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ABSTRACT

In the present paper we have made on attempt in the direction of tabulation of meijer's G-function for its different values, with the help of amplification factor and terminating value condition.

1. INTRODUCTION

The G-function defined as $G_{pq}^{mn}\left(x\,|a,b\right), G_{pq}^{mn}(x),$ or simple G(x), satisfies following differential equation.

$$[(-1)^{P-m-n}\; x\prod_{j=1}^{p}(\delta - a_j + 1)\; - \prod_{j=1}^{q}(\delta - bj)\;]y = 0 \text{ where } \quad \delta \equiv x\frac{d}{dx} \quad (1.1)$$

A survey of computational methods in special function has been presented by watter Gantschi [3]. In this further works, he [4] has dealt with minimal solution and continued fraction while dealing with difference equation of order two. S. Elaydi [1] has compiled works concerned with developments in the field of difference equations and applications. Recent developments in asymptotics of difference equation have been presented by S.Elaydi [2] in the proceeding of third International conference on Difference Equations.

Generalizing the approaches discussed by these researchers, we have come to the conclusion that the G-function is numerically evaluable according the minimal solution of (1.1). A set of minimal solution which is not empty and is one dimensional shall be a subspace of the linear space of all solutions. It can be characterized in term of the converzence of continued fraction and also can be described in the term of amplification factors for which two starting values are required.

Therefore we have considered and studied the effect of two values with a result that other values have been terminated as per prerequisite of terminating value condition.

2. Numerical Stability and Amplification Factor

Let us transform the equation (1.1) in the term of second order which shall be linear and homogeneous of the form.

$$y_2 + ay_1 + by_0 = 0, \qquad b \neq 0. \tag{2.1}$$

where a and b are coefficients, depending on parameters of the G-function.

Let us set that numerically evaluable solution of (2.1) is f_1 and let g_1 be and arbitrary second solution of (2.1) which is linearly independent of f_1 and also $f_0 \neq 0$, $f_1 \neq 0$, $g_0 \neq 0$.

If the relative errors be $\in_1$ and $\in_2$ with terminal index t, then problem amounts to identity the solution y_1 of (2.1) as

$$y_1 = f_1 (1 + \in_1), \qquad y_2 = f_2 (1 + \in_2) \tag{2.2}$$

we know that the general solution of (2.1) shall be a linear combination of two linearly independent solution say, $y_1 = c_1 f_1 + c_2 g_1$, therefore the two conditions (2.2) then serve to fix the constants c_1 and c_2, and hence the solution y_1, which can then be compared at t = n with f_t. The result is conveniently expressed in terms of the quantities.

$$\frac{f_0\, g_1}{g_0\, f_1} \tag{2.3}$$

in the form

$$\frac{y_1 - f_t}{f_t} = \frac{(g_2 f_t - g_t f_2) \in_1 f_1 - (g_1 f_t - f_1 g_t) \in_2 f_2}{(g_2 f_1 - g_1 f_2) f_t} \tag{2.4}$$

The amplification factor shall be

$$w_{1\to t} \quad : \quad = \quad \frac{| g_2 f_t - g_t f_2 | . | f_1 | - | g_1 f_t - f_1 g_t | . | f_2 |}{| g_2 f_1 - g_1 f_2 | . | f_t |} \tag{2.5}$$

which envices the amount by which initial (relative) errors $\in_1$, $\in_2$ are amplified at t. The differential equation (2.1) shall be unstable for computing f_1, if

$$\left| \frac{f_0 g_1}{g_0 f_1} \right| \quad = \quad \infty \tag{2.6}$$

The equation (2.1) shall be stable if

$$\sup \left| \frac{f_0 g_1}{g_0 f_1} \right| \quad = \quad c < \infty \tag{2.7}$$

Here it is worth to mention that if $\in_2$ = -1 which in view of (2.2) means $y_2 = 0$ then (2.4) shall become

$$\frac{f_0 y_t}{y_0} = \frac{f_t g_2 - g_t f_2}{g_2 - f_2} \tag{2.8}$$

which is independent of the error $\in_1$.

In case of backward recurrence, starting value $y_1 = 1$, $y_2 = 0$, the quantity of the left of (2.8) approximates f_t arbitrarily well for any fixed t.

We have used (2.6) and (2.3) for instability which implies that the solution f_1 of (2.1) has the properly

$$\frac{f_1}{g_1} = 0 \tag{2.9}$$

3. GENERAL FORMULATION

A particular case of G-function $[G_{02}^{10}\,(x|a,b)]$ can be obtained for m = 1, n = 0, p = 0, q = 2, which will satisfy following equation.

$$x^2 \frac{d^2 y}{dx^2} - (ax - x - x^2)\frac{dy}{dx} + (x + ab)\, y = 0 \tag{3.1}$$

assuming that

$$\frac{d^2 x}{dx^2} = y_2, \quad \frac{dy}{dx} = y_1, \quad and \quad y = y_0,$$

we easily obtain

$$x^2\, y_2 - (\, ax - x - x^2)\, y_1 + (\, x + ab)\, y_0 = 0 \tag{3.2}$$

Division of (3.2) by y_1 and then supposing.

$$\frac{y_2}{y_1} = r_1 \quad and \quad \frac{y_1}{y_0} \quad = \quad r_0$$

we obtain

$$x^2 r_1 - (ax - x - x^2) + (x + ab).\frac{1}{r_0} = 0$$

$$r_0 = \frac{(x + ab)}{\{x(a - x - 1)\} - x^2 r_1} \tag{3.3}$$

Iteration of this equation indefinitely yields formally the continued fraction:

$$\frac{f_1}{f_0} = \frac{x+ab}{\{x(a-x-1)\}-} \; \frac{x^2\{(x+1)+(a+1)(b+1)\}}{\{(x+1)(a-x-1)\}-} \; \frac{(x+1)^2\{(x+2)+(a+2)(b+2)\}}{\{(x+2)(a-x-1)\}-}$$

$$\frac{(x+2)^2\{(x+3)+(a+3)(b+3)\}}{\{(x+3)(a-x-1)\}-} \tag{3.4}$$

The continued fractions on the R.H.S. of (3.4) converges because f_1 is the minimal solution of (3.1) and limit is expressible by L.H.S. of (3.4) for different values of x, a, b as mentioned in the ensuing tables.

For numerically characterization of (3.1) we shall be use (2.5) for amplification factor and (2.6) and (2.3) for instability. Application of (2.9) shall yields minimal solution from which two values shall be considered while the remaining values shell be considered terminated.

It results to the tabulation of the function by making the substitutions.

x = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0

a = 1, 2

b = 1, 2

The values have been complied in the table (Appendix – I)

Proceeding on similar lines continued fraction of $[G_{02}^{20}(x \mid a, b)]$ can be obtained as

$$\frac{f_1}{f_0} = \frac{x + ab}{\{x\,(a - x + 1)\} +} \quad \frac{x^2\ \{(x+1) + (a+1)\,(b+1)\}}{\{(x+1)(a - x + 1)\} +} \quad \frac{(x+1)^2\{(x+2) + (a+2)(b+2)\}}{\{(x+2)(a - x + 1)\} +}$$

$$\frac{(x+2)^2\{(x+3) + (a+3)(b+3)\}}{\{(x+3)(a - x + 1)\} +} \ldots\ldots\ldots\ldots\ldots \tag{3.5}$$

where as the function can be tabulated by making substitutions.

x = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0

a = 1/2, 3/2, 2

b = 1/2, 2

The obtained values have been complied in the table (Appendix - II)

We have also obtained the continued fraction for $[G_{12}^{11}\left(x \,\middle|\, {}^{1/2}_{b,-b}\right)]$ as

$$\frac{f_1}{f_0} = \frac{x + 2b^2}{\{2x\,(x - b + 1)\} +} \quad \frac{2x^2\ \{(x+1) + 2\,(b+1)^2\}}{\{2(x+1)(x - b + 1)\} +} \quad \frac{2(x+1)^2\{(x+2) + 2(b+2)^2\}}{\{2(x+2)(x - b + 1)\} +}$$

$$\frac{2(x+2)^2\{(x+3) + 2(b+3)^2\}}{\{2(x+3)(x - b + 1)\} +} \ldots\ldots\ldots\ldots\ldots \tag{3.6}$$

for values

x = 0.4, 0.8, 1.2, 1.6, 2.0, 2.4, 2.8, 3.2, 3.6, 4.0, 4.4, 4.8, 5.2, 5.6,

6.0,

b = 3, 4, 5, 6

which has been complied in the table (Appendix - III)

Similarly the continued fraction expression for [$G_{12}^{11}\left(x\middle|\begin{matrix}a\\b,c\end{matrix}\right)$] shall be

$$\frac{f_1}{f_0} = \frac{ax - bc}{\{x\,(x - b + 1)\} +} \quad \frac{x^2\ \{(a+1)(x+1) - (b+1)\ (c+1)\}}{\{(x+1)(x-b+1)\} +}$$

$$\frac{(x+1)^2\{(a+2)(x+2)-(b+2)(c+2)\}}{\{(x+2)(x-b+1)\}+}\ \frac{(x+2)^2\{(a+3)(x+3)-(b+3)(c+3)\}}{\{(x+3)(x-b+1)\}+}\ldots\ldots\ldots\ldots \tag{3.7}$$

whereas the function can be tabulated by making the substitutions.

x = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0

a = 1.4, 1.6, 1.8

b = 1.0

c = 1.0

The values have been compiled in Appendix – IV of tables

Appendix - I

a, b / x	a = 1, b = 1	a = 2 , b = 1	a = 2, b = 2
0.2	(0) 0.1692	(-1)+ 0.3299	(-1)+ 0.3384
0.4	(0) 0.2684	(0) 0.1260	(0) 0.1073
0.6	(0) 0.3070	(0) 0.2593	(0) 0.1842
0.8	(0) 0.3183	(0) 0.4134	(0) 0.2546
1.0	(0) 0.2238	(0) 0.5767	(0) 0.2238
2.0	(0)- 0.3700	(0) 1.1588	(0) - 0.7400
3.0	(0)- 1.0928	(0) 0.9312	(0) - 3.2784
4.0	(0)- 1.5885	(0) - 0.5283	(0) - 6.3540
5.0	(0)- 1.7112	(0) - 2.2670	(0) - 8.5560
6.0	(0)- 1.4425	(0) - 2.4494	(0) - 8.6550
7.0	(0)- 0.7720	(0) - 6.3565	(0) - 5.4040
8.0	(0)- 0.2157	(0) - 7.5650	(0) - 1.7256
9.0	(0) 1.3558	(0) - 7.4704	(2) 0.1220
10.0	(0) 2.2381	(0) - 6.5802	(2) 0.2238

Appendix - II

a,b / .x	a = 1/2, b = ½		a = 3/2, b = 1/2		a = 2, b = 2	
0.1	(0)	0.4917	(0)	0.2605	(-1) +	0.1555
0.2	(0)	0.5056	(0)	0.3447	(-1) +	0.4522
0.3	(0)	0.4612	(0)	0.3611	(-1) +	0.7578
0.4	(0)	0.4028	(0)	0.3476	(0)+	0.1019
0.5	(0)	0.3445	(0)	0.3208	(0)+	0.1218
0.6	(0)	0.3312	(0)	0.3328	(0)+	0.1539
0.7	(0)	0.3145	(0)	0.3368	(0)+	0.1841
0.8	(0)	0.296	(0)	0.3349	(0)	0.2131
0.9	(0)	0.2768	(0)	0.3287	(0)	0.2363
1.0	(0)	0.2277	(0)	0.2797	(0)	0.2271
2.0	(0)	0.1239	(0)	0.2044	(0)	0.3504
3.0	(-1)	0.7606	(0)	0.14989	(0)	0.3951
4.0	(-1)	0.4463	(-1)	0.9986	(0)	0.3570
5.0	(-1)	0.3197	(-1)	0.7923	(0)	0.3575
6.0	(-1)	0.2252	(-1)	0.6066	(0)	0.3310
7.0	(-1)	0.1593	(-1)	0.4534	(0)	0.2906
8.0	(-1)	0.1085	(-1)	0.3333	(0)	0.2455
9.0	(-2)	0.7463	(-1)	0.2419	(0)	0.2015
10.0	(-2)	0.5692	(-1)	0.1938	(0)	0.1800

Appendix - III

X \ b	3	4	5	6
0.4	(-3) 0.2425	(-5) 0.6059	(-6) 0.1211	(-8) 0.2018
0.8	(-2) 0.1600	(-4) 0.7985	(-5) 0.3190	(-6) 0.1062
1.2	(-2) 0.4475	(-3) 0.3343	(-4) 0.1999	(-6) 0.9978
1.6	(-2) 0.8842	(-3) 0.8772	(-4) 0.6980	(-5) 0.4636
2.0	(-1) 0.1445	(-2) 0.1785	(-3) 0.1770	(-4) 0.1466
2.4	(-1) 0.2101	(-2) 0.3097	(-3) 0.3673	(-4) 0.3642
2.8	(-1) 0.2820	(-2) 0.4820	(-3) 0.6649	(-4) 0.7659
3.2	(-1) 0.3575	(-2) 0.6933	(-2) 0.1086	(-3) 0.1427
3.6	(-1) 0.4342	(-2) 0.9399	(-2) 0.1648	(-3) 0.2426
4.0	(-1) 0.5104	(-1) 0.1217	(-2) 0.2357	(-3) 0.3839
4.4	(-1) 0.5846	(-1) 0.1519	(-2) 0.3216	(-3) 0.5734
4.8	(-1) 0.6559	(-1) 0.1812	(-2) 0.4222	(-3) 0.8172
5.2	(-1) 0.7237	(-1) 0.2177	(-3) 0.5370	(-2) 0.1119
5.6	(-1) 0.7875	(-1) 0.2523	(-2) 0.6649	(-2) 0.1484
6.0	(-1) 0.8471	(-1) 0.2874	(-2) 0.8050	(-2) 0.1913

Appendix – IV

x \ a, b, c	a = 1.8, b = 1.0, c = 1.0	a = 1.6, b = 1.0, c = 1.0	a = 1.4, b = 1.0, c = 1.0
0.2	0.1767	0.1641	0.1588
0.4	0.3410	0.3052	0.2838
0.6	0.4951	0.4276	0.3823
0.8	0.6404	0.5347	0.4600
1.0	0.7784	0.6294	0.4952
2.0	1.3909	0.9833	0.6888
3.0	1.9257	1.2347	0.7596
4.0	2.4184	1.4434	0.8076
5.0	2.8848	1.6306	0.8514
6.0	3.3322	1.8041	0.8943
7.0	3.7647	1.9676	0.9365
8.0	4.1849	2.1229	0.9773
9.0	4.5958	2.2716	1.0168
10.0	4.9866	2.4093	1.0523

REFERECNES

[1] S. Elaydi (1996). An introduction of difference equation, Springer Verlog New York.

[2] S.Elaydi (1997). Recent Development in the asymptotics of difference equations, New development in difference Equation and Applications, Proceedings of the third Intrernational conference on difference Equations, Taipei, Republic of China sep. 1-5-1997, edited by S.S. Cheng, S.Elydi and G. Ladas, Gordon and Breach Science publishers.

[3] W. Gautschi (1975). Computational methods in special functions, A survey in Theory and application of special functions (R.A. Askey ed) Academic press New York, p.p. 1-98.

[4] W. Gautsci (1995) The computation of special function by linear difference equation. Advances in Difference Equations, Proceedings to the second International conference on Difference Equations Veszprem Hungry Aug 7-11-1995, edited by S.Elaydi, I. Gyori and G. Ladas. Garden and Breach Science publishers.

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