Omega)} = \|u\|_{\mathcal{K}^s_a(\Omega)}$.

In this section, we define the weighted Sobolev spaces $\mathcal{K}^s_a(\Omega)$ and $\mathcal{K}^s_a(\partial\Omega)$ when Ω is a *straight polyhedron* (straight polyhedra are defined below, Definition 1.2). Theorem 1.5 is valid, however, for any bounded, curvilinear polyhedral domain Ω in a smooth manifold M of dimension n. The general definitions would require us to introduce too much auxiliary material on curvilinear polyhedral domains, so we prefered to give this more general definition later. For instance, we postpone a complete, rigorous definition of curvilinear polyhedral domains until Section 6. Also, the definition of the Sobolev spaces $\mathcal{K}^s_a(\partial\Omega)$ when Ω is a *curvilinear polyhedral domain* will have to wait until Subsection 7.5 (the definition of $\mathcal{K}^s_a(\Omega)$ is essentially the same). Anticipating in this section the general definitions will allow the reader to get a very quick insight into our main results, with only minimal loss of generality. Moreover, the equivalence of the two definitions in case of straight polyhedra is a result interesting in itself.

The reader may rest assured that, although having an involved definition, a curvilinear polyhedral domain means almost exactly what one expects it to mean. See Section 3, where we describe all curvilinear polyhedral domains in $\mathbb{R}^2$, in S^2, and in $\mathbb{R}^3$, as well as the function r_Ω and the desingularization $\Sigma(\Omega)$. Note, however, that the subsection 3.3 contains some less common examples.

1.1. **Weighted Sobolev spaces.**

We shall introduce now our Sobolev spaces for "straight polyhedra." We shall need a special type of weight, that is, the "distance to the set of non-smooth boundary points," denoted $\eta_{n-2}(x)$, which we introduce in the next definition. Recall that a point $x \in \partial\Omega$ is called a *smooth boundary point of* Ω if the intersection of $\partial\Omega$ with a small neighborhood of p is a smooth manifold of dimension $n-1$. A point $p \in \partial\Omega$ that is not a smooth boundary point will be called a *non-smooth boundary point of* Ω. The set of non-smooth boundary points of Ω is denoted by $\Omega^{(n-2)}$ as before (so $\Omega^{(n-2)} \subset \partial\Omega$).

In what follows, a half-ball will be the intersection of an open ball with an open subspace whose boundary contains the center of the ball. A point $p \in \partial\Omega$ will be called a *one-sided smooth boundary point* if p has a neighborhood V_p which intersects Ω in a half-ball. A smooth boundary point $p \in \partial\Omega$ that is not a one-sided smooth boundary point will be called a *two-sided smooth boundary point of* Ω. This happens precisely when there exist points of Ω as close as desired to p on both sides of $\partial\Omega$.

Definition 1.1. Let Ω be an open set of a Riemannian manifold of dimension n. The distance $\eta_{n-2}(x)$ from x to the set $\Omega^{(n-2)}$ is

$$\eta_{n-2}(x) = \inf_{\gamma} \ell(\gamma),$$

where $\ell(\gamma)$ is the length of the curve γ, and γ ranges through all smooth curves $\gamma : [0,1] \to \overline{\Omega}$, $\gamma(0) = x$, $p := \gamma(1) \in \Omega^{(n-2)}$.

Below, by an *affine space* we shall denote the translation of a subspace of a vector space V. A *straight polyhedral domain of dimension* 1 is simply an open interval on a line (bounded or not). A *straight polyhedral domain of dimension* 2, also called a *straight polygon*, is simply a bounded, connected domain bounded by a polygon with straight edges (no self-intersections). We do not require the boundary of a straight polyhedron to be connected, as in [5].

Definition 1.2. An open subset $\Omega \subset V$ with finitely many connected components of an affine space V of dimension $n \geq 2$ will be called a *straight polyhedron* if there exist disjoint straight polyhedra $D_j \subset \partial\Omega$ of dimension $n-1$ such that $\partial\Omega = \cup \overline{D}_j$.

Examples of polyhedral domains are, in increasing generality:

(i) the simplex $\Delta_n \subset \mathbb{R}^n$, a case that was treated in [1],
(ii) a convex polytope of dimension n, that is the convex hull of a finite set of points in $\mathbb{R}^n$, provided that this set has a non-empty interior.

By contrast, a smooth, bounded domain – while being a curvilinear polyhedral domain – is not a straight polyhedron. Our definitions below, as well as the main result, Theorem 1.5, are valid for smooth, bounded domains, trivially. In fact, all these definitions and results are classical for smooth, bounded domains, and our results do not bring anything new in that case.

Let Ω be a straight polyhedron in an affine space V, which we shall identify with $\mathbb{R}^n$, for simplicity. Let f be a continuous function on Ω, $f > 0$ on the interior of Ω. We define the lth Sobolev space with weight f (and index a) by

$$\mathcal{K}^l_{a,f}(\Omega) = \{u \in L^2_{\text{loc}}(\Omega),\ f^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \text{for all } |\alpha| \leq l\}, \quad l \in \mathbb{Z}_+. \tag{5}$$

The norm on $\mathcal{K}^l_{a,f}(\Omega)$ is

$$\|u\|^2_{\mathcal{K}^l_{a,f}(\Omega)} := \sum_{|\alpha| \leq l} \|f^{|\alpha|-a}\partial^\alpha u\|^2_{L^2(\Omega)}. \tag{6}$$

Definition 1.3. Let f, g be two continuous, non-negative functions on Ω. We shall say that f and g are *equivalent* (written $f \sim g$) if there exists a constant $C > 0$ such that

$$C^{-1}f(x) \leq g(x) \leq Cf(x),$$

for all $x \in \Omega$.

Clearly, if $f \sim g$, then the norms $\|u\|_{\mathcal{K}^l_{a,f}(\Omega)}$ and $\|u\|_{\mathcal{K}^l_{a,g}(\Omega)}$ are equivalent, and hence we have $\mathcal{K}^l_{a,f}(\Omega) = \mathcal{K}^l_{a,g}(\Omega)$ as Banach spaces.

Definition 1.4. We let $\mathcal{K}^l_a(\Omega) = \mathcal{K}^l_{a,f}(\Omega)$ and $\|u\|_{\mathcal{K}^l_a(\Omega)} = \|u\|_{\mathcal{K}^l_{a,f}(\Omega)}$, where $f = \eta_{n-2}$ is the distance to $\Omega^{(n-2)}$.

For example, $\mathcal{K}^0_0(\Omega) = L^2(\Omega)$. For Ω a polygon in the plane, $\eta_{n-2}(x) = \eta_0(x)$ is the distance from x to the vertices of Ω and the resulting spaces $\mathcal{K}^l_a(\Omega)$ are the spaces introduced by Kondratiev in [25].

If $l \in \mathbb{N} = \mathbb{Z}_+ \smallsetminus \{0\}$, we define $\mathcal{K}^{-l}_{-a}(\Omega)$ to be the dual of $\mathcal{K}^l_a(\Omega)$ with pivot $\mathcal{K}^0_0(\Omega)$. That is, we define for any $u \in \mathcal{C}^\infty(\Omega)$

$$\|u\|_{\mathcal{K}^{-l}_{-a}(\Omega)} = \sup \frac{|(u,v)|}{\|v\|_{\mathcal{K}^l_a(\Omega)}}, \quad 0 \neq v \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega), \tag{7}$$

and we let $\mathcal{K}^{-l}_{-a}(\Omega)$ to be the completion of the space of smooth functions u on Ω that are such that $\|u\|_{\mathcal{K}^{-l}_{-a}(\Omega)} < \infty$. We define the spaces $\mathcal{K}^s_a(\Omega)$, $s \in \mathbb{R}$, by interpolation.

To define the spaces $\mathcal{K}^s_a(\partial\Omega)$, we need to assume at this time that Ω is a straight polyhedron and that $\partial\Omega = \partial\overline{\Omega}$. Then the definition is similar. Let D_j be as in Definition 1.2, then

$$\mathcal{K}^l_{a,f}(D_j) = \{u \in L^2_{\mathrm{loc}}(D_j),\ f^{k-a} X_1 X_2 \ldots X_k u \in L^2(D_j),\ 0 \leq k \leq l\}, \tag{8}$$

for all choices of vector fields X_j in a basis of the linear space containing D_j. Then

$$\mathcal{K}^l_a(\partial\Omega) = \{u \in L^2_{\mathrm{loc}}(\partial\Omega),\ u|_{D_j} \in \mathcal{K}^l_{a,f}(D_j),\ f = \eta_{n-2},\ \text{ for all } j\},. \tag{9}$$

Note that on each face of $\partial\Omega$, the natural weight is the distance to the boundary, not the distance to the set of singular boundary points of that face.

Similarly, the spaces $\mathcal{K}^{-l}_{-a}(\partial\Omega)$ are defined to be the duals of $\mathcal{K}^l_a(\partial\Omega)$ with pivot $L^2(\partial\Omega)$. The spaces $\mathcal{K}^s_a(\partial\Omega)$, with $s \notin \mathbb{Z}$ are defined by interpolation.

1.2. **The differential operator** P**.** We now introduce the differential operators that we study. At a first reading, the reader may consider $P = -\Delta$. Assume first that Ω is a straight polyhedron. We shall write $[D]_{pq}$ for the (p,q) entry of a $m \times m$ matrix D. The adjoint of the matrix D is denoted D^*. Hence $[D^*]_{pq} = \overline{[D]}_{qp}$. Also, we shall write $Re(z) := \frac{1}{2}(z + \overline{z})$, or simply $Re\, z$ for the real part of a complex number z. The relation $A \geq B$ means that the $m \times m$ matrices A and B satisfy $(A\xi, \xi) - (B\xi, \xi) \geq 0$ for all $\xi \in \mathbb{C}^m$. The set of smooth functions on $\overline{\Omega}$ with values $m \times m$ complex matrices will be denoted $\mathcal{C}^\infty(\overline{\Omega}, M_m(\mathbb{C}))$.

Let $u = (u_1, u_2, \ldots, u_m) \in H^2_{\mathrm{loc}}(\Omega)$. We shall study the differential operator (or matrix of differential operators)

$$Pu(x) = -\sum_{j,k=1}^n \partial_j \big[A_{jk}(x) \partial_k u(x)\big] + \sum_{j=1}^n B_j(x) \partial_j u(x) + C(x) u(x). \tag{10}$$

The coefficients A_{jk}, B_j, C are required to satisfy the following conditions, for some constant $\epsilon > 0$,

$$A_{jk},\ B_j,\ C\ \in \mathcal{C}^\infty(\overline{\Omega}, M_m(\mathbb{C})),$$

$$Re\Big(\sum_{j,k=1}^n \sum_{p,q=1}^m [A_{jk}(x)]_{pq} \xi_{jp} \bar{\xi}_{kq}\Big) \geq \epsilon \sum_{j=1}^n \sum_{p=1}^m |\xi_{jp}|^2, \quad \xi_{jp} \in \mathbb{C} \tag{11}$$

$$B_j(x)^* = B_j(x), \quad \text{for all } x \text{ and } j, \qquad \text{and}$$

(12)
$$C(x) + C(x)^* - \sum_{j=1}^{n} \partial_j B_j(x) \geq 0.$$

Let us make the following simple observation, which will be useful nevertheless in the proof of our main theorem. Namely, decompose P as $P_1 + P_2$, where P_1 involves only the terms with A_{jk} and P_2 involves only the terms with B_j and C. Then our assumptions on P are really assumptions on $P_1 + P_1^*$ and $P_2 + P_2^*$. This shows that P^*, the formal adjoint of P, is of the same type as P, that is, it also satisfies the assumptions of Equations (11) and (12).

Let us consider now the general case of a curvilinear polyhedral domain Ω. Let then E be a vector bundles on M endowed with a hermitian metric, where M is the given manifold containing Ω, of the same dimension as Ω. A coordinate free expression of the conditions in Equations (11) and (12) is obtained as follows. We assume that there exist a metric preserving connection $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$, a smooth endomorphism $A \in \mathrm{End}(E \otimes T^*M)$, and a first order differential operator $P_2 : \Gamma(E) \to \Gamma(E)$ with smooth coefficients such that

(13)
$$A + A^* \geq 2\epsilon I \text{ for some } \epsilon > 0.$$

Then we define $P_1 = \nabla^* A \nabla$ and $P = P_1 + P_2$. In particular, our operator P will be strongly elliptic in a neighborhood of Ω in M. Note that if $\Omega \subset \mathbb{R}^n$ and the vector bundle E is trivial, then the conditions (13) reduce to the conditions (11), by taking ∇ to be the trivial connection.

1.3. **The set of weights.** We first introduce the set of admissible weights. Since our definition involves the desingularization $\Sigma(\Omega)$ of Ω, which will be introduced only in Section 6 (for $n \leq 3$ in Section 3), the reader may consider only the particular admissible weight η_{n-2}^a, for $a \in \mathbb{R}$, instead of a general admissible weight, and hence ignore the next two paragraphs.

Let x_H be the defining function of the hyperface H of $\Sigma(\Omega)$. A hyperface $H \subset \Sigma(\Omega)$ will be called *a hyperface at infinity* if $\kappa(H) \subset \Omega^{(n-2)}$. A function $h = \prod_H x_H^{a_H}$, where H ranges through the set of faces at infinity of $\Sigma(\Omega)$, will be called *an admissible weight*. We endow the set of admissible weights with the topology defined by the exponents a_H.

It is instructive to mention at this time that the function $r_\Omega := \prod_H x_H$, where the product is taken over all hyperfaces at infinity, is smooth on Ω and equivalent to η_{n-2}, by Corollary 7.11. The function r_Ω, for various Ω, will play an important role in the in the inductive definition of the structural Lie algebra of vector fields $\mathcal{V}(\Omega)$ on $\Sigma(\Omega)$. In particular, r_Ω^a, $a \in \mathbb{R}$, is the most important example of an admissible weight. We also have that

(14)
$$r_\Omega^t \mathcal{K}_a^s(\Omega) = \mathcal{K}_{a+t}^s(\Omega),$$

so in a statement about the spaces $h\mathcal{K}_a^s(\Omega)$, where h is an admissible weight, we can usually assume that $a = 0$, without loss of generality.

1.4. **The main results.** We now state some of the most important results of this paper. We begin with the main theorem of this paper.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded curvilinear polyhedral domain of dimension n. Assume that the operator P satisfies the assumptions of Equations*

(11) and (12). Let $\mathcal{W}_s(\Omega)$ be the set of admissible weights h such that the map $\tilde{P}(u) := (Pu, u|_{\partial\Omega})$ establishes an isomorphism

$$\tilde{P} : h\mathcal{K}_1^{s+1}(\Omega)^m \to h\mathcal{K}_{-1}^{s-1}(\Omega)^m \oplus h\mathcal{K}_{1/2}^{s+1/2}(\partial\Omega)^m.$$

Then the set $\mathcal{W}_s(\Omega)$ is an open set containing 1.

As we have already pointed out earlier, Theorem 1.5 reduces to a well known, classical result when Ω is a smooth bounded domain. (See however Section 9 for a result on smooth domains that is not classical.) The same is true of Theorem 1.6, which works for general domains.

Theorem 1.6. *Let $\Omega \subset M$ be a bounded curvilinear polyhedral domain of dimension n. Assume that the operator P satisfies the assumptions of Equation (13). Let $\mathcal{W}'_s(\Omega)$ be the set of admissible weights h such that the map $\tilde{P}_c(u) := (Pu+cu, u|_{\partial\Omega})$ establishes an isomorphism*

$$\tilde{P}_c : h\mathcal{K}_1^{s+1}(\Omega)^m \to h\mathcal{K}_{-1}^{s-1}(\Omega)^m \oplus h\mathcal{K}_{1/2}^{s+1/2}(\partial\Omega)^m.$$

Then the set $\mathcal{W}'_s(\Omega)$ is an open set containing 1.

We obtain the following immediate consequences of Theorem 1.5. Therefore, for the rest of this section, Ω and P will be as in Theorem 1.5. Similar consequences can be obtained from Theorem 1.6, but we will not state them explicitly, since they are completely similar.

Here are two immediate consequence of the above theorem. The continuity of the inverse of $\tilde{P}$ is made explicit in the following corollary.

Corollary 1.7. *There exists $C_\Omega > 0$, depending only on Ω, $s \geq 0$, and $h \in \mathcal{W}_s(\Omega)$, such that*

$$\|u\|_{h\mathcal{K}_1^{s+1}(\Omega)} \leq C_\Omega\big(\|Pu\|_{h\mathcal{K}_{-1}^{s-1}(\Omega)} + \|u|_{\partial\Omega}\|_{h\mathcal{K}_{1/2}^{s+1/2}(\partial\Omega)}\big),$$

for any $u \in \mathcal{K}_1^1(\Omega)^m$.

From the fact that η_{n-2} is equivalent to $r_\Omega := \prod_H x_H$, where H ranges through all hyperfaces at infinity of $\Sigma(\Omega)$, by Proposition 7.9 and Corollary 7.11, we obtain the following corollary.

Corollary 1.8. *There exists $\eta > 0$ such that*

$$P : \mathcal{K}_{a+1}^{s+1}(\Omega)^m \cap \{u \in H^1_{\text{loc}}(\Omega)^m, u|_{\partial\Omega} = 0\} \to \mathcal{K}_{a-1}^{s-1}(\Omega)^m$$

is an isomorphism for all $s \geq 0$ and all $|a| < \eta$.

Proof. We know that $\mathcal{K}_{a+1}^{s+1} = r_\Omega^a \mathcal{K}_1^{s+1}$ and that r_Ω^a is an admissible weight for any $a \in \mathbb{R}$. The result then follows from the fact that $\mathcal{W}_s(\Omega)$ contains the weight 1, by Theorem 1.5 and it is an open set, by Proposition 2.6. □

Here is another corollary, in the spirit of [7]. This method was used in that paper for $n = 2$ to determine the constant η in the previous corollary as $\eta = \pi/\alpha_M$, where α_M is the largest angle of Ω. See also [26].

Corollary 1.9. *Let $h = \prod_H x_H^{a_H}$ be an admissible weight such that either all $a_H \geq 0$ or all $a_H \leq 0$. Assume that for all $\lambda \in [0, 1]$ the map*

$$P : h^\lambda \mathcal{K}_1^1(\Omega)^m \cap \{u \in H^1_{\text{loc}}(\Omega)^m, u|_{\partial\Omega} = 0\} \to \mathcal{K}_{-1}^{-1}(\Omega)^m$$

is Fredholm. Then $h \in \mathcal{W}(\Omega)$.

Proof. We proceed as in [7]. The result for an arbitrary h is equivalent to the original result for P replaced by $h^{-1}Ph$. The family $P_\lambda := h^{-\lambda}Ph^{\lambda}$ is continuous for $\lambda \in [0,1]$, consists of Fredholm operators (by the definition of $\mathcal{W}(\Omega)$), and is invertible for $\lambda = 0$. It follows that the family P_λ consists of Fredholm operators of index zero. To prove that these operators are isomorphisms, it is hence enough to prove that they are either injective or surjective. Assume first that $a_H \geq 0$ in the definition $h = \prod_H x_H^{a_H}$ of h. Then $h^\lambda \mathcal{K}_1^1(\Omega) \subset \mathcal{K}_1^1(\Omega)$. Therefore P is injective on $h^\lambda \mathcal{K}_1^1(\Omega) \cap \{u|_{\partial\Omega} = 0\}$. This, in turn, gives that P_λ is injective.

Assume that $a_H \leq 0$ and consider

$$P_\lambda : h^\lambda \mathcal{K}_1^1(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to h^\lambda \mathcal{K}_{-1}^{-1}(\Omega). \tag{15}$$

We have $(P_\lambda)^* = (P^*)_{-\lambda}$. The same argument as above shows that P_λ^* is injective, and hence that it is an isomorphism, for all $0 \leq \lambda \leq 1$. Hence P_λ is an isomorphism for all $0 \leq \lambda \leq 1$. □

2. Proof of the main result

We list in the following subsection, the stepping stones in the proof of our main result, Theorem 1.5. They are:

(i) a Hardy–Poincaré type inequality (Theorem 2.1;
(ii) a regularity result for polyhedra (Theorem 2.3); and
(iii) general properties of weighted Sobolev spaces (Theorem 2.4 and Proposition 2.5).

The Hardy–Poincaré type inequality, stated next, will be proved in Section 3 for $n = 3$ (the only missing ingredient necessary for the general case is Proposition 7.10). The other two stepping stones, follow from the results of Sections 5–7 and the results of [1], as it will be explained in Section 8.

In Subsection 2.2, we show how our main result, Theorem 1.5, can be proved using the results stated in the first subsection, Subsection 2.1. Recall that throughout this paper Ω will be an open set.

2.1. Stepping stones.

We now list the main stepping stones for the proof of Theorem 1.5. Let $dx = dx_1 dx_2 \dots dx_n$. The following result is our Hardy–Poincaré coercivity result (or inequality).

Theorem 2.1. *There exists a constant $\kappa_\Omega > 0$, depending only on Ω, such that*

$$\|u\|^2_{\mathcal{K}_1^0(\Omega)} := \int_\Omega \frac{|u(x)|^2}{\eta_{n-2}(x)^2}\, dx \leq \kappa_\Omega \int_\Omega |\nabla u(x)|^2\, dx, \tag{16}$$

for any function $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $u|_{\partial\Omega} = 0$.

From this theorem we immediately obtain the following corollary.

Corollary 2.2. *There exists a constant $\kappa'_\Omega > 0$, depending only on Ω, such that*

$$\frac{1}{\kappa'_\Omega} \|u\|^2_{\mathcal{K}_1^1(\Omega)} \leq \int_\Omega |\nabla u(x)|^2\, dx\,,$$

for any function $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $u|_{\partial\Omega} = 0$.

Proof. We have

$$\|u\|^2_{\mathcal{K}_1^1(\Omega)} = \int_\Omega \frac{|u(x)|^2}{\eta_{n-2}(x)^2}\, dx + \int_\Omega |\nabla u(x)|^2\, dx \leq (\kappa_\Omega + 1) \int_\Omega |\nabla u(x)|^2\, dx\,,$$

by Theorem 2.1 and the definition of the norm on the spaces $\mathcal{K}^l_a(\Omega)$, $l \in \mathbb{Z}_+$. $\square$

The regularity result, stated next, is of independent interest. Its proof is based to a large extend on the results of [1]. It will be proved in Section 8. Recall that h is an admissible weight if $h = \prod_H x_H^{a_H}$, $a_H \in \mathbb{R}$, where H is a hyperface at infinity of $\Sigma(\Omega)$.

Theorem 2.3. *Let h be an admissible weight. Let $u \in h\mathcal{K}^1_1(\Omega)^m$ be such that $Pu \in h\mathcal{K}^{s-1}_{-1}(\Omega)^m$ and $u|_{\partial\Omega} \in h\mathcal{K}^{s+1/2}_{1/2}(\partial\Omega)^m$, $s \in \mathbb{R}_+$. Then $u \in h\mathcal{K}^{s+1}_1(\Omega)^m$ and*

$$\|u\|_{h\mathcal{K}^{s+1}_1(\Omega)} \leq C(\|Pu\|_{h\mathcal{K}^{s-1}_{-1}(\Omega)} + \|u\|_{h\mathcal{K}^0_1(\Omega)} + \|u|_{\partial\Omega}\|_{h\mathcal{K}^{s+1/2}_{1/2}(\partial\Omega)}). \tag{17}$$

We shall need also the following result about our Sobolev spaces, which generalizes the well known results on Sobolev spaces on domains with smooth boundary. Let $\mathcal{C}^\infty_c(\Omega)$ be the space of compactly supported functions on the open set Ω.

Theorem 2.4. *The restriction $\mathcal{C}^\infty_c(\overline{\Omega} \smallsetminus \Omega^{(n-2)}) \ni u \to u|_{\partial\Omega} \in \mathcal{C}^\infty_c(\partial\Omega \smallsetminus \Omega^{(n-2)})$ extends to a continuous, surjective map*

$$\mathcal{K}^s_a(\Omega) \to \mathcal{K}^{s-1/2}_{a-1/2}(\partial\Omega), \quad s > 1/2.$$

Moreover, $\mathcal{C}^\infty_c(\Omega)$ is dense in the kernel of this map if $s = 1$.

This will follow from Theorems 3.4 and 3.7 of [1], once we shall identify the Sobolev spaces on Ω with the Sobolev spaces introduced in [1], see Proposition 8.4 and Definition 8.5. The relevant results from Theorems 3.4 and 3.7 of [1] are recalled in Proposition 5.8.

The continuity of the map $\tilde{P}$ in the above theorem is a consequence of the next proposition, which in turn is a consequence of the results of Section 7 (it is contained in Corollary 8.6).

Proposition 2.5. *Let P be a differential operator of order m on M with smooth coefficients. Then P maps $h\mathcal{K}^s_a(\Omega)$ to $h\mathcal{K}^{s-m}_{a-m}(\Omega)$ continuously, for any admissible weight h. Moreover, the resulting family $h^{-\lambda}Ph^\lambda : \mathcal{K}^s_a(\Omega) \to \mathcal{K}^{s-m}_{a-m}(\Omega)$ of bounded operators depends continuously on λ.*

The continuity (and surjectivity) of the second component in $\tilde{P}$ follows from Theorem 2.4.

Proposition 2.6. *The set $\mathcal{W}_s(\Omega)$ is open.*

Proof. This follows right away from Proposition 2.5. Indeed, the family $P : h\mathcal{K}^{s+1}_1(\Omega) \to h\mathcal{K}^{s-1}_{-1}(\Omega)$ is unitary equivalent to $h^{-1}Ph : \mathcal{K}^{s+1}_1(\Omega) \to \mathcal{K}^{s-1}_{-1}(\Omega)$. The result then follows since the set of invertible operators is open. $\square$

2.2. **Proof of the main result.** We now prove Theorem 1.5 assuming the results stated in the previous subsection, Subsection 2.1. The proof of Theorem 1.6 is completely similar.

Proof. We shall follow the pattern of proof from [15]. See also [16], where the slightly more general version of the Lax–Milgram lemma needed in this proof is proved. First, let us notice that Theorem 2.4 allows us to reduce the proof to the case when $g = 0$. Also, it is enough to prove our result for s an integer, since the general case then follows by interpolation.

The first step is to prove that $1 \in \mathcal{W}_0(\Omega)$. We shall denote by $\big(u, v\big) := \sum_{j=1}^{m} \int_\Omega u_j(x)\overline{v_j(x)}dx$ the inner product on $L^2(\Omega)^m$. Let $\mathcal{H} \subset \mathcal{K}_1^1(\Omega)$ be the subspace consisting of the functions $u \in \mathcal{K}_1^1(\Omega)$ such that $u = 0$ on $\partial\Omega$. Thus $\mathcal{H}$ is the kernel of the trace map $\mathcal{K}_1^1(\Omega) \to \mathcal{K}_{1/2}^{1/2}(\partial\Omega)$. Then

$$B(u, v) := \big(Pu, v\big) = \sum_{j,k=1}^{n} \big(A_{jk}\partial_k u, \partial_j u\big) + \sum_{j=1}^{n} \big(B_j \partial_j u, v\big) + \big(Cu, v\big),$$

for any $u, v \in \mathcal{H}$. In particular, B defines a continuous bilinear form on $\mathcal{H}$. (The easiest way to check this is to use the fact that $\mathcal{C}_c^\infty(\Omega)$ is dense in $\mathcal{H}$, by Theorem 2.4.)

Our assumptions on the coefficients A_{jk}, B_j, and C, together with Corollary 2.2 give, for the real part of the inner product $\big(Pu, u\big)$, the following inequality

$$\begin{aligned}(18)\quad Re\big(Pu, u\big) &= \int_\Omega \Big(Re \sum_{j,k=1}^{n} \sum_{p,q=1}^{m} [A_{jk}]_{pq} \partial_k u_q \overline{\partial_j u_p}\Big)\, dx \\ &+ \big((C + C^* - \sum_j \partial_j B_j)u, u\big)/2 \geq \epsilon \sum_{j=1}^{n} \sum_{p=1}^{m} \|\partial_j u_p\|^2 = \epsilon \sum_{p=1}^{m} \|\nabla u_p\|^2_{L^2(\Omega)} \\ &\geq \epsilon \sum_{p=1}^{m} \|u_p\|^2_{\mathcal{K}_1^1(\Omega)} =: \epsilon \|u\|^2_{\mathcal{K}_1^1(\Omega)}.\end{aligned}$$

Hence the bilinear form B is also coercive.

The assumptions of the Lax-Milgram lemma are therefore satisfied, see [16], and hence $P : \mathcal{H} \to \mathcal{H}^* = \mathcal{K}_{-1}^{-1}(\Omega)$ is an isomorphism (by this we understand that P is continuous with continuous inverse). This proves that $1 \in \mathcal{W}_0(\Omega)$.

Let us next prove that $\mathcal{W}_0 \subset \mathcal{W}_s(\Omega)$ for any $s \in \mathbb{R}_+$, and hence, in particular, $1 \in \mathcal{W}_s(\Omega)$. Let $h \in \mathcal{W}_0(\Omega)$. Then Theorem 2.3 gives that the map

$$(19)\qquad P : h\mathcal{K}_1^{s+2}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to h\mathcal{K}_{-1}^{s}(\Omega)$$

is surjective. Since this map is also continuous (Proposition 2.5) and injective (because $h \in \mathcal{W}_0(\Omega)$), it is an isomorphism by the open mapping theorem. This shows that $\mathcal{W}_s(\Omega) \subset \mathcal{W}_0(\Omega)$, for any $s \in \mathbb{R}_+$.

Since we have already proven that $\mathcal{W}_s(\Omega)$ is open, the proof is complete. □

3. Curvilinear polyhedral domains in 1, 2, and 3 dimensions

In this section we give some examples of curvilinear polyhedral domains Ω in $\mathbb{R}^2$, in S^2, or in $\mathbb{R}^3$. These examples are crucial in understanding the definition of curvlinear polyhedral domains in arbitrary dimensions, which will be given in Section 6. See the paper of Babuška and Guo [5], the paper of Mazya and Rossman [31], and the papers and Verchota and Vogel [44, 43] for related definitions. Let us stress that all curvilinear polyhedral domains will be open sets.

For completeness, let us mention that a subset $\Omega \subset \mathbb{R}$ or $\Omega \subset S^1$ is a *curvilinear polyhedral domain* if, and only if, it is a finite union of open intervals.

3.1. **Some curvilinear polyhedral domains in dimension 2 and 3.**

The definition of a curvilinear polyhedral domain is local, by induction. In this section we specialize the general definition, Definition 6.1, to the cases $n = 2$ and $n = 3$.

The desingularizations $\Sigma(\Omega)$ and the functions r_Ω will be introduced in the next subsection.

Definition 3.1. A bounded subset $\Omega \subset \mathbb{R}^2$ will be called *a curvilinear polyhedral domain* if, for every point of the boundary $p \in \partial\Omega$, there exists a neighborhood $V_p \subset \mathbb{R}^2$ of p and a diffeomorphism $\phi_p : V_p \to U \subset \mathbb{R}^2$, $\phi_p(p) = 0$, where $U \subset \mathbb{R}^2$ is an open subset, such that one of the following conditions is satisfied

(a) $\phi_p(V_p \cap \Omega) = \{(r\cos\theta, r\sin\theta),\, 0 < r < 1, \theta \in \omega_p\}$, where ω_p is a union of open intervals of the unit circle, different from an interval of length π;
(b) $\phi_p(V_p \cap \Omega) = \{(x, y),\, 0 < x < 1, -1 < y < 1\}$; or
(c) $\phi_p(V_p \cap \Omega) = \{(x, y),\, -1 < x < 1, x \neq 0, -1 < y < 1\}$.

The condition that ω_p be not an interval of length π of S^1 above is included to make sure that no point p will satisfy more than one of the conditions (a), (b), or (c) above. The set ω_p is called *the link of* Ω *at* p. The definition of a curvilinear polyhedral domain $\Omega \subset M$, where M is a manifold of dimension 2, is completely similar.

Example 3.2. Let us define first a bounded, curvilinear polygon $\Omega = \mathbb{P} \subset \mathbb{R}^2$ with k vertices. The cases $k = 0$ and $k = 1$ are also allowed.

If $k = 0$, then $\mathbb{P}$ is, by definition, a connected, bounded domain with smooth boundary. Let $k \geq 1$. Then we assume $\mathbb{P}$ to be an open set whose boundary is a closed (not necessarily connected), piecewise smooth curve, consisting of a union of curves $\gamma_j : [0, 1] \to \mathbb{R}^2$, $j = 1, 2, \ldots, k$. We require that the curves γ_j, have no common interior points. When $k = 1$, we have only one curve $\gamma_1 : [0, a] \to \mathbb{R}^2$ satisfying $\gamma_1(0) = \gamma_1(a)$ and $\gamma_1'(0) \neq \gamma_1'(a)$.

Let $\{A_1, A_2, \ldots, A_k\}$ be the vertices of $\mathbb{P}$, defined as the set of end points of the curves γ_j. Let $r_j(x)$ be the distance from x to A_j. Denote $V_j := \{r_j < 2\epsilon\}$. Then we require that there exist $\alpha_j \in (0, 2\pi)$, $\alpha_j \neq \pi$, and a diffeomorphisms $\phi_j : V_j \to B^2$, $\phi_j(A_j) = 0$, such that

$$\phi_j(V_j \cap \mathbb{P}) = \{(r\cos\theta, r\sin\theta),\, 0 < r < 1,\, 0 < \theta < \alpha_j\}, \quad 1 \leq j \leq k.$$

If the curves γ_j are straight lines (that is, if $\mathbb{P}$ is a straight polygon), then we can take ϕ_j to be a composition of constant translation and constant rotation. In particular, a vertex belongs to exactly two of the curves γ_j.

A bounded, connected domain $\mathbb{P}$ satisfying the conditions above will be called *a curvilinear polygonal domain* (with k vertices).

We carry the above definition to domains on spheres, using diffeomorphisms.

Example 3.3. An open subset $\Omega \subset M$ of a two dimensional manifold M will be called a *curvilinear polygonal domain* there exists a bounded, curvilinear polygonal domain $\Omega_0 \subset \mathbb{R}^2$ and a diffeomorphism $\phi : U_0 \to U$, from a neighborhood U_0 of $\overline{\Omega}_0$ in $\mathbb{R}^2$ to a neighborhood U of $\overline{\Omega}$ in M, such that

$$\phi(\Omega_0) = \Omega.$$

The domains Ω and Ω_0 will have the same number of vertices, and these vertices will correspond under ϕ.

It follows from the definitions that every vertex p of a two dimensional curvilinear polygonal domain $\Omega \subset M$ will have a neighborhood $V_p \in M$ for which there exists a diffeomorphism $\phi_p : V_p \to B^2$, $\phi_p(p) = 0$, such that

(20) $$\phi_j(V_p \cap \Omega) = \{(r\cos\theta, r\sin\theta),\, 0 < r < 1,\, 0 < \theta < \alpha_j\}, \quad 1 \leq j \leq k.$$

Therefore, a polygonal domain is a curvilinear polyhedral domain where $\omega_p = (0, \alpha_p)$, $\alpha_p \neq \pi$.

We shall denote by B^k the open unit ball in $\mathbb{R}^k$ and by S^{k-1} its boundary, the unit sphere in $\mathbb{R}^l$. To discuss curvilinear polyhedral domains in $\mathbb{R}^3$, we shall need the concept of an irreducible subset of S^2. For further reference, we include the definition of an irreducible subset of S^{k-1}.

Definition 3.4. A subset $\omega \subset S^{k-1} := \partial B^k$, the unit sphere in $\mathbb{R}^k$ will be called irreducible if $\mathbb{R}_+\omega := \{rx',\, r > 0, x' \in \omega\}$ cannot be written as $V + V'$ for a linear subspace $V \subset \mathbb{R}^k$ of dimension ≥ 1.

For example, $(0, \alpha) \subset S^1$ is irreducible if, and only if, $\alpha \neq \pi$. A subset $\omega \subset S^{k-1}$ strictly contained in an open half-space is irreducible, but the intersection of S^{k-1}, $k \geq 2$, with an open half-space is not irreducible.

Definition 3.5. A bounded subset $\Omega \subset \mathbb{R}^3$ will be called *a curvilinear polyhedral domain* if, for every point of the boundary $p \in \partial\Omega$, there exists a neighborhood $V_p \subset \mathbb{R}^3$ of p and a diffeomorphism $\phi_p : V_p \to U \subset \mathbb{R}^3$, $\phi_p(p) = 0$, where $U \subset \mathbb{R}^3$ is an open subset, such that one of the following conditions is satisfied

(a) $\phi_p(V_p \cap \Omega) = \{rx',\, 0 < r < 1, x' \in \omega_p\}$, where $\omega_p \subset S^2$ is an irreducible curvilinear polyhedral domain;
(b) $\phi_p(V_p \cap \Omega) = \{(r\cos\theta, r\sin\theta, z),\, 0 < r < 1, \theta \in \omega_p, 0 < z < 1\}$, where $\omega_p \subset S^1$ is an irreducible subset;
(c) $\phi_p(V_p \cap \Omega) = \{(x, y, z),\, 0 < x < 1, -1 < y, z < 1\}$; or
(d) $\phi_p(V_p \cap \Omega) = \{(x, y, z),\, -1 < x < 1, x \neq 0, -1 < y, z < 1\}$

Anticipating, the definition of a curvilinear polyhedral domain in $\mathbb{R}^n$ is by induction, closely based on the model of the last example. In fact, generalizations of the diffeomorphisms ϕ_p used above will provide the crucial ingredient in the definition of a curvilinear polyhedral domain, Definition 6.1. If p is as in (c) above, we denote by $\omega_p = \{+1\} \in S^0 = \{\pm 1\}$. If p is as in (d) above, we denote by $\omega_p = S^0 = \{\pm 1\}$. The set ω_p will be called the link of Ω at p, as before. Then condition that the link ω_p be irreducible guarantees that no point satisfies more than one of the conditions (a), (b), (c), and (d).

If the point $p \in \overline{\Omega}$ satisfies Condition (a) but does not satisfy condition (b) of the last definition, then we say that p is a *vertex of* Ω. If p satisfies condition (b) of the above example, then we say that p is an *edge point of* Ω. Recall the discussion before Definition 1.1 of (one-sided, two-sided) smooth boundary points. The point p satisfies Condition (c) or Condition (d) above if, and only if, p is a smooth boundary point of Ω. If (c) is satisfied, then p is a one-sided smooth boundary point of Ω. If (d) is satisfied then $\partial\Omega$ is a two-sided smooth boundary point of Ω.

The following subsection contains several examples.

3.2. **Definition of $\Sigma(\Omega)$ and of r_Ω if $n = 2$ or $n = 3$.** We now introduce the desingularization $\Sigma(\Omega)$ and the function r_Ω for some of the typical examples of curvilinear polyhedral domains in $n = 2$ or $n = 3$. We also frame these definitions as examples. The general case (of which the examples considered here are particular cases) is in Section 7. The reader can skip this subsection at a first reading.

The case $n = 2$ and of a polygonal domain $\mathbb{P}$ particularly simple. We use the notation in Example 3.2.

Example 3.6. The desingularization $\Sigma(\mathbb{P})$ of $\mathbb{P}$ will replace each of the vertices A_j, $j = 1, \ldots, k$ of $\mathbb{P}$ with a segment of length $\alpha_j > 0$, where α_j is the magnitude of the angle at A_j. We can realize $\Sigma(\mathbb{P})$ in three dimensions as follows. Let θ_j be the angle in a polar coordinates system centered at A_j. Let ϕ_j be a smooth function on $\mathbb{P}$ that is equal to 1 on $\{r_j < \epsilon\}$ and vanishes outside $V_j := \{r_j < 2\epsilon\}$. By choosing $\epsilon > 0$ small enough, we can arrange that the sets V_j do not intersect. We define then

$$\Phi : \overline{\mathbb{P}} \smallsetminus \{A_1, A_2, \ldots, A_k\} \to \mathbb{P} \times \mathbb{R} \subset \mathbb{R}^3$$

by $\Phi(p) = (p, \sum \phi_j(p)\theta_j(p))$. Then $\Sigma(\mathbb{P})$ is (up to a diffeomorphism) the closure in $\mathbb{R}^3$ of $\Phi(\mathbb{P})$. The desingularization map is $\kappa(p, z) = p$. The structural Lie algebra of vector fields $\mathcal{V}(\mathbb{P})$ on $\Sigma(\mathbb{P})$ is given by (the lifts of) the smooth vector fields X on $\overline{\mathbb{P}} \smallsetminus \{A_1, A_2, \ldots, A_k\}$ that on $V_j = \{r_j < 2\epsilon\}$ can be written as

$$X = a_r(r_j, \theta_j) r_j \partial_{r_j} + a_\theta(r_j, \theta_j)\partial_{\theta_j}, \tag{21}$$

with a_r and a_θ smooth functions of (r_j, θ_j), including $r = 0$. We can take $r_\Omega(x) := \psi(x) \prod_{j=1}^k r_j(x)$, where ψ is a smooth, nowhere vanishing function on $\Sigma(\Omega)$. (A factor ψ like that can always be introduced, and the function r_Ω is determined only up to this factor. We shall omit this factor in the examples below.)

The example of an edge, is one of the most instructive.

Example 3.7. Let us consider first the case when Ω is the wedge

$$\mathbb{W} := \{(r\cos\theta, r\sin\theta, z),\ 0 < r, 0 < \theta < \alpha, z \in \mathbb{R}\}, \tag{22}$$

where $0 < \alpha < 2\pi$, $\alpha \neq \pi$ and $x = r\cos\theta$ and $y = r\sin\theta$ define the usual cylindrical coordinates (r, θ, z), with $(r, \theta, z) \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}$. Then the manifold of generalized cylindrical coordinates is, in this case, just the domain of the cylindrical coordinates on $\overline{\mathbb{W}}$:

$$\Sigma(\mathbb{W}) = [0, \infty) \times [0, \alpha] \times \mathbb{R}.$$

The desingularization map is $\kappa(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ and the structural Lie algebra of vector fields of $\Sigma(\mathbb{W})$ is

$$a_r(r, \theta, z) r\partial_r + a_\theta(r, \theta, z)\partial_\theta + a_z(r, \theta, z) r\partial_z,$$

where a_r, a_z, and a_θ are smooth functions on $\Sigma(\mathbb{W})$. Note that the vector fields in $\mathcal{V}(\mathbb{W})$ may not extend to the closure $\overline{\mathbb{W}}$. We can take $r_\Omega = r$, the distance to the Oz-axis.

The following example, that of a domain with one conical point, is an example of a curvilinear polyhedral domain that is not a straight polyhedral domain (and is not even diffeomorphic to one).

Example 3.8. Let us now consider the case of *a domain with one conical point.* That is, we assume that Ω is bounded and that $\partial\Omega$ is smooth, except at a point $p \in \partial\Omega$, which is a *conical point*, in the sense that there exists a neighborhood V_p of p such that, up to a local change of coordinates,

$$V_p \cap \Omega = \{rx',\ 0 \leq r < \epsilon, x' \in \omega\}, \tag{23}$$

for some smooth, connected domain $\omega \subset S^{n-1} := \partial B^n$. Then we can realize $\Sigma(\Omega)$ in $\mathbb{R}^{2n}$ as follows. Assume $p = 0$, the origin, for simplicity. We define $\Phi(x) = (x, |x|^{-1}x)$ for $x \neq p$, where $|x|$ is the distance from x to the origin (*i.e.,* to p). Then $\Sigma(\Omega)$ is defined to be the closure of the range of Φ. The map κ is the projection onto the first n components. Then κ is one-to-one, except that $\kappa^{-1}(p) = \{p\} \times \omega$. We can take $r_\Omega(x) = |x|$. The Lie algebra of vector fields $\mathcal{V}(\Omega)$ consists of the vector fields on $\Sigma(\Omega)$ that are tangent to $\kappa^{-1}(p)$. This example is due to Melrose [33].

Example 3.9. Let $\Omega \subset \mathbb{R}^3$ be a straight polyhedral domain. To construct $\Sigma(\Omega)$, we combine the ideas used in the previous examples. First, for each edge e we define (r_e, θ_e, z_e) to be a coordinate system aligned to that edge and such that $\theta_e \in (0, \alpha_e)$, as in Example 3.7. Let $v_1, v_2, \dots, v_b$ be the set of vertices of Ω. Then, for x not on any edge of Ω, we define $\Phi(x) \in \mathbb{R}^{3+a+b}$ by

$$\Phi(x) = \big(x,\, \theta_{e_1}, \theta_{e_2}, \dots, \theta_{e_a}, |x - v_1|^{-1}(x - v_1), \dots, |x - v_b|^{-1}(x - v_b)\big)\,.$$

The desingularization $\Sigma(\Omega)$ is defined as the closure of the range of Φ. The resulting set will be a manifold with corners with several different types of hyperfaces (a *hyperface* is a proper face of maximal dimension). Namely, the manifold $\Sigma(\Omega)$ will have a hyperface for each face of Ω, a hyperface for each edge of Ω, and, finally, a hyperface for each vertex of Ω. The last two types of hyperfaces are the hyperfaces at infinity of $\Sigma(\Omega)$. Let x_H be the distance to the hyperface H. We can take then $r_\Omega = \prod_H x_H$, where H ranges through the hyperfaces at infinity of $\Sigma(\Omega)$.

We can imagine $\Sigma(\Omega)$ as follows. Let $\epsilon > 0$. Remove the sets $\{x \in \Omega, |x - v_j| \leq \epsilon\}$ and $\{x \in \Omega, |x - e_k| \leq \epsilon^2\}$. Call the resulting set Ω_ϵ. Then, for ϵ small enough, the closure of Ω_ϵ is diffeomorphic to $\Sigma(\Omega)$.

3.3. **Exotic examples.** We include now some non-standard examples of curvilinear polyhedral domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. They will show how complicated curvilinear polyhedral domains can be.

Example 3.10. Let $\mathbb{P}$ be a straight polygonal domain. Remove from the interior of $\mathbb{P}$ some short segments, each starting at some vertex of $\mathbb{P}$. We assume that the interior of the segments that we remove have no common points and that they do not intersect the boundary. The resulting set Ω will be a curvilinear polyhedral domain.

Example 3.11. Let Ω be obtained from the unit ball B^3 by removing a segment joining two opposite points of the boundary. Then the resulting set Ω will be a curvilinear polyhedral domain. The same will be true if we remove a disk whose boundary is a great circle of S^2. The resulting set is not connected though.

Example 3.12. We start with a connected polygonal domain $\mathbb{P}$ with connected boundary and we deform it until one, and exactly on of the vertices, say A, touches the interior of another edge, say $[B, C]$. Let Ω be the resulting connected open set. Therefore that $\partial\Omega$ is a union of straight segments, but is not a straight polygonal domain. Nevertheless, Ω will be a curvilinear polyhedral domain. We define the set $\Sigma(\Omega)$ as for the polygonal domain $\mathbb{P}$, but by introducing polar coordinates in the whole neighborhood of the point A.

If we deform $\mathbb{P}$ to Ω, $\Sigma(\mathbb{P})$ will deform continuously to a space $\Sigma'(\Omega)$, different from $\Sigma(\Omega)$. For certain purposes, the desingularization $\Sigma'(\Omega)$ is better suited than $\Sigma(\Omega)$.

4. A Poincaré type inequality

We now prove the Hardy–Poincaré type inequality stated in Theorem 2.1, provided that the dimension is $n = 2$ or $n = 3$. The general case proceeds by induction, and the inductive step is very similar to the proof of the case $n = 3$. There is only a small detail that requires, however, the results of the subsequent sections, namely Proposition 7.10, which is proved in the last section, Section 7.

If u/η_{n-2} is, in fact, not square integrable (*i.e.,* $\int_\Omega |u(x)|^2 \eta_{n-2}(x)^{-2}\, dx = \infty$), then we understand the Theorem 2.1 to mean that ∇u is not square integrable either.

4.1. **The case** $n = 2$**.**

A first step is provided by the following lemma. By abuse of notation, we shall write $u(r, \theta) := u(r\cos\theta, r\sin\theta)$ for a function $u(x_1, x_2)$ expressed in polar coordinates. Below, $dx = dx_1 dx_2 \dots dx_n$.

Lemma 4.1. *Let* $\mathcal{C} = \mathcal{C}_R(\alpha, \beta) := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2, 0 < r < R, \beta < \theta < \alpha\}$, $0 \le \alpha - \beta \le 2\pi$*. Then*

$$\int_{\mathcal{C}} \frac{|u|^2}{r^2}\, dx \le \frac{\pi^2}{\alpha^2} \int_{\mathcal{C}} |\nabla u|^2 dx$$

for any $u \in H^1_{\mathrm{loc}}(\mathcal{C})$ *satisfying* $u(r, \theta) = 0$ *if* $\theta = 0$ *or* $\theta = \alpha$*. The same result holds if* $\mathcal{C}$ *is the disjoint union of domains* $\mathcal{C}_R(\alpha, \beta)$*, for different values of* R*,* α*, and* β*.*

Again, if u/r is not square integrable, the statement of the Lemma above is understood to mean that ∇u is not square integrable either.

Proof. We include here the elementary proof from [36][Subsection 2.3.1]. See also [24]. Let $\partial_\theta u(r, \theta)$ be the weak derivative of u, which exists and is a locally integrable function since $u \in H^1_{\mathrm{loc}}(\mathcal{C})$. We can assume that $\beta = 0$.

Fubini's theorem gives that the function $u(r, \theta)$ is a Lebesgue measurable function of θ, except maybe for r in a set of measure zero (*i.e.,* almost everywhere in r). Since $\mathcal{C}^\infty_{\mathrm{c}}(U)$ has a countable dense subsequence, for any open set $U \subset \mathbb{R}^n$, we obtain that, for r fixed outside a set of measure zero, the functions $u(r, \theta)$ and $\partial_\theta u(r, \theta)$ are measurable in θ, are locally square integrable, and the later is the weak derivative in θ of the first one.

The one dimensional Poincaré inequality then applies and gives

$$\int_0^\alpha |u|^2\, d\theta \le \frac{\pi^2}{\alpha^2} \int_0^\alpha |\partial_\theta u|^2\, d\theta,$$

almost everywhere in r. Moreover, both sides of the above inequality define measurable functions of r (we extend them with zero where they are not defined). Using again Fubini's theorem, we obtain

$$\int_{\mathcal{C}} \frac{|u|^2}{r^2}\, dx = \int_0^R \left(\int_0^\alpha |u|^2 d\theta \right) r^{-1} dr \le \frac{\pi^2}{\alpha^2} \int_0^R \left(\int_0^\alpha |\partial_\theta u|^2 r^{-1} d\theta \right) dr$$
$$\le \frac{\pi^2}{\alpha^2} \int_0^R \int_0^\alpha \left(\frac{|\partial_\theta u|^2}{r^2} + |\partial_r u|^2 \right) r d\theta dr = \frac{\pi^2}{\alpha^2} \int_{\mathcal{C}} |\nabla u|^2 dx.$$

The proof when $\mathcal{C}$ is a union of domains as above is obtained by adding up the inequalities for the individual domains. □

The following lemma verifies the case $n = 2$ in Theorem 2.1, the first step in our induction proof.

Lemma 4.2. *Let Ω be a curvilinear polygonal domain in a two dimensional manifold M. Fix an arbitrary metric g on M and let $\eta_0(z)$ be the distance from z to the vertices of Ω. Then there exists a constant $\kappa_\Omega > 0$ such that*

$$\int_\Omega \frac{|u(z)|^2}{\eta_0(z)^2}\, dz \leq \kappa_\Omega \int_\Omega |\nabla u(z)|^2 dz$$

for any $u \in H^1_{\text{loc}}(\Omega)$ satisfying $u = 0$ on $\partial\Omega$.

Proof. Let us fix, for any vertex $p \in \overline{\Omega}$, a diffeomorphism $\phi_p : V_p \to B^2$ such that $\phi_p(V_p \cap \Omega) = \mathcal{C}$, as in Definition (3.1)(a). By decreasing V_p, if necessary, we may assume that ϕ_p extends to a diffeomorphism defined in a neighborhood of $\overline{V}_p$ in M. Note that $\mathcal{C}$ satisfies the assumptions of Lemma 4.1.

Let $r_p(z) = r(\phi_p(z))$ be the distance from $\phi_p(z)$ to $0 = \phi_p(p) \in \mathbb{R}^2$. Then $\eta_0(z)/r_p(z)$ is continuous on $V_p \smallsetminus \{p\}$ and

$$\eta_0(z)/r_p(z) \text{ is bounded as } z \to p.$$

If follows that $\eta_0(z)/r_p(z)$ is bounded on V_p. Similarly, the norms of the differentials $D\phi_p$ and $D\phi_p^{-1}$ are bounded on V_p, by continuity. Hence the Jacobians $\det(D\phi_p) = |\frac{\partial x}{\partial z}|$ and $\det(D\phi_p^{-1}) = |\frac{\partial z}{\partial x}|$ are bounded on V_p.

Let $u \in H^1_{\text{loc}}(\Omega)$, $u = 0$ on $\partial\Omega$. Then Lemma 4.1 gives

$$\begin{aligned}(24)\quad \int_{V_p\cap\Omega} \frac{|u(z)|^2}{\eta_0(z)^2}\, dz &= \int_{\mathcal{C}} \frac{|u(z)|^2}{r_p(z)^2}\, \frac{r_p(z)^2}{\eta_0(z)^2}\, dz \leq C \int_{\mathcal{C}} \frac{|u(z)|^2}{r_p(z)^2}\, dz \\ = C \int_{\mathcal{C}} \frac{|u(\phi_p^{-1}(x))|^2}{r^2} \left|\frac{\partial z}{\partial x}\right| dx &\leq C \int_{\mathcal{C}} \frac{|u(\phi_p^{-1}(x))|^2}{r^2}\, dx \leq C \int_{\mathcal{C}} |\nabla u(\phi_p^{-1}(x))|^2\, dx \\ &\leq C \int_{V_p\cap\Omega} |\nabla u(z)|^2\, dz.\end{aligned}$$

In the last inequality we have used the fact that the differential $D\phi_p$ is bounded in norm. (We have used above the usual convention that C denotes a generic constant that may be different in each inequality.)

We shall also need the usual Poincaré inequality

$$(25)\qquad \int_\Omega |u(z)|^2 dz \leq C \int_\Omega |\nabla u(z)|^2 dz.$$

By decreasing the neighborhoods V_p, we can assume that they are disjoint. Adding up then the inequalities (24) for all p and Equation (25), we obtain

$$\int_\Omega f(z)|u(z)|^2 dz \leq C \int_\Omega |\nabla u(z)|^2 dz,$$

where $f(z) = 1$ if $z \notin V_p$ and $f(z) = 1 + r_p^{-2}$ if $z \in V_p$. We claim that the function $\eta_0(z)^2 f(z)$ is bounded. Indeed, the function $\eta_0^2(z)$ is bounded on Ω. Outside all V_p, $f(z)$ is bounded as well. Finally, on V_p, we have already noticed that the product $\eta_0(z)/r_p(z)$ is bounded.

We therefore obtain $\eta_0^{-2}(z) \leq Cf(z)$, and hence

$$\int_\Omega |u(z)|^2 \eta_0(z)^{-2} dz \leq C \int_\Omega |\nabla u(z)|^2 dz.$$

This completes the proof. □

4.2. **The case $n = 3$.** We are ready now for the proof of Theorem 2.1 for $n = 3$. Our proof combines the methods used in the previous two Lemmata and the inequality for the case $n = 2$. The general case will be completed once we will have proven Proposition 7.10.

Proof. Recall that $\eta_{n-2}(x) = \eta_1(x)$ denotes the distance from x to the set $\Omega^{(n-2)}$ as in Definition 1.1 (also discussed in the Introduction). The set $\Omega^{(n-2)}$ consists of the non-smooth boundary points of Ω (*i.e.,* those points $p \in \partial\Omega$ where $\partial\Omega$ is not smooth).

Let us denote by B^l the ball of radius 1 in $\mathbb{R}^l$ centered at the origin, as above. Also, let us fix, for any $p \in \partial\Omega$, a neighborhood V_p of p in M and a diffeomorphism $\phi_p : V_p \to U = B^{3-l} \times B^l$ as in Definition 3.5. Let $\epsilon > 0$. We may assume that V_p lies at distance at most ϵ^{1+l} to $\partial\Omega$. We shall use the notation ω_p introduced in that definition. The value $l = 0$ corresponds to the case (a), *i.e.,* when p is a vertex, whereas the value $l = 1$ corresponds to the case (b), when p is an edge point. (No point p corresponds to more than one value of l.) We need only treat these two cases. By decreasing V_p, if necessary, we may assume that ϕ_p extends to a diffeomorhism defined in a neighborhood of $\overline{V}_p$ in $\mathbb{R}^3$.

Assume $l = 0$. Then we denote by $\psi_0(x')$ the distance from a point $x' \in \omega_p \subset S^2$ to the vertices of ω_p and let $r_p(z) = \rho\psi_0(x')$, if $\phi_p(z) = \rho x'$, where $0 < \rho$ and $x' \in \omega_p$.

Assume $l = 1$. Then we let $r_p(z) = r$ if $\phi_p(z) = (r\cos\theta, r\sin\theta, z)$, where $0 < r$, $0 < \theta < \alpha$, and $z \in \mathbb{R}$.

We claim that the function $\eta_1(x)/r_p(x)$ is bounded for any p, provided that we choose ϵ small enough, independent of p. Assume p is a vertex, the other case being easier. Then, for $\epsilon > 0$ small enough, the quotient $\eta_1(x)/r_p(x)$ is homogeneous in x, in the sense that $\eta_1(x)/r_p(x) = \eta_1(tx)/r_p(tx)$ for $1 \geq t > 0$. We can therefore assume that the length $|x|$ is constant. (This argument will be discussed again, at great length, in Proposition 7.10.)

Recall that the set $\mathcal{C} := \phi_p(V_p \cap \Omega)$ is assumed to be as in Definition 3.5. We shall consider separately the cases $l = 1$ and $l = 0$.

If $l = 1$, $\mathcal{C} = \mathcal{C}' \times (-1, 1)$, and hence we obtain, as in the proof of Lemma 4.2 (especially Equation (24)), the following inequalities. We shall write $u(x)$ instead of $u(\phi_p^{-1}(x))$, by abuse of notation. We first proceed as in Equation (24) to obtain.

$$\int_{V_p\cap\Omega} \frac{|u(z)|^2}{\eta_1(z)^2}\, dz = \int_{\mathcal{C}} \frac{|u(z)|^2}{r_p(z)^2}\, \frac{r_p(z)^2}{\eta_1(z)^2}\, dz \leq C \int_{\mathcal{C}} \frac{|u(z)|^2}{r_p(z)^2}\, dz$$
$$= C \int_{\mathcal{C}} \frac{|u(x)|^2}{r^2} \left|\frac{\partial z}{\partial x}\right| dx \leq C \int_{\mathcal{C}} \frac{|u(x)|^2}{r^2}\, dx. \qquad (26)$$

Then we use again, as in Equation (26), the Hardy–Poincaré type inequality in an angle proved in Lemma 4.1, to obtain

$$\begin{aligned}(27)\quad \int_{\mathcal{C}} \frac{|u(x)|^2}{r^2}\,dx &= \int_{-1}^{1}\Big(\int_{\mathcal{C}'} \frac{|u(x)|^2}{r^2}\,dx_1dx_2\Big)\,dx_3\\ \le \int_{-1}^{1}\Big(\int_{\mathcal{C}'} |\nabla_{(x_1,x_2)}u(x)|^2\,dx_1dx_2\Big)\,dx_3 &\le \int_{-1}^{1}\Big(\int_{\mathcal{C}'} |\nabla u(x)|^2\,dx_1dx_2\Big)\,dx_3\\ &= C\int_{\mathcal{C}} |\nabla u(x)|^2\,dx \le C\int_{V_p\cap\Omega} |\nabla u(z)|^2\,dz.\end{aligned}$$

We perform a similar calculation on $V_p\cap\Omega$ when $l=0$. Recall that $\phi_p(V_p\cap\Omega) = \{\rho x',\, 0<\rho<1,\, x'\in\omega_p\} =: \mathcal{C}$. The inequality

$$(28)\qquad \int_{V_p\cap\Omega} \frac{|u(z)|^2}{\eta_1(z)^2}\,dz \le C\int_{\mathcal{C}} \frac{|u(x)|^2}{\rho^2\psi_0(x')^2}\,dx,\quad x=\rho x',\ |x'|=1,$$

is proved as in Equation (26) above (which in turn had followed the pattern of Equation (24)), using also the inequality $C\eta_1(x)\ge\rho\psi_0(x)$. Next, let ∇' be the gradient of a function defined on ω_p, so that

$$\nabla u(\rho x') = \rho^{-1}\nabla' u(\rho x') + \partial_\rho u(\rho x').$$

(The analogue of this decomposition in two dimensions is $\nabla u = r^{-1}\partial_\theta u + \partial_r u$.) Hence

$$|\nabla' u(\rho x')|^2 \le \rho^2|\nabla u(\rho x')|,$$

which gives

$$\begin{aligned}(29)\quad \int_{\mathcal{C}} \frac{|u(x)|^2}{\rho^2\psi_0(x)^2}\,dx &= \int_0^1\Big(\int_{\omega_p} \frac{|u(\rho x')|^2}{\psi_0^2}dx'\Big)\,d\rho\\ \le C\int_0^1\Big(\int_{\omega_p} |\nabla' u(\rho x')|^2dx'\Big)\,d\rho &\le C\int_0^1\Big(\int_{\omega_p} \rho^2|\nabla u(\rho x')|^2dx'\Big)\,d\rho\\ &= C\int_{\mathcal{C}} |\nabla u(x)|^2\,dx \le C\int_{V_p\cap\Omega} |\nabla u(z)|^2\,dz.\end{aligned}$$

Putting together the Equations (26) and (27) when $l=1$ and the Equations (28) and (29) when $l=0$, we obtain the following inequality

$$(30)\qquad \int_{V_p\cap\Omega} \frac{|u(z)|^2}{\eta_1(z)^2}\,dz \le C_p\int_{V_p\cap\Omega} |\nabla u(z)|^2\,dz \le C_p\int_{\Omega} |\nabla u(z)|^2\,dz.$$

where the constant C_p depends on the point $p\in\Omega^{(1)}$ but not on the function $u\in H^1_{\text{loc}}(\Omega)$ satisfying $u=0$ on $\partial\Omega$. We shall also need the usual Poincaré inequality

$$(31)\qquad \int_{\Omega} |u(z)|^2\,dz \le C_\Omega\int_{\Omega} |\nabla u(z)|^2\,dz.$$

We now conclude the proof of the case $n=3$ following the same argument used to complete the proof of Lemma 4.2. Let us cover the set $\Omega^{(1)}$ of points of $\partial\Omega$ where $\partial\Omega$ is not smooth with finitely many sets of the form V_p. Call these sets V_{p_k}. Let

$C_0 > \eta_0^{-2}$ *outside* the union of the sets V_{p_k}. Let $\kappa_\Omega = C_0 C_\Omega + \sum C_{p_k}$. We add all inequalities (30) for $p = p_k$ and the inequality (31), multiplied by C_0. This gives

$$\int_\Omega f(z)|u(z)|^2 dz \leq \kappa_\Omega \int_\Omega |\nabla u(z)|^2 dz, \tag{32}$$

where $f(z) \geq \eta_1(z)^{-2}$ on V_{p_j} and $f(z) \geq C_0 \geq \eta_1(z)^{-2}$ *outside* the union of the sets V_{p_k}. Hence $f(z) \geq \eta_1(z)^{-2}$ on Ω. The proof of Theorem 2.1 is now complete for $n = 3$.

For $n > 3$, the only things that still require to be taken care of are: the definition of curvilinear polyhedral domains and the definition and boundedness of $\eta_{n-2}(x)/r_p(x)$. This will be done in the next sections. For instance, the boundedness is proved in Proposition 7.10. (Note that r_p is denoted r_α in that proposition.) □

5. Lie manifolds with boundary

The results that remain to be proved are, to a certain extend, particular cases of more general results on manifolds with a Lie structure at infinity. In order to make this paper as self-contained as possible, we now recall the definition of a Lie manifold from [2] and of a Lie manifold with boundary from [1]. We also recall a few other needed definitions and results from those papers.

5.1. **Definition.** Let us recall that a topological space $\mathfrak{M}$ is a, by definition, a manifold with corners if every point $p \in \mathfrak{M}$ has a coordinate neighborhood homeomorphic to $[0,1)^k \times (-1,1)^{n-k}$ such that the transition functions are smooth (including on the boundary). It therefore makes sense to talk about smooth functions on $\mathfrak{M}$, these being the functions that correspond to smooth functions on $[0,1)^k \times (-1,1)^{n-k}$. We denote by $\mathcal{C}^\infty(\mathfrak{M})$ the set of smooth functions on a manifold with corners $\mathfrak{M}$. The least integer k with this property is called the *depth* of p.

Throughout this paper, $\mathfrak{M}$ will denote a manifold with corners, not necessarily compact. We shall denote by $\mathfrak{M}_0$ the interior of $\mathfrak{M}$ and by $\partial\mathfrak{M} = \mathfrak{M} \smallsetminus \mathfrak{M}_0$ the boundary of $\mathfrak{M}$, *i.e.,* $\partial\mathfrak{M}$ is the union of all hyperfaces of $\mathfrak{M}$. The set $\mathfrak{M}_0$ consists of the set of points of depth zero of $\mathfrak{M}$. It is usually no loss of generality to assume that $\mathfrak{M}_0$ is connected. Let $\mathfrak{M}_k$ denote the set of points of $\mathfrak{M}$ of depth k and F_0 be a connected component of $\mathfrak{M}_k$. We shall call F_0 an *open face of codimension k* of $\mathfrak{M}$ and $F := \overline{F}_0$ a *face of codimension k* of $\mathfrak{M}$. A face of codimension 1 will be called a *hyperface* of $\mathfrak{M}$. In general, a face of $\mathfrak{M}$ need not be a smooth manifold (with or without corners). A face $F \subset \mathfrak{M}$ which is a submanifold with corners of $\mathfrak{M}$ will be called an *embedded face.*

Anticipating, a Lie manifold with boundary $\mathfrak{M}_0$ is the interior of a manifold with corners $\mathfrak{M}$ together with a Lie algebra of vector fields $\mathcal{V}$ on $\mathfrak{M}$ satisfying certain conditions. To state these conditions, it will be convenient of first introduce a few other concepts.

Definition 5.1. Let $\mathfrak{M}$ be a manifold with corners and $\mathcal{V} = \mathcal{C}^\infty(\mathfrak{M})\mathcal{V}$ be a space of vector fields on $\mathfrak{M}$. Let $U \subset \mathfrak{M}$ be an open set and $Y_1, Y_2, \ldots, Y_k$ be vector fields on $U \cap \mathfrak{M}_0$. We shall say that $Y_1, Y_2, \ldots, Y_k$ form a *local basis of $\mathcal{V}$ on U* if the following three conditions are satisfied:

(i) there exist vector fields $X_1, X_2, \ldots, X_k \in \mathcal{V}$, $Y_j = X_j$ on $U \cap \mathfrak{M}_0$;

(ii) for any $X \in \mathcal{V}$, there exist smooth functions $\phi_1, \phi_2, \ldots, \phi_k \in \mathcal{C}^\infty(\mathfrak{M}_0)$ such that

$$X = \phi_1 X_1 + \phi_2 X_2 + \ldots + \phi_k X_k \quad \text{on } U \cap \mathfrak{M}_0\,; \tag{33}$$

and

(iii) if $\phi_1, \phi_2, \ldots, \phi_k \in \mathcal{C}^\infty(\mathfrak{M})$ and $\phi_1 X_1 + \phi_2 X_2 + \ldots + \phi_k X_k = 0$ on $U \cap \mathfrak{M}_0$, then $\phi_1 = \phi_2 = \ldots = \phi_k = 0$ on U.

We now recall structural Lie algebras of vector fields from [2].

Definition 5.2. A subspace $\mathcal{V} \subseteq \Gamma(\mathfrak{M}, T\mathfrak{M})$ of the Lie algebra of all smooth vector fields on $\mathfrak{M}$ is said to be a *structural Lie algebra of vector fields on* $\mathfrak{M}$ provided that the following conditions are satisfied:

(i) $\mathcal{V}$ is closed under the Lie bracket of vector fields;
(ii) every vector field $X \in \mathcal{V}$ is tangent to all hyperfaces of $\mathfrak{M}$;
(iii) $\mathcal{C}^\infty(\mathfrak{M})\mathcal{V} = \mathcal{V}$; and
(iv) for each point $p \in \mathfrak{M}$ there exist a neighborhood U_p of p in $\mathfrak{M}$ and a local basis of $\mathcal{V}$ on U_p.

The concept of Lie structure at infinity, defined next, is from [2].

Definition 5.3. A *Lie structure at infinity* on a smooth manifold $\mathfrak{M}_0$ is a pair $(\mathfrak{M}, \mathcal{V})$, where $\mathfrak{M}$ is a compact manifold, possibly with corners, and $\mathcal{V} \subset \Gamma(\mathfrak{M}, T\mathfrak{M})$ is a structural Lie algebra of vector fields on $\mathfrak{M}$ with the following properties:

(i) $\mathfrak{M}_0 = \mathfrak{M} \smallsetminus \partial\mathfrak{M}$, the interior of $\mathfrak{M}$, and
(ii) If $p \in \mathfrak{M}_0$, then any local basis of $\mathcal{V}$ in a neigborhood of p is also a local basis of the tangent space to $\mathfrak{M}_0$. (In particular, the constant k of Equation (33) equals n, the dimension of $\mathfrak{M}_0$.)

A *manifold with a Lie structure at infinity* (or, simply, a *Lie manifold*) is a manifold $\mathfrak{M}_0$ together with a Lie structure at infinity $(\mathfrak{M}, \mathcal{V})$ on $\mathfrak{M}_0$. We shall sometimes denote a Lie manifold as above by $(\mathfrak{M}_0, \mathfrak{M}, \mathcal{V})$, or, simply, by $(\mathfrak{M}, \mathcal{V})$, because $\mathfrak{M}_0$ is determined as the interior of $\mathfrak{M}$.

Let $\mathcal{V}_b$ be the set of vector fields on $\mathfrak{M}$ that are tangent to all faces of $\mathfrak{M}$. Then $(\mathfrak{M}, \mathcal{V}_b)$ is a Lie manifold [33, 32]. See [27, 2, 1] for more examples.

5.2. **Riemannian metric.** Let $(\mathfrak{M}, \mathcal{V})$ be a Lie manifold and g a Riemannian metric on $\mathfrak{M}_0 := \mathfrak{M} \smallsetminus \partial\mathfrak{M}$. We shall say that g *is compatible* (with the Lie structure at infinity $(\mathfrak{M}, \mathcal{V})$) if, for any $p \in \mathfrak{M}$, there exist a neighborhood U_p of p in $\mathfrak{M}$ and a local basis $X_1, X_2, \ldots, X_n$ of $\mathcal{V}$ on U_p that is orthonormal with respect to g on U_p.

It was proved in [2] that $(\mathfrak{M}_0, g_0)$ is necessarily of infinite volume and complete. Moreover, all the covariant derivatives of the Riemannian curvature tensor of g are bounded. Under additional mild assumptions, we also know that the injectivity radius is bounded from below by a positive constant, *i.e.,* $(\mathfrak{M}_0, g_0)$ is of bounded geometry. (A *manifold with bounded geometry* is a Riemannian manifold with positive injectivity radius and with bounded covariant derivatives of the curvature tensor, see for example [8] or [40] and references therein). *We assume from now on that* $r_{\text{inj}}(\mathfrak{M}_0)$*, the injectivity radius of* $(\mathfrak{M}_0, g_0)$*, is positive.*

5.3. **$\mathcal{V}$-differential operators.** We are especially interested in the analysis of the differential operators generated using only derivatives in $\mathcal{V}$. Let $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ be the algebra of differential operators on $\mathfrak{M}$ generated by multiplication with functions in $\mathcal{C}^\infty(\mathfrak{M})$ and by differentiation with vector fields $X \in \mathcal{V}$. The space of order m differential operators in $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ will be denoted $\mathrm{Diff}^m_{\mathcal{V}}(\mathfrak{M})$. A differential operator in $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ will be called a $\mathcal{V}$-differential operator. We define the set $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M}; E, F)$ of $\mathcal{V}$-differential operators acting between sections of smooth vector bundles $E, F \to \mathfrak{M}$ in the usual way [1, 2].

A simple but useful property of the differential operator in $\mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ is that

$$x^s P x^{-s} \in \mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M}) \tag{34}$$

for any $P \in \mathrm{Diff}^*_{\mathcal{V}}(\mathfrak{M})$ and any defining function x of some hyperface of $\mathfrak{M}$ [32, 3]. This is easily proved using the fact that X is tangent to the hyperface defined by x, for any $X \in \mathcal{V}$ (a proof of a slightly more general fact is included in Corollary 8.6).

5.4. **Lie manifolds with boundary.** A subset $\mathfrak{N} \subset \mathfrak{M}$ is called a *submanifold with corners of* $\mathfrak{M}$ if $\mathfrak{N}$ is a closed submanifold of $\mathfrak{M}$ such that $\mathfrak{N}$ is transverse to all faces of $\mathfrak{M}$ and any face of $\mathfrak{N}$ is a component of $\mathfrak{N} \cap F$ for some face F of $\mathfrak{M}$.

The following definition is a reformulation of a definition of [1].

Definition 5.4. Let $(\mathfrak{N}, \mathcal{W})$ and $(\mathfrak{M}, \mathcal{V})$ be Lie manifolds, where $\mathfrak{N} \subset \mathfrak{M}$ is a submanifold with corners and

$$\mathcal{W} = \{X|_{\mathfrak{N}},\ X \in \mathcal{V}, X|_{\mathfrak{N}} \text{ tangent to } \mathfrak{N}\}.$$

We shall say that $(\mathfrak{N}, \mathcal{W})$ is a *tame* submanifold of $(\mathfrak{M}, \mathcal{V})$ if, for any $p \in \partial\mathfrak{N}$ and any $X \in T_p\mathfrak{M}$, there exist $Y \in \mathcal{V}$ and $Z \in T_p\mathfrak{N}$ such that $X = Y(p) + Z$.

Let $\mathfrak{N} \subset \mathfrak{M}$ be a submanifold with corners. We assume that $\mathfrak{M}$ and $\mathfrak{N}$ are endowed with the Lie structures $(\mathfrak{N}, \mathcal{W})$ and $(\mathfrak{M}, \mathcal{V})$. We shall say that $\mathfrak{N}$ is a *regular* submanifold of $(\mathfrak{M}, \mathcal{V})$ if we can choose a tubular neighborhood V of $\mathfrak{N}_0 := \mathfrak{N} \smallsetminus \partial\mathfrak{N} = \mathfrak{N} \cap \mathfrak{M}_0$ in $\mathfrak{M}_0$, a compatible metric g_1 on $\mathfrak{N}_0$, a product-type metric g_1 on V that reduces to g_1 on $\mathfrak{N}_0$, and a compatible metric on $\mathfrak{M}_0$ that restricts to g_1 on V. Theorem 5.8 of [2] states that every tame submanifold is regular. The point of this result is that it is much easier to check that a submanifold is tame than to check that it is regular.

In the case when $\mathfrak{N}$ is of codimension one in $\mathfrak{M}$, the condition that $\mathfrak{N}$ be tame is equivalent to the fact that there exists a vector field $X \in \mathcal{V}$ that restricts to a normal vector of $\mathfrak{N}$ in $\mathfrak{M}$. The neighborhood V will then be of the form $V \simeq (\partial\mathfrak{N}_0) \times (-\varepsilon_0, \varepsilon_0)$. Moreover, there will exit a compatible metric on $\mathfrak{M}_0$ that restricts to the product metric $g_1 + dt^2$ on V, where g_1 is a compatible metric on $\mathfrak{N}_0$.

Let $\mathbb{D} \subset \mathfrak{M}$ be an open subset. We shall denote by $\partial_{\mathfrak{M}} A$ the boundary of a subset $A \subset \mathfrak{M}$ (computed within $\mathfrak{M}$). This should not be confused with ∂A, the boundary of A as a manifold with corners, if A happens to be one. We say that $\mathbb{D}$ is a *Lie domain* in $\mathfrak{M}$ if, and only if, $\partial_{\mathfrak{M}}\mathbb{D} = \partial_{\mathfrak{M}}\overline{\mathbb{D}}$ and $\partial_{\mathfrak{M}}\mathbb{D}$ is a regular submanifold of $\mathfrak{M}$. A typical example of a Lie domain $\mathbb{D} \subset \mathfrak{M}$ is obtained by considering a regular sumbanifold with corners $\mathfrak{N} \subset \mathfrak{M}$ of codimension one with the property that $\mathfrak{M} \smallsetminus \mathfrak{N}$ consists of two connected components. Any of these two components will be a Lie domain.

Let $\mathbb{D}$ be a *Lie domain* in $\mathfrak{M}$ and $\mathbb{D}_0 = \mathbb{D} \cap \mathfrak{M}_0$. Also, let $\mathfrak{O} := \mathbb{D}_0 \cup \partial_{\mathfrak{M}_0}\mathbb{D}$. Then $\mathfrak{O}$ is a manifold with boundary $\partial\mathfrak{O} = \partial_{\mathfrak{M}_0}\mathbb{D}$, a submanifold of codimension one of $\mathfrak{M}_0$. Note that unlike our usual conventions, $\mathfrak{O}_0$ *is not* the interior of $\mathfrak{O}$.

Definition 5.5. A *Lie manifold with boundary* is a triple $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V}')$, where $\mathfrak{O}_0$ is a smooth manifold with boundary, $\mathfrak{O}$ is a compact manifold with corners containing $\mathfrak{O}_0$ as an open subset, and $\mathcal{V}'$ is a Lie algebra of vector fields on $\mathfrak{O}$ with the property that there exists a Lie manifold $(\mathfrak{M}_0, \mathfrak{M}, \mathcal{V})$, a Lie domain $\mathbb{D}$ in $\mathfrak{M}$ and a diffeomorphism $\phi : \mathfrak{O} \to \overline{\mathbb{D}}$ such that $\phi(\mathfrak{O}_0) = \overline{\mathbb{D}} \cap \mathfrak{M}_0$ and $D\phi(\mathcal{V}|_{\mathbb{D}}) = \mathcal{V}'$.

We continue with some simple observations. First note that if $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V})$ is a Lie manifold with boundary, then $\mathfrak{O}_0$ is determined by $(\mathfrak{O}, \mathcal{V})$. Indeed, if we remove from $\mathfrak{O}$ the hyperfaces H with the property that $\mathcal{V}$ consists only of vectors tangent to H, then the resulting set is $\mathfrak{O}_0$. Therefore, we can denote the Lie manifold with boundary $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V})$ simply by $(\mathfrak{O}, \mathcal{V})$.

Another observation is that $\partial\mathfrak{O}_0$, the boundary of $\mathfrak{O}_0$ has a canonical structure of Lie manifold $(\partial\mathfrak{O}_0, D = \partial_{\mathfrak{M}}\mathbb{D}, \mathcal{W})$, where $\mathcal{W} = \{X|_D,\ X \in \mathcal{V}, X|_D$ is tangent to $D\}$. The compactification D is the closure of $\partial\mathfrak{O}_0$ in $\mathfrak{O}$.

5.5. **Sobolev spaces.** The main reason for considering Lie manifolds (with or without boundary) in our setting is that they carry some naturally defined Sobolev spaces and these Sobolev spaces behave almost exactly like the Sobolev spaces on a compact manifold with a smooth boundary. Let us recall one of the equivalent definitions in [1].

Definition 5.6. Fix a Lie manifold $(\mathfrak{M}, \mathcal{V})$. The spaces $L^2(\mathfrak{M}_0) = L^2(\mathfrak{M}_0)$ are defined using the natural volume form on $\mathfrak{M}_0$ given by an arbitrary compatible metric g on $\mathfrak{M}_0$ (*i.e.,* compatible with the Lie structure at infinity). All such volume forms are known to define the same space $L^2(\mathfrak{M})$, but with possible different norms. Let $k \in \mathbb{Z}_+$. Choose a finite set of vector fields $\mathcal{X} \subset \mathcal{V}$ such that $\mathcal{C}^\infty(\mathfrak{M})\mathcal{X} = \mathcal{V}$. This condition is equivalent to the fact that the set $\{X(p), X \in \mathcal{X}\}$ generates A_p linearly, for any $p \in \mathfrak{M}$. Then the system $\mathcal{X}$ provides us with the norm

$$\|u\|^2_{\mathcal{X},\Omega} := \sum \|X_1 X_2 \ldots X_l u\|^2_{L^2(\Omega)}\,, \quad 1 \le p < \infty, \tag{35}$$

the sum being over all possible choices of $0 \le l \le k$ and all possible choices of not necessarily distinct vector fields $X_1, X_2, \ldots, X_l \in \mathcal{X}$. Then

$$H^k(\mathfrak{M}_0) = H^k(\mathfrak{M}) := \{u \in L^2(\mathfrak{M}),\ \|u\|_{\mathcal{X},\mathfrak{M}} < \infty\}.$$

The spaces $H^s(\mathfrak{M}_0) = H^s(\mathfrak{M})$ are defined by duality (with pivot $L^2(\mathfrak{M}_0)$) when $-s \in \mathbb{Z}_+$, and then by interpolation, as above.

Let $(\mathfrak{O}_0, \mathfrak{O}, \mathcal{V})$ be a Lie manifold with boundary. We shall assume that $\mathfrak{O}$ is the closure of a Lie domain $\mathbb{D}$ of the Lie manifold $\mathfrak{M}$. The Sobolev spaces $H^s(\mathfrak{O}_0)$ are defined as the set of restrictions to $\mathfrak{O}_0$ of distributions $u \in H^s(\mathfrak{M}_0)$, using the notation of Definition 5.5. In particular, we obtain the following description of $H^k(\mathfrak{O}_0)$ from [1].

Lemma 5.7. *We have that*

$$H^k(\mathfrak{O}_0) = \{u \in L^2(\mathfrak{M}),\ \|u\|_{\mathcal{X}} < \infty\}.$$

The hyperfaces of $\mathfrak{O}$ that do not intersect the boundary $\partial\mathfrak{O}_0$ of the manifold with boundary $\mathfrak{O}_0$ will be called *hyperfaces at infinity.* (This is compatible with our previous use of the concept of hyperface at infinity, as we shall see later on.) Let x_H be a defining function of the hyperface H of $\mathfrak{O}$. As before, an admissible weight on $\mathfrak{O}$ will be a function of the form $h = \prod x_H^{a_H}$, where H ranges through the set of hyperfaces at infinity of $\mathfrak{O}$ and $a_H \in \mathbb{R}$.

The following proposition, which summarizes the relevant results from Theorem 3.4 and 3.7 from [1], will imply Theorem 2.4, once we will have identified the spaces $\mathcal{K}_a^s$ with suitable spaces hH^s in Proposition 8.4 Definition 8.5.

Proposition 5.8. *The restriction to the boundary extends to a continuous, surjective map $hH^s(\mathfrak{O}_0) \to hH^{s-1/2}(\partial\mathfrak{O}_0)$, for any $s > 1/2$ and any admissible weight h. The kernel of this map, for $s = 1$, consists of the closure of $\mathcal{C}_c^\infty(\mathfrak{O}_0)$ in $hH^1(\mathfrak{O}_0)$.*

For $\mathbb{D}$, $\mathfrak{O}$, $\mathfrak{O}_0$ as in the proposition above, $hH^s(\mathbb{D})$, $hH^s(\mathfrak{O})$, and $hH^s(\mathfrak{O}_0)$ will all denote the same space.

6. Polyhedral domains

We shall define a curvilinear polyhedral domain in $\mathbb{R}^n$, or, more generally, in a smooth manifold M of dimension n, by induction on $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. The reader interested only in the case $n = 3$ should skip this section at a first reading and only refer back when necessary.

We first agree that a curvilinear polyhedral domain of dimension $n = 0$ is simply a finite set of points. Assume now that we have defined curvilinear polyhedral domains in dimension $\leq n - 1$. Let us define a curvilinear polyhedral domain in a manifold M of dimension n. We continue to denote by B^l the open unit ball in $\mathbb{R}^l$ and by $S^{l-1} := \partial B^l$ its boundary. We shall use in the following definition the concept of an irreducible subset of S^{l-1}, introduced in Definition 3.4.

Definition 6.1. Let M be a smooth manifold of dimension $n \geq 1$. Let $\Omega \subset M$ be an open subset. Then $\Omega \subset M$ is a *curvilinear polyhedral domain* if, for every point $p \in \partial\Omega$, there exist:

(i) an irreducible curvilinear polyhedral domain $\omega_p \subset S^{n-l-1}$;
(ii) a neighborhood V_p of p in M; and
(iii) a diffeomorphism $\phi_p : V_p \to B^{n-l} \times B^l$ such that

$$\phi_p(\Omega \cap V_p) = \{rx',\, 0 < r < 1,\, x' \in \omega_p\} \times B^l. \tag{36}$$

We continue with some comments on the definition. Denote by tB^l the ball of radius t in $\mathbb{R}^l$, $l \in \mathbb{N}$ centered at the origin. We also let tB^0 to be a point independent of t. Sometimes it is convenient to replace Condition (36) with the equivalent condition that there exist $t > 0$ such that

$$\phi_p(\Omega \cap V_p) = \{rx',\, 0 < r < t, x' \in \omega_p\} \times tB^l. \tag{37}$$

We shall interchange conditions (36) and (37) at will from now on.

The condition that ω_p be irreducible implies that

$$\mathbb{R}_+\overline{\omega}_p := \{rx',\, 0 \leq r < \infty, x' \in \overline{\omega}_p\}$$

contains no nonzero subspace of $\mathbb{R}^{n-l}$. This also shows that l is the largest integer with the property that p belongs to a submanifold of dimension l of $\partial\Omega$. We shall denote by $\ell(p) = l$ this integer. This integer is the same for both the pair (Ω, p) and for the pair $(\mathbb{R}_+\omega_p, p)$. This is not true if the condition that $\overline{\omega}_p$ not contain diametrally opposite points is dropped.

We have the following simple result that is an immediate consequence of the definitions.

Proposition 6.2. *Let $\psi : M \to M'$ be a diffeomorphism and let $\Omega \subset M$ be a curvilinear polyhedral domain. Then $\psi(\Omega)$ is also a curvilinear polyhedral domain.*

If $\ell(p) = 0$, then we shall call p a *vertex* of Ω and we shall interpret the condition (36) as saying that ϕ_p defines a diffeomorphism such that

$$\phi_p(\Omega \cap V_p) = \{rx',\, 0 < r < t, x' \in \omega_p\}. \tag{38}$$

This is consistent with our convention that B^0 is a point. If $\ell(p) = n-1$, then we shall interpret the condition (36) as saying that ϕ_p defines a diffeomorphism

$$\phi_p(\Omega \cap V_p) = (0,t) \times B^{n-1}. \tag{39}$$

We shall then say that p is a *smooth boundary point* of Ω. This is consistent with our convention that ω_p must be a point, since it is a polyhedral domain of dimension 0. If $p \in \Omega$, we set $\ell(p) = n$.

Notations 6.3. *From now on, Ω will denote a curvilinear polyhedral domain in a manifold M of dimension n. Also, we shall denote $\Omega^{(k)} := \{p \in \overline{\Omega}, \ell(p) \leq k\}$. All this notation will remain fixed throughout this paper.*

It follows from the definition that if $\ell(p) = l$, then there exists a manifold S_p of dimension l such that $p \in S_p \subset \partial\Omega$. If ω_p is connected, then there exists no manifold of dimension $l+1$ with this property.

It follows from the definition that if $q \in V_p$, $q \neq p$, then $\ell(q) > \ell(p)$. Therefore

$$\Omega^{(0)} \subset \Omega^{(1)} \subset \ldots \subset \Omega^{(n)} = \overline{\Omega},$$

are closed subsets such that $\Omega^{(d)} \smallsetminus \Omega^{(d-1)}$ is a (usually disconnected) smooth manifold of dimension d, for any d. It is possible that this manifold is either empty or disconnected. In particular, the set $\Omega^{(n-2)}$ consists of the non-smooth boundary points of Ω, as before. Similarly, we have $\Omega^{(n-1)} = \partial\Omega$. Also, the set $\Omega^{(0)}$, the set of vertices of Ω, consists of finitely many points or is empty. We agree that $\Omega^{(k)} = \emptyset$ if $k < 0$.

See Section 3 for the explicit description of some of the most common curvilinear polyhedral domains in $\mathbb{R}^2$, S^2, and $\mathbb{R}^3$.

Recall that a straight polyhedral domain were defined in Definition 1.2. We claim that every straight polyhedral domain is a curvilinear polyhedral domain. We shall prove this, as well as the following preliminary lemma, by induction.

Lemma 6.4. *Let Ω be a straight polyhedral domain and Π an affine subspace. Then the intersection $\Pi \cap \Omega$ is a finite union of straight polyhedral domains.*

Proof. We can assume that Π is of maximal dimension. The intersection $\omega := \Omega \cap \Pi$ is an open subset of Π and the intersection $\overline{\Omega} \cap \Pi$ is a closed subset of Π. Therefore the boundary of the set ω in Π is contained in $\partial\Omega \cap \Pi = (\overline{\Omega} \smallsetminus \Omega) \cap \Pi$. We shall use the notation of Definition 1.2. By induction, the sets $D_j \cap \Pi$ are disjoint unions of straight polyhedral domains. The union of the closures of these polyhedral domains then contains the boundary of ω. □

Proposition 6.5. *Let Ω be an open subset of a manifold M of dimension n. Assume that, for every $p \in \overline{\Omega}$, there exists a diffeomorphism $\psi_p : U_p \to B^n$, for some neighborhood U_p of p in M, such that $\psi_p(U_p \cap \Omega) = B^n \cap \Omega_p$, for some straight polyhedron Ω_p. Then Ω is a curvilinear polyhedral domain.*

Proof. We shall use the notation of Definitions 1.2 and 6.1. Fix $p \in \partial\Omega$ arbitrary. To check the condition (37) shall chose $V_p \subset U_p$ and ϕ_p the restriction of ψ_p to V_p. Since the problem is local in p and invariant under diffeomorphisms, we can assume that $\Omega = \Omega_p$ and that $p = 0$.

By relabeling the faces of Ω_p, we can assume that the faces of Ω_p containing $p = 0$ are denoted $D_1, D_2, \ldots, D_k$. By changing scale, if necessary, we can assume that these are the only faces of Ω_p intersecting $2B^n$ (the ball of radius 2 in $\mathbb{R}^n$). By changing the system of coordinates, we can then assume that the intersection of the hyperplanes V_j containing D_j, $j = 1, 2, \ldots, k$, is $\mathbb{R}^l \subset \mathbb{R}^n$. This l will have the same meaning as the l in Definition 6.1. Denote by $\mathbb{R}^{n-l}$ the orthogonal complement of $\mathbb{R}^l$ in $\mathbb{R}^n$. Since the only faces D_j intersecting $2B^n$ are the ones with $j = 1, 2, \ldots, k$, we obtain that $(B^n \cap \Omega_p) + v \subset \Omega_p$, for any $v \in B^n \cap \mathbb{R}^l$.

The result will follow once we prove that ω_p is an irreducible curvilinear polyhedral domain of S^{n-l-1}. Let $q \in \omega_p$ be a point on the boundary and Π_q be the hyperplane through q perpendicular to q (*i.e.,* Π_q is the tangent plane through q to the unit sphere). The intersection ω between Π_q and Ω is then a polyhedral domain, by Lemma 6.4. The orthogonal projection onto ω_q defines a diffeomorphism of a neighbhorhood of q in $\overline{\omega}_q$ onto a neighborhood of q in $\overline{\omega}$. We can therefore use induction to conclude that ω_p is a curvilinear polyhedral domain.

To prove that ω_p is irreducible, let us notice that any subspace $V \subset \mathbb{R}^{n-l}$ such that $\mathbb{R}_+\omega_p = V + V'$ satisfies $V \subset V_j \cap \mathbb{R}^{n-1}$, where V_j is, as before, the linear subspace containing the face D_j. □

7. Desingularization of polyhedra

In this section, we introduce a desingularization procedure that we shall use for studying curvilinear polyhedral domains. The desingularization will carry a natural structure of Lie manifold with boundary. This will allow us to study curvilinear polyhedral domains using Lie manifolds with boundary.

As before, $\Omega \subset M$ denotes a curvilinear polyhedral domain in an n-dimensional manifold M. We shall construct by induction on n a *canonical* manifold with corners $\Sigma(\Omega)$ and a differentiable map $\kappa : \Sigma(\Omega) \to \overline{\Omega}$ that is a diffeomorphism from the interior of $\Sigma(\Omega)$ to Ω. In particular, the map κ allows us to identify Ω with a subset of $\Sigma(\Omega)$. We shall also construct a canonical Lie algebra of vector fields $\mathcal{V}(\Omega)$ on $\Sigma(\Omega)$. The manifold $\Sigma(\Omega)$ will be called the *desingularization of* Ω, the map κ will be called the *desingularization* map, and the Lie algebra of vector fields will be called the *structural Lie algebra of vector fields of* $\Sigma(\Omega)$. We shall also introduce in this section a smooth weight function r_Ω equivalent to η_{n-2}.

The space $\Sigma(\Omega)$ that we construct is not optimal if the links ω_p are not connected. A better desingularization would be obtained if one considers a diffeomorphism ϕ_{pC} for each connected component C of $V_p \cap \Omega$ that maps C to a conic set of the form $(0,1)\omega_{pC} \times B^\lambda$, with λ largest possible. The difference between these two constructions is seen by looking at the Example 3.12.

Notations 7.1. *From now on V_p and $\phi_p : V_p \to tB^{n-l} \times tB^l$, $l = \ell(p)$, will denote a neighborhood of $p \in \partial\Omega$ in $M \supset \Omega$ and ϕ_p will be a diffeomorphism satisfying the conditions of Definition* (6.1). *Also, $\omega_p \subset S^{n-l-1}$ is the curvilinear polyhedron such that*

$$\phi_p(V_p \cap \Omega) = \{(rx', x'')\},\ r \in (0,t), x' \in \omega_p \textit{ and } x'' \in tB^l\}.$$

All this notation will remain fixed throughout the paper.

If $l = 0$ in the notation above, then B^l is reduced to a point, and we just drop x'' from the notation above. We will assume that ϕ_p extends to the closure of V_p, if necessary.

7.1. **The desingularization** $\Sigma(\Omega)$**.** We now define the canonical desingularization of a curvilinear polyhedral domain Ω. For $n = 0$, Ω consists of finitely many points. Then we define $\Sigma(\Omega) = \Omega$ and $\kappa = id$. To define $\Sigma(\Omega)$ for general Ω, we shall proceed by induction.

We need first to make the important observation that the set ω_p, $p \in \partial\Omega$, of Definition 6.1 is determined up to a linear isomorphism of $\mathbb{R}^{n-l-1}$. Indeed, let $p \in S_p \subset \partial\Omega$ be maximal connected manifold of dimension $\ell(p)$ passing through p (this was defined in Section 6 as the connected component of $\Omega^{(l)} \smallsetminus \Omega^{(l-1)}$ containing p). Let $(T_pS_p)^\perp = T_pM/T_pS_p$. We first define a canonical set $\mathcal{N}_p \subset (T_pS_p)^\perp$ such that the differential $D\phi_p : T_pM \to \mathbb{R}^n = T_0\mathbb{R}^n$ of the map ϕ_p at p is such that $D\phi_p(T_pS_p) = T_0\mathbb{R}^l$, $D\phi_p\big((T_pS_p)^\perp\big) = T_0\mathbb{R}^n/T_0\mathbb{R}^l = \mathbb{R}^{n-l}$, and, most importantly,

$$D\phi_p(\mathcal{N}_p) \simeq \mathbb{R}_+\omega_p.$$

Since the definition $\mathcal{N}_p$, which follows next, is independent of any choices used in the definition of a polyhedral domain, it follows that ω_p is unique, up to a linear isomorphism of $\mathbb{R}^n$. It remains to define the set $\mathcal{N}_p$ with the desired independence property. It is enough to define the *complement* of $\mathcal{N}_p$. This complement is the projection onto $(T_pS_p)^\perp = T_pM/T_pS_p$ of the set $\gamma'(0) \in T_pM$, where γ ranges through the set of smooth curves $\gamma : [0,1] \to \partial\Omega$, $\gamma(t) \in \partial\Omega$, $\gamma(0) = p$.

We let then $\sigma_p := \mathcal{N}_p/\mathbb{R}_+$, the set of rays in $\mathcal{N}_p$, for $p \in \partial\Omega$. Any choice of a metric on $T_pM/T_pS_p \supset \mathcal{N}_p$ will give identify σ_p with a subset of the unit sphere of T_pM/T_pS_p, which depends however on the metric. In particular, $D\phi_p : \sigma_p \to \omega_p$ is a diffeomorphism. If p is a smooth boundary point of Ω, then σ_p consists of one or two points single point (one point for a one-sided smooth boundary point and two points for two sided boundary points). The map κ is the projection onto the second component and is one-to-one above Ω and above the set of one-sided smooth boundary points.

We now proceed with the induction step. Assume $\Sigma(\Omega_0)$ and $\kappa : \Sigma(\Omega_0) \to \overline{\Omega}_0$ have been constructed for all curvilinear polyhedral domains Ω_0 of dimension at most $n-1$. Let Ω be an arbitrary curvilinear polyhedral domain of dimension n. We define

$$\Sigma(\Omega, M) := \cup_p \{p\} \times \Sigma(\sigma_p, S^{k-1}), \quad p \in \overline{\Omega}. \tag{40}$$

In particular, if Ω is a bounded domain with smooth boundary, then $\Sigma(\Omega) = \{0\} \times \Omega \simeq \Omega$. This definition makes sense since ω_p is a curvilinear polyhedral domain of dimension at most $n-1$. Since M in $\Sigma(\Omega, M)$ is most of the time fixed, we will sometimes omit it from the notation. For example, we shall write $\Sigma(\sigma_p) = \Sigma(\sigma_p, S^{k-1})$.

Below, an open embedding will mean a diffeomorphism onto an open subset of the codomain.

Proposition 7.2. *Let $\Omega \subset M$ and $\Omega_0 \subset M_0$ be curvilinear polyhedral domains and $\phi : M \to M_0$ be an open embedding such that $\phi(\Omega)$ is a union of connected* *components of* $\Omega_0 \cap \phi(M)$. *Then* ϕ *defines a canonical map* $\Sigma(\phi) : \Sigma(\Omega, M) \to \Sigma(\Omega_0, M_0)$ *such that*

$$\Sigma(\phi \circ \phi') = \Sigma(\phi) \circ \Sigma(\phi'),$$

for all open embeddings ϕ *and* ϕ' *for which all the terms are defined.*

Proof. The proof is by induction. There is nothing to prove for $n = 0$. Let $p \in \overline{\Omega}$. We have that $\phi(\overline{\Omega}) \subset \overline{\Omega}_0$, and hence $p \in \overline{\Omega}_0$, as well. Let V_{0p} be an open neighborhood of p such that there exists a diffeomophism $\phi_{0p} : V_{0p} \to B^{n-l} \times B^l$ satisfying the condition (36) of the definition of a polyhedral domain (*i.e.,* $\phi_{0p}(\Omega_0 \cap V_{0p})$ is the edge $\mathbb{R}_+\omega_{0p} \times B^l$, for some curvilinear polyhedral domain $\omega_{0p} \subset S^{n-l-1}$). By decreasing V_{0p}, if necessary, we can assume that $V_{0p} \subset \phi(M)$. Then $V_{0p} \cap \phi(\Omega)$ is a union of connected components of $V_{0p} \cap \Omega_0$. Therefore ω_{0p} is a union of connected components of ω_p, where $\omega_p \subset S^{n-l-1}$ is associated to $p \in \overline{\Omega}$ in the same way as ω_{0p} was associated to $p \in \overline{\Omega}_0$. The induction hypothesis then gives rise to a canonical, injective map $\Sigma(\omega_p, S^{n-l-1}) \to \Sigma(\omega_{0p}, S^{n-l-1})$. The map $\Sigma(\phi)$ is obtained by putting together all these maps.

The functoriality (*i.e.,* the relation $\Sigma(\phi \circ \phi') = \Sigma(\phi) \circ \Sigma(\phi')$) is proved similarly by induction. □

Here is a corollary of the above proof.

Corollary 7.3. *If* $\Omega = \Omega' \cup \Omega''$ *is the disjoint union of two open sets, then the inclusions* $\Sigma(\Omega', M) \subset \Sigma(\Omega, M)$ *and* $\Sigma(\Omega', M) \subset \Sigma(\Omega, M)$ *defined by Proposition 7.2 realize* $\Sigma(\Omega, M) = \Sigma(\Omega', M) \cup \Sigma(\Omega'', M)$, *where the union is a disjoint union.*

Proof. We use the same argument as in the proof of Proposition 7.2. □

The desingularization has a simple behavior with respect to products.

Lemma 7.4. *We have a canonical identification*

$$\Sigma(M_0 \times \Omega, M_0 \times M) = M_0 \times \Sigma(\Omega, M),$$

for any smooth manifolds M *and* M_0 *and any curvilinear polyhedral domain* $\Omega \subset M$.

Proof. This will follow from the canonical bijection $\sigma_{(p,q)} \simeq \sigma_p$ for any $p \in \overline{\Omega}$ and any $q \in M_0$ (so (p, q) is in the closure of $\Omega \times M_0$ in $M \times M_0$). Indeed, since M_0 is smooth, we can chose the structural local diffeomorphism $\phi_{(p,q)}$ to be given by $\phi_p \times \psi_q$, where ψ_q is a local coordinate chart defined in a neighborhood of $q \in M_0$. Then

$$\Sigma(M_0 \times \Omega) := \cup_{q,p}\{(q, p)\} \times \Sigma(\sigma_{(q,p)}) = \cup_{q,p}\{(q, p)\} \times \Sigma(\sigma_p) = M_0 \times \Sigma(\Omega), \tag{41}$$

where $p \in \overline{\Omega}$ and $q \in M_0$. □

It remains to define the topology and differentiable structure on $\Sigma(\Omega)$. These definitions will again be canonical if we require that the map of the above lemma, as well as the maps κ and $\Sigma(\phi)$, be differentiable, for any open embedding ϕ.

Let $V_p \subset M$ and ϕ_p be as in Equation (37). By Proposition 7.2, we may assume that ϕ_p is the identity, so that $p = 0$, $V_p = B^{n-l} \times B^l$, and $V_p \cap \Omega = (0, 1)\omega_p \times B^l$. Let $(\Sigma(\omega_p), \kappa_p')$ be the canonical desingularization of ω_p. We shall need the following lemma.

Lemma 7.5. *We have a canonical identification*

$$\Sigma(V_p \cap \Omega, M) = [0,1) \times \Sigma(\omega_p, S^{n-l-1}) \times B^l$$

and the desingularization map

$$\kappa_p : [0,1) \times \Sigma(\omega_p) \times B^l \to \overline{V_p \cap \Omega} \subset B^{n-l} \times B^l \tag{42}$$

is given by $\kappa_p(r, x', y) = (r\kappa_p'(x'), y)$.

Proof. We shall replace p with 0 below, since we have assumed $p = 0$. This will simplify the notation. The closure of $V_0 \cap \Omega$ in $V_0 = V_p$ is $\{0\} \times B^l \cup (0,1)\overline{\omega}_0 \times B^l$ (disjoint union). Then we decompose the union of $\Sigma(\sigma_q)$ defining $\Sigma(V_0 \cap \Omega, M)$, where $q \in \overline{V_0 \cap \Omega}$, as a union of two disjoint sets, corresponding to this disjoint union decomposition of the closure of $V_0 \cap \Omega$. Using also Lemma 7.4, we then obtain

$$\begin{aligned}\Sigma(V_0 \cap \Omega, M) &= \Sigma\big(V_0 \cap \Omega, M \smallsetminus (\{0\} \times B^l)\big) \cup \bigcup_{q \in B^l} \{(0,q)\} \times \Sigma(\omega_0) \\ &= \Sigma\big((0,1) \times \omega_0 \times B^l, (0,1) \times S^{n-l-1} \times B^l\big) \cup \bigcup_{q \in B^l} \{(0,q)\} \times \Sigma(\omega_0) \\ &= (0,1) \times \Sigma(\omega_0) \times B^l \cup \{0\} \times \Sigma(\omega_0) \times B^l \\ &= [0,1) \times \Sigma(\omega_0) \times B^l.\end{aligned}$$

The formula for κ_0 follows from the definition. □

Since $\Sigma(\Omega, M)$ is the union of all the sets $\Sigma(V_p \cap \Omega, M)$, with V_p in the covering above, we can define the topology and smooth structure on $\Sigma(\Omega, M)$ as follows.

Definition 7.6. Let $\phi_p : V_p \to B^{n-l} \times B^l$ and ω_p be as in Definition 6.1. The topology and smooth structure on $\Sigma(\Omega, M)$ are such that the induced structures on $\Sigma(V_p \cap \Omega, M)$ is the same as the one obtained from the canonical identification $\Sigma(V_p \cap \Omega, M) = [0,1) \times \Sigma(\omega_p) \times B^l$ of Lemma 7.5.

We need to prove that the transition functions are smooth. This follows from the fact that the maps ϕ_p are diffeomorphisms.

We have therefore completed our definition of the desingularization $\Sigma(\Omega, M)$ and of its smooth structure.

7.2. **The distance to singular boundary points.** We continue with a study of the geometric and, especially, metric properties of $\Sigma(\Omega, M)$. Since M will be fixed from now on, we shall write $\Sigma(\Omega) = \Sigma(\Omega, M)$. We first argue that $\Sigma(\Omega)$ has embedded faces and hence that every hyperface of $\Sigma(\Omega)$ has a defining function.

Let F_0 be an open hyperface of a manifold with corners $\mathfrak{M}$. Then F_0 is a manifold of dimension $n-1$. Its closure F, in general, will not necessarily be a manifold, because it may have self-intersections. (An example typical is the boundary of a curvilinear polygonal domain with only one vertex, the "tear drop domain.") By induction, however, it follows that $F \cap V_p$ will be a manifold, for any p. In particular, we obtain that all (closed) faces of $\Sigma(\Omega)$ are embedded submanifolds of $\Sigma(\Omega)$. Let H be a hyperface of $\Sigma(\Omega)$, since H is an embedded submanifold of codimension 1, there will exist a function $x_H > 0$ on Ω, $H = \{x_H = 0\}$, and $dx_H \neq 0$ on H. A function x_H with this property is called a *defining function of* H, see Melrose's book [33].

One of the main reasons for introducing the desingularization space $\Sigma(\Omega)$ is the following result.

Proposition 7.7. *Let Ω be a curvilinear polyhedral domain and g_1 and g_2 be two smooth Riemannian metrics on M. Fix k and let $f_j(x)$ be the distance from $x \in \overline{\Omega}$ to the set $\Omega^{(k)} \neq \emptyset$ in the metric g_j, computed within $\overline{\Omega}$. Then the quotient f_2/f_1 extends to a continuous function on $\Sigma(\Omega)$.*

Proof. By replacing V_p with a smaller neighborhood of p, if necessary, we can also assume that $g_2(\xi) \leq Cg_1(\xi)$, which implies that $f_2 \leq Cf_1$, and hence that f_2/f_1 is bounded.

We shall prove the statement by induction on n. In the case $n = 1$, the only possibility is that $k = 0$, or otherwise $\Omega^{(k)} = \emptyset$. Then $f(x)$ is the distance to the vertices of Ω. If say $\Omega = [a, b]$, then close to a, $f_j(x) = a_j(x)(x-a)$, with a_j smooth near a and $a_j(a) \neq 0$. The same situation holds at b. This proves our result in the case $n = 1$. We now proceed with the induction step.

The function f_1/f_2 is clearly continuous on the open set Ω. Fix $p \in \partial\Omega$. We shall construct an open neighborhood U_p of $p \in \overline{\Omega}$ such that f_1/f_2 extends to a continuous function on $\kappa^{-1}(U_p)$. Let V_p be as in the definition of polyhedral domains (Definition 6.1). We shall identify V_p with $(0, t) \times \omega_p \times B^l$ using the diffeomorphism ϕ_p of Equation (37). If $l > k$, then both f_1 and f_2 extend to continuous, non-vanishing functions on $V_p \cap \overline{\Omega}$, which can be lifted to continuous, non-vanishing functions on $\kappa^{-1}(V_p)$. We shall assume hence that $k \geq l$.

On a smaller neighborhood $V' \subset V_p$ if necessary, we can arrange that the function f_1 gives the distance to $V_p^{(k)}$, that is, that the point of $\Omega^{(k)}$ closest to $x \in V' \cap \Omega$ is, in fact, in V_p. By decreasing V' even further, we can further arrange that the same holds for f_2. Then we shall take $U_p := V'$.

To prove that f_2/f_1 extends to a continuous function on $\kappa^{-1}(U_p)$, it is enough to do that in the case $\Omega = V_p$, because the quotient f_2/f_1 does not change on $U_p \cap \Omega$ if we replace Ω with V_p, as explained in the paragraph above. We can also assume that g_2 is the standard Euclidean metric, but then we have to prove that f_1/f_2 extends to a *nowhere vanishing* continuous function on $\Sigma(\Omega)$.

The scaling property of the Euclidean metric and our assumption that $k \geq l$ imply that

$$f_2(rx', x'') = rf_2(x', x''),$$

for any $r \in [0, 1]$. Let g_0 be a constant metric on $\mathbb{R}^n$ that coincides with g_1 at the origin.

Let f_0 be associated to g_0 in the same way as f_j is associated to g_j, for $j = 1, 2$. Similarly, we have that $f_0(rx', x'') = rf_0(x', x'')$. This shows that the quotient $f_0(rx', x'')/f_1(rx', x'')$ does not depend on r. We can therefore fix $r = 1$. This means that we can restrict ourselves to the lower dimensional polyhedral domain $\omega := \omega_p \times B^l$ and prove that f_0/f_1 extends by continuity to $\Sigma(\omega)$. It remains to see that we can use induction to prove the existence of this extension. Let f_1' be the distance function to $\omega^{(k-1)} = \omega^{(k-l-1)} \times B^l$ on ω (*i.e.,* computed *within* $\overline{\omega}$, with respect to the metric induced by g_1, as in the statement of Proposition 7.7). We let $f_1' = 1$ if $\omega_p^{(k-l-1)} = \emptyset$, for example if $k - l - 1 < 0$.

Define f_0' similarly. The inductive hypothesis guarantees that f_0'/f_1' extends to a continuous function on $\Sigma(\omega) = \Sigma(\omega_p) \times B^l$. On the other hand, it is easy to see that both f_1/f_1' and f_1'/f_1 extend to continuous functions on $\overline{\omega}$ if we set them to be equal to 1 on $\omega^{(k-1)}$. The same is true of f_0/f_0' and f_0'/f_0. Putting all this together, it follows that

$$f_0/f_1 = (f_0/f_0')(f_0'/f_1')(f_1'/f_1)$$

extends to a continuous, nowhere vanishing function on $\Sigma(\omega)$.

Let us tackle now the case g_2 arbitrary. Let f_0 be defined as before. We then have that $f_2(rx', x'') = rf_0(x', x'') + r^2h(rx', x'')$, with h a continuous function on $\Sigma(V_p \times \Omega)$ that vanishes on $\Omega^{(k)}$. Then

$$\frac{f_2}{f_1} = \frac{f_0}{f_1} + r\frac{h(rx', x'')}{f_1(x', x'')}.$$

The function f_0/f_1 was already shown to extend by continuity to $\Sigma(\Omega)$. The same argument as above shows that h/f_1 extends by continuity to a nowhere vanishing function on

$$[\epsilon, 1) \times \Sigma(\omega_p) \times B^l \subset (0, 1) \times \Sigma(\omega_p) \times B^l =: \Sigma(\Omega).$$

The continuity of f_2/f_1 then follows from the boundedness of f_2/f_1.

The resulting function does not vanish at $r = 0$, because it is equal to f_0/f_1 there. It was already proved that it does not vanish for $\epsilon > 0$. The proof is complete. □

We shall need also the following corollary of the above proof.

Corollary 7.8. *Identify V_p with $(0, a) \times \omega_p \times B^l$ using the diffeomorphism ϕ_p, $l = \ell(p)$. Let g be a smooth metric on V_p, let $f(x)$ be the distance from x to $\Omega^{(k)}$, $k \geq l$, and $f'(x', x'')$ be the distance from $(x', x'') \in \omega := \omega_p \times B^l$ to $\omega^{(k-1)}$ (within $\overline{\omega}$, as in Proposition 7.7) if $\omega^{(k-1)} \neq \emptyset$, and $f'(x', x'') = 1$ otherwise. Assume ω_p connected. Then*

$$f(rx', x'')/rf'(x', x'')$$

extends to a continuous, nowhere vanishing function on $\Sigma(\Omega) = [0, a) \times \Sigma(\omega_p) \times B^l$.

Proof. Assume first that $\omega^{(k-1)} \neq \emptyset$. Let f_0 and f_0' be defined in the same way f and f' were defined, but replacing g with a constant metric g_0. Then the proof of Proposition 7.7 gives that $f_0(rx', x'')/rf_0'(x', x'')$ is independent of r. Hence $f_0(rx', x'')/rf_0'(x', x'')$ extends to a continuous, nowhere vanishing function on $\Sigma(\Omega)$, because we can do that for $r = 1$, as it was shown in the proof of Proposition 7.7. Then

$$\frac{f(rx', x'')}{rf'(x', x'')} = \frac{f(rx', x'')}{f_0(rx', x'')} \times \frac{f_0(rx', x'')}{rf_0'(rx', x'')} \times \frac{f_0'(x', x'')}{f'(x', x'')}.$$

We have just argued that the middle quotient in the above product extends to a continuous function on $\Sigma(\Omega)$. The other two quotients also extend to continuous functions on $\Sigma(\Omega)$, by Proposition 7.7 applied to Ω and ω.

Let us assume now that $\omega^{(k-1)} = \emptyset$. Then the same proof will work in this case, except that the reason why the quotient f_0'/f' extends to a continuous function on $\Sigma(\Omega)$ is that $f_0'/f' = 1$. □

7.3. **The weight function r_Ω.** Recall that $\eta_{n-2}(x)$ denotes the distance from $x \in \overline{\Omega}$ to the set $\Omega^{(n-2)} := \{p,\ \ell(p) \leq n-2\}$. The set $\Omega^{(n-2)}$ consists of the non-smooth boundary points of Ω. For instance, if $n = 2$, the set $\Omega^{(0)}$ corresponds to the set of points satisfying (a) in the Definition 3.1, that is, it is the set of vertices of Ω. If $n = 3$, the set $\Omega^{(1)}$ corresponds to the set of points satisfying (a) or (b) in the Definition 3.5. Also, recall the definition of the distance function $\eta_{n-2}(x)$ (Definition 1.1, also discussed in the Introduction).

The main goal of this subsection is to define on any curvilinear polyhedral domain Ω a function

$$r_\Omega : \overline{\Omega} \to [0,\infty)$$

that is *smooth* on Ω and is *equivalent* to η_{n-2} when Ω is bounded. (Additional properties of r_Ω will be established later on.) This will lead to a definition of the Sobolev spaces $\mathcal{K}^l_a(\Omega)$ as weighted Sobolev spaces on Lie manifolds with boundary, Proposition 8.4. We again proceed by induction on n.

We define $r_\Omega = 1$ if $n = 1$ or if Ω is a smooth manifold, possibly with boundary (that is, if $\Omega^{(n-2)} = \emptyset$). (Note that if Ω is connected and $n = 1$, then Ω is a smooth manifold with boundary.)

Assume now that a function r_ω was defined for all curvilinear polyhedral domains ω of dimension $< n$ and let us define it for a given bounded n-dimensional curvilinear polyhedral domain Ω.

Let us consider first the case $\Omega = V_p$, for some $p \in \partial\Omega$, and let the polyhedral domain $\omega_p \subset S^{n-l-1}$, $n-l-1 \geq 1$, and the diffeomorphism $\phi_p : V_p \to B^{n-l} \times B^l$ be as in the definition of a curvilinear polyhedral domain (*i.e.,* , satisfying $V_p \cap \Omega \simeq (0,\epsilon) \times \omega_p \times B^l$, see Condition (36)). We can assume that ϕ_p is the identity map. Replace in what follows V_p with $\phi_p^{-1}\big(\frac{1}{2}B^{n-l} \times \frac{1}{2}B^l\big)$. Let r_{ω_p} the function associated to the curvilinear polyhedral domain ω_p. Then we define

$$r_{V_P}(rx', x'') := r r_{\omega_p}(x', x''), \quad (rx', x'') \in \Omega \subset V_p, \tag{43}$$

if $x' \in \omega_p$, $x'' \in B^l$, and $1 \leq l = \ell(p) \leq n-2$. Following our usual procedures, we set $r_{V_p}(rx') = r r_{\omega_p}(x')$ if $l = 0$. The case $l = n-1$, excluded by the inequality $n-l-1 \geq 1$ need not be considered because it corresponds to the case when $\Omega = V_p$ is a smooth manifold with boundary, and hence $r_\Omega = 1$ was already defined.

Let us consider now a locally finite covering of $\overline{\Omega}$ with open sets U_α of one of the three following forms

(i) $U_\alpha = \Omega$;
(ii) $U_\alpha = V_p$ with $\ell(p) = n-1$ (*i.e.,* p is a smooth boundary point of $\overline{\Omega}$); or
(iii) such that for any $x \in U_\alpha \cap \Omega$, the point of $\Omega^{(n-2)}$ closest to x is in V_p and

$$p \in U_\alpha \subset \overline{U}_\alpha \subset V_p. \tag{44}$$

A condition similar to (iii) above was already used in the proof of Proposition 7.7. The conditions (i) and (ii) above correspond exactly to the case when ∂U_α is smooth (this includes the case when ∂U_α is empty).

We then set

$$r_\alpha = \begin{cases} 1 & \text{if } \partial U_\alpha \text{ is smooth} \\ r_{V_p} & \text{if } U_\alpha \text{ is as in (44).} \end{cases} \tag{45}$$

Let r_α be the function defined for each $V_p \supset U_\alpha$ by Equation (43) above, if U_α is as in Equation (44). Let φ_α be a partition of unity subordinated to U_α and consisting of smooth functions. Finally, we define

$$r_\Omega = \sum_\alpha \varphi_\alpha r_\alpha. \tag{46}$$

We notice that the definition of r_Ω is not canonical, because it depends on a choice of a covering (U_α) of $\overline{\Omega}$ as above and a choice of a subordinated partition of unity.

Proposition 7.9. *Let Ω be a curvilinear polyhedral domain of dimension $n \geq 2$. Then r_Ω of Equation (46) is continuous on $\overline{\Omega}$ and $r_\Omega \circ \kappa$ is smooth on $\Sigma(\Omega)$. Moreover, η_{n-2}/r_Ω extends to a continuous, nowhere vanishing function on $\Sigma(\Omega)$ and r_α/r_Ω extends to a smooth function on $\Sigma(V_p)$.*

Proof. Let $\eta_{-1} := 1$ for the inductive step. We shall prove the statement on η_{n-2}/r_Ω by induction on $n \geq 1$. Since $r_\Omega = 1$ for polyhedral domains of dimension $n = 1$, the result is obviously true for $n = 1$. We now proceed with the inductive step.

We shall use the above results, in particular, Proposition 7.7, for $k = n - 2 \geq 0$. Thus $f = \eta_{n-2}$ in the notation of Proposition 7.7. Let $f_\alpha(x)$ be the distance from $x \in V_p$ to $V_p \cap \Omega^{(n-2)}$, if $U_\alpha \subset V_p$ is as in Equation (44) (so $\ell(p) \leq n-2$ in this case). Thus $f_\alpha = f$ on $U_\alpha \cap \Omega$, by the construction of U_α. We identify V_p with $(0, \epsilon) \times \omega_p \times B^l$, $l = \ell(p)$, using the diffeomorphism ϕ_p. Let $\omega := \omega_p \times B^l$. Also, for any $x \in \omega$, let $f'_\alpha(x)$ be the distance from x to the set of $p \in \partial\omega$ where $\partial\omega$ is not smooth (*i.e.,* to $\omega^{(n-3)}$) if $\omega^{(n-3)} \neq \emptyset$, $f'_\alpha(x) = 0$ otherwise. Let r_α be as in the definition of r_Ω, Equation (46). Then

$$\frac{f_\alpha(rx', x'')}{r_\alpha(rx', x'')} = \frac{f_\alpha(rx', x'')}{rf'_\alpha(x', x'')}\,\frac{f_\alpha(x', x'')}{r_{\omega_p}(x', x'')}\,, \qquad \text{for } (rx', x'') \in V_p \cap \Omega.$$

The quotient $f_\alpha(rx', x'')/rf'_\alpha(x', x'')$ extends to a continuous, nowhere vanishing function on $\Sigma(V_p)$, by Corollary 7.8. By the induction hypothesis, the quotient $f_\alpha(x', x'')/r_{\omega_p}(x', x'')$ also extends to a continuous, nowhere vanishing function on $\Sigma(\omega) = \Sigma(\omega_p) \times B^l$. Since this quotient is independent of r, it also extends to a continuous, nowhere vanishing function on $\Sigma(V_p)$. Hence f_α/r_α extends to a continuous, nowhere vanishing function on $\Sigma(V_p)$. Therefore

$$r/f = \sum_\alpha \varphi_\alpha r_\alpha / f = \sum_\alpha \varphi_\alpha r_\alpha / f_\alpha$$

extends to a continuous function on $\Sigma(\Omega)$.

The quotient r/f is immediately seen to be non-zero everywhere, from the definition. Hence f/r also extends to a continuous function on $\Sigma(\Omega)$.

We have already noticed that r_α/f extends to a continuous, nowhere vanishing function on $\Sigma(V_p)$. Hence $r_\alpha/r_\Omega = (r_\alpha/f)(f/r_\Omega)$ extends to a continuous, nowhere vanishing function on $\Sigma(V_p)$. Since both r_α and r_Ω are smooth on $\Sigma(V_p)$ and the set of zeroes of r_Ω is the union of transversal manifolds on which r_Ω has simple zeroes, it follows that r_α/r_Ω extends to a smooth function on $\Sigma(V_p)$. Since $\overline{U}_\alpha \subset V_p$ is compact, it follows from a compactness argument that r_α and r are equivalent on U_α. The proof is complete. □

We can now prove the following result. As we have observed in the proof of Theorem 2.1, the following proposition completes the proof of that theorem.

Proposition 7.10. *Let Ω be a bounded, curvilinear polyhedral domain. Let r' be another choice for r_Ω. In other words, r' is constructed in the same way r_Ω was constructed, but with possibly other choices of ϕ_p, U_α, φ_α, r_{ω_p}. Then r'/r_Ω extends to a smooth, nowhere vanishing function on $\Sigma(\Omega)$.*

Proof. We know from Proposition 7.9, that f/r' and f/r_Ω extend to continuous, nowhere vanishing functions on $\Sigma(\Omega)$. Hence r'/r_Ω extends to a continuous, nowhere vanishing function on $\Sigma(\Omega)$. Since both r' and r_Ω are smooth functions on $\Sigma(\Omega)$ and the set of zeroes of r_Ω is a union of transverse manifolds, each a set of simple zeroes of r_Ω, it follows that the quotient r'/r_Ω is smooth on $\Sigma(\Omega)$. $\square$

We obtain the following corollary. Let $H \subset \Sigma(\Omega)$ be a hyperface (*i.e.,* face of maximal dimension) of $\Sigma(\Omega)$. Recall that *a defining function of H* is a smooth function $x_H \geq 0$ defined on $\Sigma(\Omega)$, such that $H = \{x = 0\}$ and $dx_H \neq 0$ on H. All the faces of $\Sigma(M)$ are closed subsets of $\Sigma(M)$, by definition. Also, we have already noticed that any face of $\Sigma(\Omega)$ has a defining function. We then have the following corollary.

Corollary 7.11. *Let $\eta = \prod_H x_H$, where H ranges through the set of hyperfaces of $\Sigma(\Omega)$ that do not intersect $\partial\Omega \smallsetminus \Omega^{(n-2)}$ (*i.e., *through the set of hyperfaces at infinity). Then η/r_Ω extends to a smooth, nowhere vanishing function on $\Sigma(\Omega)$.*

Proof. This is a local statement that can be checked by induction, as in the previous proofs. $\square$

In particular, since the function r_Ω is anyway determined only up to a factor of $h \in \mathcal{C}^\infty(\Sigma(\Omega))$, $h \neq 0$, we obtain that we could take $r_\Omega = \prod_H x_H$, where H ranges through the set of hyperfaces at infinity of $\Sigma(\Omega)$. (Recall from Subsection 1.3 that a hyperface $H \subset \Sigma(\Omega)$ is called a hyperface at infinity if $\kappa(H) \subset \Omega^{(n-2)}$.)

7.4. **The structural Lie algebra of vector fields.** We now proceed to define by induction a canonical Lie algebra of vector fields $\mathcal{V}(\Omega)$ on $\Sigma(\Omega)$, for Ω a curvilinear polyhedral domain of dimension $n \geq 1$. In view of Corollary 7.3, we can assume that Ω is connected.

We let

$$\mathcal{V}(\Omega) = \mathcal{X}(\overline{\Omega}) = \mathcal{X}(\Sigma(\Omega)) \quad \text{if } n = 1. \tag{47}$$

In other words, there is no restriction on the vector fields $X \in \mathcal{V}(\Omega)$, if Ω has dimension one.

Assume now that the Lie algebra of vector fields $\mathcal{V}(\omega)$ has been defined on $\Sigma(\omega)$ for all curvilinear polyhedral domains ω of dimension $1 \leq k \leq n-1$ and let us define $\mathcal{V}(\Omega)$ for a curvilinear polyhedral domain of dimension n. Let us denote by

$$\mathcal{X}(M) := \Gamma\big(M; TM\big)$$

the space of vector fields on a manifold M. Fix $p \in \partial\Omega$ and let V_p and ϕ_p be as in Definition 6.1, as usual. We identify V_p with $(0,1) \times \omega_p \times B^l$ using ϕ_p. Assume $1 \leq \ell(p) \leq n-2$, so that in particular ω_p is a curvilinear polyhedral domain of dimension ≥ 1. We notice that

$$T([0,1) \times \Sigma(\omega_p) \times B^l) = T([0,1)) \times T\Sigma(\omega_p) \times TB^l$$

and hence

$$\mathcal{X}\big([0,1) \times \Sigma(\omega_p) \times B^l\big) = \mathcal{X}\big([0,1)\big) \times \mathcal{X}\big(\Sigma(\omega_p)\big) \times \mathcal{X}\big(B^l\big).$$

Then we define

$$\mathcal{V}(V_p \cap \Omega) = \{X = (X_1, X_2, X_3) \in \mathcal{X}([0,1)) \times \mathcal{X}(\Sigma(\omega_p)) \times \mathcal{X}(B^l)\} \tag{48}$$

where X_1, X_2, and X_3 are required to satisfy the following two conditions:

$$Y_1 := r_\Omega^{-1} X_1 \ \text{ and } \ Y_3 := r_\Omega^{-1} X_3 \ \text{ are smooth} \tag{49}$$

and

(50) $$X_2(t, x', x'') \in \mathcal{V}\big(\{t\} \times \omega_p \times \{x''\}\big) = \mathcal{V}(\omega_p), \quad \text{for any fixed } t, y.$$

In Condition (49) above, smooth means, smooth *including at* $r = 0$. If $\ell(p) = 0$, then we just drop the component X_3, but keep the same conditions on X_1 and X_2. By Proposition 7.10, the definition of $\mathcal{V}(V_p \cap \Omega)$ is independent of the choice of r_Ω. All vector fields are assumed to be smooth.

Finally, we define $\mathcal{V}(\Omega)$ to consist of the vector fields $X \in \mathcal{X}(\Sigma(\Omega))$ such that $X|_{V_p \cap \Omega} \in \mathcal{V}(V_p \cap \Omega)$ for all $p \in \Omega^{(n-2)}$. In particular, only the smoothness condition is imposed on our vector fields at the smooth points of $\partial\Omega$. Note that the vector fields in $\mathcal{V}(\Omega)$ may not extend to the closure $\overline{\Omega}$, in general. This was seen in Example 3.7.

7.5. **Lie manifolds with boundary.** We now proceed to show that the pair $(\Sigma(\Omega), \mathcal{V}(\Omega))$ defines a Lie manifold with boundary. Lie manifolds with boundary were defined in [1]; their definition will be recalled below.

We first establish some lemmata.

Lemma 7.12. *Let* $X \in \mathcal{X}(\Sigma(\Omega))$ *be such that* $X = 0$ *in a neighborhood of the boundary of* $\Sigma(\Omega)$. *Then* $X \in \mathcal{V}(\Omega)$.

Proof. This follows right away by induction from the definition of $\mathcal{V}(\Omega)$. □

We also get the following simple fact.

Lemma 7.13. *If* $f : \Sigma(\Omega) \to \mathbb{C}$ *is a smooth function and* $X \in \mathcal{V}(\Omega)$, *then* $X(f)$ *is a smooth function on* $\Sigma(\Omega)$ *and* $fX \in \mathcal{V}$.

Proof. The vector field X is smooth on $\Sigma(\Omega)$, hence $X(f)$ is smooth on $\Sigma(\Omega)$. The second statement is local, so it is enough to check it on Ω and on each V_p. This is then immediate using the definition and induction. □

Lemma 7.14. *For any* $X \in \mathcal{V}(\Omega)$ *and any continuous function* $f : \overline{\Omega} \to \mathbb{C}$ *such that* $f \circ \kappa$ *is smooth on* $\Sigma(\Omega)$, *we have*

$$X(f) = f' r_\Omega,$$

where f' *is a smooth function on* $\Sigma(\Omega)$. *In particular,* $X(r_\Omega) = f_X r_\Omega$, *where* f_X *is a smooth function on* $\Sigma(\Omega)$.

Proof. This is a local statement that can be checked by induction in any neighborhood $V_p = (0, \epsilon) \times \omega_p \times \mathbb{R}^l$, using the definition, as follows. Let us use the notation of Equation (48), and the equation immediately after it, and write

$$X = (X_1, 0, 0) + (0, X_2, 0) + (0, 0, X_3).$$

We shall write, by abuse of notation, $X_1 = (X_1, 0, 0)$. Define X_2 and X_3 similarly. It is enough to check that $X_j f(rx', x'')$ is of the indicated form, for $j = 1, 2, 3$. We have $X_1 = r_\Omega Y_1$ and $X_3 = r_\Omega Y_3$, where Y_1 and Y_3 are smooth (in appropriate spaces), by Equation (49). This proves our lemma if $X = X_1$ or $X = X_3$. If $X = X_2$, then we have

(51) $$(Xf)(r, x', x'') = X_2(f(rx', x'')) = r_{\omega_p} f_1(r, x', x''),$$

with f_1 a smooth function on $\Sigma(V_p) = [0, \epsilon) \times \Sigma(\omega_p) \times \mathbb{R}^l$, by the induction hypothesis. Moreover, from the explicit formula for Xf, we see that $Xf(0, x', x'') = 0$, (due to the "r" factor). Therefore $Xf = r r_{\omega_p} f'$, for some smooth function f'

on $\Sigma(V_p)$. Let us denote $r_\alpha = r r_{\omega_p}$, as in Equation (45) and in Proposition 7.9. Proposition 7.9 gives that r_α/r_Ω is smooth on its domain of definition. Hence $Xf = r_\alpha f_1 = r_\Omega(r_\alpha/r_\Omega)f_1 = r_\Omega f'$, with f' smooth on each $\Sigma(V_p)$. Hence f' is smooth on $\Sigma(\Omega)$. □

Let us now check that the statement "the vector field X on Ω is the restriction to Ω of a vector field in $\mathcal{V}(\Omega)$" is a local statement.

Lemma 7.15. *Let Y be a vector field on Ω with the property that every point $p \in \overline{\Omega}$ has a neighborhood U_p in M such that $Y = X_U$ on $U \cap \Omega$, for some $X_U \in \mathcal{V}(\Omega)$. Then there exists $X \in \mathcal{V}(\Omega)$ such that Y is the restriction of X to Ω.*

Proof. Let us cover $\overline{\Omega}$ with a locally finite family of sets U_p, $p \in B \subset \overline{\Omega}$. Let ψ_p, $p \in B$ be a subordinated partition of unity.

We claim that $X = \sum_{p \in B} \psi_p X_{U_p} \in \mathcal{V}(\Omega)$ satisfies $X(x) = Y(x)$, $x \in \Omega$. Indeed, $X(x) = \sum_{p\in B} \psi_p(x) X_{U_p}(x) = \sum \psi_p(x) Y(x)$. □

We can now prove the following lemma.

Lemma 7.16. *Let Y be a smooth vector field on $\overline{\Omega}$. Then $r_\Omega Y$ is the restriction to $\Omega \subset \Sigma(\Omega)$ of a vector field X in $\mathcal{V}(\Omega)$.*

Proof. Let us notice that this is a local statement by Lemma 7.15.

To check this statement on a neighoborhood V_p of some $p \in \partial\Omega$, we shall proceed by induction. We can assume that $V_p = B^{n-l} \times B^l$ and that $V_p \cap \Omega \simeq (0,1)\omega_p \times B^l$. Assume first that $Y = \partial_j$ is a constant vector field on V_p. Let $\alpha_t(x', x'') = (tx', x'')$. Then $D\alpha_t(\partial_j) = t\partial_j$. Therefore,

$$D\alpha_t(X) = X, \tag{52}$$

where $X = r_\Omega \partial_j$, where r_Ω can be taken, on V_p, to be given by $r r_{\omega_p}$. Let us decompose $\partial_j = (Y_1, Y_2, Y_3)$ on V_p using the notation of Equation (48). Then Y_3 is constant. In fact, either $Y_3 = \partial_j$ or $Y_3 = 0$. In any instance, if we write $X = (X_1, X_2, X_3)$, then $X_3 = r_\Omega Y_3$ satisfies the condition of Equation (49). The relation (52) gives that $Y_1(r, x', x'') = a_1(x')\partial_r$ and $Y_2(r, x', x'') = r^{-1}Z(x')$, with a_1 a smooth function and Z a smooth vector field on $\overline{\omega}_p$. Clearly $X_1 = r_\Omega Y_1$ will satisfy the conditions of Equation (49). The induction hypothesis then gives that $X_2(r, x', x'') = r_\Omega Y_2(r, x', x'') = r_{\omega_p}(x')Z(x')$ is the restriction to $\Omega = V_p$ of a smooth vector field in $\mathcal{V}(V_p)$. (This vector field depends only on the second factor in $\Sigma(V_p) = [0,1) \times \omega_p \times \mathbb{R}^l$.) □

We now identify a canonical metric on the vector fields $\mathcal{V}$. Recall that the concept of local basis of a space of vector fields was defined in Definition 5.1.

Proposition 7.17. *Let us fix a metric h on $M \supset \Omega$. Let $q \in \Sigma(\Omega)$ be arbitrary. Then there exists a neighborhood U of q in $\Sigma(\Omega)$ and $X_1, X_2, \ldots, X_n \in \mathcal{V}(\Omega)$ that form a local basis of $\mathcal{V}(\Omega)$ on U and satisfy*

$$h(X_j, X_k) = r_\Omega^2 \delta_{jk}.$$

In other words, the vectors $X_1, X_2, \ldots, X_n$ form an othonormal system on $\Omega \cap U$ for the metric $r_\Omega^{-2} h$. A local basis $X_1, X_2, \ldots, X_n$ with this property will be called *a local orthonormal basis of $\mathcal{V}(\Omega)$ over U.*

Proof. If $q \in \Omega \subset \Sigma(\Omega)$, the result follows from Lemma 7.12. Let $p = \kappa(q)$. We shall hence assume that $p \in \partial\Omega$. This is again a local statement in $p \in \partial\Omega$. We can therefore proceed by induction. If the dimension n of Ω is 1, then there is nothing to prove because $r_\Omega = 1$ in this case.

Let $\phi_p : V_p \to B^{n-l} \times B^l$ and ω_p be as in Definition 6.1. We can assume that ϕ_p is the identity map. If we can prove the result for the function $r = r_\Omega$, then we can prove it for the function $r' = f'r$, where $f', 1/f' \in \mathcal{C}^\infty(\Sigma(\Omega))$, simply by replacing X_j with $f'X_j$. By Proposition 7.9, we can therefore assume that $r_\Omega = rr_{\omega_p}$ on $V_p \cap \Omega$. Let $q = (0, x', x'') \in [0,1) \times \Sigma(\omega_p) \times B^l$.

Let h_0 be the standard metric on V_l. For the induction hypothesis, we shall need that the metric h_0 is given by

$$h_0(r, x', x'') = (dr)^2 + r^2(dx')^2 + (dx'')^2 \tag{53}$$

on $\Omega \cap V_p = (0,1) \times \omega_p \times \mathbb{R}^l$. Here $(dx')^2$ denotes the metric on ω_p induced by the Euclidean metric on the sphere S^{n-l-1}. In other words, if $X = (X_1, X_2, X_3)$ is a vector field on $V_p \cap \Omega$, written using the product decomposition explained above (or in the Equation (48)), then

$$h_0(X) = \|X_1\|^2 + r^2\|X_2\|^2 + \|X_3\|^2$$

where the norms come from the standard metrics on $T[0,1)$, on $TS^{n-l-1} \supset T\omega_p$, and, respectively, $T\mathbb{R}^l$.

Let us assume first that $h = h_0$, the standard metric on $\mathbb{R}^n$. By the induction hypothesis, we can construct $Y_2, \ldots, Y_{n-l} \in \mathcal{V}(\omega_p)$, a local orthonormal basis of $\mathcal{V}$ over some small neighborhood U' of x' in $\Sigma(\omega_p)$ (*i.e.,* $Y_2, \ldots, Y_{n-l} \in \mathcal{V}(\omega_p)$ is orthonormal with respect to the metric $r_{\omega_p}^{-2}(dx')^2$). Here $(dx')^2$ denotes the metric on ω_p induced by the Euclidean metric on the sphere S^{n-l-1}, as above. Let $Y_1 = r_\Omega \partial_r$ and $Y_j = r_\Omega \partial_j$, $j = n-l+1, \ldots, n$, where ∂_j is the constant vectors on $\mathbb{R}^n$ in the direction of the jth coordinate. Then we claim that we can take $U = [0,1) \times U' \times \mathbb{R}^l$ and

$$\{X_1, X_2, \ldots, X_n\} = \{Y_1\} \cup \{Y_2, \ldots, Y_{n-l}\} \cup \{Y_{n-l+1}, \ldots, Y_n\}. \tag{54}$$

(If $n - l = 1$, then the second set in the above union is empty. If $l = 0$, then the third set in the above union is empty.) Indeed, $\{X_1, \ldots, X_n\}$ is a local basis by construction and by the local definition of $\mathcal{V}(\Omega)$ in Equation (48). Let us check that this is an orthonormal local basis. To this end, we shall use the form of the standard metric h_0 given in Equation (53), to obtain

$$h_0(X_1) = r_\Omega^2 \|\partial_r\|^2 = r_\Omega^2\,, \quad h_0(X_{n-l+1}) = \ldots = h_0(X_n) = r_\Omega^2$$
$$\text{and } \; h_0(X_2) = \ldots = h_0(X_{n-l}) = r^2\|X_2\|^2 = r^2 r_{\omega_p}^2 = r_\Omega^2\,.$$

It is also clear that $\{X_1, X_2, \ldots, X_n\}$ is an orthogonal system. This completes the induction step if $h = h_0$, the standard metric on $\mathbb{R}^n$.

If h is not the standard metric on V_l, we can nevertheless chose a matrix valued function T defined on a neighborhood of q in U such that $h(T\xi, T\eta) = h_0(\xi, \eta)$. We then let $X_j = TY_j$ and replace U with this smaller neighborhood. □

This lemma gives the following corollary.

Corollary 7.18. *Let $X, Y \in \mathcal{V}(\Omega)$ and h be a fixed metric on M. Then the function $r_\Omega^{-2} h(X, Y)$, defined first on Ω, extends to a smooth function on $\Sigma(\Omega)$.*

Proof. This is a local statement in the neighborhood of each point $q \in \Sigma(\Omega)$. Let $X_1, X_2, \ldots, X_n$ be a local basis of $\mathcal{V}$ on a neighborhood U of q in $\Sigma(\Omega)$ satisfying the conditions of Proposition 7.17 (*i.e.,* orthogonal with respect to $r_\Omega^{-2}h$). Let $X = \sum \phi_j X_j$ and $Y = \sum \psi_j X_j$ on $U \cap \Omega$, where ϕ_j, ψ_j are smooth functions on $\Sigma(\Omega)$. Then $r_\Omega^{-2}h(X,Y) = \sum \phi_j \overline{\psi}_j$ is smooth on U. □

Lemma 7.19. *Let $p \in \partial\Omega$ and $X_1, X_2, \ldots, X_n$ be vector fields on $\overline{\Omega}$ that define a local basis of TM on $\overline{U}$, for some neighborhood U of p. Then $r_\Omega X_1, r_\Omega X_2, \ldots, r_\Omega X_n$ is a local basis of $\mathcal{V}(\Omega)$ on U.*

Conversely, if a vector field Y on Ω satisfies Condition (55) *for any p and any local basis $X_1, \ldots, X_n$ of TM at p, then Y is the restriction to Ω of a vector field in $\mathcal{V}(\Omega)$.*

In particular, for any $Y \in \mathcal{V}(\Omega)$, there exist smooth function $\phi_1, \phi_2, \ldots, \phi_n$ on $\Sigma(\Omega)$ satisfying

$$Y = \phi_1 r_\Omega X_1 + \phi_2 r_\Omega X_2 + \ldots + \phi_n r_\Omega X_n \quad \text{on } U \cap \Omega \subset \Sigma(\Omega). \tag{55}$$

Proof. The converse part is easier, so let us prove it first. Let Y be a vector field on Ω that satisfies Condition (55) for any p and any local basis $X_1, \ldots, X_n$ of TM at p. Fix an arbitrary $p \in \Omega$. Lemmata 7.13 and 7.16 give that $\phi_j r_\Omega X_j$ is the restriction to Ω of a vector field in $\mathcal{V}(\Omega)$. Hence on each $U \cap \Omega$, Y is the restriction of a vector field $Y_U \in \mathcal{V}(\Omega)$. Lemma 7.15 then gives the converse part of our lemma.

We now prove the direct part of the lemma. We can assume that the vector fields $X_1, \ldots, X_n$ form an orthonormal system on U with respect to some fixed metric h on M. We know from Lemma 7.16 that $r_\Omega X_j \in \mathcal{V}(\Omega)$.

Let then $Y \in \mathcal{V}(\Omega)$ and $\phi_j = r_\Omega^{-1} h(Y, X_j) = r_\Omega^{-2} h(Y, r_\Omega X_j) \in \mathcal{C}^\infty(\Sigma(\Omega))$, by Corollary 7.18. Then $Y = \sum_{j=1}^n \phi_j r_\Omega X_j$ on $U \cap \Omega$. The local uniqueness of the functions ϕ_j follows from the fact that $r_\Omega X_1, r_\Omega X_2, \ldots, r_\Omega X_n$ also form a local basis of $T\Omega$ on $U \cap \Omega$. □

We are now ready to prove the following caracterizations of $\mathcal{V}(\Omega)$. We notice that the restriction map $\mathcal{V}(\Omega) \ni X \to X|_\Omega$ is injective, so we may identify $\mathcal{V}(\Omega)$ with a subspace of the space $\Gamma(\Omega, TM)$ of vector fields on Ω.

Proposition 7.20. *Let $\Omega \subset M$ be a curvilinear polyhedral domain of dimension n and let X be a smooth vector field on Ω. Fix an arbitrary metric h on M. Then $X \in \mathcal{V}(\Omega)$ if, and only if, $r_\Omega^{-1}h(X,Y)$ extends to a smooth function on $\Sigma(\Omega)$ for any smooth vector field Y on $\overline{\Omega}$.*

Proof. In one direction the result follows from Lemma 7.16 and Corollary 7.18. Indeed, let $X \in \mathcal{V}(\Omega)$ and Y be a smooth vector field on $\overline{\Omega}$. Then $r_\Omega Y \in \mathcal{V}(\Omega)$ by Lemma 7.16 and hence $r_\Omega^{-1}h(X,Y) = r_\Omega^{-2}h(X, r_\Omega Y)$ extends to a smooth function on $\Sigma(\Omega)$ by Corollary 7.18. (We have already used this argument in the proof of the previous lemma.)

Conversely, assume that $r_\Omega^{-1}h(X,Y)$ extends to a smooth function on $\Sigma(\Omega)$ for any smooth vector field on $\overline{\Omega}$. The statement that $X \in \mathcal{V}(\Omega)$ is a local statement, by Lemma 7.15. So let $p \in \overline{\Omega}$ and U an arbitrary neighborhood of p. Chose smooth vector fields defined in a neighborhood of $\overline{\Omega}$ in M such that $X_1, X_2, \ldots, X_n$ is a local orthonormal basis on U (orthonormal with respect to h). Let

$$\phi_j = r_\Omega^{-1} h(Y, X_j) \in \mathcal{C}^\infty(\Sigma(\Omega)),$$

by assumption. Then $Y = \sum_{j=1}^{n} \phi_j X_j$ on $U \cap \Omega$ and $\sum_{j=1}^{n} \phi_j X_j \in \mathcal{V}(\Omega)$. Lemma 7.15 then shows that $X \in \mathcal{V}(\Omega)$. □

We now prove the main characterization of the structural Lie algebra of vector fields $\mathcal{V}(\Omega)$.

Theorem 7.21. *Let $\Omega \subset M$ be a bounded curvilinear polyhedral domain of dimension n. Then the vector space $\mathcal{V}(\Omega)$ is generated as a vector space by the vector fields of the form $\phi r_\Omega X$, where $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and X is a smooth vector field on $\overline{\Omega}$.*

Proof. We know that $\phi r_\Omega X \in \mathcal{V}(\Omega)$ whenever X is a smooth vector field on $\overline{\Omega}$, by Lemmata 7.13 and 7.16. This shows that the linear span of vectors of the form $\phi r_\Omega X$, where $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and X is a smooth vector field in a neighborhood of Σ, is contained in $\mathcal{V}(\Omega)$.

Conversely, let $Y \in \mathcal{V}(\Omega)$. Then Lemma 7.19 shows that we can find, in the neighbhorhood U_p of any point $p \in \overline{\Omega}$ vector fields $X_{1p}, X_{2p}, \ldots, X_{np}$ and smooth functions ϕ_{jp} such that $Y = \sum \phi_{jp} r_\Omega X_{jp}$ on U_p. The result then follows using a finite partition of unity on $\Sigma(\Omega)$ subordinated to the covering U_p. □

If we drop the condition that Ω be bounded, we obtain the following result, which was proved in the first half of the above proof.

Proposition 7.22. *Let $\Omega \subset M$ be a curvilinear polyhedral domain of dimension n. Then $\mathcal{V}(\Omega)$ consists of the set of vector fields that locally can be written as linear combinations of vector fields of the form $\phi r_\Omega X$, where $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and X is a smooth vector field on $\overline{\Omega}$.*

Recall from Subsection 1.3 that a hyperface $H \subset \Sigma(\Omega)$ is called a hyperface at infinity if $\kappa(H) \subset \Omega^{(n-2)}$. Let $\partial'\Sigma(\Omega)$ be the union of the interiors H_0 of the hyperfaces of $\Sigma(\Omega)$ that are not at infinity and $\partial''\Sigma(\Omega)$ be the union of the hyperfaces of $\Sigma(\Omega)$ that are at infinity. Then $\partial'\Sigma(\Omega) = \partial\Sigma(\Omega) \smallsetminus \partial''\Sigma(\Omega)$.

Theorem 7.23. *Let Ω be a bounded curvilinear polyhedral domain and*

$$\mathfrak{D}_0 := \Sigma(\Omega) \smallsetminus \partial''\Sigma(\Omega) = \Omega \cup \partial'\Sigma(\Omega) = \kappa^{-1}(\overline{\Omega} \smallsetminus \Omega^{(n-2)}).$$

Then $(\mathfrak{D}_0, \Sigma(\Omega), \mathcal{V}(\Omega))$ is a Lie manifold with boundary $\partial'\Sigma(\Omega)$. We have that $\kappa : \mathfrak{D}_0 \to \overline{\Omega} \smallsetminus \Omega^{(n-2)}$ is such that $\kappa^{-1}(p)$ consists of two points if $p \in \overline{\Omega} \smallsetminus \Omega^{(n-2)}$ is a two-sided smooth boundary point and consists of exactly one point otherwise.

Proof. The last statement (on the number of elements in $\kappa^{-1}(p)$, $p \in \overline{\Omega} \smallsetminus \Omega^{(n-2)}$ follows from the definition. Therefore, to prove the proposition, we need, using the notation of Definition 5.5, to construct a compactification $\mathfrak{D}$ of $\mathfrak{D}_0$ that identifies with the closure of a Lie domain in a Lie manifold $\mathfrak{M}$.

We shall choose then $\mathfrak{D} = \Sigma(\Omega)$. Then we shall let $\mathfrak{M}$ be the "double" of $\Sigma(\Omega)$, also denoted ${}^d\Sigma(\Omega)$. More precisely, $\mathfrak{M}$ is obtained from the disjoint union of two copies of $\Sigma(\Omega)$ by identifying the hyperfaces that are not at infinity. We let $\mathcal{V}$ to be the set of vector fields on $\mathfrak{M}$ such that the restriction to either copy of $\Sigma(\Omega)$ is in $\mathcal{V}(\Omega)$.

Let $\mathbb{D}$ be obtained from the closure of Ω in $\mathfrak{M}$ by removing the closure of $\partial'\Sigma(\Omega)$. Then $\mathbb{D}$ is an open subset of $\mathfrak{M}$ whose closure is $\Sigma(\Omega)$. Moreover, $\partial_{\mathfrak{M}}\mathbb{D}$ (the boundary of $\mathbb{D}$ regarded as a subset of $\mathfrak{M}$) is the closure of $\partial'\Sigma(\Omega)$. To prove our theorem, we shall check that $\mathfrak{M}$ is a manifold with corners, that $(\mathfrak{M}, \mathcal{V})$ is a Lie manifold, and that ∂DD is a regular submanifold of $\mathfrak{M}$. Each of these properties is local, so it can be checked in the neighborhood of a point of $\mathfrak{M}$.

Fix $V_p = (0, \epsilon) \times \omega_p \times B^l$. Then the union of the two copies of $\Sigma(V_p)$ is the double ${}^d\Sigma(V_p)$ of $\Sigma(V_p)$. Denote by ${}^d\omega_p$ the double of ω_p. Then

$$ {}^d\Sigma(V_p) = [0, \epsilon) \times {}^d\omega_p \times B^l. $$

An inductive argument then shows that ${}^d\Sigma(\Omega)$ is a manifold with corners and that $\partial\mathbb{D}$ is a regular submanifold of $\mathfrak{M}$.

Let us check that $\mathcal{V}$ satisfies the conditions defining a Lie manifold structure on $\mathfrak{M}$. It follows from Theorem 7.21 that $\mathcal{V}$ is a $\mathcal{C}^\infty(\mathfrak{M})$–module (this checks condition (iii) of Definition 5.2). Theorem 7.21 and Lemmata 7.14 and 7.13 show that $\mathcal{V}$ is closed under Lie brackets (this checks condition (i) of Definition 5.2). Condition (ii) of that definition follows from the definition of $\mathcal{V}(\Omega)$. Condition (iv) of Definition 5.2 as well as Condition (ii) of Definition 5.3 were proved in Lemma 7.19. This shows that $(\mathfrak{M}, \mathcal{V})$ is a Lie manifold. □

An immediate consequence of the above Proposition is that the boundary $\partial\mathfrak{O}_0 = \partial'\Sigma(\Omega)$ of $\mathfrak{O}_0 = \Sigma(\Omega) \smallsetminus \partial''\Sigma(\Omega)$ will aquire the structure of a Lie manifold, as explained after the definition of a Lie manifold with boundary, Definition 5.5. Let D be the closure of $\partial\mathfrak{O}_0$ in $\mathfrak{O}$. Then the Lie structure at infinity is $(\partial\mathfrak{O}_0, D, \mathcal{W})$, where

$$ \mathcal{W} = \{X|_D,\ X \in \mathcal{V}, X|_D \text{ is tangent to} D\}. \tag{56} $$

As always, $X \in \mathcal{W}$, is completely determined by its restriction to $\mathfrak{O}_0$.

8. End of proofs

We now explain how the general results on Lie manifolds and Lie domains, together with the results of the previous three sections, allows us to complete the proof of our main result.

8.1. **Sobolev spaces and Lie manifolds.** We now identify the weighted Sobolev space $\mathcal{K}^l_a(\Omega)$ with $hH^l(\Sigma(\Omega))$, for a suitable admissible weight h (more precisely, $h = r_\Omega^{a-n/2}$. The following description of $\mathcal{V}(\Omega)$ for Ω a curvilinear polyhedral domain in $\mathbb{R}^n$ will be useful.

Corollary 8.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded curvilinear, polyhedral domain. Then*

$$ \mathcal{V}(\Omega) = \{\phi_1 r_\Omega \partial_1 + \phi_2 r_\Omega \partial_2 + \ldots + \phi_n r_\Omega \partial_n,\ \textit{where } \phi_j \in \mathcal{C}^\infty(\Sigma(\Omega))\}. $$

We shall denote by

$$ \operatorname{Diff}^k_\Omega := \operatorname{Diff}_{\mathcal{V}(\Omega)}(\Sigma(\Omega)) \tag{57} $$

the space of differential operators with smooth coefficients of order $\leq k$ on $\Sigma(\Omega)$ generated by $\mathcal{V}(\Omega)$. The algebra of differential operators $\operatorname{Diff}^\infty_\Omega$ is an example of the algebra of differential operators considered in 5.3. From the last corollary, we obtain right away the following lemma.

Lemma 8.2. *Let $X_1, X_2, \ldots, X_k$ be smooth vector fields on M. Then*

$$ P := r^k_\Omega X_1 X_2 \ldots X_k \in \operatorname{Diff}^k_\Omega, $$

and ϕP, with P as above and $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ generate $\operatorname{Diff}^k_\Omega$ linearly.

Proof. For $k = 1$, this follows from Lemma 7.16. Next, we have

$$r_\Omega^{k+1} X_0 X_1 \dots X_k = r_\Omega X_0 r_\Omega^k X_1 \dots X_k - k X_0(r_\Omega) r_\Omega^k X_1 \dots X_k.$$

The fact that $P \in \mathrm{Diff}_\Omega^k$ then follows by induction, since $X_0(r_\Omega) \in \mathcal{C}^\infty(\Sigma(\Omega))$, by Lemma 7.14.

Conversely, we can similarly check by induction (using the same identity above) that the product $r_\Omega X_1 r_\Omega X_2 \dots r_\Omega X_k$ can be written as a linearly combination of differential operators of the form ϕP, with $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$ and P as above. Since $r_\Omega X$ generate $\mathcal{V}(\Omega)$ as a $\mathcal{C}^\infty(\Sigma(\Omega))$–module (see the second part of Theorem 7.21), the result follows. □

We next provide a different description of the weighted Sobolev spaces $\mathcal{K}_a^l(\Omega)$, $l \in \mathbb{Z}_+$. Denote

$$(r_\Omega \partial)^\alpha := (r_\Omega \partial_1)^{\alpha_1} (r_\Omega \partial_2)^{\alpha_2} \dots (r_\Omega \partial_n)^{\alpha_n}. \tag{58}$$

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded curvilinear polyhedral domain and*

$$\|u\|_{l,a}^2 := \sum_{|\alpha| \le l} \|r_\Omega^{-a} (r_\Omega \partial_1)^{\alpha_1} (r_\Omega \partial_2)^{\alpha_2} \dots (r_\Omega \partial_n)^{\alpha_n} u\|_{L^2(\Omega)}^2.$$

Then $\|u\|_{l,a}$ is equivalent to $\|u\|_{\mathcal{K}_a^l(\Omega)}$ of Definition 1.4. In particular,

$$\mathcal{K}_a^l(\Omega) = \{u, \, \|u\|_{l,a} < \infty\}.$$

Proof. We have that

$$\begin{aligned} u \in \mathcal{K}_a^l(\Omega) \;&\Leftrightarrow \eta_{n-2}^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega) \quad \text{for all } |\alpha| \le l \\ &\Leftrightarrow r_\Omega^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega) \quad \text{for all } |\alpha| \le l \quad \text{ by Proposition 7.9} \\ &\Leftrightarrow r_\Omega^{-a} (r_\Omega \partial)^\alpha u \in L^2(\Omega) \quad \text{for all } |\alpha| \le l \quad \text{ by Proposition 8.2.} \end{aligned}$$

Above the corresponding square integrability conditions define the topology on the indicated spaces. Therefore $\Leftrightarrow$ also means that the topologies are the same. □

Let Ω be a bounded curvilinear polyhedral domain and

$$\mathfrak{O}_0 := \Sigma(\Omega) \smallsetminus \partial'' \Sigma(\Omega) = \Omega \cup \partial' \Sigma(\Omega) = \kappa^{-1}(\overline{\Omega} \smallsetminus \Omega^{(n-2)}),$$

as in Theorem 7.23. Since $(\mathfrak{O}_0, \mathfrak{O} := \Sigma(\Omega), \mathcal{V}(\Omega)$ is a Lie manifold with boundary (Theorem 7.23), the definitions of Sobolev spaces on Lie manifolds (with or without boundary) of Subsection 5.5 provide us with natural spaces $H^s(\Sigma(\Omega))$ and $H^s(\partial' \Sigma(\Omega)) = H^s(\partial \mathfrak{O}_0)$. The last equality is due to the fact that the boundary of the manifold with boundary $\mathfrak{O}_0$ is $\partial' \Sigma(\Omega)$.

Proposition 8.4. *Let Ω be a bounded curvilinear polyhedral domain and let h be an admissible weight on Ω. We have an equality*

$$h \mathcal{K}_a^s(\Omega) = h r_\Omega^{a-n/2} H^s(\Sigma(\Omega)).$$

If Ω is a straight polyhedral domain and $\partial \Omega = \partial \overline{\Omega}$, then

$$h \mathcal{K}_a^s(\partial \Omega) = h r_\Omega^{a-(n-1)/2} H^s(\partial' \Sigma(\Omega)) = h r_\Omega^{a-(n-1)/2} H^s(\partial \mathfrak{O}_0).$$

Proof. This is again a local statement. We can therefore assume that $\Omega \subset \mathbb{R}^n$. It is enough to prove our statement in the case $h = 1$ and $s \in \mathbb{Z}_+$. (The other cases follow by duality and interpolation.) Equation (14) and Proposition 7.9 show that we can also assume $a = 0$. Recall from Lemma 5.7 that the spaces $H^k(\Sigma(\Sigma(\Omega)))$ are defined using $L^2(\Sigma(\Omega))$. In turn, $L^2(\Sigma(\Omega))$ is defined using the volume element of a compatible metric. A typical compatible metric is $r_\Omega^{-2}h$, where h is the Euclidean metric. Therefore the volume element on $\Sigma(\Omega)$ is $r_\Omega^{-n}dx$, where dx is the Euclidean volume element. In particular, $v \in L^2(\Omega) \Leftrightarrow v \in r_\Omega^{-n/2}L^2(\Sigma(\Omega))$.

Let us also notice next that $r_\Omega^{-t}(r_\Omega\partial)^\alpha r_\Omega^t - (r_\Omega\partial)^\alpha$ is a linear combination with $\mathcal{C}^\infty(\Sigma(\Omega))$–coefficients of monomials $(r_\Omega\partial)^\beta$, with $|\beta| < |\alpha|$, by the second part of Lemma 7.14.

From this we obtain

$$\begin{aligned} u \in \mathcal{K}_0^l(\Omega) \ &\Leftrightarrow (r_\Omega\partial)^\alpha u \in L^2(\Omega) \quad \text{for all } |\alpha| \le l \quad \text{by Theorem 8.3} \\ &\Leftrightarrow (r_\Omega\partial)^\alpha u \in r_\Omega^{-n/2}L^2(\Sigma(\Omega)) \quad \text{for all } |\alpha| \le l \\ &\Leftrightarrow (r_\Omega\partial)^\alpha r_\Omega^{n/2} u \in L^2(\Sigma(\Omega)) \quad \text{for all } |\alpha| \le l \Leftrightarrow u \in r_\Omega^{-n/2}H^l(\Sigma(\Omega)). \end{aligned}$$

For the second part we shall use the notation of Definition 1.2. The condition $\partial\Omega = \partial\overline{\Omega}$ guarantees that there are no two-sided smooth boundary points, and hence $\partial\mathfrak{D}_0 = \cup D_j$, disjoint union, since $\kappa : \mathfrak{D}_0 \to \overline{\Omega} \smallsetminus \Omega^{(n-2)}$ is one to one. The proof of $h\mathcal{K}_a^s(\partial\Omega) = hr_\Omega^{a-(n-1)/2}H^s(\partial'\Sigma(\Omega)) = H^s(\partial\mathfrak{D}_0)$ is then completely similar. First, we observe that we can assume $h = 1$ and $a = 0$, as above. Then we use the description in Equation (56) of the Lie structure at infinity on the boundary. In particular, Theorem 7.21 shows that the Lie structure at infinity on $\partial\mathfrak{D}_0$ is linearly generated by the vector fields X on $\partial\mathfrak{D}_0 = \cup D_j$ that vanish on all D_j, except maybe one, and $X = \phi r_\Omega \partial_k$, where $\phi \in \mathcal{C}^\infty(\partial\mathfrak{D})$ and ∂_k are the constant vector fields along the subspace containing D_j, the face on which X is not zero. □

8.2. **Proof of Theorem 2.4.** First we define the Sobolev spaces on the boundary in general.

Definition 8.5. Let Ω be a bounded, curvilinear polyhedral domain. Then we define

$$h\mathcal{K}_a^s(\partial\Omega) := hr_\Omega^{a-(n-1)/2}H^s(\partial'\Sigma(\Omega)),$$

for any admissible weight h.

We are ready now to prove Theorem 2.4.

Proof. The map $H^s(\Sigma(\Omega)) \to H^{s-1/2}(\partial'\Sigma(\Omega))$ is well defined, continuous, and surjective by Proposition 5.8. Proposition 8.4 then shows that the map

$$h\mathcal{K}_a^s(\Omega) = hr_\Omega^{a-n/2}H^s(\Omega) \to hr_\Omega^{a-n/2}H^{s-1/2}(\partial\Omega) = h\mathcal{K}_{a-1/2}^{s-1/2}(\Omega)$$

is also well defined, continuous, and surjective.

The density of $\mathcal{C}_c^\infty(\Omega)$ in the subspace of elements in $h\mathcal{K}_a^1(\Omega)$ with trace zero also follows from Proposition 5.8 and Proposition 8.4. □

8.3. **Proof of Proposition 2.5.** We also have the following corollary, which implies right away Proposition 2.5 of Section 2. This corollary can be proved directly in dimension 2 and 3, using polar, cylindrical, or spherical coordinates. Recall that the algebra $\mathrm{Diff}_\Omega^\infty$ is the natural algebra of differential operators on Ω associated to the Lie algebra of vector fields $\mathcal{V}(\Omega)$, namely, it is generated as an algebra by $X \in \mathcal{V}(\Omega)$ and $\phi \in \mathcal{C}^\infty(\Sigma(\Omega))$. (This algebra was used also in Equation (57) and in Subsection 5.3.)

For the following result, let us recall that if $\Omega \subset \mathbb{R}^n$, the principal symbol of $(r_\Omega \partial)^\alpha$ is ξ^α. This follows from the definition of the principal symbol in [2, 1] and from Corollary 8.1. (The reader can just take this as the definition of the principal symbol of a differential operator in $\mathrm{Diff}_\Omega^\infty$.)

Corollary 8.6. *Let P be a differential operator of order m on M with smooth coefficients. Then*

(i) $r_\Omega^m P \in \mathrm{Diff}_\Omega^m$*;*
(ii) P is (strongly) elliptic if, and only if, r_Ω^m is (stronly) elliptic in Diff_Ω^m*;*
(iii) $h^\lambda P h^{-\lambda}$ depends continuously on λ;
(iv) P maps $h\mathcal{K}_a^s(\Omega) \to h\mathcal{K}_{a-m}^{s-m}(\Omega)$ continuously;

Proof. The relation $r_\Omega^m P \in \mathrm{Diff}_\Omega^m$ was proved as part of Lemma 8.2. Strong ellipticity is a local property, so we can assume $\Omega \subset \mathbb{R}^n$. The proof of Lemma 8.2 shows that P and $r_\Omega^m P$ have the same principal symbol, and they are at the same time elliptic or strongly elliptic.

For any $X \in \mathcal{V}$ and any defining function x of some hyperface at infinity of $\Sigma(\Omega)$, we have that $x^\lambda X x^{-\lambda} = X - \lambda x^{-1} X(x)$. Since $x^{-1}X(x)$ is a smooth function (this follows from the fact that X is tangent to the face defined by x), we see that $x^\lambda X x^{-\lambda} \in \mathrm{Diff}_\Omega^1$ and depends continuously on λ. This proves (iii). It also shows, in particular, that Diff_Ω^k is conjugation invariant with respect to defining functions of hyperfaces at infinity (Equation (34)).

We can therefore assume that $h = 1$. Since $(\Sigma(\Omega), \mathcal{V}(\Omega))$ is a Lie manifold with boundary (Theorem 7.23) any $Q \in \mathrm{Diff}_\Omega^k$ maps $H^s(\Sigma(\Omega)) \to H^{s-k}(\Sigma(\Omega))$ continuously. (This simple property, proved in [1], follows right away from the definitions.) The continuity of $P : \mathcal{K}_a^s(\Omega) \to \mathcal{K}_{a-m}^{s-m}(\Omega)$ then follows using also the fact that multiplication defines an isometry $r_\Omega^a : \mathcal{K}_a^{s-m}(\Omega) \to \mathcal{K}_{a-m}^{s-m}(\Omega)$. ☐

8.4. **Proof of Theorem 2.3.** We now show how the proof of Theorem 2.3 can be obtained from the results of [1] and the theory developped in Section 7. The following result was proved in [1].

Theorem 8.7. *Let $(\mathfrak{M}, \mathcal{V})$ be a Lie manifold with boundary and $P_0 \in \mathrm{Diff}^m(\mathfrak{M})$ be a second order, strongly elliptic operator. Let h be an admissible weight and $u \in hH^1(\mathfrak{M})$ be such that $Pu \in hH^{s-1}(\mathfrak{M})$ and $u|_{\partial\mathfrak{M}} \in hH^{s+1/2}(\partial\mathfrak{M})$, $s \in \mathbb{R}_+$. Then $u \in hH^{s+1}(\mathfrak{M})$ and*

$$\|u\|_{hH^{s+1}(\mathfrak{M})} \leq C(\|P_0 u\|_{hH^{s-1}(\mathfrak{M})} + \|u\|_{hH^0(\mathfrak{M})} + \|u|_{\partial\Omega}\|_{hH^{s+1/2}(\partial\mathfrak{M})}). \tag{59}$$

The same result holds for systems.

Theorem 2.3 then follows by applying the above theorem to $P_0 := r_\Omega^2 P$, which is strongly elliptic by Corollary 8.6(ii). The exact statement is obtained by using also the identifications of Proposition 8.4 and Definition 8.5.

9. Extensions

It seems that it would be more natural to work in the framework of stratified spaces than in the framework of polyhedral domains. For example, if we consider a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ and a submanifold $X \subset \partial\Omega$ of lower dimension, then we can consider $\eta_{n-2}(x)$ to be the distance from x to X. Then Theorem 1.5 remains true, with essentially the same proof, by taking $\Omega^{(n-2)} := X$ in this case.

The method of layer potentials seems also to give more precise results, but is less elementary [22, 34, 35].

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E-mail address: `bacuta@math.psu.edu`

E-mail address: `nistor@math.psu.edu`

E-mail address: `ltz@math.psu.edu`

Pennsylvania State University, Math. Dept., University Park, PA 16802