ON THE FOUNDATIONS OF VISION MODELING V. NONCOMMUTATIVE MONOIDS OF OCCLUSIVE PREIMAGES

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On the Foundations of Vision Modeling V. Noncommutative Monoids of Occlusive Preimages

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Abstract

A significant cue for visual perception is the occlusion pattern in 2-D images projected onto biological or digital retinas, which allows humans or robots to successfully sense and navigate the 3-D environments. There have been many works on modeling and studying the role of occlusion in image analysis and visual perception, mostly from analytical or statistical points of view. The current paper presents a new theory of occlusion based on simple topological definitions of *preimages* and a *binary operation* on them called "occlu." We study many topological as well as algebraic structures of the resultant preimage monoids (a monoid is a semigroup with identity). The current paper is intended to foster the connection between mathematical ways of abstract thinking and realistic modeling of human and computer vision.

Key words: Vision, depth, occlusion, preimages, segmentation, noncommutative, monoids, topology, invariants, diffeomorphism, knot theory

To my beloved mentor Gian-Carlo Rota for his 5th year above in the Heaven.

1 Introduction

In human and computer vision, the occlusion phenomenon plays a key role in successful retrieval of 3-D structural information from 2-D images projected

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onto the retinas. Its importance has been repeatedly emphasized by many giants in vision sciences, including David Marr in computer vision and artificial intelligence [1], Gaetano Kanizsa in Gestalt and cognitive vision [2], and David Mumford in mathematical and statistical modeling of visual perception [3–5].

The left panel of Figure 1 shows a popular image in vision research, for which the lack of occlusion cues causes a perceptual illusion, i.e., two distinct 3-D scenes can be interpreted from the same image. The right panel on the other hand shows the role of occlusion cues in the visual perception of 3-D knots as in knot theory [6].

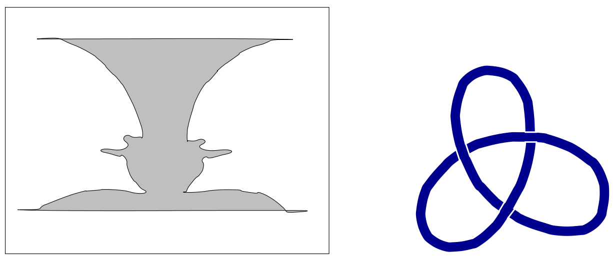

Fig. 1. Role of occlusion in visual perception. Left panel: a gray vase against a bright background or two opposite human faces against a gray background? This is a typical illusion problem caused by the lack of occlusion cues; Right panel (a trefoil): the role of occlusion in knot theory [6].

The occlusion phenomenon has motivated numerous research works in mathematical image and vision analysis, especially by David Mumford, together with his students and colleagues. In the monograph [5], Nitzberg, Mumford, and Shiota invented variational models for the disocclusion task as well as boundary interpolation, which has also inspired the recent wave of interest in image inpainting and geometric interpolation [7–10]. In [3], Lee, Mumford, and Huang employed the occlusion mechanism to simulate image formation of natural scenes, and discovered the remarkable agreement, in terms of both qualitative and quantitative properties (e.g., scale invariance), between such simulated images and complex natural ones. Occlusion has also led to the logical faith in the literature that natural images are non-stationary random fields or non-Sobolev functions globally. As a result, mixture image models have been proposed to handle such images in the celebrated works of Geman and Geman [11], and Mumford and Shah [4], especially for the tasks of image restoration and segmentation.

The present work attempts to take a different approach toward occlusion. Instead of the aforementioned analytical, functional, or stochastic methods, we propose a simple occlusion model based on point-set topology, and study both its *topological* and *algebraic* properties. **Our major discovery is**: with proper topological definitions of preliminary images (or *preimages*, sketched images before the applications of colors or shades), as well as a binary operation called "occlu" on them (denoted by $A \dashv B$), the set of all preimages constitute into a noncommutative monoid, i.e., a noncommutative semigroup with an identity.

In this paper, we gradually develop this new topological as well as algebraic theory of occlusion, by properly defining concepts as inspired by human and computer vision, and revealing many basic properties via rigorous mathematical formulations and proofs.

In spirit, the current work has more or less been inspired by knot theory [6], where algebraic structures (such as braid groups and Jones' polynomials) allow to study knots algebraically, instead of merely visually watching and drawing them.

The organization goes as follows. Section 2 briefly reviews the notions of groups, semigroups, and monoids, mainly intended for potential readers from the community of computer vision or artificial intelligence. In Section 3, we introduce the topological definitions of preimages and the occlusion operation "$\dashv$," and establish the important fact that the occlusion operation is associative, which naturally leads to the monoid structure of preimages. In Section 4, we discuss generators for preimage monoids, and particularly focus on minimal generators as well as prime (or atomic) preimages. A prime preimage is analogous to a prime number for the commutative monoid of natural numbers under multiplication. In Section 5, we model and study one of the most important notions in human and machine vision - *depth*, in the context of preimage monoids. Section 6 concerns the equivalence of preimages under topological transforms such as homeomorphisms and diffeomorphisms, and their invariants. Section 7 closes the presentation with a brief conclusion.

2 Groups, Semigroups, and Monoids

We first briefly review these three concepts in abstract algebra, mainly for potential readers from the community of vision sciences.

A nonempty set G with a binary operation

$$* : (g, f) \to g * f \in G, \qquad \forall g, f \in G,$$

is said to be *semigroup*, if the binary operation is *associative*:

$$g * (f * h) = (g * f) * h, \qquad \forall g, f, h \in G.$$

A semigroup $(G, *)$ is said to be a *monoid* if it contains an identity element $e \in G$:

$$e * g = g * e = g, \qquad \forall g \in G.$$

Notice that if an identity element exists it has to be unique since

$$e' = e * e' = e, \qquad \text{if } e \text{ and } e' \text{ are both identities.}$$

A monoid $(G, *, e)$ is said to be a *group*, if for each element $g \in G$, there exists an element, denoted by g^{-1}, such that

$$g * g^{-1} = g^{-1} * g = e.$$

Similarly by the association axiom, g^{-1} has to be unique and thus will be called the *inverse* element of g.

The center $C(G)$ of a semigroup $(G, *)$ is defined by

$$C(G) = \{g \in G \mid g * f = f * g, \quad \forall f \in G\}.$$

That is, the center consists of all elements that communicate with any other element in G. For a monoid $(G, *, e)$, it is trivial to see that $e \in C(G)$.

A semigroup is said to be *commutative* if $C(G) = G$. A commutative group is often said to be *Abelian*.

For example, the integer set $\mathbb{Z}$ is an Abelian group under the ordinary summation operation, while is only a commutative monoid under the ordinary multiplication operation.

In the next section, we shall present an interesting monoid structure underlying many important works of computer vision [2,3,1,4,5].

3 Monoids of Occlusive Preimages

Definition 1 (Preimages) *A preliminary image (in $\mathbb{R}^2$), abbreviated to "a preimage," is a pair (a, γ) that satisfies*

[1] a and γ are both closed *subsets of* $\mathbb{R}^2$ *(but unnecessarily being compact);*
[2] $\partial a \subseteq \gamma \subseteq a$; and
[3] γ is σ-finite under 1-dimensional Hausdorff measure $\mathcal{H}^1$.

Here by the σ-finiteness, we mean that for any compact subset K of $\mathbb{R}^2$,

$$\mathcal{H}^1(\gamma \cap K) < \infty.$$

It requires γ not to be too complex locally, as inspired by the celebrated variational segmentation model of Mumford and Shah [4].

For convenience, sometimes we shall use the notation a_γ, b_δ for preimages $(a, \gamma), (b, \delta)$, or capital letters A, B if the topological details are less concerned than their algebraic relations.

Definition 2 (Support and Pattern) *For a preimage (a, γ), a is called its* support, *and γ its* pattern.

We now define a binary operation called "occlu," which is denoted by $\dashv$ and pronounced as in the English words "occlusion" or "occlude."

Definition 3 ((a, γ) occlu (b, δ)) *Let $a_\gamma = (a, \gamma)$ and $b_\delta = (b, \delta)$ be any two preimages. Then a_γ occlu b_δ is defined as*

$$a_\gamma \dashv b_\delta = (a \cup b, \gamma \cup (\delta \setminus a^\circ)) := (c, \eta) = c_\eta, \tag{1}$$

where a° denotes the topological interior of the set a. We shall call $a_\gamma \dashv b_\delta$ the occlusion *of a_γ* to *b_δ.*

Theorem 1 *The occlusion of two preimages is still a preimage.*

Proof. Recall in topology [12], the union and joint of any two closed sets are still closed. Thus c and η in (1) are both closed, since

$$\delta \setminus a^\circ = \delta \cap (a^\circ)^c,$$

where the superscript c stands for the complement operation. Thus Condition [1] is satisfied in the definition of preimages.

One also has in general topology, for any two sets a and b,

$$\partial(a \cup b) \subseteq \partial a \cup \partial b.$$

Furthermore, since $a^\circ \subseteq (a \cup b)^\circ$, and $\partial a \cap a^\circ = \phi$,

$$\partial(a \cup b) \subseteq (\partial a \cup \partial b) \setminus a^\circ = \partial a \cup (\partial b \setminus a^\circ).$$

Therefore,

$$\partial c \subseteq \gamma \cup (\delta \setminus a^\circ)(= \eta) \subseteq a \cup b = c.$$

This verifies Condition [2] in the definition of preimages.

Finally by the subadditivity of general measures, for any compact set K,

$$\mathcal{H}^1(\eta \cap K) \leq \mathcal{H}^1((\gamma \cup \delta) \cap K) \leq \mathcal{H}^1(\gamma \cap K) + \mathcal{H}^1(\delta \cap K) < \infty,$$

which verifies Condition [3]. $\square$

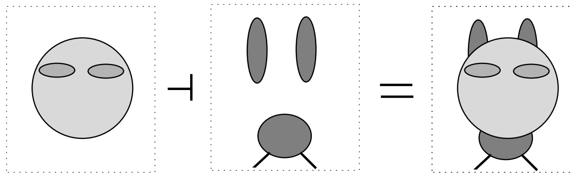

Fig. 2. An example of two preimages a_γ and b_δ and their occlusion $a_\gamma \dashv b_\delta$. Complex image patterns in the real world often originate from simple objects via occlusion.

Theorem 2 *The occlusion operation is associative, i.e., for any three preimages* $a_\gamma = (a, \gamma), b_\delta = (b, \delta),$ *and* $c_\eta = (c, \eta)$,

$$a_\gamma \dashv (b_\delta \dashv c_\eta) = (a_\gamma \dashv b_\delta) \dashv c_\eta.$$

Proof. By definition,

$$a_\gamma \dashv (b_\delta \dashv c_\eta) = (a \cup b \cup c,\ \gamma \cup (\delta \cup \eta \setminus b^\circ) \setminus a^\circ) \tag{2}$$
$$(a_\gamma \dashv b_\delta) \dashv c_\eta = (a \cup b \cup c,\ \gamma \cup (\delta \setminus a^\circ) \cup \eta \setminus (a \cup b)^\circ). \tag{3}$$

(To avoid too many parentheses, the set subtraction operation "$\setminus$" is assumed to have first priority in computation compared with other set operations such as union $\cup$ and joint $\cap$.) It then suffices to show that the two patterns are identical. For the first pattern, one has

$$\gamma \cup (\delta \cup \eta \setminus b^\circ) \setminus a^\circ = \gamma \cup (\delta \setminus a^\circ) \cup (\eta \setminus b^\circ) \setminus a^\circ, \tag{4}$$

where the last group in the unions

$$(\eta \setminus b^\circ) \setminus a^\circ = \eta \setminus (b^\circ \cup a^\circ) \supseteq \eta \setminus (a \cup b)^\circ, \tag{5}$$

since $a^\circ \cup b^\circ \subseteq (a \cup b)^\circ$. This completes half of the proof, i.e., the pattern of the right hand side contains that of the left in the theorem.

For the other half, let us investigate how much on earth could be possibly left out in the inclusion relation (5). Applying the set identity

$$(X \setminus A) \setminus (X \setminus B) = (X \cap B) \setminus A,$$

one has

$$(\eta \setminus (b^\circ \cup a^\circ)) \setminus (\eta \setminus (a \cup b)^\circ) = (\eta \cap (a \cup b)^\circ) \setminus (a^\circ \cup b^\circ) \subseteq (a \cup b)^\circ \setminus (a^\circ \cup b^\circ).$$

For this last set,

$$\begin{aligned}(a \cup b)^\circ \setminus (a^\circ \cup b^\circ) &\subseteq \overline{a \cup b} \setminus (a^\circ \cup b^\circ) = (\bar{a} \cup \bar{b}) \setminus (a^\circ \cup b^\circ) \\ &\subseteq (\partial a \cup \partial b) \setminus (a^\circ \cup b^\circ) \subseteq (\partial a \cup \partial b) \setminus a^\circ \\ &= \partial a \cup (\partial b) \setminus a^\circ \subseteq \gamma \cup (\delta \setminus a^\circ).\end{aligned}$$

Thus what could be potentially missing in the inclusion relation (5), is already contained in the first two commonly shared terms in (3) and (4), which completes the proof. ☐

Lemma 1 *Two preimages a_γ and b_δ satisfies*

$$a_\gamma \dashv b_\delta = a_\gamma,$$

if and only if $b \subseteq a$.

Proof. If the occlusion identity holds, then in terms of the supports, $a \cup b = a$, which immediately implies $b \subseteq a$.

Now suppose $b \subseteq a$. Then in terms of the patterns,

$$\gamma \subseteq \gamma \cup (\delta \setminus a^\circ) \subseteq \gamma \cup (b \setminus a^\circ) \subseteq \gamma \cup (a \setminus a^\circ) = \gamma \cup \partial a = \gamma.$$

This establishes the occlusion identity. ☐

Lemma 2 *A universal right identity element c_η, i.e., for any preimage a_η in $\mathbb{R}^2$,*

$$a_\gamma \dashv c_\eta = a_\gamma,$$

has to be the empty preimage (ϕ, ϕ), or simply denoted by ϕ.

Proof. By the preceding lemma, a right identity element $c_\eta = (c, \eta)$ has to satisfy

$$c \subseteq a, \quad \forall \text{ closed set } a \text{ in } \mathbb{R}^2.$$

In particular, $a = \phi$ implies $c = \phi$. Then $\eta = \phi$ since $\eta \subseteq c = \phi$. ☐

It is also easy to verify that the empty preimage $\phi = (\phi, \phi)$ is a left identity as well. Therefore, it is a genuine (double-sided) identity element.

To conclude, we have established the following theorem.

Theorem 3 (Monoids of Occlusive Preimages) *Any set $\mathcal{I}$ of preimages is a semigroup if it is closed under the occlusion operation, i.e., for any $a_\gamma, b_\delta \in \mathcal{I}$, $a_\gamma \dashv b_\delta \in \mathcal{I}$. In addition, if $\mathcal{I}$ contains the empty preimage $\phi = (\phi, \phi)$, it is a monoid.*

In particular, the set of all preimages on $\mathbb{R}^2$ forms a monoid, which is apparently the largest preimage monoid, and will be called the *universal monoid* of preimages.

It is easy to see that generally a preimage monoid is non-commutative, which is mainly due to the pattern part since the support part is commutative (i.e., $a \cup b = b \cup a$). For instance, for any two preimages a_γ and b_δ with $b \subseteq a$,

$$a_\gamma \dashv b_\delta = (a, \gamma), \text{ and } b_\delta \dashv a_\gamma = (a, \delta \cup \gamma \setminus b^\circ).$$

Thus as long as $\delta \subsetneq \gamma$, a_γ and b_δ are not commutative.

It is also trivial to show that for the universal monoid of preimages, the center C only consists of the identity element ϕ.

We know discuss an identity that most explicitly reveals the algebraic structure of preimage monoids.

Theorem 4 (Fundamental Identity of Occlusion) *For any two preimages A and B, one has:*

$$A \dashv B \dashv A = A \dashv B. \tag{6}$$

Proof. Suppose $A = (a, \gamma)$ and $B = (b, \delta)$. Then by definition,

$$A \dashv B = (a \cup b, \gamma \cup \delta \setminus a^\circ).$$

Since $a \subseteq a \cup b$, by Lemma 1,

$$(A \dashv B) \dashv A = A \dashv B.$$

□

Unlike the identity in Lemma 1 which involves the topological constraint $b \subseteq a$, the fundamental identity can be understood as a purely algebraic identity. Thus it could play a role similar to, for example,

$$Q^T Q = I_n, \quad \text{for the } n \times n \text{ rotaion group;} \quad \text{or,}$$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \quad \text{for a Lie algebra.}$$

In particular we have the following.

Corollary 1 *The following identities hold for any preimages $A, B, \cdots$:*

(i) ***(Projection)*** $A \dashv A = A$.

(ii) ***(Non-Repetition)***

$$A \dashv \cdots \dashv B \dashv \cdots \dashv C \dashv B = A \dashv \cdots \dashv B \dashv \cdots \dashv C.$$

Recall in linear algebra that any square matrix P satisfying $P^2 = P$ is called a projection, which has inspired the name of the first identity. On the other hand, from vision point of view, occlusion is indeed a genuine projection, i.e., objects on the foreground are projected onto those on the background.

4 Generators: Minimal, Prime (or Atomic), and Non-Hiding

Definition 4 (Generators) *Let $\mathcal{I}$ be a preimage monoid. A subset $S \subseteq \mathcal{I}$ is said to be a generator of the monoid if for any $a_\gamma \in \mathcal{I}$, there exists a* finite *sequence of preimages*

$$b_\delta, \cdots, c_\eta \in S,$$

such that

$$a_\gamma = b_\delta \dashv \cdots \dashv c_\eta.$$

If so, $\mathcal{I}$ is said to be generated from S.

Theorem 5 *Given any set S of preimages, there exists a* unique *preimage monoid $\mathcal{I}_* = \mathcal{I}_*(S)$ that is generated from S, and is* minimal *in the sense that*

$$\mathcal{I}_*(S) \subseteq \mathcal{I}, \qquad \textit{for any preimage monoid } \mathcal{I} \textit{ that contains } S.$$

Proof. Uniqueness is trivial since the minimality condition requires that any two minimal monoids contain each other.

Existence is guaranteed by the following construction, which is standard in abstract algebra. Define

$$\mathcal{I}_* = \{(b_1, \gamma_1) \dashv \cdots \dashv (b_n, \gamma_n) \mid n = 0, 1, \cdots, \ (b_k, \gamma_k) \in S, k = 1 : n\},$$

with the agreement that $n = 0$ corresponds to the identity element $\phi = (\phi, \phi)$. It is then easy to verify that $\mathcal{I}_*$ is closed under the occlusion operation and hence is indeed a preimage monoid. Its minimality is also crystal clear since any of its elements belongs to any monoid that contains S. □

Theorem 6 (Size Bound of Preimage Monoids) *Suppose S is a finite set of preimages with $\#S = n$ elements. Then the preimage monoid $\mathcal{I}_*(S)$ generated by S is a finite set as well, and*

$$\#\mathcal{I}_*(S) \leq \sum_{k=0}^{n} (n)_k,$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ with $(n)_0 = 1$.

Proof. According to the properties of projection and non-repetition in Corollary 1, and the construction in the preceding theorem, any element $Z \in \mathcal{I}_*$ can be written in the form of

$$Z = A \dashv B \dashv \cdots \dashv C, \tag{7}$$

with preimages $A, B, \cdots C$ belonging to S and being *distinct*.

For each integer k, there are $\binom{n}{k}$ different ways of selecting k preimages from S, followed by $k! = k(k-1)\cdots 1$ permutations for each selection to assemble a general preimage in $\mathcal{I}_*(S)$ by (7). Therefore

$$\#\mathcal{I}_* \leq \sum_{k=0}^{n} \binom{n}{k} k! = \sum_{k=0}^{n} (n)_k.$$

□

We now define some interesting classes of preimage monoids based on their generators.

[a] (Diskette Preimage Monoid) Define the generator set

$$S_d = \{(B_r(x), \partial B_r(x)) \mid r \geq 0, x = (x_1, x_2) \in \mathbb{R}^2\},$$

consisting of all closed disks on the plane. Then $\mathcal{I}_{\text{disk}} = \mathcal{I}_*(S_d)$ shall be called the *diskette preimage monoid.*

[b] (Planar Preimage Monoid) Define the generator set

$$S_p = \{(P, \partial P) \mid P = \{y \in \mathbb{R}^2 \mid n \cdot (y - x) \geq 0\}, n \in S^1, x \in \mathbb{R}^2\},$$

consisting of all half-planes. Then $\mathcal{I}_{\text{plane}} = \mathcal{I}_*(S_p)$ shall be called the *planar preimage monoid.* (Here S^1 denotes the unit circle.)

[c] (Convex Preimage Monoid) Define the generator set

$$S_c = \{(E, \partial E) \mid E \subseteq \mathbb{R}^2 \text{ is convex}\}.$$

Then $\mathcal{I}_{\text{conv}} = \mathcal{I}_*(S_c)$ shall be called the *convex preimage monoid.* Both the diskette and planar monoids are submonoids of the convex monoid.

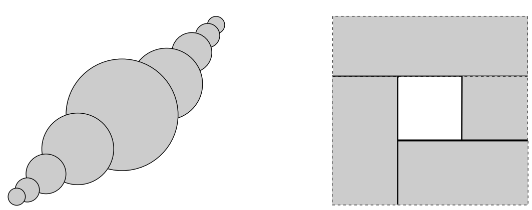

Fig. 3. Left: a diskette preimage; Right: a planar preimage.

[d] (Smooth Preimage Monoid) Define the generator set

$$S_s = \{(M, \partial M) \mid M \text{ is a connected } C^{(2)} \text{ submanifold with borders}\}. \quad (8)$$

That is, as in differential topology [13], for each boundary point $x = (x_1, x_2) \in \partial M$, there is an open neighborhood U_x and a C^2 invertible map ϕ that maps U_x onto $\mathbb{R}^2$, so that

$$\phi(U_x \cap M) = \{(x_1, x_2) \mid x_2 \geq 0\}, \quad \text{i.e., the upper half-plane.}$$

Then we shall call $\mathcal{I}_{\text{smooth}} = \mathcal{I}_*(S_s)$ the *smooth preimage monoid*. The C^2 regularity condition has been imposed so that curvatures along the boundaries are well defined ($C^{1,1}$ also suffices as in Mumford and Shah [4])
.

Notice that although the above generator sets look quite regular, they can already generate very complex preimages and patterns. The underlying motivation and philosophy are therefore clearly rooted in the fundamental fact of physics: the complex physical world arises from its simple generators of basic particles such as quarks and protons.

Definition 5 (Minimal Generating Sets) *Suppose S generates a preimage monoid $\mathcal{I}$. Then S is said to be* minimal *if*

$$A \notin \mathcal{I}_*(S \setminus \{A\}), \qquad \forall A \in S.$$

That is, no proper subset of S alone can re-generate the entire monoid $\mathcal{I}$.

Definition 6 (Prime or Atomic Preimages) *An element a_γ in a preimage monoid $\mathcal{I}$ is said to be prime or atomic, if*

$$a_\gamma \notin \mathcal{I}_*(\mathcal{I} \setminus \{a_\gamma\}).$$

Let $\mathcal{A}(\mathcal{I})$ denotes the collection of all prime or atomic preimages, and will be called the atomic set of $\mathcal{I}$.

Notice that if $\mathcal{I}$ is compared to the (commutative) monoid of natural numbers under multiplication, a prime or atomic preimage is then analogous to a prime number. The following proposition is evident from the above definitions.

Proposition 1 *For a given preimage monoid, it is always true that $\mathcal{A}(\mathcal{I}) \subseteq S(\mathcal{I})$ for any minimal generating set $S(\mathcal{I})$.*

Definition 7 (Regular Preimages) *A preimage (a, γ) is said to be* regular *if its 2-D Hausdorff measure (identical to the 2-D Lebesgue measure in $\mathbb{R}^2$) $\mathcal{H}^2(a)$ is nonzero, and $\partial a = \partial(a^\circ)$. Otherwise it is said to be* degenerate.

Theorem 7 *A preimage (a, γ) is regular if and only if $a \neq \phi$ and $a = \overline{a^\circ}$.*

Proof. First suppose (a, γ) is regular. Then $a \neq \phi$ since $\mathcal{H}^2(a) > 0$, and

$$a = \partial a \cup a^\circ = \partial a^\circ \cup a^\circ = \overline{a^\circ}.$$

Conversely, suppose $a \neq \phi$ and $a = \overline{a^\circ}$. Then $a^\circ \neq \phi$, implying that

$$\mathcal{H}^2(a) \geq \mathcal{H}^2(a^\circ) > 0.$$

Furthermore, since

$$a = \partial a \cup a^\circ \quad \text{and} \quad \overline{a^\circ} = a^\circ \cup \partial a^\circ,$$

and $\partial a \cap a^\circ = \phi$, $a = \overline{a^\circ}$ must imply that

$$\partial a \subseteq \partial a^\circ \subseteq \partial a,$$

which concludes the proof. □

Theorem 8 *Suppose a is a closed set and ∂a is σ-finite in $\mathcal{H}^1$. If the preimage $(a, \partial a)$ is regular and the interior a° is connected, it must be prime in* any *preimage monoid $\mathcal{I}$ that contains it.*

Proof. Otherwise, there would exist a finite number of nonempty preimages

$$(b, \delta), \cdots, (e, \pi), (d, \gamma), \cdots, (c, \eta),$$

each of which is distinct from $(a, \partial a)$, such that

$$(a, \partial a) = (b, \delta) \dashv \cdots \dashv (e, \pi) \dashv (d, \gamma) \dashv \cdots \dashv (c, \eta).$$

Since $\mathcal{H}^2(a) > 0$, and $a = b \cup \cdots \cup e \cup d \cup \cdots c$, at least one of the preimages on the right hand side whose support must have nonzero $\mathcal{H}^2$ measure. Assume for instance that (d, γ) is the *first* such preimage from the left to the right. Then,

$$\mathcal{H}^2(b \cup \cdots \cup e) \leq \mathcal{H}^2(b) + \cdots + \mathcal{H}^2(e) = 0,$$

implying that $(b \cup \cdots \cup e)^\circ = \phi$. As a result,

$$\partial a \supseteq \gamma \setminus (b \cup \cdots \cup e)^\circ = \gamma. \tag{9}$$

In particular, we have

$$d \subseteq a \quad \text{and} \quad \partial d \subseteq \gamma \subseteq \partial a.$$

Therefore,

$$a^\circ \cap d = a^\circ \cap d^\circ = d^\circ \quad \text{must be open.}$$

On the other hand,

$$a^\circ \setminus d = a^\circ \cap d^c \quad \text{is also open.}$$

Since a° is connected, and d° is nonempty (otherwise $\mathcal{H}^2(d) = \mathcal{H}^2(\partial d) = 0$ since $\partial d \subseteq \gamma$ is σ-finite in $\mathcal{H}^1$), one must have

$$a^\circ = d^\circ \neq \phi.$$

Then by the regularity assumption on $(a, \partial a)$,

$$a = \overline{a^\circ} = \overline{d^\circ} \subseteq d.$$

Since $d \subseteq a$, we conclude that $a = d$. Thus,

$$\partial a = \partial d \subseteq \gamma,$$

and in combination with (9), one must have $\partial a = \gamma$.

In summary, it has been shown that $(a, \partial a) = (d, \gamma)$, which contradicts the earlier assumption that the two are distinct. Therefore, $(a, \partial a)$ has to be prime. □

Figure 4 shows an example in which the preimage is even simply connected but not atomic. Therefore the condition that a° is connected in the preceding theorem is necessary.

Fig. 4. The necessity for a° being connected in order to become atomic: The preimage A on the left is even simply connected, but not atomic since $A = B \dashv C$. Notice that a° of $A = (a, \gamma)$ has two connected components ($\gamma = \partial a$ in this example).

Definition 8 (Non-Hiding) *A generating set $S(\mathcal{I})$ of a preimage monoid $\mathcal{I}$ is said to be* non-hiding *if*

$$a \nsubseteq b \quad \textit{for any two distinct preimages } a_\gamma, b_\delta \in S(\mathcal{I}).$$

Theorem 9 *A non-hiding generating set $S(\mathcal{I})$ of a preimage monoid $\mathcal{I}$ must be minimal, and if indeed existing, it must be unique.*

Proof. Suppose otherwise there exists a generating set S which is non-hiding but not minimal. Then there must exist some preimage $a_\gamma \in S$, such that

$$a_\gamma \in I_*(S \setminus \{a_\gamma\}),$$

i.e.,

$$a_\gamma = b_\delta \dashv \cdots \dashv c_\eta,$$

for some preimages $b_\delta, \cdots, c_\eta \in S \setminus \{a_\gamma\}$. Therefore

$$a = b \cup \cdots \cup c, \qquad \text{implying that } b \subseteq a,$$

which is contradictory to the non-hiding assumption.

For uniqueness, it suffices to show that for any two non-hiding generating sets S and T, one must have $S \subseteq T$. Let $A = a_\gamma \in S$, we now show that $A \in T$ as well.

Since T is a generating set, there exist preimages $B, \cdots, C \in T$ such that

$$A = B \dashv \cdots \dashv C.$$

Since S is a generating set as well, we must have

$$\begin{aligned} B &= A_1^b \dashv \cdots \dashv A_n^b \\ \vdots &= \quad \vdots \\ C &= A_1^c \dashv \cdots \dashv A_m^c, \end{aligned}$$

for some $A_1^b, \cdots, A_m^c \in S$. Therefore,

$$A = A_1^b \dashv \cdots \dashv A_n^b \dashv \cdots \dashv A_1^c \dashv \cdots A_m^c.$$

Assume that $A_i^b = (a_i^b, \gamma_i^b), \cdots, A_j^c = (a_j^c, \gamma_j^c)$. Then

$$a = a_1^b \cup \cdots \cup a_n^b \cup \cdots \cup a_1^c \cdots \cup a_m^c.$$

In particular,

$$a_1^b,\ \cdots,\ a_m^c \subseteq a.$$

Since S is non-hiding, this necessary implies that

$$A_1^b,\ \cdots,\ A_m^c = A.$$

As a result, $A = B = \cdots = C \in T$, which completes the proof. □

The second half of the proof also establishes the following result.

Theorem 10 (Uniqueness Under the Non-Hiding Condition) *Let S and T be a non-hiding generating set and an ordinary generating set for a given preimage monoid $\mathcal{I}$. Then $S \subseteq T$. In particular, S is the unique minimal generating set.*

5 Depths and Layers

In this section we shall define one of the most basic feature in computational and biological vision: depth.

Definition 9 (Compositional Depth) *Suppose S is a minimal generating set for a preimage monoid $\mathcal{I}$. Then the* compositional depth *of any preimage*

$A \in \mathcal{I}$ relative to S is defined by

$$\operatorname{dep}_S^c(A) = \min\{n \mid A = A_1 \dashv A_2 \dashv \cdots \dashv A_n,\ A_k \in S, k = 1 : n\}.$$

If A is the empty preimage, define $\operatorname{dep}_S^c(A) = 0$.

We leave the proof of the following properties to our readers.

[a] For any preimage A in $\mathcal{I}$, $\operatorname{dep}_S^c(A) < \infty$.
[b] $A \in S$ if and only if $\operatorname{dep}_S^c(A) = 1$, and $A = \phi$ if and only if $\operatorname{dep}_S^c(A) = 0$.
[c] Suppose S is finite. Then $\operatorname{dep}_S(A) \leq \#S$, for any $A \in \mathcal{I}$.
[d] (Subadditivity) For any two preimages $A, B \in \mathcal{I}$,

$$\operatorname{dep}_S^c(A \dashv B) \leq \operatorname{dep}_S^c(A) + \operatorname{dep}_S^c(B).$$

[e] The following is equivalent to the above definition of compositional depth:

$$\operatorname{dep}_S^c(A) = \min\{n \mid A = A_1 \dashv \cdots \dashv A_n, A_1, \cdots, A_n \in S \text{ are distinct}\}.$$

Definition 10 (Transparent Layers and Scenes) *Let $\mathcal{I}$ be a preimage monoid. A finite subset $L \subseteq \mathcal{I}$ is called a* transparent layer *if any two of its preimages commute: $A \dashv B = B \dashv A$, $\forall A, B \in L$. If L is a transparent layer, define its* scene *$[L]$ by*

$$[L] = \dashv_{A \in L} A.$$

Notice that due to the commutativity condition, the definition of a scene is indeed independent of the particular order of the successive occlusions. Next we define a second notion for *depth*, which is motivated by vision research.

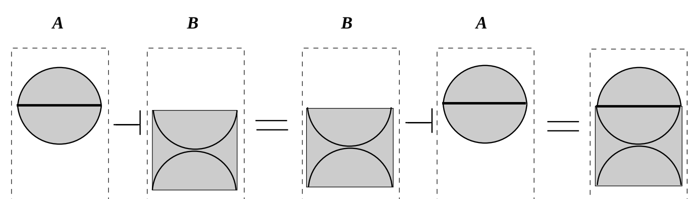

Fig. 5. A layer $L = \{A, B\}$ and its scene $[L] = A \dashv B = B \dashv A$.

Definition 11 (Layered Depth) *Suppose S is a minimal generating set for a preimage monoid $\mathcal{I}$. Define the* layered depth *of any preimage A in $\mathcal{I}$ with respect to S by*

$$\operatorname{dep}_S(A) = \{m \mid A = [L_1] \dashv [L_2] \dashv \cdots \dashv [L_m], L_k \subseteq S \text{ are layers}\}.$$

It is easy to establish the following results.

[a] For any $A \in \mathcal{I}$, $\operatorname{dep}_S(A) \leq \operatorname{dep}_S^c(A)$.

[b] (Subadditivity) For any $A, B \in \mathcal{I}$,

$$\mathrm{dep}_S(A \dashv B) \leq \mathrm{dep}_S(A) + \mathrm{dep}_S(B).$$

[c] Suppose S_1 and S_2 are minimal generating sets for preimage monoids $\mathcal{I}_1$ and $\mathcal{I}_2$. If $S_1 \subseteq S_2$, then $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and for any $A \in \mathcal{I}_1$,

$$\mathrm{dep}_{S_2}(A) \leq \mathrm{deg}_{S_1}(A).$$

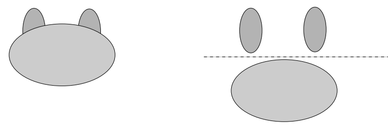

Fig. 6. The difference between compositional and layered depths: for the preimage A on the left, the compositional depth $\mathrm{deg}^c_{S_c}(A) = 3$ with respect to the convex generators S_c (or the smooth generators S_s; see Section 4), while its layered depth $\mathrm{det}_{S_c}(A) = 2$, as given by the decomposition on the right.

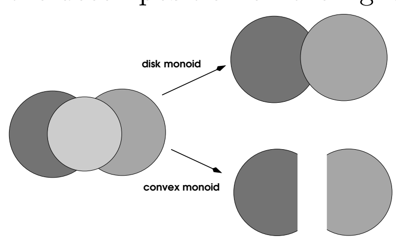

Fig. 7. Dependence of the layered depth on the generator set S: for the preimage A on the left, $\mathrm{deg}_{S_d}(A) = 3$ with respect to the diskette generators S_d as illustrated in the upper right panel, while $\mathrm{deg}_{S_c}(A) = 2$ with respect to the convex generators S_c as illustrated in the lower right panel.

6 Homeomorphism, Diffeomorphism, and Their Invariants

In this section, we discuss topological transforms of preimage monoids and their invariants.

Definition 12 (Homeomorphism) *Two preimages $a_\gamma = (a, \gamma)$ and $b_\delta = (b, \delta)$ are said to be homeomorphic to each other if there exists a homeomorphism (i.e. a continuously invertible and surjective continuous map) $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, such that*

$$\phi(a) = b \;\; \text{and} \;\; \phi(\gamma) = \delta.$$

Definition 13 (Segments and $\mathrm{seg}(a_\gamma)$) *For a preimage $a_\gamma = (a, \gamma)$, each connected component of $a \setminus \gamma$ is called a* segment *of the preimage. The number of segments of a preimage a_γ is denoted by $\mathrm{seg}(a_\gamma)$.*

Here the name "segment" has been motivated by the segmentation problem in image processing and computer vision [4].

Since $\partial a \subseteq \gamma$ and $\partial a = a^\circ \cup \partial a$, one has

$$a \setminus \gamma = a^\circ \setminus \gamma = a^\circ \cap \gamma^c,$$

which is an open set since γ is closed. As a result, each segment of a preimage must be open.

Notice that for a general preimage a_γ, even with a compact support a, it could still happen that $\mathrm{seg}(a_\gamma) = +\infty$.

Theorem 11 *Suppose a_γ and b_δ are homeomorphic. Then*

$$\mathrm{seg}(a_\gamma) = \mathrm{seg}(b_\delta).$$

Proof. Suppose a_γ and b_δ are homeomorphic under a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$. Then $b = \phi(a)$ and $\delta = \phi(\gamma)$. Consequently,

$$b \setminus \delta = \phi(a) \setminus \phi(\gamma) = \phi(a \setminus \gamma).$$

On the other hand, connectivity of a set is preserved under a homeomorphism. Therefore, any connected component of $a \setminus \gamma$ is mapped onto a connected component of $\beta \setminus \delta$, implying that the two preimages have the same number of segments: $\mathrm{seg}(a_\gamma) = \mathrm{seg}(b_\delta)$. □

Definition 14 (C^2-Diffeomorphism) *Two preimages a_γ and b_δ are said to be C^2-diffeomorphic to each other if there exists a C^2 diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\phi(a) = b \ \textit{ and } \ \phi(\gamma) = \delta.$$

Definition 15 (T-Junctions) *A T-junction of a given preimage (a, γ) is a point $t = (t_1, t_2) \in \gamma$ where there exists an open neighborhood U_t of t in $\mathbb{R}^2$ and some C^2-diffeomorphism $\Phi : U_t \to \mathbb{R}^2$, such that*

$$\Phi(t) = (0, 0) \ \textit{ and } \ \Phi(\gamma \cap U_t) = \mathbb{R} \times \{0\} \ \cup \ \{0\} \times \mathbb{R}^+.$$

Let $\mathrm{jun}(a_\gamma)$ *denotes the total number of T-junctions of a given preimage a_γ.*

Theorem 12 *Suppose two preimages a_γ and b_δ are diffeomorphic. Then the numbers of T-junctions must be identical:* $\mathrm{jun}(a_\gamma) = \mathrm{jun}(b_\delta)$.

Proof. Simply notice that a T-junction is invariant under diffeomorphisms. □

Notice that among the four classes of preimage monoids defined in Section 4: the diskette monoid $\mathcal{I}_{\mathrm{disk}}$, the planar monoid $\mathcal{I}_{\mathrm{plane}}$, the convex monoid $\mathcal{I}_{\mathrm{conv}}$, and the smooth monoid $\mathcal{I}_{\mathrm{smooth}}$, not all are suitable for the discussion on homeomorphism or diffeomorphism. The planar and convex monoids are defined via linear properties and thus only invariant under linear transforms, not topological homeomorphisms or diffeomorphisms. The diskette monoid is defined by Euclidean transforms (i.e., rotation, reflection, and translation), and thus is variant too under homeomorphisms or diffeomorphisms. In particular, only the smooth preimage monoid $\mathcal{I}_{\text{smooth}}$ is closed under diffeomorphisms, and consequently diffeomorphic invariants such as T-junctions become meaningful features to study. In fact, for quite a long time T-junctions have been believed to be crucial visual cues for human vision to successfully interpret the 3-D world [2,5].

Theorem 13 *Let S_s be the generating set of preimages defined in (8), and $\mathcal{I}_{\text{smooth}}$ the resultant smooth preimage monoid. Suppose two preimages $A, B \in \mathcal{I}_{\text{smooth}}$ are diffeomorphic. Then*

$$\text{dep}^c_{S_s}(A) = \text{dep}^c_{S_s}(B) \quad \textit{and} \quad \text{dep}_{S_s}(A) = \text{dep}_{S_s}(B).$$

Proof. Suppose A and B are diffeomorphic under diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$. Then $S_s = \phi(S_s)$, and for any two smooth preimages $a_\gamma, b_\delta \in S_s$,

$$\phi(a_\gamma \dashv b_\delta) = \phi(a_\gamma) \dashv \phi(b_\delta).$$

Consequently, a_γ and b_δ commutate if and only if $\phi(a_\gamma)$ and $\phi(b_\delta)$ do, and $L \subseteq S_s$ is a transparent layer if and only if $\phi(L) \subseteq S_s$ is. Furthermore,

$$A = a_\gamma \dashv b_\delta \dashv \cdots \dashv c_\eta, \quad a_\gamma, \cdots c_\eta \in S_s,$$

if and only if

$$B = \phi(A) = \phi(a_\gamma) \dashv \phi(b_\delta) \dashv \cdots \dashv \phi(c_\eta), \quad \phi(a_\gamma), \cdots, \phi(c_\eta) \in S_s.$$

Therefore by the definitions of compositional and layered depths in the preceding section, one must have

$$\text{dep}^c_{S_s}(A) = \text{dep}^c_{S_s}(B) \text{ and } \text{dep}_{S_s}(A) = \text{dep}_{S_s}(B).$$

$\square$

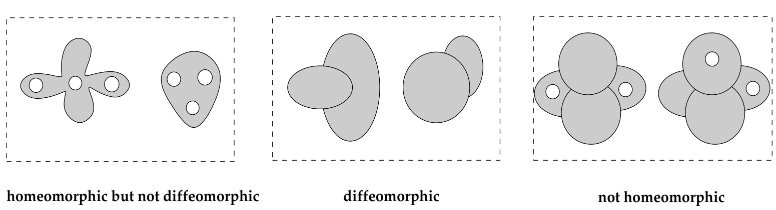

Fig. 8. Three preimage pairs on homeomorphism and diffeomorphism.

7 Conclusion

Motivated by human and machine vision, the current paper has proposed an abstract occlusion model based on both point-set topology and algebra of noncommutative monoids. Many topological as well as algebraic properties are defined and developed. Further study on the finer structures of the preimage monoids is still wide open to the mathematics community, for whom the current novel work has been intended to serve as a good bridge (as similar to the mission of Shen [14], the first work in this series of "On the Foundations of Vision Modeling.").

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