

ANALYSIS OF GRADIENT FLOW OF A REGULARIZED MUMFORD-SHAH FUNCTIONAL FOR IMAGE SEGMENTATION AND IMAGE INPAINTING *

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Abstract. This paper studies the gradient flow of a regularized Mumford-Shah functional proposed by Ambrosio and Tortorelli [3, 4] for image segmentation, and adopted by Esedoglu and Shen [20] for image inpainting. It is shown that the gradient flow with $L^2 \times L^\infty$ initial data possesses a global weak solution, and it has a unique global in time strong solution, which has at most finite number of point singularities in the space-time, when the initial data are in $H^1 \times H^1 \cap L^\infty$. A family of fully discrete approximation schemes using low order finite elements is proposed for the gradient flow. Convergence of a subsequence (resp. the whole sequence) of the numerical solutions to a weak solution (resp. the strong solution) of the gradient flow is established as the mesh sizes tend to zero, and optimal and suboptimal order error estimates, which depend on $\frac{1}{\varepsilon}$ and $\frac{1}{k\varepsilon}$ only in low polynomial order, are derived for the proposed fully discrete schemes under the mesh relation $k = o(h^{\frac{1}{2}})$.

Key words. Image segmentation and inpainting, Mumford-Shah model, elliptic approximation, gradient flow, a priori estimates, finite element method, error analysis

AMS subject classifications. 35K55, 65M12, 65M15, 68U10, 94A08

1. Introduction. Image segmentation in computer vision aims at automatic partitioning of a given image on $\Omega \subset \mathbf{R}^N$ ($N = 2, 3$) into regions where the gray-level function $u : \Omega \rightarrow \mathbf{R}$ is smooth, having only discontinuities across edges. A variational model was proposed by Mumford and Shah [31] to segment the image into as few and simple regions as possible and thus detect essential structures of the image. Following earlier discrete models proposed by D. Geman and S. Geman [23] and by Blake and Zisserman [6], they proposed to find

$$(v, \Gamma) := \underset{\substack{K \subset \Omega \text{ closed} \\ u \in H^1(\Omega \setminus K)}}{\operatorname{argmin}} E(u, K),$$

for given $g \in L^2(\Omega)$, $\alpha, \beta, \gamma \geq 0$. Where

$$(1.1) \quad E(u, K) := \frac{\alpha}{2} \int_{\Omega \setminus K} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} \gamma |u - g|^2 dx + \beta \mathcal{H}^{N-1}(K),$$

and $\mathcal{H}^{N-1}(K)$ denotes the $(N - 1)$ -dimensional Hausdorff measure of K , which measures the “length” of the set K .

Heuristically, we expect solutions to this problem to be smooth and close to the image g at places $x \notin \Gamma$, and Γ constitutes edges of the image. To show existence of solutions to the above problem, a weak formulation was proposed by De Giorgi, Carriero and Leaci [16] by dropping the requirement that K is a closed set, and to allow it to be the jump set of an *SBV* (Special Bounded Variation) function u . So the idea was to find

$$(\tilde{v}, S_{\tilde{v}}) := \underset{u \in SBV(\Omega)}{\operatorname{argmin}} \tilde{E}(u),$$

where

$$(1.2) \quad \tilde{E}(u) := \frac{\alpha}{2} \int_{\Omega \setminus S_u} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} \gamma |u - g|^2 dx + \beta \mathcal{H}^{n-1}(S_u).$$

SBV(Ω) denotes the set of special bounded variation functions, and S_u stands for the jump set of u , the complement of the set of all Lebesgue-points of u in Ω (cf. [2]). The existence of solutions to problem (1.2) was established in [16], and $(v, \Gamma) = (\tilde{v}, \overline{S_{\tilde{v}}})$ was proved for a large range of applications.

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Since the above variational problems require the computation of geometrical properties of the unknown set of discontinuity boundaries, this results in considerable difficulties to numerically compute the solutions. In fact, exact computation of solutions of this type free discontinuity problems can be very rarely performed, with the exception of situations where some symmetries allow to reduce the problem to a one-dimensional problem. To approximate and compute solutions to variational problems (1.1) and (1.2), the most popular and successful approach is to use the theory of Γ -convergence [15]. This theory, introduced by De Giorgi and Franzoni in [17], is designed to approximate a variational problem by a sequence of different (usually, regularized) variational problems and ensures convergence of extremal values to extremal values and of minimizers to minimizers.

Several types of approximate variational problems for (1.1) and (1.2) have been studied extensively in the literature. At least three classes of Γ -convergent approximations to the functional (1.2) were proposed and analyzed in the literature. The first class is based on introducing higher-order singular perturbation terms (cf. [8] and the references therein); the second class approximates the functional (1.2) by non-local functionals, i.e., density functions are non-local (cf. [9]). The idea of the third class of approximations, which was proposed by De Giorgi and developed by Gobbi (cf. [25] and reference therein), is to average the difference quotients among all possible directions in the functional. One such example is

$$(1.3) \quad J_\varepsilon(u) = \frac{1}{\varepsilon} \int_{\Omega} \arctan\left(\frac{|u(x+\varepsilon) - u(x)|^2}{\varepsilon}\right) dx + \frac{1}{2} \int_{\Omega} \gamma |u - g|^2 dx.$$

Following an earlier idea of Blake and Zissermann [6], Chambolle [12, 13] proposed a discrete finite-difference approximation and showed its Γ -convergence to the Mumford-Shah functional (1.1) in the one-dimensional case and to an anisotropic version of (1.1) in the two-dimensional case. Probably, the best known and commonly used approximation to the Mumford-Shah functional (1.1) is the following elliptic approximation due to Ambrosio and Tortorelli [3, 4]:

$$(1.4) \quad AT_\varepsilon(u, \varphi) = \frac{\alpha}{2} \int_{\Omega} (k_\varepsilon + \varphi^2) |\nabla u|^2 dx + \beta \int_{\Omega} \left[\varepsilon |\nabla \varphi|^2 + \frac{1}{4\varepsilon} f(\varphi) \right] dx + \frac{1}{2} \int_{\Omega} \gamma |u - g|^2 dx,$$

for $f(\varphi) = (1 - \varphi)^2$, and $0 < k_\varepsilon = o(\varepsilon)$. The Γ -convergence of $AT_\varepsilon(u, \varphi)$ to $E(u, K)$ was established in [3, 4]. The motivation for this approximation came from the Modica-Mortola theorem ([28, 29]) which enables the variational approximation of $P(E, \Omega)$, the perimeter of E in Ω , by the quadratic and elliptic Cahn-Hilliard functional [11]. A key idea of Ambrosio and Tortorelli's approximation is to replace the original "double well" potential $p(z) = z^2(1 - z)^2$ in the Cahn-Hilliard functional by the quadratic "single well" potential $f(z) = (1 - z)^2$ in order to approximate $\mathcal{H}^{N-1}(K)$ in (1.1). The function φ in (1.4) can be regarded as a "phase function" so K is roughly indicated by the set $\{\varphi \approx 0\}$, whereas its complement in Ω corresponds to $\{\varphi \approx 1\}$. Borrowing terminologies from phase transition in materials science, $AT_\varepsilon(u, \varphi)$ can be regarded as a *phase field (diffuse interface)* model to the Mumford-Shah (*sharp interface*) model $E(u, K)$.

Finite element approximations to the Mumford-Shah functional (1.1) based on the functional $AT_\varepsilon(u, \varphi)$ were first carried out by Bellettini and Coscia in [5]. It was shown that $AT_\varepsilon : V^h(\Omega) \times V^h(\Omega, [0, 1]) \rightarrow \overline{\mathbf{R}}$ Γ -converges to $E : H^1(\Omega) \times L^\infty(\Omega) \rightarrow \overline{\mathbf{R}}$ under the condition that the mesh size $h = o(k_\varepsilon)$, provided that S_u is piecewise smooth. Here $V^h(\Omega)$ denotes the continuous piecewise linear finite element space (cf. Section 3), and $V^h(\Omega, [0, 1]) = \{v_h \in V^h(\Omega); 0 \leq v_h(x) \leq 1 \forall x \in \Omega\}$. Later, Bourdin [7] extended the result of [5] to general S_u by using a result of Dibos and for approximating the set S_u by piecewise smooth hypersurfaces. Moreover, he showed that restricting functions $\varphi_h \in V^h(\Omega)$ to values in $[0, 1]$ is not necessary since this constraint is already implicitly satisfied.

The main goal of this paper is to analyze the following gradient flow of the functional AT_ε :

$$(1.5) \quad u_t - \alpha \operatorname{div}((k_\varepsilon + \varphi^2)\nabla u) + \gamma(u - g) = 0 \quad \text{in } \Omega_T := (0, T) \times \Omega,$$

$$(1.6) \quad \varphi_t - 2\beta\varepsilon\Delta\varphi + \alpha|\nabla u|^2\varphi + \frac{\beta}{2\varepsilon}(\varphi - 1) = 0 \quad \text{in } \Omega_T,$$

$$(1.7) \quad \frac{\partial u}{\partial n} = \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial\Omega_T := (0, T) \times \partial\Omega,$$

$$(1.8) \quad u(\cdot, 0) = u_0(\cdot), \quad \varphi(\cdot, 0) = \varphi_0(\cdot) \quad \text{in } \Omega,$$

and its fully discrete finite element approximations. The primary motivation for considering the above gradient flow is to study the steepest descent method for minimizing the functional (1.4), which is often used in practice. In addition, a good understanding of the above gradient flow paves the way for us to analyze the gradient flow of the Mumford-Shah functional [22], in particular, in high dimensions. Numerical results of [27, 20] have indicated various interesting stability and instability properties of the solutions of the gradient flow (1.5)-(1.8), for certain choices of parameters $\alpha, \beta, \gamma, \varepsilon, k_\varepsilon > 0$. Another goal of this work is to understand these observations qualitatively by studying regularity and stability of the solutions of the above initial-boundary value problem, as well as error estimates of their finite element approximations.

We note that in the one-dimensional case, the gradient flow for the Mumford-Shah functional was studied by Gobbi in [25], *based on the nonlocal approximate functional* (1.3). He established Γ -convergence of $J_\varepsilon(\cdot)$ to $\tilde{E}(\cdot)$ with respect to L^2 -topology, derived the gradient flow for $\tilde{E}(\cdot)$ by setting $\varepsilon \rightarrow 0$, and proved that solutions to the limiting problem solve local heat equations, separated by an invariant set of jumps, i.e., $S(u(t)) \subseteq S(u_0)$, for all $t \geq 0$.

We also note that a slightly modified Mumford-Shah model has recently been proposed by Esedoglu and Shen [20] as an image inpainting model, and Ambrosio and Tortorelli's elliptic approximation was also used as a vehicle for numerical simulations. The proposed image inpainting model has exactly the same form as the Mumford-Shah model (1.1), and the proposed elliptic approximation has exactly the same form as the Ambrosio and Tortorelli's approximation. The only difference is that the parameter γ in (1.1) and (1.4) now stands for the *indicator function* $\gamma_D(x)$ of $\Omega \setminus D$, where D denotes the inpainted region of an image. We like to remark that, as a by-product, the results of the present paper also apply to this image inpainting model.

The paper is organized as follows. Section 2 devotes to analyzing the initial-boundary value problem (1.5)-(1.8). We first establish existence of weak solutions for u_0, g in $L^2(\Omega)$ and $\varphi_0 \in L^\infty(\Omega)$, then prove that the system has a unique global in time strong solution, which has at most finite number of point singularities in the space-time, when the initial data $(u_0, \varphi_0) \in H^1(\Omega) \times H^1(\Omega) \cap L^\infty(\Omega)$. Our proof is based on a local energy idea due to Struwe [34]. A priori solution estimates are established in various norms, especially, by tracing their precise dependence on data (u_0, φ_0) and g , as well as on parameters $\varepsilon, \alpha, \beta$ and γ . These results play a crucial role for understanding stability properties of the flow and for establishing error estimates for finite element approximations in the next section. Based on the analytical results of Section 2, Section 3 studies qualitatively finite element approximations of the gradient flow (1.5)-(1.8). We formulate and analyze a family of fully discrete finite element approximations using implicit Euler discretization in time, and continuous, piecewise linear finite element discretization in space. Optimal and suboptimal order error bounds, which show dependence on $\frac{1}{\varepsilon}$ and $\frac{1}{k_\varepsilon}$ only in low polynomial order, are derived for the proposed fully discrete schemes under the mesh relation $k = o(h^{\frac{1}{2}})$. It is shown that semi-implicit treatment of the nonlinear term in (1.5) or in (1.6) results in schemes which satisfy a discrete energy law, while the same discrete energy law may not hold for the fully implicit scheme. On the other hand, as expected, the fully implicit scheme produces smaller errors than its semi-implicit counterparts do, although they have same asymptotic rate of convergence.

2. Analysis of the initial-boundary value problem (1.5)-(1.8). The first goal of this section is to establish existence, uniqueness, stability and regularity properties for the gradient flow (1.5)-(1.8). The second goal is to derive a priori estimates, by tracing dependence on the parameters $\varepsilon, k_\varepsilon, \alpha, \beta$ and γ (or γ_D). The results of this section will serve as the theoretical foundation for analyzing space-time discretizations of (1.5)-(1.8) in the next section, they will also play a crucial role for studying the gradient flow of the Mumford-Shah functional in [22].

Standard function space and norm notation are used in this paper, we refer to [1, 14, 26, 33] for their precise definitions. Throughout this paper, unless stated otherwise, C will be used to denote a generic positive constant, which is independent of the parameters $\varepsilon, k_\varepsilon, \alpha, \beta$ and γ , as well as data u_0, φ_0, g , and solution (u, φ) .

2.1. Existence of weak solutions. We begin this subsection with the definition of weak solutions to (1.5)-(1.8).

DEFINITION 2.1. *Let $\Omega \subset \mathbf{R}^N$ ($N = 2, 3$) be a domain with Lipschitz boundary $\partial\Omega$. For given data $(u_0, \varphi_0) \in [L^2(\Omega)]^2$ with $0 \leq \varphi_0 \leq 1$ a.e. in Ω , and $g \in L^2((0, T); L^2(\Omega))$, a pair of functions (u, φ) is said to be a weak solution to (1.5)-(1.8) if $(u, \varphi) \in [L^\infty((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))]^2$ satisfies (1.5)-(1.8)*

in distribution sense, and $0 \leq \varphi \leq 1$ a.e. in Ω_T .

REMARK 2.1. *It is not hard to check that a weak solution (u, φ) also satisfies $(u_t, \varphi_t) \in [L^2((0, T); H^{-2}(\Omega))]^2$. An application of Aubin-Lions embedding lemma [26, 33] implies that $(u, \varphi) \in [C^0((0, T); L^2(\Omega))]^2$, hence, two equations in (1.8) are well-defined.*

The existence of a weak solution is given by the following theorem.

THEOREM 2.2. *Given $(u_0, \varphi_0) \in [L^2(\Omega)]^2$ with $0 \leq \varphi_0 \leq 1$ a.e. in Ω , and $g \in L^2((0, T); L^2(\Omega))$, let $\tau(t)$ be any positive C^1 function on $[0, T]$ such that $\tau(0) = 0$. Then, (1.5)-(1.8) possesses a weak solution (u, φ) which satisfies for $s \in [0, T]$*

$$(2.1) \quad \|u(s)\|_{L^2}^2 + \int_0^s \left[2\alpha(k_\varepsilon \|\nabla u\|_{L^2}^2 + \|\varphi \nabla u\|_{L^2}^2) + \gamma \|u\|_{L^2}^2 \right] dt \leq \mathcal{B}_0$$

$$(2.2) \quad \|\varphi(s)\|_{L^2}^2 + \int_0^s \left[4\beta\varepsilon \|\nabla \varphi\|_{L^2}^2 + 2\alpha \|\varphi \nabla u\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|f(\varphi)\|_{L^1} \right] dt \leq \mathcal{B}_1,$$

$$(2.3) \quad \tau(s) AT_\varepsilon(u(s), \varphi(s)) + \int_0^s \tau(t) \left[\|u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2 \right] dt = \int_0^s \tau'(t) AT_\varepsilon(u, \varphi) dt,$$

where

$$\mathcal{B}_0 := \|u_0\|_{L^2}^2 + \|\sqrt{\gamma}g(t)\|_{L^2(L^2)}^2, \quad \mathcal{B}_1 := \|\varphi_0\|_{L^2}^2 + \beta|\Omega|\varepsilon^{-1}.$$

Moreover, if $u_0 \in L^\infty(\Omega)$ and $g \in L^\infty(\Omega_T)$, then u satisfies the following weak maximum principle,

$$(2.4) \quad \min \left\{ \min_{\Omega_T} g, \min_{\Omega} u_0 \right\} \leq u(x, t) \leq \max \left\{ \max_{\Omega_T} g, \max_{\Omega} u_0 \right\} \quad \text{a.e. } (x, t) \in \Omega_T.$$

Proof. Since solvability follows easily from applying the standard energy method [26] provided that a priori estimates (2.1)-(2.3) and $0 \leq \varphi \leq 1$ can be verified, we only give a proof for these estimates in the following.

Since φ and $\psi := \varphi - 1$ satisfy, respectively,

$$\begin{aligned} \varphi_t - 2\beta\varepsilon\Delta\varphi + \left[\alpha|\nabla u|^2 + \frac{\beta}{2\varepsilon} \right] \varphi &\geq 0, & \varphi(0) = \varphi_0 \geq 0, \\ \psi_t - 2\beta\varepsilon\Delta\psi + \left[\alpha|\nabla u|^2 + \frac{\beta}{2\varepsilon} \right] \psi &\leq 0, & \psi(0) = \varphi_0 - 1 \leq 0, \end{aligned}$$

testing the first equation by $\varphi^- := \max\{-\varphi, 0\}$ and the second by $\psi^+ := \max\{\psi, 0\}$, and using the fact that $\varphi(0)^- = 0$ and $\psi(0)^+ = 0$ immediately yields $0 \leq \varphi(t, x) \leq 1$, a.e. $(t, x) \in \Omega_T$.

Next, (2.1) and (2.2) follows directly from testing (1.5) by u and (1.6) by φ , respectively.

To show (2.3), testing (1.5) by $\tau(t)u_t$ and (1.6) by $\tau(t)\varphi_t$, and adding the resulted equations we get

$$(2.5) \quad \frac{d}{dt} \left[\tau(t) AT_\varepsilon(u(t), \varphi(t)) \right] + \tau(t) \left[\|u_t(t)\|_{L^2}^2 + \|\varphi_t(t)\|_{L^2}^2 \right] = \tau'(t) AT_\varepsilon(u(t), \varphi(t)).$$

Then (2.3) follows from integrating (2.5) in t from 0 to s and using the assumption $\tau(0) = 0$.

Finally, it remains to verify (2.4). Define

$$\kappa_1 := \min \left\{ \min_{\Omega_T} g, \min_{\Omega} u_0 \right\}, \quad \kappa_2 := \max \left\{ \max_{\Omega_T} g, \max_{\Omega} u_0 \right\},$$

The assertion immediately follows from testing (1.5) by $(u - \kappa_1)^-$ and $(u - \kappa_2)^+$, respectively. The proof is complete. \square

REMARK 2.2. (a). *Note that \mathcal{B}_0 is independent of ε , but \mathcal{B}_0 grows linearly in ε^{-1} . This is expectable since φ should have large gradient (corresponding to small ε) near the edges of an image. Since (2.2) implies that $\|\nabla \varphi\|_{L^2(\Omega_T)} = O(\varepsilon^{-1})$, we conclude that the width of smeared edges of Ambrosio-Tortorelli's approximation model (1.4) is not bigger than $O(\varepsilon)$ order.*

(b). *Since $u_0 \in L^\infty(\Omega)$ and $g \in L^\infty(\Omega_T)$ are satisfied for all image applications, practically, (2.4) holds in general.*

2.2. Regularity and uniqueness properties of weak solutions. In this section we address regularity and uniqueness properties of weak solutions to (1.5)–(1.8) with more regular datum functions. Special attention will be given on deriving a priori estimates with explicit dependence on the parameters $\varepsilon, k_\varepsilon, \alpha, \beta, \gamma$, and the data (u_0, φ_0) and g . Such a priori estimates will be useful for finite element error analysis in the next section. It turns out that estimates in $L^2(\Omega_T)$ of all first order derivatives of the solutions are easy to get due to the fact that there is an underlying *energy law* for every gradient flow. However, it is far from straightforward to derive a priori estimates for the second order derivatives of the solution due to the strong nonlinearity. To overcome the difficulty, we use an idea of Struwe [34] to examine the solution's behavior based on a *local energy law*, and show that the system has a unique global in time strong solution which is (strongly) differentiable in space-time, away from at most finitely many points $\{(\bar{x}_\ell, \bar{t}_\ell)\}_{\ell=1}^K$ in Ω_T .

We start with the energy law for the gradient flow (1.5)–(1.8).

THEOREM 2.3. *In addition to assumptions of Theorem 2.2, suppose that $(u_0, \varphi_0) \in [H^1(\Omega)]^2$, and $g \in L^2(\Omega)$ is independent of t . Then, weak solutions (u, φ) to (1.5)–(1.8) satisfy $(u, \varphi) \in [L^\infty((0, T); H^1(\Omega)) \cap H^1((0, T); L^2(\Omega))]^2$ and the energy law*

$$(2.6) \quad \operatorname{ess\,sup}_{t \in [0, T]} AT_\varepsilon(u(t), \varphi(t)) + \int_0^T \left[\|\varphi_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 \right] dt = AT_\varepsilon(u_0, \varphi_0).$$

Proof. It suffices to prove (2.6), which follows in the same way as for (2.3) (cf. (2.5)) with $\tau(t) \equiv 1$. Since $AT_\varepsilon(u_0, \varphi_0) < \infty$, (2.6) follows from integrating (2.5) in t from 0 to $s \in [0, T]$. The proof is complete. \square

DEFINITION 2.4. *A weak solution (u, φ) is called a quasi-strong solution if $(u, \varphi) \in [H^1((0, T); L^2(\Omega)) \cap L^\infty((0, T); H^1(\Omega))]^2$.*

For $x_0 \in \Omega$, let $B_R(x_0)$ denote the ball of radius $R (> 0)$ centered at x_0 . Define the local energy over $B_R(x_0)$ as

$$(2.7) \quad AT_\varepsilon((u(t), \varphi(t)); B_R(x_0)) = \frac{\alpha}{2} \int_{B_R(x_0)} (k_\varepsilon + \varphi^2) |\nabla u|^2 dx \\ + \beta \int_{B_R(x_0)} \left[\varepsilon |\nabla \varphi|^2 + \frac{1}{4\varepsilon} f(\varphi) \right] dx + \frac{1}{2} \int_{B_R(x_0)} \gamma |u - g|^2 dx.$$

Next, we show that a local version of (2.6), referred as *the local energy law*, also holds for the gradient flow (1.5)–(1.8).

LEMMA 2.5. *Let $(u_0, \varphi_0) \in [H^1(\Omega)]^2$. For $x_0 \in \Omega$, let $B_{2R_0}(x_0)$ be the largest ball contained in Ω with the center at x_0 . Then, for all $0 < R \leq 2R_0$, there holds*

$$AT_\varepsilon((u(T), \varphi(T)); B_R(x_0)) \leq AT_\varepsilon((u_0, \varphi_0); B_{2R}(x_0)) + \frac{\mathcal{B}_2 T}{R^2} AT_\varepsilon(u_0, \varphi_0) \quad \forall T \geq 0,$$

where $\mathcal{B}_2 := 16[\alpha(1 + k_\varepsilon) + \beta\varepsilon]$.

Proof. Let ϕ be a cutoff function satisfying $\phi \in C_0^\infty(B_{2R}(x_0))$, $0 \leq \phi \leq 1$ in Ω , $\phi \equiv 1$ in $B_R(x_0)$, and $|\nabla \phi| \leq \frac{2}{R}$. Testing (1.5)–(1.6) with $(\phi^2 u_t, \phi^2 \varphi_t)$ and applying Young's inequality lead to

$$\begin{aligned}
& \int_{\Omega} [|u_t|^2 + |\varphi_t|^2] \phi^2 dx + \frac{d}{dt} \left[\frac{k_\varepsilon \alpha}{2} \int_{\Omega} |\nabla u|^2 \phi^2 dx + \beta \varepsilon \int_{\Omega} |\nabla \varphi|^2 \phi^2 dx \right. \\
& \quad \left. + \frac{\alpha}{2} \int_{\Omega} |\nabla u|^2 \varphi^2 \phi^2 dx + \frac{1}{2} \int_{\Omega} \gamma |u - g|^2 \phi^2 dx + \frac{\beta}{4\varepsilon} \int_{\Omega} |\varphi - 1|^2 \phi^2 dx \right] \\
& \leq 2k_\varepsilon \alpha |(\nabla u, \phi \nabla \phi u_t)| + 4\beta \varepsilon |(\nabla \varphi, \phi \nabla \phi \varphi_t)| + 2\alpha |(\varphi^2 \nabla u, \phi \nabla \phi u_t)| \\
& \leq 2k_\varepsilon^2 \alpha^2 \int_{\Omega} |\nabla u|^2 |\nabla \varphi|^2 dx + 4\beta^2 \varepsilon^2 \int_{\Omega} |\nabla \varphi|^2 |\nabla \phi|^2 dx \\
& \quad + 2\alpha^2 \int_{\Omega} \varphi^4 |\nabla u|^2 |\nabla \phi|^2 dx + \int_{\Omega} [|u_t|^2 + |\varphi_t|^2] \phi^2 dx \\
& \leq \frac{16}{R^2} [k_\varepsilon \alpha + \alpha + \beta \varepsilon] AT_\varepsilon(u_0, \varphi_0) + \int_{\Omega} [|u_t|^2 + |\varphi_t|^2] \phi^2 dx.
\end{aligned}$$

Integration over time then implies the assertion. \square

To proceed, we need some notation. Suppose $(u_0, \varphi_0) \in [H^1(\Omega)]^2$, for any given $\varepsilon_1 > 0$, let $R_1 > 0$ be the maximal number such that

$$(2.8) \quad \sup_{x_0 \in \Omega} AT_\varepsilon((u_0, \varphi_0); B_{2R_1}(x_0)) < \varepsilon_1,$$

and let $T_1 > 0$ be a number such that any weak solution (u, φ) of (1.5)–(1.8) taking the initial value (u_0, φ_0) satisfies

$$(2.9) \quad \sup_{x_0 \in \Omega} AT_\varepsilon((u(t), \varphi(t)); B_{R_1}(x_0)) < 2\varepsilon_1.$$

Note that, in view of the local energy law of Lemma 2.5, we may let $T_1 = \frac{\varepsilon_1 R_1^2}{\mathcal{B}_2 AT_\varepsilon(u_0, \varphi_0)}$.

THEOREM 2.6. *Let $N = 2$, suppose that $AT_\varepsilon(u_0, \varepsilon_0) < \infty$. For sufficiently small $\varepsilon_1 = O(\varepsilon k_\varepsilon^2)$, choose T_1 as above, (1.5)–(1.8) has a unique strong solution $(u, \varphi) \in [L^\infty((0, T_1); H^1(\Omega)) \cap L^2((0, T_1); H^2(\Omega))]^2$ which satisfies*

$$\begin{aligned}
& \max_{[0, T_1]} (\| \nabla u \|_{L^2}^2 + \| \nabla \varphi \|_{L^2}^2) + \int_0^{T_1} (\alpha \| \sqrt{k_\varepsilon^2 + \varphi^2} \Delta u \|_{L^2}^2 + \beta \varepsilon \| \Delta \varphi \|_{L^2}^2) ds \\
& \leq C_0 := \frac{C \varepsilon k_\varepsilon}{\mathcal{B}_2} \left[\frac{R_1^2}{AT_\varepsilon(u_0, \varphi_0)} \| \gamma g \|_{L^2}^2 + k_\varepsilon \right].
\end{aligned}$$

Proof. It suffices to derive the desired a priori error estimate and prove the uniqueness. We divide the proof into three steps.

Step 1: Existence: a first a priori bound. Testing equations (1.5)–(1.6) with $(-\Delta u, -\Delta \varphi)$ gives

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\| \nabla u \|_{L^2}^2 + \| \nabla \varphi \|_{L^2}^2) + \frac{\alpha}{2} \| \sqrt{k_\varepsilon^2 + \varphi^2} \Delta u \|_{L^2}^2 + 2\beta \varepsilon \| \Delta \varphi \|_{L^2}^2 + \frac{\beta}{2\varepsilon} \| \nabla \varphi \|_{L^2}^2 + \| \sqrt{\gamma} \nabla u \|_{L^2}^2 \\
(2.10) \quad & \leq \frac{1}{2\alpha k_\varepsilon} \| \gamma g \|_{L^2}^2 + 2\alpha |(\varphi \nabla \varphi, \nabla u \Delta u)| + \alpha (| \nabla u |^2 \varphi, \Delta \varphi) |.
\end{aligned}$$

The last two terms can be bounded as follows

$$\begin{aligned}
(2.11) \quad & 2\alpha |(\varphi \nabla \varphi, \nabla u \Delta u)| + \alpha (| \nabla u |^2 \varphi, \Delta \varphi) | \leq \frac{\alpha}{4} \| \varphi \Delta u \|_{L^2}^2 + \beta \varepsilon \| \Delta \varphi \|_{L^2}^2 \\
& \quad + 2\alpha \| \nabla \varphi \|_{L^4}^4 + \left[2\alpha + \frac{\alpha^2}{4\beta \varepsilon} \right] \| \nabla u \|_{L^4}^4.
\end{aligned}$$

Step 2: Existence: control of $(\nabla u, \nabla \varphi)$. Let $\{\phi_i\}_{i \geq 1}$ be a smooth partition of unity subordinate to a cover of Ω by balls $\{B_{R_1}(x_i)\}_{i \geq 1}$ with finite overlaps satisfying $0 \leq \phi_i \leq 1$, $|\nabla \phi_i| \leq \frac{2}{R_1}$, and $\sum_{i \geq 1} \phi_i^2 = 1$. Then, interpolating L^4 norm by L^2 and H^1 norms and using (2.9) we have

$$\begin{aligned}
(2.12) \quad \|\nabla\varphi(t)\|_{L^4}^4 &= \sum_{i \geq 1} \int_{\Omega} |\nabla\varphi(t)|^4 \phi_i^2 dx \\
&\leq C \sum_{i \geq 1} \|\nabla\varphi\|_{L^2(B_{R_1}(x_i))}^2 \left[\|\nabla^2\varphi\|_{L^2(B_{R_1}(x_i))}^2 + \|\nabla\varphi\|_{L^2(B_{R_1}(x_i))} \|\nabla\phi_i\|_{L^2(B_{R_1}(x_i))} \right] \\
&\leq \frac{C}{\beta\varepsilon} \sup_i AT_\varepsilon\left((u(t), \varphi(t)); B_{R_1}(x_i)\right) \left[\|\nabla^2\varphi(t)\|_{L^2}^2 + \frac{1}{\beta\varepsilon R_1^2} AT_\varepsilon(u_0, \varphi_0) \right] \\
&\leq \frac{C\varepsilon_1}{\beta\varepsilon} \left[\|\Delta\varphi(t)\|_{L^2}^2 + \frac{1}{\beta\varepsilon R_1^2} AT_\varepsilon(u_0, \varphi_0) \right] \quad \forall t \in [0, T_1].
\end{aligned}$$

Here we have used the Calderón-Zygmund inequality to get the last inequality. Similarly, we have

$$(2.13) \quad \|\nabla u(t)\|_{L^4}^4 \leq \frac{C\varepsilon_1}{k_\varepsilon} \left[\|\Delta u(t)\|_{L^2}^2 + \frac{1}{k_\varepsilon R_1^2} AT_\varepsilon(u_0, \varphi_0) \right] \quad \forall t \in [0, T_1].$$

We recall that T_1 depends on ε_1 linearly in the form $T_1 = \frac{\varepsilon_1 R_1^2}{B_2 AT_\varepsilon(u_0, \varphi_0)}$.

Substituting (2.11)-(2.13) into (2.10) yields

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|\nabla u\|^2 + \|\nabla\varphi\|^2) + \frac{\alpha}{4} \|\sqrt{k_\varepsilon + \varphi^2} \Delta u\|_{L^2}^2 + \beta\varepsilon \|\Delta\varphi\|_{L^2}^2 + \|\sqrt{\gamma} \nabla u\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|\nabla\varphi\|_{L^2}^2 \\
&\leq \frac{1}{2\alpha k_\varepsilon} \|\gamma g\|_{L^2}^2 + \frac{2\alpha\varepsilon_1 C}{\beta\varepsilon} \left[\|\Delta\varphi(t)\|_{L^2}^2 + \frac{1}{\beta\varepsilon R_1^2} AT_\varepsilon(u_0, \varphi_0) \right] \\
&\quad + \left(\frac{\alpha(8\beta\varepsilon + \alpha)\varepsilon_1 C}{4\beta\varepsilon k_\varepsilon} \right) \left[\|\Delta u(t)\|_{L^2}^2 + \frac{1}{k_\varepsilon R_1^2} AT_\varepsilon(u_0, \varphi_0) \right].
\end{aligned}$$

Then, the desired estimate follows from integrating in t from 0 to T_1 after choosing $\varepsilon_1 = O(\varepsilon k_\varepsilon^2)$.

Step 3: Uniqueness: We now show that the space $[L^\infty((0, T); H^1(\Omega)) \cap L^2((0, T); H^2(\Omega))]^2$ is a uniqueness class. Suppose (u_i, φ_i) for $i = 1, 2$ are two strong solutions corresponding to the same datum functions (u_0, φ_0) , and g . Let $e = u_1 - u_2$ and $\eta = \varphi_1 - \varphi_2$, subtracting equations satisfied by (u_i, φ_i) leads to the following ‘error’ equations which hold in distributional sense,

$$(2.14) \quad e_t - \alpha \operatorname{div}((k_\varepsilon + \varphi_1^2) \nabla e + \eta(\varphi_1 + \varphi_2) \nabla u_2) + \gamma e = 0 \quad \text{in } \Omega_T,$$

$$(2.15) \quad \eta_t - 2\beta\varepsilon \Delta \eta + \left(\alpha |\nabla u_1|^2 + \frac{\beta}{2\varepsilon} \right) \eta + \alpha (|\nabla u_1|^2 - |\nabla u_2|^2) \varphi_2 = 0 \quad \text{in } \Omega_T,$$

$$(2.16) \quad \frac{\partial e}{\partial n} = \frac{\partial \eta}{\partial n} = 0 \quad \text{on } \partial\Omega_T,$$

$$(2.17) \quad e(\cdot, 0) = \eta(\cdot, 0) = 0 \quad \text{in } \Omega.$$

Testing (2.14) by e and (2.15) by η we obtain

$$(2.18) \quad \frac{1}{2} \frac{d}{dt} \|e\|_{L^2}^2 + \alpha k_\varepsilon \|\nabla e\|_{L^2}^2 + \alpha \|\varphi_1 \nabla e\|_{L^2}^2 + \|\sqrt{\gamma} e\|_{L^2}^2 = -\alpha ((\varphi_1 + \varphi_2) \nabla u_2 \eta, \nabla e),$$

$$(2.19) \quad \frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + 2\beta\varepsilon \|\nabla \eta\|_{L^2}^2 + \alpha \|\nabla u_1 \eta\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|\eta\|_{L^2}^2 \\ = -\alpha ((|\nabla u_1|^2 - |\nabla u_2|^2) \varphi_2, \eta).$$

Reversing the roles of (u_1, φ_1) and (u_2, φ_2) in the above derivation then gives

$$(2.20) \quad \frac{1}{2} \frac{d}{dt} \|e\|_{L^2}^2 + \alpha k_\varepsilon \|\nabla e\|_{L^2}^2 + \alpha \|\varphi_2 \nabla e\|_{L^2}^2 + \|\sqrt{\gamma} e\|_{L^2}^2 = -\alpha ((\varphi_1 + \varphi_2) \nabla u_1 \eta, \nabla e),$$

$$(2.21) \quad \frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + 2\beta\varepsilon \|\nabla \eta\|_{L^2}^2 + \alpha \|\nabla u_2 \eta\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|\eta\|_{L^2}^2$$

$$(2.22) \quad = -\alpha ((|\nabla u_1|^2 - |\nabla u_2|^2) \varphi_1, \eta).$$

Now, adding (2.20) to (2.18), and (2.21) to (2.19), we finally get

$$(2.23) \quad \begin{aligned} \frac{d}{dt} \|e\|_{L^2}^2 + 2\alpha k_\varepsilon \|\nabla e\|_{L^2}^2 + \alpha [\|\varphi_1 \nabla e\|_{L^2}^2 + \|\varphi_2 \nabla e\|_{L^2}^2] + 2\|\sqrt{\gamma}e\|_{L^2}^2 \\ = -\alpha((\varphi_1 + \varphi_2)(\nabla u_1 + \nabla u_2)\eta, \nabla e), \end{aligned}$$

$$(2.24) \quad \begin{aligned} \frac{d}{dt} \|\eta\|_{L^2}^2 + 4\beta\varepsilon \|\nabla \eta\|_{L^2}^2 + \alpha [\|\nabla u_1 \eta\|_{L^2}^2 + \|\nabla u_2 \eta\|_{L^2}^2] + \frac{\beta}{\varepsilon} \|\eta\|_{L^2}^2 \\ = -\alpha(|\nabla u_1|^2 - |\nabla u_2|^2)(\varphi_1 + \varphi_2), \eta) \\ = -\alpha((\varphi_1 + \varphi_2)(\nabla u_1 + \nabla u_2)\eta, \nabla e). \end{aligned}$$

We bound the term on the right hand sides of (2.23) and (2.24) as follows,

$$(2.25) \quad \begin{aligned} |2\alpha((\varphi_1 + \varphi_2)(\nabla u_1 + \nabla u_2)\eta, \nabla e)| &\leq 2\alpha\|\varphi_1 + \varphi_2\|_{L^\infty} \|\nabla e\|_{L^2} \|\nabla(u_1 + u_2)\|_{L^4} \|\eta\|_{L^4} \\ &\leq \frac{8\alpha}{k_\varepsilon} \|\nabla(u_1 + u_2)\|_{L^4}^2 \|\eta\|_{L^4}^2 + \frac{\alpha k_\varepsilon}{2} \|\nabla e\|_{L^2}^2 \\ &\leq \frac{8C\alpha^2}{\beta\varepsilon k_\varepsilon^2} \|\nabla(u_1 + u_2)\|_{L^4}^4 \|\eta\|_{L^2}^2 + 2\beta\varepsilon \|\nabla \eta\|_{L^2}^2 + \frac{\alpha k_\varepsilon}{2} \|\nabla e\|_{L^2}^2. \end{aligned}$$

Since $\nabla u_i \in L^2((0, T); H^2(\Omega)) \subset L^4((0, T); W^{1,4}(\Omega))$, it follows from adding (2.23) to (2.24), using (2.25) and the Gronwall's inequality that

$$\|e(t)\|_{L^2}^2 + \|\eta(t)\|_{L^2}^2 = 0 \quad \forall t \in [0, T^*].$$

Hence, uniqueness follows. \square

REMARK 2.3. *Step 3 above actually shows that $[L^\infty((0, T); H^1(\Omega)) \cap L^4((0, T); W^{1,4}(\Omega))]^2$ is a uniqueness class in both cases $N = 2$ and $N = 3$.*

The following lemma is now a consequence of Theorem 2.6.

COROLLARY 2.7. *Let $N = 2$, and ε_1, T_1 be same as in Theorem 2.6. Suppose $(u_0, \varphi_0) \in [H^2(\Omega)]^2$. Then, $(u, \varphi) \in [L^\infty((0, T_1); H^2(\Omega))]^2$, and*

$$(i) \operatorname{ess\,sup}_{[0, T_1]} \left(\|u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2 \right) + \int_0^{T_1} \left[\alpha \|\sqrt{k_\varepsilon + \varphi^2} \nabla u_t\|_{L^2}^2 + \alpha \|\varphi_t \nabla u\|_{L^2}^2 + \beta\varepsilon \|\nabla \varphi_t\|_{L^2}^2 \right] ds \\ \leq C_1 := \exp\left(\frac{\alpha(\mathcal{B}_2 C_0 + \varepsilon_1)C}{\beta\mathcal{B}_2}\right) \left[\|u_t(0)\|_{L^2}^2 + \|\varphi_t(0)\|_{L^2}^2 \right].$$

$$(ii) \operatorname{ess\,sup}_{[0, T_1]} \left(\alpha^2 k_\varepsilon \|\sqrt{k_\varepsilon + \varphi^2} \Delta u\|_{L^2}^2 + \beta^2 \varepsilon^2 \|\Delta \varphi\|_{L^2}^2 \right) \leq C_2 := C_1 + \frac{\alpha^2 \varepsilon (1 + k_\varepsilon) C}{R_1^2},$$

$$(iii) \int_0^{T_1} \|u_{tt}\|_{H^{-1}}^2 ds \leq C_3 := 2C_1 + 2\|\gamma\|_{L^\infty} AT_\varepsilon(u_0, \varphi_0),$$

$$(iii) \int_0^{T_1} \|\varphi_{tt}\|_{H^{-1}}^2 ds \leq C_4 := C C_1 \left[8\beta(\varepsilon + \beta k_\varepsilon) + \sqrt{\frac{C_2 AT_\varepsilon(u_0, \varphi_0)}{\alpha k_\varepsilon^3}} \right].$$

Proof. (i). Differentiating (1.5) and (1.6) in t yields

$$(2.26) \quad u_{tt} - \alpha \operatorname{div}((k_\varepsilon + \varphi^2)\nabla u_t + 2\varphi\varphi_t \nabla u) + \gamma u_t = 0 \quad \text{in } \Omega_T,$$

$$(2.27) \quad \varphi_{tt} - 2\beta\varepsilon \Delta \varphi_t + 2\alpha(\nabla u, \nabla u_t)\varphi + \alpha|\nabla u|^2 \varphi_t + \frac{\beta}{2\varepsilon} \varphi_t = 0 \quad \text{in } \Omega_T.$$

Testing (2.26) with u_t and (2.27) with φ_t , and adding the resulting equations lead to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2) + \alpha \|\sqrt{k_\varepsilon + \varphi^2} \nabla u_t\|_{L^2}^2 \\ + \alpha \|\varphi_t \nabla u\|_{L^2}^2 + 2\beta\varepsilon \|\nabla \varphi_t\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|\varphi_t\|_{L^2}^2 + \|\sqrt{\gamma} u_t\|_{L^2}^2 \\ \leq \frac{C\alpha^2}{\beta\varepsilon} \|\nabla u\|_{L^4}^4 \|\varphi_t\|_{L^2}^2 + \beta\varepsilon \|\nabla \varphi_t\|_{L^2}^2 + \frac{\alpha}{2} \|\varphi \nabla u_t\|_{L^2}^2 \end{aligned}$$

By (2.13), the first term on the right hand side can be bounded as

$$\frac{C\alpha^2}{\beta\varepsilon} \|\nabla u\|_{L^4}^4 \|\varphi_t\|_{L^2}^2 \leq \frac{C\alpha^2\varepsilon_1}{\beta\varepsilon k_\varepsilon} \left[\|\Delta u(t)\|_{L^2}^2 + \frac{1}{k_\varepsilon R_1^2} AT_\varepsilon(u_0, \varphi_0) \right] \|\varphi_t\|_{L^2}^2.$$

The regularity of the initial data ensures the existence of $\lim_{s \rightarrow 0} \|u_t(s)\|_{L^2}$ and $\lim_{s \rightarrow 0} \|\varphi_t(s)\|_{L^2}$. Recall that $\varepsilon_1 = O(\varepsilon k_\varepsilon^2)$, the assertion (i) then follows from applying the Gronwall's lemma and Theorem 2.6.

(ii). First, testing (1.6) by $-\beta\varepsilon\Delta\varphi$ and using (2.13) we get

$$(2.28) \quad \beta^2\varepsilon^2 \|\Delta\varphi\|_{L^2}^2 \leq \|\varphi_t\|_{L^2}^2 + \alpha^2 \|\nabla u\|_{L^4}^4 \leq \|\varphi_t\|_{L^2}^2 + \frac{\alpha^2\varepsilon_1 C}{k_\varepsilon} \|\Delta u\|_{L^2}^2 + \frac{\alpha^2\varepsilon_1 C}{k_\varepsilon^2 R_1^2} AT_\varepsilon(u_0, \varphi_0).$$

Then, testing (1.5) by $-\alpha k_\varepsilon \Delta u$ and using (2.12) and (2.13) we obtain

$$\begin{aligned} \alpha^2 k_\varepsilon \|\sqrt{k_\varepsilon + \varphi^2} \Delta u\|_{L^2}^2 &\leq \|u_t\|_{L^2}^2 + \alpha^2 k_\varepsilon (\|\nabla u\|_{L^4}^4 + \|\nabla \varphi\|_{L^4}^4) \\ &\leq \|u_t\|_{L^2}^2 + \frac{\alpha^2 \varepsilon_1 k_\varepsilon C}{\beta \varepsilon} \|\Delta \varphi\|_{L^2}^2 + \frac{\alpha^2 \varepsilon_1 (1 + k_\varepsilon) C}{k_\varepsilon} \|\Delta u\|_{L^2}^2 \\ &\quad + \left[\frac{\alpha^2 \varepsilon_1 (\beta^2 \varepsilon^2 + k_\varepsilon^3) C}{\beta^2 \varepsilon^2 k_\varepsilon^2 R_1^2} \right] AT_\varepsilon(u_0, \varphi_0). \end{aligned}$$

(iii) From (2.26) we have

$$\|u_{tt}\|_{H^{-1}}^2 \leq 2\alpha \left[\|\sqrt{k_\varepsilon + \varphi^2} \nabla u_t\|_{L^2}^2 + \|\varphi_t \nabla u\|_{L^2}^2 \right] + 2\|\gamma\|_{L^\infty}^2 \|u_t\|_{L^2}^2.$$

The assertion follows from integrating the above inequality in t from 0 to T_1 and using (i) and (2.6).

(iv) From (2.27), the assertions (i)-(ii) and (2.6) we conclude

$$\begin{aligned} \int_0^{T_1} \|\varphi_{tt}\|_{H^{-1}}^2 ds &\leq 8 \int_0^{T_1} \left[\beta^2 \varepsilon^2 \|\nabla \varphi_t\|_{L^2}^2 + \alpha \|\nabla u \cdot \nabla u_t \varphi\|_{H^{-1}}^2 + \alpha^2 \|\nabla u\|^2 \|\varphi_t\|_{H^{-1}}^2 \right. \\ &\quad \left. + \frac{\beta^2}{\varepsilon^2} \|\varphi_t\|_{L^2}^2 \right] ds \\ &\leq 8C_1 \beta \left(\varepsilon + \frac{\beta T_1}{\varepsilon^2} \right) + C\alpha^2 \int_0^{T_1} \|\nabla u\|_{L^4}^2 \left[\|\varphi \nabla u_t\|_{L^2}^2 + \|\nabla u \varphi_t\|_{L^2}^2 \right] ds \\ &\leq 8C_1 \beta \left(\varepsilon + \frac{\beta k_\varepsilon^2}{\varepsilon} \right) + C\alpha \int_0^{T_1} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2} \left[\alpha \|\varphi \nabla u_t\|_{L^2}^2 + \alpha \|\nabla u \varphi_t\|_{L^2}^2 \right] ds \\ &\leq 8C_1 \beta \left(\varepsilon + \beta k_\varepsilon \right) + \frac{C\sqrt{C_2} AT_\varepsilon(u_0, \varphi_0)}{\sqrt{\alpha} k_\varepsilon^{\frac{3}{2}}} C_1. \end{aligned}$$

The proof is complete. \square

REMARK 2.4. We note that by Morrey's lemma, a maximum radius $R_1 = R_1(\varepsilon, k_\varepsilon; \alpha, \beta)$ in (2.8) can be computed explicitly for a given ε_1 when the initial data $(u_0, \varphi_0) \in [H^2(\Omega)]^2$.

The local strong solution exists until the first time $T = t_1$ such that

$$\limsup_{t \rightarrow T} \left(AT_\varepsilon((u(t), \varphi(t)); B_R(x_0)) \right) \geq \varepsilon_1 \quad \forall R > 0.$$

By Theorem 2.3, $u_t, \varphi_t \in L^2(\Omega_T)$, hence the L^2 -limits $u_1 = \lim_{t \rightarrow T} u(t)$ and $\varphi_1 = \lim_{t \rightarrow T} \varphi(t)$ exist. Let (v, ψ) be a local strong solution of (1.5)-(1.8) over the time interval $[T, \bar{T})$ with initial data (u_1, φ_1) at time T . The composed function

$$(\tilde{u}(t), \tilde{\varphi}(t)) = \begin{cases} (u(t), \varphi(t)) & \text{for } 0 \leq t \leq T, \\ (v(t), \psi(t)) & \text{for } T \leq t \leq \bar{T}, \end{cases}$$

then actually is a quasi-strong solution of (1.5)-(1.8) over the time interval $[0, \bar{T}]$. It is easy to see that this extension process can be continued until $\bar{T} = \infty$.

We now prove that there are at most *finite number* of restarts in the above extension process. Moreover, we show that such a ‘piecewise’ (in t) strong solution has at most *finite number of point singularities* in space-time where the solution ceases to be strong, and it is unique in this class. To state the result, we need the following notation

$$\mathcal{Y}^T(\Omega) := \left\{ \chi \in L^\infty((0, T); H^1(\Omega)) : \exists (t_k, x_k)_{k=1}^K : \Delta\chi \in L^2(\Omega_T \setminus \bigcup_{1 \leq k \leq K} \overline{B_\delta(t_k, x_k)}), \forall \delta > 0 \right\}.$$

THEOREM 2.8. *Let $N = 2$ and $T = \infty$. Suppose $AT_\varepsilon(u_0, \varphi_0) < \infty$ and $g \in L^2(\Omega)$. Then, there exist at most finite number space-time points (x_k, t_k) , $1 \leq k \leq K$ such that (1.5)-(1.8) has a unique solution in $[\mathcal{Y}^T(\Omega)]^2$.*

Proof. Since existence has been showed above, it suffices to prove that the singularity set must be finite in space-time and uniqueness. The idea of the proof is borrowed from [34].

Let $\bar{t} = \bar{t}_1$ be the first singular time, and

$$\text{Sing}(\bar{t}) = \left\{ x_0 \in \Omega : \forall R > 0 : \limsup_{t \rightarrow \bar{t}} AT_\varepsilon((u(t), \varphi(t)); B_R(x_0)) > \varepsilon_1 \right\}.$$

We want to show that $\text{Sing}(\bar{t})$ is finite: suppose $x_1, \dots, x_K \in \text{Sing}(\bar{t})$. Choose $R > 0$ such that $B_{2R}(x_i) \cap B_{2R}(x_j) = \emptyset$ ($i \neq j$), and fix $\tau \in [\bar{t} - \frac{\varepsilon_1 R^2}{2\mathcal{B}_2 AT_\varepsilon(u_0, \varphi_0)}, \bar{t}]$, where \mathcal{B}_2 is the constant in Lemma 2.5 and $\varepsilon_1 = O(\varepsilon k_\varepsilon^2)$ same as in Theorem 2.3. According to Lemma 2.5,

$$\begin{aligned} K\varepsilon_1 &\leq \sum_{k=1}^K \limsup_{t \rightarrow \bar{t}} AT_\varepsilon((u(t), \varphi(t)); B_R(x_k)) \\ &\leq \sum_{i=1}^K \left[AT_\varepsilon(u(\tau), \varphi(\tau); B_{2R}(x_i)) + \frac{\varepsilon_1}{2} \right] \leq AT_\varepsilon(u_0, \varphi_0) + \frac{K\varepsilon_1}{2}, \end{aligned}$$

and hence $K = K_1 = \text{card}(\text{Sing}(\bar{t}_1)) \leq 2AT_\varepsilon(u_0, \varphi_0)\varepsilon_1^{-1}$. Moreover, for $(u_1, \varphi_1) = \lim_{t \rightarrow \bar{t}_1} (u(t), \varphi(t))$ we have

$$\begin{aligned} AT_\varepsilon(u_1, \varphi_1) &= \lim_{R \rightarrow 0} AT_\varepsilon\left((u_1, \varphi_1); \Omega \setminus \bigcup_{i=1}^{K_1} \overline{B_{2R}(x_i)}\right) \\ &\leq \lim_{R \rightarrow 0} \limsup_{t \rightarrow \bar{t}_1} AT_\varepsilon\left((u(t), \varphi(t)); \Omega \setminus \bigcup_{i=1}^{K_1} \overline{B_{2R}(x_i)}\right) \\ &\leq \lim_{R \rightarrow 0} \limsup_{t \rightarrow \bar{t}_1} \left[AT_\varepsilon(u(t), \varphi(t)) - \sum_{i=1}^{K_1} AT_\varepsilon((u(t), \varphi(t)); B_{2R}(x_i)) \right] \\ &\leq \lim_{R \rightarrow 0} \left[AT_\varepsilon(u_0, \varphi_0) - \liminf_{t \rightarrow \bar{t}_1} \sum_{i=1}^{K_1} AT_\varepsilon((u(t), \varphi(t)); B_{2R}(x_i)) \right] \\ &\leq AT_\varepsilon(u_0, \varphi_0) - \sum_{i=1}^{K_1} \lim_{R \rightarrow 0} \limsup_{t \rightarrow \bar{t}_1} AT_\varepsilon((u(t), \varphi(t)); B_R(x_i)) \\ &\leq AT_\varepsilon(u_0, \varphi_0) - K_1\varepsilon_1. \end{aligned}$$

Similarly, let K_2, K_3, \dots be the number of singular points at consecutive times $\bar{t}_2 < \bar{t}_3 < \dots$, and let $(u_j, \varphi_j) = \lim_{t \rightarrow \bar{t}_j} (u(t), \varphi(t))$, for $j = 2, 3, \dots$. By induction, we have

$$AT_\varepsilon(u_j, \varphi_j) \leq AT_\varepsilon(u_{j-1}, \varphi_{j-1}) - K_j\varepsilon_1 \leq \dots \leq AT_\varepsilon(u_0, \varphi_0) - (K_1 + \dots + K_j)\varepsilon_1.$$

This implies that the total number K of singular points, hence also the number of singular times \bar{t}_j is finite. Indeed, $K \leq AT_\varepsilon(u_0, \varphi_0)\varepsilon_1^{-1}$.

The finite number of singular times together with Theorem 2.5 implies uniqueness of solutions in the class \mathcal{Y}^T . \square

REMARK 2.5. (a). *It is not clear whether singularity really exists for a fixed ε although we believe so, the above theorem describes the worst possible scenario when it does.*

(b). *The estimate $K \leq AT_\varepsilon(u_0, \varepsilon_0)\varepsilon_1^{-1}$ indicates that K could tend to infinity very fast as $\varepsilon \rightarrow 0$.*

3. Analysis of fully discrete finite element approximations. In this section, we shall propose a family of fully discrete finite element discretizations for the gradient flow (1.5)-(1.8), including mass lumping strategies. Three (semi-)implicit schemes for both ‘consistent’ and mass lumped spatial discretizations are analyzed in detail. We prove convergence of solutions to these discrete schemes (‘mass lumped version’) to weak and (piecewise) strong solutions of (1.5)-(1.8), and establish optimal order error estimates for the numerical solutions (‘consistent version’) in the case that (1.5)-(1.8) possesses strong solutions.

3.1. Fully discrete schemes. In the sequel, we assume that $\Omega \subset \mathbf{R}^N$ is a polygonal domain. Let \mathcal{T}_h be a quasi-uniform triangulation of Ω ($K \in \mathcal{T}_h$ are tetrahedrons when $N = 3$) with mesh size $h \in (0, 1)$. Let V^h denote the finite element space of continuous, piecewise linear functions associated with \mathcal{T}_h , that is,

$$V^h = V^h(\Omega) := \{ v_h \in C^0(\overline{\Omega}); v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h \}.$$

Let $\{t_m\}_{m=0}^M$ be an equidistant partition of $[0, T]$ of mesh size $k \in (0, 1)$ and introduce the notation $d_t u^m := (u^m - u^{m-1})/k$. We now define the following family of consistent fully discrete finite element discretizations for the gradient flow (1.5)-(1.8): Find $(u_h^m, \varphi_h^m) \in [V^h]^2$ for $m = 0, 1, 2, \dots, M$ such that

$$(3.1) \quad (d_t u_h^m, v_h) + \alpha(k_\varepsilon \nabla u_h^m + |\varphi_h^{m,\mu}|^2 \nabla u_h^m, \nabla v_h) + (\gamma[u_h^m - g^m], v_h) = 0 \forall v_h \in V^h,$$

$$(3.2) \quad (d_t \varphi_h^m, \psi_h) + 2\beta\varepsilon(\nabla \varphi_h^m, \nabla \psi_h) + \alpha(|\nabla u_h^{m,\nu}|^2 \varphi_h^m, \psi_h) + \frac{\beta}{2\varepsilon}(\varphi_h^m - 1, \psi_h) = 0 \forall \psi_h \in V^h,$$

with some starting value $(u_h^0, \varphi_h^0) \in [V^h]^2$.

Approximation of weak solutions requires a modification of this scheme (‘mass lumped version’) that has a maximum principle for discrete solutions $\{\varphi_h^m\}$: Find $(u_h^m, \varphi_h^m) \in [V^h]^2$ for $m = 0, 1, 2, \dots, M$ such that

$$(3.3) \quad (d_t u_h^m, v_h) + \alpha k_\varepsilon(\nabla u_h^m, \nabla v_h) + \alpha(|\varphi_h^{m,\mu}|^2 \nabla u_h^m, \nabla v_h)_h + (\gamma[u_h^m - g^m], v_h) = 0 \forall v_h \in V^h,$$

$$(3.4) \quad (d_t \varphi_h^m, \psi_h) + 2\beta\varepsilon(\nabla \varphi_h^m, \nabla \psi_h) + \alpha(|\nabla u_h^{m,\nu}|^2 \varphi_h^m, \psi_h)_h + \frac{\beta}{2\varepsilon}(\varphi_h^m - 1, \psi_h)_h = 0 \forall \psi_h \in V^h,$$

with some starting value $(u_h^0, \varphi_h^0) \in [V^h]^2$.

In both above discretizations, (\cdot, \cdot) denotes the standard inner product of $L^2(\Omega)$, and

$$\begin{aligned} \varphi_h^{m,\mu} &:= \mu \varphi^m + (1 - \mu) \varphi^{m-1} & \forall \mu \in [0, 1], \\ u_h^{m,\nu} &:= \nu u^m + (1 - \nu) u^{m-1} & \forall \nu \in [0, 1]. \end{aligned}$$

Moreover, lumped mass integration in (3.3)–(3.4) is defined by

$$(3.5) \quad (\chi, \eta)_h = \sum_{K \in \mathcal{T}_h} \int_K I_h(\chi\eta) \, dx \quad \forall \chi, \eta \in C(\overline{\Omega}),$$

where $I_h : C(\overline{\Omega}) \rightarrow V^h$ denotes the (linear) Lagrange interpolation operator on \mathcal{T}_h . We recall the following properties of lumped mass integration (cf. appendix of [19]),

$$(3.6) \quad |(\chi_h, \eta_h)_h - (\chi_h, \eta_h)| \leq Ch^2 \|\chi_h\|_{H^1} \|\eta_h\|_{H^1} \quad \forall \chi_h, \eta_h \in V^h,$$

$$(3.7) \quad |(\chi_h, \eta)_h - (\chi_h, \eta)| \leq Ch^2 \|\chi_h\|_{H^1} \|\eta\|_{H^2} \quad \forall \chi_h \in V^h, \quad \forall \eta \in H^2(\Omega),$$

$$(3.8) \quad c_0 \|\chi_h\|_{L^2} \leq \|\chi_h\|_h := ((\chi_h, \chi_h)_h)^{\frac{1}{2}} \leq c_1 \|\chi_h\|_{L^2} \quad \forall \chi_h \in V^h,$$

where $c_i > 0$, $i = 0, 1$ does not depend on h .

REMARK 3.1. *If $\mu\nu \neq 0$, the above family of schemes is fully implicit, and if $\mu\nu = 0$, the family is semi-implicit. In the rest of this section, we will mainly consider the following cases of (μ, ν) . (i) $(\mu, \nu) = (0, 1)$, (ii) $(\mu, \nu) = (1, 0)$, and (iii) $(\mu, \nu) = (1, 1)$. We also remark that the unique solvability of (3.1)–(3.2), and (3.3)–(3.4) is trivial for the cases (i) and (ii). For the case (iii), it can be shown by using a fixed point argumentation in view of the a priori estimates to be given in Lemma 3.2 below.*

3.2. Convergence of the fully discrete schemes. In this subsection we shall establish convergence of a subsequence (resp. the whole sequence) of the fully discrete solution defined by (3.3)-(3.4) to a weak solution (resp. the strong solution) of (1.5)-(1.8) as $h, k \rightarrow 0$. We begin the subsection with a definition of acute triangulations [14], which will be imposed to ensure that the discrete maximum principle holds for the solution $\{\varphi_h^m\}$.

DEFINITION 3.1. A triangulation \mathcal{T}_h is said to be an acute triangulation if there exists $\theta_0 > 0$ such that every interior angle of all elements in \mathcal{T}_h is less than or equal to $\frac{\pi}{2} - \theta_0$.

LEMMA 3.2. The solutions (u_h^m, φ_h^m) of both (3.1)-(3.2) and (3.3)-(3.4) satisfy the following estimates:

$$(3.9) \quad \max_{0 \leq m \leq M} \|u_h^m\|_{L^2}^2 + k \sum_{m=1}^M \left\{ k \|d_t u_h^m\|_{L^2}^2 + \alpha k_\varepsilon \|\nabla u_h^m\|_{L^2}^2 + \alpha \|\varphi^{m,\mu} \nabla u_h^m\|_h^2 \right. \\ \left. + \alpha c_0^2 \|\varphi^{m,\mu} \nabla u_h^m\|_{L^2}^2 + \|\sqrt{\gamma} u_h^m\|_{L^2}^2 \right\} \leq \|u_h^0\|_{L^2}^2 + k \sum_{m=1}^M \|\sqrt{\gamma} g^m\|_{L^2}^2,$$

$$(3.10) \quad \max_{0 \leq m \leq M} \|\varphi_h^m\|_{L^2}^2 + k \sum_{m=1}^M \left\{ k \|d_t \varphi_h^m\|_{L^2}^2 + 4\beta\varepsilon \|\nabla \varphi_h^m\|_{L^2}^2 + \alpha \|\varphi_h^m \nabla u^{m,\nu}\|_h^2 \right. \\ \left. + \alpha c_0^2 \|\varphi_h^m \nabla u^{m,\nu}\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|\varphi_h^m - 1\|_{L^2}^2 \right\} \leq \|\varphi_h^0\|_{L^2}^2 + \frac{\beta T}{2\varepsilon} |\Omega|.$$

In the case (3.3)-(3.4), and for an acute triangulation \mathcal{T}_h of Ω , $h^2 \leq C_{\theta_0} \beta \varepsilon k$, and $0 \leq \varphi_h^0 \leq 1$, then

$$(3.11) \quad 0 \leq \varphi_h^m \leq 1 \quad \forall 0 \leq m \leq M, \text{ a.e. } x \in \Omega.$$

Where C_{θ_0} is positive constant which depends on θ_0 but is independent of h .

Proof. The proof for (3.1)-(3.2) is in the same line as that for Theorem 2.2. (3.9) and (3.10) are obtained by testing (3.1) by u_h^m and testing (3.2) by φ_h^m , respectively, and applying the operator $k \sum_{m=1}^\ell$ ($\ell \leq M$) to the resulting inequalities. For (3.3)-(3.4), since I_h is linear and $\|\nabla u_h^m\|_K \equiv \text{const}$ for any $K \in \mathcal{T}_h$, elementwise application of (3.8) then yields assertions (3.9) and (3.10), respectively.

To show (3.11), let $\{\phi_i\}_{i=1}^I$ denote the canonical basis of V^h , $K := \{k_{ij}\}_{i,j=1}^I$ be the stiffness matrix, $M := \{m_{ij}\}_{i,j=1}^I$ the mass matrix, $M^h := \{m_{ij}^h\}_{i,j=1}^I$ the lumped mass matrix, and $M_\beta^h = \{(m_\beta^h)_{ij}\}_{i,j=1}^I$, $\beta \in \mathbb{R}^I$ a scaled lumped mass matrix, that is

$$k_{ij} := (\nabla \phi_i, \nabla \phi_j), \quad m_{ij} := (\phi_i, \phi_j), \quad m_{ij}^h = (\phi_i, \phi_j)_h = \int_\Omega I_h(\phi_i \phi_j) dx, \quad (m_{|\beta|}^h)_{ij} = |\beta_i|^2 m_{ij}^h.$$

Consider $a_{ij} := \frac{1}{k} m_{ij} + \beta \varepsilon k_{ij} + \alpha (m_{\nabla u_h^{m,\nu}}^h)_{ij} + \frac{\beta}{2\varepsilon} m_{ij}^h$. Since \mathcal{T}_h is acute, there exists $C_{\theta_0} > 0$, such that $k_{ij} \leq -C_{\theta_0} < 0$ uniformly in h , for any pair of adjacent nodes. This implies $a_{ij} \leq 0$, provided $i \neq j$ and $0 \leq k^{-1} \leq C_{\theta_0} h^{-2}$; $\{a_{ij}\}_{i,j=1}^I$ is thus an M -matrix.

Now, the desired discrete maximum principle follows from testing (3.2) by $[\varphi_h^m]^- := I_h(\varphi_h^m)^-$ and $[\varphi_h^m - 1]^+ := I_h(\varphi_h^m - 1)^+$, as well as utilizing an inductive argument used in the proofs of Lemmas 4.1 and 4.2 of [32].

□

We introduce the following modified energy $\overline{AT}_\varepsilon : V^h \times V^h \rightarrow \mathbf{R}$,

$$(3.12) \quad \overline{AT}_\varepsilon(u_h^m, \varphi_h^m) = \frac{\alpha}{2} (k_\varepsilon \|\nabla u_h^m\|_{L^2}^2 + \|\varphi_h^m \nabla u_h^m\|_h^2) + \beta \varepsilon \|\nabla \varphi_h^m\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|\varphi_h^m - 1\|_h^2 + \frac{\gamma}{2} \|u_h^m - g\|_{L^2}^2.$$

It follows from (3.8) that

$$(3.13) \quad \min\{c_0, 1\} AT_\varepsilon(u_h^m, \varphi_h^m) \leq \overline{AT}_\varepsilon(u_h^m, \varphi_h^m) \leq \max\{c_1, 1\} AT_\varepsilon(u_h^m, \varphi_h^m).$$

LEMMA 3.3. Suppose that g is independent of t , then the solution (u_h^m, φ_h^m) of (3.3)-(3.4) satisfies the following equations.

(i). For $(\mu, \nu) = (1, 0)$ and $(\mu, \nu) = (0, 1)$, there holds for all $1 \leq \ell \leq M$

$$(3.14) \quad \begin{aligned} \overline{AT}_\varepsilon(u_h^\ell, \varphi_h^\ell) + k \sum_{m=1}^{\ell} \left[\|d_t u_h^m\|_{L^2}^2 + \|d_t \varphi_h^m\|_{L^2}^2 \right] + \frac{k^2}{2} \sum_{m=1}^{\ell} \left[\alpha k_\varepsilon \|d_t \nabla u_h^m\|_{L^2}^2 \right. \\ \left. + \alpha \|\varphi^{m,\mu} d_t \nabla u_h^m\|_h^2 + \|\sqrt{\gamma} d_t u_h^m\|_{L^2}^2 + 2\beta \varepsilon \|d_t \nabla \varphi_h^m\|_{L^2}^2 \right. \\ \left. + \alpha \|\nabla u^{m,\nu} d_t \varphi_h^m\|_h^2 + \frac{\beta}{2\varepsilon} \|d_t \varphi_h^m\|_h^2 \right] = \overline{AT}_\varepsilon(u_h^0, \varphi_h^0). \end{aligned}$$

(ii). For $(\mu, \nu) = (1, 1)$, there holds for all $1 \leq \ell \leq M$

$$(3.15) \quad \begin{aligned} \overline{AT}_\varepsilon(u_h^\ell, \varphi_h^\ell) + k \sum_{m=1}^{\ell} \left[\|d_t u_h^m\|_{L^2}^2 + \|d_t \varphi_h^m\|_{L^2}^2 \right] + \frac{k^2}{2} \sum_{m=1}^{\ell} \left[\alpha k_\varepsilon \|d_t \nabla u_h^m\|_{L^2}^2 \right. \\ \left. + \alpha \|\varphi^{m,\mu} d_t \nabla u_h^m\|_h^2 + \|\sqrt{\gamma} d_t u_h^m\|_{L^2}^2 + 2\beta \varepsilon \|d_t \nabla \varphi_h^m\|_{L^2}^2 \right. \\ \left. + \alpha \|\nabla u^{m,\nu} d_t \varphi_h^m\|_h^2 + \frac{\beta}{2\varepsilon} \|d_t \varphi_h^m\|_h^2 + \alpha (d_t |\varphi_h^m|^2, d_t |\nabla u_h^m|^2)_h \right] = \overline{AT}_\varepsilon(u_h^0, \varphi_h^0). \end{aligned}$$

Moreover, statements (i) and (ii) also hold for the solution of (3.1)-(3.2) with $AT_\varepsilon(\cdot, \cdot)$ and $\|\cdot\|_{L^2}$ being in place of $\overline{AT}_\varepsilon(\cdot, \cdot)$ and $\|\cdot\|_h$, respectively.

Proof. The proof of (3.14) is in the same line as that of (2.6). That is, the assertion follows from testing (3.3) by $d_t u_h^m$ and (3.4) by $d_t \varphi_h^m$, using the algebraic formulas

$$\begin{aligned} (d_t \xi^m) \xi^m &= \frac{1}{2} d_t |\xi^m|^2 + \frac{k}{2} |d_t \xi^m|^2, \\ d_t (\xi^m \psi^m) &= \xi^m d_t \psi^m + \psi^{m-1} d_t \xi^m = \psi^m d_t \xi^m + \xi^{m-1} d_t \psi^m, \end{aligned}$$

adding two resulting equations and applying the operator $k \sum_{m=1}^{\ell}$ ($\ell \leq M$) to the sum.

(3.15) follows from repeating the above procedure. The only difference is that we now need to use the following algebraic formula to handle the nonlinear terms

$$\xi^m d_t \psi^m + \psi^m d_t \xi^m = d_t (\xi^m \psi^m) + k d_t \xi^m d_t \psi^m.$$

The proof is complete. \square

REMARK 3.2. If $g \in L^2((0, T); L^2(\Omega))$, similar estimates to (3.14) and (3.15) hold for (u_h^m, φ_h^m) . Since in this case we need to control the term $(g, d_t u_h^m)$ by $\frac{1}{2} \|g\|_{L^2}^2 + \frac{1}{2} \|d_t u_h^m\|_{L^2}^2$, hence, we do not get equalities like (3.14) and (3.15), instead, we now obtain inequalities with a term involves g^m on the right-hand sides.

Corresponding estimates to (3.14) and (3.15) for (3.3)-(3.4) in terms of $AT_\varepsilon(\cdot)$ and mesh-independent norms immediately follow from (3.13) and (3.8).

We remark that estimate (3.15) is not in a closed form, since the last term on the left-hand side does not have a fixed sign. In fact, it is not clear if that term can be absorbed by the positive terms that proceed it, allowing that h, k and \mathcal{T}_h satisfy some constraints.

For the fully discrete finite element solutions $\{u_h^m\}$ and $\{\varphi_h^m\}$, we define their constant and linear interpolations in t as follows:

$$(3.16) \quad \overline{u}_h(\cdot, t) := u_h^m, \quad \underline{u}_h(\cdot, t) := u_h^{m-1} \quad \forall t \in (t_{m-1}, t_m), \quad 1 \leq m \leq M,$$

$$(3.17) \quad \overline{\overline{u}}_h(\cdot, t) := \frac{t - t_{m-1}}{k} u_h^m + \frac{t_m - t}{k} u_h^{m-1} \quad \forall t \in [t_{m-1}, t_m], \quad 1 \leq m \leq M;$$

$$(3.18) \quad \overline{\varphi}_h(\cdot, t) := \varphi_h^m, \quad \underline{\varphi}_h(\cdot, t) := \varphi_h^{m-1} \quad \forall t \in (t_{m-1}, t_m), \quad 1 \leq m \leq M,$$

$$(3.19) \quad \overline{\overline{\varphi}}_h(\cdot, t) := \frac{t - t_{m-1}}{k} \varphi_h^m + \frac{t_m - t}{k} \varphi_h^{m-1} \quad \forall t \in [t_{m-1}, t_m], \quad 1 \leq m \leq M,$$

$$(3.20) \quad \overline{u}_h(\cdot, 0) = \underline{u}_h(\cdot, 0) = \overline{\overline{u}}_h(\cdot, 0) := u_h^0, \quad \overline{\varphi}_h(\cdot, 0) = \underline{\varphi}_h(\cdot, 0) = \overline{\overline{\varphi}}_h(\cdot, 0) := \varphi_h^0.$$

Clearly, $\overline{u}_h, \underline{u}_h, \overline{\varphi}_h$ and $\underline{\varphi}_h$ are continuous in x but discontinuous in t . On the other hand, $\overline{\overline{u}}_h$ and $\overline{\overline{\varphi}}_h$ are continuous in both x and t .

THEOREM 3.4. *Let (u_h, φ_h) be the solution of (3.3)-(3.4), which is defined on an acute triangulation \mathcal{T}_h , and $h^2 \leq C_{\theta_0} \beta \varepsilon k$. Suppose that the starting value (u_h^0, φ_h^0) satisfies*

$$(3.21) \quad \lim_{h,k \rightarrow 0} (\|u_h^0 - u_0\|_{L^2} + \|\varphi_h^0 - \varphi_0\|_{L^2}) = 0.$$

Let $(\mu, \nu) = (1, 1)$. Then there exists a subsequence of $\{(u_h, \varphi_h)\}$ (still denoted by the same notation) and a solution (u, φ) of (1.5)-(1.8) such that

$$(3.22) \quad \lim_{h,k \rightarrow 0} (\|\bar{u}_h - u\|_{L^2(L^2)} + \|\bar{\varphi}_h - \varphi\|_{L^2(L^2)}) = 0,$$

$$(3.23) \quad \lim_{h,k \rightarrow 0} (\|\underline{u}_h - u\|_{L^2(L^2)} + \|\underline{\varphi}_h - \varphi\|_{L^2(L^2)}) = 0,$$

$$(3.24) \quad \lim_{h,k \rightarrow 0} (\|\bar{\bar{u}}_h - u\|_{L^2(L^2)} + \|\bar{\bar{\varphi}}_h - \varphi\|_{L^2(L^2)}) = 0.$$

Proof. First we notice that (3.9)-(3.11) imply the following (uniform in h and k) estimates:

$$(3.25) \quad \|\nabla \bar{u}_h\|_{L^2(L^2)} \leq 2\|\nabla \underline{u}_h\|_{L^2(L^2)} = 2\|\nabla \underline{u}_h\|_{L^2(L^2)} = 2\left(k \sum_{m=1}^M \|\nabla u_h^m\|_{L^2}^2\right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{k_\varepsilon}},$$

$$(3.26) \quad \|\bar{u}_h\|_{L^\infty(L^2)} \leq \|\underline{u}_h\|_{L^\infty(L^2)} = \|\underline{u}_h\|_{L^\infty(L^2)} = \max_{0 \leq m \leq M} \|u_h^m\|_{L^2} \leq C,$$

$$(3.27) \quad \sum_{i=1}^M \|u_h^m - u_h^{m-1}\|_{L^2}^2 \leq C,$$

$$(3.28) \quad \|\nabla \bar{\bar{\varphi}}_h\|_{L^2(L^2)} \leq 2\|\nabla \underline{\varphi}_h\|_{L^2(L^2)} = 2\|\nabla \underline{\varphi}_h\|_{L^2(L^2)} = 2\left(k \sum_{m=1}^M \|\nabla \varphi_h^m\|_{L^2}^2\right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{\varepsilon}},$$

$$(3.29) \quad 0 \leq \min_{0 \leq m \leq M} \varphi_h^m \leq \bar{\bar{\varphi}}_h(x, t), \quad \bar{\varphi}_h(x, t), \quad \underline{\varphi}_h(x, t) \leq \max_{0 \leq m \leq M} \varphi_h^m \leq 1 \quad (x, t) \in \Omega_T,$$

$$(3.30) \quad \sum_{i=1}^M \|\varphi_h^m - \varphi_h^{m-1}\|_{L^2}^2 \leq C.$$

Then there exist convergent subsequences of $\{\bar{u}_h\}$ and $\{\bar{\varphi}_h\}$ (denoted by the same notation) and a pair of functions $(\hat{u}, \hat{\varphi}) \in [L^\infty((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))]^2$ such that as $h, k \rightarrow 0$

$$(3.31) \quad \begin{aligned} \bar{u}_h &\longrightarrow \hat{u} && \text{weakly}^\star \text{ in } L^\infty((0, T); L^2(\Omega)), \\ &&& \text{weakly in } L^2((0, T); H^1(\Omega)) \end{aligned}$$

$$(3.32) \quad \begin{aligned} \bar{\varphi}_h &\longrightarrow \hat{\varphi} && \text{weakly}^\star \text{ in } L^\infty(\Omega_T), \\ &&& \text{weakly in } L^2((0, T); H^1(\Omega)) \end{aligned}$$

Following a standard procedure (cf. [24]), it can be shown that

$$(3.33) \quad \bar{u}_h \longrightarrow \hat{u}, \quad \bar{\varphi}_h \longrightarrow \hat{\varphi} \quad \text{strongly in } L^2(\Omega_T),$$

by first proving $\{(\bar{u}_h, \bar{\varphi}_h)\}$ is a Cauchy sequence in $[L^2(\Omega_T)]^2$ for almost every $t \in [0, T]$. We remark that (3.21) is required in this step.

Hence, (3.33) implies that there exist subsequences of $\{\bar{u}_h\}$ and $\{\bar{\varphi}_h\}$ (still denoted by the same notation) such that

$$(3.34) \quad \bar{u}_h \longrightarrow \hat{u}, \quad \bar{\varphi}_h \longrightarrow \hat{\varphi} \quad \text{a.e. in } \Omega_T,$$

which and (3.29) immediately imply

$$(3.35) \quad 0 \leq \hat{\varphi}(x, t) \leq 1 \quad \text{for a.e. } (x, t) \in \Omega_T.$$

From (3.27) we have

$$\begin{aligned}
(3.36) \quad \|\bar{u}_h - \underline{u}_h\|_{L^2(L^2)}^2 &= \int_0^T \|\bar{u}_h - \underline{u}_h\|_{L^2}^2 dt \\
&= \sum_{m=1}^M \|u_h^m - u_h^{m-1}\|_{L^2}^2 \int_{t_{m-1}}^{t_m} \left(\frac{t_m - t}{k}\right)^2 dt \\
&= \frac{k}{3} \sum_{m=1}^M \|u_h^m - u_h^{m-1}\|_{L^2}^2 \leq C k.
\end{aligned}$$

Similarly,

$$(3.37) \quad \|\bar{u}_h - \underline{u}_h\|_{L^2(L^2)}^2 + \|\bar{\varphi}_h - \underline{\varphi}_h\|_{L^2(L^2)}^2 + \|\bar{\varphi}_h - \underline{\varphi}_h\|_{L^2(L^2)}^2 \leq C k.$$

In view of (3.33), (3.36)-(3.37), we conclude that (3.22)-(3.24) hold provided that we can show $(\hat{u}, \hat{\varphi})$ is a weak solution of (1.5)-(1.8).

Let $\phi \in C_0^\infty((0, T))$ and $(v, \psi) \in [C^\infty(\Omega)]^2$. It follows from the finite element theory [10, 14] that there exists a sequence $\{(v_h, \psi_h)\} \in [V^h]^2$ which converges to (v, ψ) strongly in $[W^{1,\infty}(\Omega)]^2$. Now multiply (3.3) and (3.4) by $\phi(t)$ and integrate in t from 0 to T to get

$$\begin{aligned}
&\int_0^T \left[(\bar{u}_h(t), \phi'(t)v_h) - \alpha([k_\varepsilon + |\bar{\varphi}_h|^2] \nabla \bar{u}_h, \nabla(\phi(t)v_h)) - (\gamma[\bar{u}_h - \bar{g}_h], \phi(t)v_h) \right] dt \\
&= \int_0^T \int_\Omega \left(I_h[|\bar{\varphi}_h|^2 \nabla \bar{u}_h \nabla(\phi(t)v_h)] - |\bar{\varphi}_h|^2 \nabla \bar{u}_h \nabla(\phi(t)v_h) \right) dx dt, \\
&\int_0^T \left[(\bar{\varphi}_h, \phi'(t)\psi_h) - 2\beta\varepsilon(\nabla \bar{\varphi}_h, \nabla(\phi(t)\psi_h)) - \alpha(|\nabla \bar{u}_h|^2 \bar{\varphi}_h, \phi(t)\psi_h) - \frac{\beta}{2\varepsilon}(\bar{\varphi}_h - 1, \phi(t)\psi_h) \right] dt \\
&= \alpha \int_0^T \int_\Omega \left(I_h[|\nabla \bar{u}_h|^2 \bar{\varphi}_h \phi(t)\psi_h] - |\nabla \bar{u}_h|^2 \bar{\varphi}_h \phi(t)\psi_h \right) dx dt \\
&\quad + \frac{\beta}{2\varepsilon} \int_0^T \int_\Omega \left(I_h[(\bar{\varphi}_h - 1)\phi(t)\psi_h] - (\bar{\varphi}_h - 1)\phi(t)\psi_h \right) dx dt,
\end{aligned}$$

where \bar{g}_h denotes the (upper) constant interpolation of $\{g^m\}$ in t (cf. (3.16)).

Sending $h, k \rightarrow 0$ in the last two inequalities, it follows from (3.31)-(3.36) and a separability argument for $L^2((0, T)(0, T); ; L^1(\Omega))$ that

$$\begin{aligned}
&\int_0^T \left[(\hat{u}(t), \phi'(t)v) - \alpha([k_\varepsilon - |\hat{\varphi}|^2] \nabla \hat{u}, \nabla(\phi(t)v)) - (\gamma[\hat{u} - g], \phi(t)v) \right] dt = 0, \\
&\int_0^T \left[(\hat{\varphi}, \phi'(t)\psi) - 2\beta\varepsilon(\nabla \hat{\varphi}, \nabla(\phi(t)\psi)) - \alpha(|\nabla \hat{u}|^2 \hat{\varphi}, \phi(t)\psi) \right. \\
&\quad \left. + \frac{\beta}{2\varepsilon}(\hat{\varphi} - 1, \phi(t)\psi) \right] dt = 0,
\end{aligned}$$

that is, $(\hat{u}, \hat{\varphi})$ satisfies (1.5)-(1.7) in the distribution sense.

Finally, it remains to show $\hat{u}(0) = u_0$ and $\hat{\varphi}(0) = \varphi_0$. This follows the same guidelines. The main difference is that now we choose $\phi \in C^\infty([0, T])$ such that $\phi(0) = 1$ and $\phi(T) = 0$, and use the integration by parts formulas

$$\begin{aligned}
&\int_0^T (\bar{u}_h'(t), \phi(t)v_h) dt = - \int_0^T (\bar{u}_h(t), \phi'(t)v_h) dt + (u_h^0, \phi(0)v_h), \\
&\int_0^T (\bar{\varphi}_h'(t), \phi(t)v_h) dt = - \int_0^T (\bar{\varphi}_h(t), \phi'(t)v_h) dt + (\varphi_h^0, \phi(0)v_h)
\end{aligned}$$

and (3.21). The proof is complete. \square

For the cases $(\mu, \nu) = (1, 0)$ and $(\mu, \nu) = (0, 1)$, to show that $(\bar{u}_h, \bar{\varphi}_h)$ has a subsequence satisfying (3.22)-(3.24), we need to use the stronger estimate (3.14) in order to ensure $\nabla \bar{u}_h$, $\nabla \underline{u}_h$ and $\nabla \underline{u}_h$, respectively, $\nabla \bar{\varphi}_h$, $\nabla \underline{\varphi}_h$ and $\nabla \underline{\varphi}_h$ converge (weakly) to the same limit.

THEOREM 3.5. *Let (u_h, φ_h) be the solution of (3.3)-(3.4) which is defined on an acute triangulation \mathcal{T}_h , and $h^2 \leq C_{\theta_0} \beta \varepsilon k$. Suppose that the starting value (u_h^0, φ_h^0) satisfies (3.21) and the constraint $AT_\varepsilon(u_h^0, \varphi_h^0) < \infty$. Then, the conclusions of Theorem 3.4 hold for the semi-implicit schemes corresponding to $(\mu, \nu) = (1, 0)$ and $(\mu, \nu) = (0, 1)$.*

Proof. Since the proof follows the exact same lines as that of Theorem 3.4, we only sketch the main differences.

In addition to the estimates (3.25)–(3.30), from (3.14) we also get

$$(3.38) \quad \|(\bar{u}_h)_t\|_{L^2(L^2)} = \left(k \sum_{m=1}^M \|d_t u_h^m\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C,$$

$$(3.39) \quad \|\nabla \bar{u}_h\|_{L^\infty(L^2)} \leq \|\nabla \bar{u}_h\|_{L^\infty(L^2)} = \|\nabla \underline{u}_h\|_{L^\infty(L^2)} = \max_{0 \leq m \leq M} \|\nabla u_h^m\|_{L^2} \leq \frac{C}{\sqrt{k_\varepsilon}},$$

$$(3.40) \quad \sum_{i=1}^M \|\nabla u_h^m - \nabla u_h^{m-1}\|_{L^2}^2 \leq \frac{C}{\sqrt{k_\varepsilon}},$$

$$(3.41) \quad \|(\bar{\varphi}_h)_t\|_{L^2(L^2)} = \left(k \sum_{m=1}^M \|d_t \varphi_h^m\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C,$$

$$(3.42) \quad \|\nabla \bar{\varphi}_h\|_{L^\infty(L^2)} \leq \|\nabla \bar{\varphi}_h\|_{L^\infty(L^2)} = \|\nabla \underline{\varphi}_h\|_{L^\infty(L^2)} = \max_{0 \leq m \leq M} \|\nabla \varphi_h^m\|_{L^2} \leq \frac{C}{\sqrt{\varepsilon}},$$

$$(3.43) \quad \sum_{i=1}^M \|\nabla \varphi_h^m - \nabla \varphi_h^{m-1}\|_{L^2}^2 \leq \frac{C}{\sqrt{\varepsilon}}.$$

From (3.40) we have

$$(3.44) \quad \begin{aligned} \|\nabla \bar{u}_h - \nabla \underline{u}_h\|_{L^2(L^2)}^2 &= \int_0^T \|\nabla \bar{u}_h - \nabla \underline{u}_h\|_{L^2}^2 dt \\ &= \sum_{m=1}^M \|\nabla u_h^m - \nabla u_h^{m-1}\|_{L^2}^2 \int_{t_{m-1}}^{t_m} \left(\frac{t_m - t}{k} \right)^2 dt \\ &= \frac{k}{3} \sum_{m=1}^M \|\nabla u_h^m - \nabla u_h^{m-1}\|_{L^2}^2 \leq \frac{Ck}{\sqrt{k_\varepsilon}}. \end{aligned}$$

Similarly,

$$(3.45) \quad \|\nabla \bar{u}_h - \nabla \underline{u}_h\|_{L^2(L^2)}^2 \leq \frac{Ck}{\sqrt{k_\varepsilon}},$$

$$(3.46) \quad \|\nabla \bar{\varphi}_h - \nabla \underline{\varphi}_h\|_{L^2(L^2)}^2 \leq \frac{Ck}{\sqrt{\varepsilon}},$$

$$(3.47) \quad \|\nabla \bar{\varphi}_h - \nabla \underline{\varphi}_h\|_{L^2(L^2)}^2 \leq \frac{Ck}{\sqrt{\varepsilon}}.$$

The remaining part of the proof is to extract a convergent subsequence of $\{(u_h^m, \varphi_h^m)\}$ (still denoted by the same notation) and then pass to the limit. Due to the stronger estimates in (3.38)-(3.47), it is easier to pass to the limits because now not only \bar{u}_h , \underline{u}_h and \underline{u}_h , respectively, $\bar{\varphi}_h$, $\underline{\varphi}_h$ and $\underline{\varphi}_h$, converge strongly to the same limit in $L^2(\Omega_T)$, but also their respective gradients converge weakly* to the same limit in $L^\infty((0, T); H^1(\Omega))$. Also, in view of (3.38) and (3.41), (3.33) follows directly from applying an Aubin-Lions compactness lemma [26, 33]. \square

For $(u_0, \varphi_0) \in [H^1(\Omega)]^2$, since (piecewise) strong solutions are unique in the sense of Theorem 2.8, then we have

COROLLARY 3.6. *Suppose $AT_\varepsilon(u_0, \varphi_0) < \infty$ and $g \in L^2(\Omega)$. Let \mathcal{T}_h be an acute triangulation of Ω , and $h^2 \leq C_{\theta_0} \beta \varepsilon k$ in scheme (3.3)-(3.4). Then, the whole sequence $\{(u_h, \varphi_h)\}$ of each of the fully implicit and semi-implicit finite element solutions satisfies (3.22)-(3.24).*

REMARK 3.3. *We conclude this subsection by commenting on the choices of the starting value (u_h^0, φ_h^0) for the fully discrete finite element schemes in (3.3)-(3.4). It follows from [14] that if we set $u_h^0 = Q_h u_0$ and $\varphi_h^0 = Q_h \varphi_0$, the L^2 -projections of u_0 and φ_0 in V^h , respectively, then all the requirements of Theorems 3.4 and 3.5 for (u_h^0, φ_h^0) are satisfied. We also note that $u_h^0 = I_h u_0$ and $\varphi_h^0 = I_h \varphi_0$, the piecewise linear interpolations of u_0 and φ_0 into V^h , satisfy the constraints of Theorem 3.4, but may fail to satisfy the requirement “ $AT_\varepsilon(I_h u_0, I_h \varphi_0) < \infty$ uniformly in h ” of Theorem 3.5.*

3.3. Error estimates for the fully discrete schemes. In this last subsection, we shall derive some error estimates for the fully discrete finite element solutions defined in (3.1)-(3.2) in the two dimensional case (i.e. $N = 2$), provided that the solution (u, φ) of (1.5)-(1.8) belongs to $[L^\infty((0, T); W^{1,\infty}(\Omega)) \cap L^2((0, T); H^2(\Omega)) \cap H^2((0, T); H^{-1}(\Omega))]^2$ satisfying assumption (3.54) below. The analysis relies on regularity of the strong solution (u, φ) and thus allows to use (3.1)-(3.2) rather than its lumped version which satisfies a discrete maximum principle.

We establish practically useful error estimates which depends on ε^{-1} and k_ε^{-1} in some low polynomial order. Our analysis also carries over to solutions in $[L^2((0, T); H^2(\Omega)) \cap H^2((0, T); H^{-1}(\Omega))]^2$ as well, at the expense that error constant depends exponentially on ε^{-1} and k_ε^{-1} (cf. (e) of Remark 3.4).

We begin with definitions of two projection operators. First, let $Q_h : L^2(\Omega) \rightarrow V^h$ denote the L^2 projection operator

$$(3.48) \quad (w - Q_h w, v_h) = 0 \quad \forall v_h \in V^h.$$

Second, let $P_h : H^1(\Omega) \rightarrow V^h \cap H_0^1(\Omega)$ denote the usual elliptic projection operator

$$(3.49) \quad (\nabla(w - P_h w), \nabla v_h) = 0 \quad \forall v_h \in V^h \cap H_0^1(\Omega).$$

It is well-known that the above projection operators have the following approximation properties [10, 14].

LEMMA 3.7. *There exists a positive constant C such that for $h \in (0, 1)$*

$$(3.50) \quad \|v - Q_h v\|_{L^2} + h \|\nabla(v - Q_h v)\|_{L^2} \leq C h \|v\|_{H^1} \quad \forall v \in H^1(\Omega),$$

$$(3.51) \quad \|v - Q_h v\|_{L^2} \leq C h^2 \|v\|_{H^2} \quad \forall v \in H^2(\Omega),$$

$$(3.52) \quad \|v - P_h v\|_{L^2} + h \|\nabla(v - P_h v)\|_{L^2} \leq C h^2 \|v\|_{H^2} \quad \forall v \in H^2(\Omega),$$

$$(3.53) \quad \|v - P_h v\|_{L^\infty} \leq C h |\ln h| \|\nabla v\|_{L^\infty} \quad \forall v \in W^{1,\infty}(\Omega).$$

THEOREM 3.8. *Let $(u_0, \varphi_0) \in [H^2(\Omega)]^2$. Let \mathcal{T}_h be a quasi-uniform triangulation of Ω , and $(u, \varphi) \in [L^\infty((0, T); W^{1,\infty}(\Omega)) \cap L^2((0, T); H^2(\Omega)) \cap H^2((0, T); H^{-1}(\Omega))]^2$ satisfies the growth condition*

$$(3.54) \quad C^* := \|\nabla u\|_{L^\infty(L^\infty)} + \|\nabla \varphi\|_{L^\infty(L^\infty)} = O(\varepsilon^{-\frac{1}{2}}) \quad (\text{for } \varepsilon \ll 1).$$

For each of three pairs of (μ, ν) :

$$(i) (\mu, \nu) = (0, 1), \quad (ii) (\mu, \nu) = (1, 0), \quad (iii) (\mu, \nu) = (1, 1),$$

in (3.1)-(3.2), let $E_u^m := u(t_m) - u_h^m$ and $E_\varphi^m := \varphi(t_m) - \varphi_h^m$. Then, under the following starting value and mesh constraints

$$(3.55) \quad \|u_h^0 - u_0\|_{L^2} + \|\varphi_h^0 - \varphi_0\|_{L^2} \leq C h,$$

$$(3.56) \quad h |\ln h| < \frac{\alpha k_\varepsilon}{C^*} = O(\varepsilon^{\frac{1}{2}} k_\varepsilon),$$

$$(3.57) \quad k = o(h^{\frac{1}{2}}),$$

the global errors E_u^m and E_φ^m satisfy

$$(3.58) \quad \max_{0 \leq m \leq M} [\|E_u^m\|_{L^2} + \|E_\varphi^m\|_{L^2}] + \left\{ k \sum_{m=1}^M \left[k \|d_t E_u^m\|_{L^2}^2 + k \|d_t E_\varphi^m\|_{L^2}^2 \right. \right. \\ \left. \left. + \alpha k_\varepsilon \|\nabla E_u^m\|_{L^2}^2 + \alpha \|\varphi(t_m) \nabla E_u^m\|_{L^2}^2 + \|\sqrt{\gamma} E_u^m\|_{L^2}^2 + \beta \varepsilon \|\nabla E_\varphi^m\|_{L^2}^2 \right. \right. \\ \left. \left. + \alpha \|\nabla u(t_m) E_\varphi^m\|_{L^2}^2 + \frac{\beta}{\varepsilon} \|E_\varphi^m\|_{L^2}^2 \right] \right\}^{\frac{1}{2}} \leq \hat{C} (h + k),$$

where \hat{C} is some positive constant which depends linearly on $\frac{1}{\sqrt{\varepsilon}}$ and $\frac{1}{\sqrt{k_\varepsilon}}$, and on C^* , and C_0 to C_4 , which are defined in Theorem 2.6 and Corollary 2.7, in some low polynomial order.

Proof. We shall prove the above error estimate for the three fully discrete schemes corresponding to (i) $(\mu, \nu) = (0, 1)$, (ii) $(\mu, \nu) = (1, 0)$, and (iii) $(\mu, \nu) = (1, 1)$ together. For the purpose, we introduce the notation

$$\begin{aligned} \varphi^{m,\mu} &:= \mu \varphi(t_m) + (1 - \mu) \varphi(t_{m-1}), & u^{m,\nu} &:= \nu u(t_m) + (1 - \nu) u(t_{m-1}), \\ E_u^{m,\nu} &:= \nu E_u^m + (1 - \nu) E_u^{m-1}, & E_\varphi^{m,\mu} &:= \mu E_\varphi^m + (1 - \mu) E_\varphi^{m-1}. \end{aligned}$$

Since the proof is long and technical, we divide it into four steps.

Step 1: Test (1.5) by $v_h \in V^h$ and (1.6) by $\psi_h \in V^h$, respectively, and subtract the resulting equations from (3.1) and (3.2), respectively yield the following error equations:

$$(3.59) \quad (d_t E_u^m, v_h) + \alpha k_\varepsilon (\nabla E_u^m, \nabla v_h) + \alpha (|\varphi(t_m)|^2 \nabla u(t_m) - |\varphi_h^{m,\mu}|^2 \nabla u_h^m, \nabla v_h) \\ + (\gamma E_u^m, v_h) = (\mathcal{R}_u^m, v_h) \quad \forall v_h \in V^h,$$

$$(3.60) \quad (d_t E_\varphi^m, \psi_h) + 2\beta \varepsilon (\nabla E_\varphi^m, \nabla \psi_h) + \alpha (|\nabla u(t_m)|^2 \varphi(t_m) - |\nabla u_h^{m,\nu}|^2 \varphi_h^m, \psi_h) \\ + \frac{\beta}{2\varepsilon} (E_\varphi^m, \psi_h) = (\mathcal{R}_\varphi^m, \psi_h) \quad \forall \psi_h \in V^h,$$

where

$$\mathcal{R}_u^m := u_t(t_m) - d_t u(t_m), \quad \mathcal{R}_\varphi^m := \varphi_t(t_m) - d_t \varphi(t_m).$$

Let

$$\Theta_u^m := u(t_m) - P_h u(t_m), \quad \Theta_\varphi^m := \varphi(t_m) - P_h \varphi(t_m).$$

Taking

$$v_h = P_h u(t_m) - u_h^m = E_u^m - \Theta_u^m \quad \text{and} \quad \psi_h = P_h \varphi(t_m) - \varphi_h^m = E_\varphi^m - \Theta_\varphi^m$$

in (3.59) and (3.60), respectively, we have

$$(3.61) \quad \frac{1}{2} d_t \|E_u^m\|_{L^2}^2 + \frac{k}{2} \|d_t E_u^m\|_{L^2}^2 + \alpha k_\varepsilon \|\nabla E_u^m\|_{L^2}^2 + \|\sqrt{\gamma} E_u^m\|_{L^2}^2 \\ + \alpha (|\varphi(t_m)|^2 \nabla u(t_m) - |\varphi_h^{m,\mu}|^2 \nabla u_h^m, \nabla E_u^m) \\ = (\mathcal{R}_u^m, \Theta_u^m - E_u^m) + (d_t E_u^m, \Theta_u^m) + \alpha k_\varepsilon (\nabla E_u^m, \nabla \Theta_u^m) + (\gamma E_u^m, \Theta_u^m) \\ + \alpha (|\varphi(t_m)|^2 \nabla u(t_m) - |\varphi_h^{m,\mu}|^2 \nabla u_h^m, \nabla \Theta_u^m),$$

$$(3.62) \quad \frac{1}{2} d_t \|E_\varphi^m\|_{L^2}^2 + \frac{k}{2} \|d_t E_\varphi^m\|_{L^2}^2 + 2\beta \varepsilon \|\nabla E_\varphi^m\|_{L^2}^2 + \frac{\beta}{2\varepsilon} \|E_\varphi^m\|_{L^2}^2 \\ + \alpha (|\nabla u(t_m)|^2 \varphi(t_m) - |\nabla u_h^{m,\nu}|^2 \varphi_h^m, E_\varphi^m) \\ = (\mathcal{R}_\varphi^m, \Theta_\varphi^m - E_\varphi^m) + (d_t E_\varphi^m, \Theta_\varphi^m) + 2\beta \varepsilon (\nabla E_\varphi^m, \nabla \Theta_\varphi^m) + \frac{\beta}{2\varepsilon} (E_\varphi^m, \Theta_\varphi^m) \\ + \alpha (|\nabla u(t_m)|^2 \varphi(t_m) - |\nabla u_h^{m,\nu}|^2 \varphi_h^m, \Theta_\varphi^m).$$

Step 2: It remains to bound all inner product terms in (3.61) and (3.62). Clearly, each linear term can be easily bounded from above using standard techniques. On the other hand, bounding four nonlinear terms is more complicated, they need to be handled carefully in order to obtain the desired error estimate.

Using the algebraic relations

$$\begin{aligned}\varphi(t_m) - \varphi_h^{m,\mu} &= (1 - \mu)k d_t \varphi(t_m) + E_\varphi^{m,\mu}, \\ \varphi(t_m) + \varphi_h^{m,\mu} &= 2\varphi(t_m) - (1 - \mu)k d_t \varphi(t_m) - E_\varphi^{m,\mu},\end{aligned}$$

we have

$$\begin{aligned}\mathcal{A}_1 &:= |\varphi(t_m)|^2 \nabla u(t_m) - |\varphi_h^{m,\mu}|^2 \nabla u_h^m \\ &= |\varphi(t_m)|^2 \nabla E_u^m + [\varphi(t_m) - \varphi_h^{m,\mu}] [\varphi(t_m) + \varphi_h^{m,\mu}] \nabla u_h^m \\ &= |\varphi(t_m)|^2 \nabla E_u^m + [2(1 - \mu)k d_t \varphi(t_m) \varphi(t_m) - (1 - \mu)^2 k^2 |d_t \varphi(t_m)|^2 \\ &\quad - 2(1 - \mu)k d_t \varphi(t_m) E_\varphi^{m,\mu} + 2\varphi(t_m) E_\varphi^{m,\mu} - |E_\varphi^{m,\mu}|^2] [\nabla u(t_m) - \nabla E_u^m].\end{aligned}$$

Then, it follows from Schwarz inequality and the fact that $0 \leq \varphi(t_m) \leq 1$

$$\begin{aligned}(3.63) \quad (\mathcal{A}_1, \nabla E_u^m) &\geq \|\varphi(t_m) \nabla E_u^m\|_{L^2}^2 - \|\nabla u(t_m)\|_{L^\infty} \left\{ 2(1 - \mu)k \|d_t \varphi(t_m)\|_{L^2} \|\varphi(t_m) \nabla E_u^m\|_{L^2} \right. \\ &\quad \left. + (1 - \mu)^2 k^2 \|d_t \varphi(t_m)\|_{L^4}^2 \|\nabla E_u^m\|_{L^2} + 2(1 - \mu)k \|d_t \varphi(t_m)\|_{L^2} \|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2} \right. \\ &\quad \left. + 2\|\varphi(t_m) \nabla E_u^m\|_{L^2} \|E_\varphi^{m,\mu}\|_{L^2} + \|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2} \|E_\varphi^{m,\mu}\|_{L^2} \right\} \\ &\quad - \|\nabla E_u^m\|_{L^2}^2 \left\{ 2(1 - \mu)k \|d_t \varphi(t_m)\|_{L^\infty} + (1 - \mu)^2 k^2 \|d_t \varphi(t_m)\|_{L^\infty}^2 \right\} \\ &\quad - 2(1 - \mu)k \|d_t \varphi(t_m)\|_{L^\infty} \|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2} \|\nabla E_u^m\|_{L^2} \\ &\quad - 2\|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2} \|\varphi(t_m) \nabla E_u^m\|_{L^2} + \|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2}^2 \\ &\geq \frac{1 + \mu}{4} \|\varphi(t_m) \nabla E_u^m\|_{L^2}^2 - \left[\frac{(1 - \mu)^2 k_\varepsilon}{4} + 2(1 - \mu)k \|d_t \varphi(t_m)\|_{L^\infty} \right. \\ &\quad \left. + 2(1 - \mu)^2 k^2 \|d_t \varphi(t_m)\|_{L^\infty}^2 \right] \|\nabla E_u^m\|_{L^2}^2 - 5\|\nabla u(t_m)\|_{L^\infty} \|E_\varphi^{m,\mu}\|_{L^2}^2 \\ &\quad - (5 + \mu) \|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2}^2 - k^2 \left[5(1 - \mu) \|d_t \varphi(t_m)\|_{L^2}^2 \right. \\ &\quad \left. + \frac{(1 - \mu)^2 k^2}{k_\varepsilon} \|d_t \varphi(t_m)\|_{L^4}^4 \right] \|\nabla u(t_m)\|_{L^\infty}^2,\end{aligned}$$

and

$$\begin{aligned}(3.64) \quad |(\mathcal{A}_1, \nabla \Theta_u^m)| &\leq \frac{1}{16} \|\varphi(t_m) \nabla E_u^m\|_{L^2}^2 + \left[3 + \frac{1}{\varepsilon} \|\nabla u(t_m)\|_{L^\infty}^2 \|\nabla \Theta_u^m\|_{L^2}^2 \right] \|E_\varphi^{m,\mu}\|_{L^2}^2 \\ &\quad + \frac{\varepsilon}{16} \|\nabla E_\varphi^{m,\mu}\|_{L^2}^2 + 4(1 - \mu)^4 k^4 \|d_t \varphi(t_m)\|_{L^\infty}^4 \|\nabla E_u^m\|_{L^2}^2 \\ &\quad + 2 \left[1 + \|\nabla \Theta_u^m\|_{L^\infty}^2 \right] \|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2}^2 + 12 \|\nabla \Theta_u^m\|_{L^2}^2 + (1 - \mu)k^2 \|d_t \varphi(t_m)\|_{L^2}^2.\end{aligned}$$

Similarly, using the algebraic relations

$$\begin{aligned}\nabla u(t_m) - \nabla u_h^{m,\nu} &= (1 - \nu)k d_t \nabla u(t_m) + E_u^{m,\nu}, \\ \nabla u_h^{m,\nu} &= \nabla u(t_m) - (1 - \nu)k d_t \nabla u(t_m) - \nabla E_u^{m,\nu}, \\ \varphi_h^m &= \varphi(t_m) - E_\varphi^m,\end{aligned}$$

we have

$$\begin{aligned}\mathcal{A}_2 &:= |\nabla u(t_m)|^2 \varphi(t_m) - |\nabla u_h^{m,\nu}|^2 \varphi_h^m \\ &= |\nabla u(t_m)|^2 E_\varphi^m + (\nabla u(t_m) - \nabla u_h^{m,\nu}, \nabla u(t_m) + \nabla u_h^{m,\nu}) \varphi_h^m \\ &= |\nabla u(t_m)|^2 E_\varphi^m + ((1 - \nu)k d_t \nabla u(t_m) + E_u^{m,\nu}, 2\nabla u(t_m) - (1 - \nu)k d_t \nabla u(t_m) - \nabla E_u^{m,\nu}) \\ &\quad \times (\varphi(t_m) - E_\varphi^m).\end{aligned}$$

Then

$$\begin{aligned}
(3.65) \quad (\mathcal{A}_2, E_\varphi^m) &\geq \|\nabla u(t_m) E_\varphi^m\|_{L^2}^2 - (1-\nu)k \|d_t \nabla u(t_m)\|_{L^2} \left\{ 2\|\nabla u(t_m) E_\varphi^m\|_{L^2} \right. \\
&\quad \left. + (1-\nu)k \|d_t \nabla u(t_m)\|_{L^\infty} \|E_\varphi^m\|_{L^2} + 2\|E_\varphi^m \nabla E_u^{m,\nu}\|_{L^2} \right\} \\
&\quad - \|E_\varphi^m \nabla E_u^{m,\nu}\|_{L^2} \left\{ (1-\nu)k \|d_t \nabla u(t_m)\|_{L^2} + \|\varphi(t_m) \nabla E_u^{m,\nu}\|_{L^2} \right\} \\
&\quad - 2(1-\nu)k \|d_t \nabla u(t_m)\|_{L^\infty} \|E_\varphi^m\|_{L^2} \left\{ \|\nabla u(t_m) E_\varphi^m\|_{L^2} + \|E_\varphi^m \nabla E_u^{m,\nu}\|_{L^2} \right\} \\
&\quad + (1-\nu)^2 k^2 \|d_t \nabla u(t_m) E_\varphi^m\|_{L^2}^2 + \|E_\varphi^m \nabla E_u^{m,\nu}\|_{L^2}^2 \\
&\geq \frac{1}{2} \|\nabla u(t_m) E_\varphi^m\|_{L^2}^2 - \frac{1}{16} \|\varphi(t_m) \nabla E_u^{m,\nu}\|_{L^2}^2 - 12 \|E_\varphi^m \nabla E_u^{m,\nu}\|_{L^2}^2 \\
&\quad - 4 \left\{ (1-\nu)k \|d_t \nabla u(t_m)\|_{L^\infty}^2 + \|\nabla u(t_m)\|_{L^\infty}^2 \right\} \|E_\varphi^m\|_{L^2}^2 - 4(1-\nu)^2 k^2 \|d_t \nabla u(t_m)\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
(3.66) \quad |(\mathcal{A}_2, \Theta_\varphi^m)| &\leq \frac{1}{4} \|\nabla u(t_m) E_\varphi^m\|_{L^2}^2 + \frac{1}{16} \|\varphi(t_m) \nabla E_u^{m,\nu}\|_{L^2}^2 + 8 \|\Theta_\varphi^m\|_{L^\infty}^2 \left[\|E_\varphi^m\|_{L^2}^2 + \|\nabla E_u^{m,\nu}\|_{L^2}^2 \right] \\
&\quad + 8 \left[1 + \|\nabla u(t_m)\|_{L^\infty}^2 \right] \|\Theta_\varphi^m\|_{L^2}^2 + (1-\nu)^4 k^4 \|d_t \nabla u(t_m)\|_{L^2}^4 \\
&\quad + 8(1-\nu)^2 k^2 \left[1 + \|\Theta_\varphi^m\|_{L^\infty}^2 \right] \|d_t \nabla u(t_m)\|_{L^2}^2.
\end{aligned}$$

Next, we need to bound the factor $k \|d_t \varphi(t_m)\|_{L^\infty}$ which appears in (3.63)-(3.65). Using a Nirenberg-Gagliardo inequality [1] we have

$$\begin{aligned}
(3.67) \quad k \|d_t \varphi(t_m)\|_{L^\infty} &\leq k \left[\|d_t \varphi(t_m)\|_{L^2}^{\frac{1}{2}} \|\nabla d_t \varphi(t_m)\|_{L^\infty}^{\frac{1}{2}} + \|d_t \varphi(t_m)\|_{L^2} \right] \\
&\leq k^{\frac{1}{2}} \|d_t \varphi(t_m)\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi(t_m) - \nabla \varphi(t_{m-1})\|_{L^\infty}^{\frac{1}{2}} + k \|d_t \varphi(t_m)\|_{L^2}, \\
&\leq 3k^{\frac{1}{2}} \|\nabla \varphi\|_{L^\infty(\Omega_T)}^{\frac{1}{2}}.
\end{aligned}$$

Finally, using the relations

$$\begin{aligned}
(d_t E_u^m, \Theta_u^m) &= d_t(E_u^m, \Theta_u^m) - (E_u^{m-1}, d_t \Theta_u^m), \\
(d_t E_\varphi^m, \Theta_\varphi^m) &= d_t(E_\varphi^m, \Theta_\varphi^m) - (E_\varphi^{m-1}, d_t \Theta_\varphi^m),
\end{aligned}$$

we can bound the linear terms in (3.61) and (3.62) follows:

$$\begin{aligned}
(3.68) \quad |(\mathcal{R}_u^m, \Theta_u^m - E_u^m) - (E_u^{m-1}, d_t \Theta_u^m) + \alpha k_\varepsilon (\nabla E_u^m, \nabla \Theta_u^m) + (\gamma E_u^m, \Theta_u^m)| \\
\leq \frac{\alpha k_\varepsilon}{4} [\|\nabla E_u^{m-1}\|_{L^2}^2 + \|\nabla E_u^m\|_{L^2}^2] + \frac{1}{2} \|\sqrt{\gamma} E_u^m\|_{L^2}^2 \\
+ \frac{C}{\alpha k_\varepsilon} [\|\mathcal{R}_u^m\|_{H^{-1}}^2 + \|d_t \Theta_u^m\|_{H^{-1}}^2] + C [\alpha k_\varepsilon \|\nabla \Theta_u^m\|_{L^2}^2 + \|\sqrt{\gamma} \Theta_u^m\|_{L^2}^2],
\end{aligned}$$

$$\begin{aligned}
(3.69) \quad |(\mathcal{R}_\varphi^m, \Theta_\varphi^m - E_\varphi^m) - (E_\varphi^{m-1}, d_t \Theta_\varphi^m) + 2\beta\varepsilon (\nabla E_\varphi^m, \nabla \Theta_\varphi^m) + \frac{\beta}{2\varepsilon} (E_\varphi^m, \Theta_\varphi^m)| \\
\leq \beta\varepsilon [\|\nabla E_\varphi^{m-1}\|_{L^2}^2 + \|\nabla E_\varphi^m\|_{L^2}^2] + \frac{\beta}{4\varepsilon} \|E_\varphi^m\|_{L^2}^2 \\
+ \frac{C}{\beta\varepsilon} [\|\mathcal{R}_\varphi^m\|_{H^{-1}}^2 + \|d_t \Theta_\varphi^m\|_{H^{-1}}^2] + C [\beta\varepsilon \|\nabla \Theta_\varphi^m\|_{L^2}^2 + \frac{\beta}{4\varepsilon} \|\Theta_\varphi^m\|_{L^2}^2].
\end{aligned}$$

Since \mathcal{R}_u^m and \mathcal{R}_φ^m can be written as

$$\mathcal{R}_u^m = \frac{1}{k} \int_{t_m}^{t_{m+1}} (s - t_m) u_{tt}(s) ds, \quad \mathcal{R}_\varphi^m = \frac{1}{k} \int_{t_m}^{t_{m+1}} (s - t_m) \varphi_{tt}(s) ds,$$

from Lemma 2.7 we have

$$(3.70) \quad k \sum_{m=0}^M \|\mathcal{R}_u^m\|_{H^{-1}}^2 \leq k \sum_{m=0}^M \left[\int_{t_m}^{t_{m+1}} (s-t_m)^2 ds \right] \left[\int_{t_m}^{t_{m+1}} \|u_{tt}(s)\|_{H^{-1}}^2 ds \right] \leq C_3 k^2,$$

$$(3.71) \quad k \sum_{m=0}^M \|\mathcal{R}_\varphi^m\|_{H^{-1}}^2 \leq k \sum_{m=0}^M \left[\int_{t_m}^{t_{m+1}} (s-t_m)^2 ds \right] \left[\int_{t_m}^{t_{m+1}} \|\varphi_{tt}(s)\|_{H^{-1}}^2 ds \right] \leq C_3 k^2.$$

Step 3: Now, substituting the estimates (3.63)-(3.71) into (3.61) and (3.62), respectively, adding the resulted inequalities and applying the operator $k \sum_{k=1}^\ell$ ($\ell \leq M$) we get

$$(3.72) \quad \begin{aligned} & \|E_u^\ell\|_{L^2}^2 + \|E_\varphi^\ell\|_{L^2}^2 + k \sum_{m=1}^\ell \left\{ k \|d_t E_u^m\|_{L^2}^2 + k \|d_t E_\varphi^m\|_{L^2}^2 \right. \\ & \quad + \left[\alpha k_\varepsilon - c_0(1-\mu)k \|\nabla \varphi\|_{L^\infty(L^\infty)} - c_0(1-\nu) \|\Theta_\varphi\|_{L^\infty(L^\infty)}^2 \right] \|\nabla E_u^m\|_{L^2}^2 \\ & \quad + \alpha \|\varphi(t_m) \nabla E_u^m\|_{L^2}^2 + \|\sqrt{\gamma} E_u^m\|_{L^2}^2 + \beta \varepsilon \|\nabla E_\varphi^m\|_{L^2}^2 + \alpha \|\nabla u(t_m) E_\varphi^m\|_{L^2}^2 \\ & \quad \left. + \left[\frac{\beta}{\varepsilon} - c_0 \|\nabla u\|_{L^\infty(L^\infty)}^2 - \frac{c_0}{\varepsilon} \|\nabla u\|_{L^\infty(L^\infty)} \|\nabla \Theta_u\|_{L^\infty(L^2)}^2 - c_0 \|\Theta_\varphi\|_{L^\infty(L^\infty)}^2 \right] \|E_\varphi^m\|_{L^2}^2 \right\} \\ & \leq C k \sum_{m=1}^\ell \left\{ \|E_u^m\|_{L^2}^2 + \|E_\varphi^{m,\mu}\|_{L^2}^2 + (1 + \|\nabla \Theta_u^m\|_{L^\infty}^2) \|E_\varphi^{m,\mu} \nabla E_u^m\|_{L^2}^2 + \|E_\varphi^m \nabla E_u^{m,\nu}\|_{L^2}^2 \right\} \\ & \quad + \frac{k}{4} \sum_{m=1}^\ell \left\{ (1-\mu) \|\nabla u(t_m)\|_{L^\infty}^2 \|E_\varphi^{m,\mu}\|_{L^2}^2 + \alpha(1-\nu) \|\varphi(t_m) \nabla E_u^{m,\nu}\|_{L^2}^2 + \beta \varepsilon (1-\mu) \|\nabla E_\varphi^{m,\mu}\|_{L^2}^2 \right\} \\ & \quad + C k^2 (1-\nu) k \sum_{m=1}^\ell \|\Theta_\varphi^m\|_{L^\infty}^2 \|d_t \nabla u(t_m)\|_{L^2}^2 + C k^2 (1-\mu) k \sum_{m=1}^\ell \|\nabla u(t_m)\|_{L^\infty}^2 \|d_t \varphi(t_m)\|_{L^2}^2 \\ & \quad + C k^5 \sum_{m=1}^\ell \left\{ \frac{1-\mu}{k_\varepsilon} \|\nabla u(t_m)\|_{L^\infty}^2 \|d_t \varphi(t_m)\|_{L^4}^4 + (1-\nu) \|d_t \nabla u(t_m)\|_{L^4}^4 \right\} \\ & \quad + C k \sum_{m=1}^\ell \left[\|\nabla \Theta_u^m\|_{L^2}^2 + \|\nabla u\|_{L^\infty(L^\infty)}^2 \|\Theta_\varphi^m\|_{L^2}^2 \right] \\ & \quad + C k \sum_{m=1}^\ell \left\{ \frac{1}{\alpha k_\varepsilon} \|d_t \Theta_u^m\|_{H^{-1}}^2 + \frac{1}{\beta \varepsilon} \|d_t \Theta_\varphi^m\|_{H^{-1}}^2 + \alpha k_\varepsilon \|\nabla \Theta_u^m\|_{L^2}^2 \right. \\ & \quad \left. + \beta \varepsilon \|\nabla \Theta_\varphi^m\|_{L^2}^2 + \frac{\beta}{4\varepsilon} \|\Theta_\varphi^m\|_{L^2}^2 \right\} + \|E_u^0\|_{L^2}^2 + \|E_\varphi^0\|_{L^2}^2 \end{aligned}$$

for some positive constant c_0 which is independent of ε , k_ε , u and φ .

In view of (3.53) and the mesh condition (3.56), we have

$$c^* := \alpha k_\varepsilon - c_0(1-\mu)k \|\nabla \varphi\|_{L^\infty(L^\infty)} - c_0(1-\nu) \|\Theta_\varphi\|_{L^\infty(L^\infty)}^2 > 0.$$

Also, the assumption (3.54) and the mesh condition (3.56) imply that

$$\frac{\beta}{\varepsilon} - c_0 \|\nabla u\|_{L^\infty(L^\infty)}^2 - \frac{c_0}{\varepsilon} \|\nabla u\|_{L^\infty(L^\infty)} \|\nabla \Theta_u\|_{L^\infty(L^2)}^2 - c_0 \|\Theta_\varphi\|_{L^\infty(L^\infty)}^2 > 0.$$

Hence, all terms (in the sum) on the left-hand side of (3.72) are positive. We also remark that when $\mu \neq 1$ and $\nu \neq 1$, the second sum on the right-hand side of (3.72) can be absorbed by the sum on the left-hand side.

Step 4: We now conclude the proof using a fixed point argumentation. Suppose that

$$(3.73) \quad \max_{0 \leq m \leq M} [\|E_u^m\|_{L^\infty}^2 + \|E_\varphi^m\|_{L^\infty}^2] < \frac{c^*}{4},$$

then the third and fourth terms in the first sum on the right-hand side of (3.72) can be absorbed by the fifth term in the sum on the left-hand side. Applying the discrete Gronwall's inequality and Lemma 3.7 immediately yields the desired estimate (3.58). In particular, we have

$$\max_{0 \leq m \leq M} [\|E_u^m\|_{L^2} + \|E_\varphi^m\|_{L^2}] \leq C(k+h),$$

hence,

$$\max_{0 \leq m \leq M} [\|P_h u(t_m) - u_h^m\|_{L^2} + \|P_h \varphi(t_m) - \varphi_h^m\|_{L^2}] \leq C(k+h).$$

By the inverse inequality bounding $L^\infty(\Omega)$ norm in terms of $L^2(\Omega)$ norm [10, 14], we get

$$\begin{aligned} \max_{0 \leq m \leq M} [\|P_h u(t_m) - u_h^m\|_{L^\infty} + \|P_h \varphi(t_m) - \varphi_h^m\|_{L^\infty}] \\ \leq C h^{-\frac{1}{2}} \max_{0 \leq m \leq M} [\|P_h u(t_m) - u_h^m\|_{L^2} + \|P_h \varphi(t_m) - \varphi_h^m\|_{L^2}] \\ \leq C(kh^{-\frac{1}{2}} + h^{\frac{1}{2}}), \end{aligned}$$

which and (3.53) imply that

$$(3.74) \quad \max_{0 \leq m \leq M} [\|E_u^m\|_{L^\infty} + \|E_\varphi^m\|_{L^\infty}] \leq C(\varepsilon)(h|\ln h| + kh^{-\frac{1}{2}} + h^{\frac{1}{2}}).$$

Clearly, (3.74) implies the induction assumption (3.73) for sufficient small h , and provided that h, k satisfy the mesh relation $k = o(h^{\frac{1}{2}})$. The proof is complete. \square

We conclude this section with a few remarks.

REMARK 3.4. (a). *The error estimates in (3.58) are optimal in $L^2((0, T); H^1(\Omega))$ -norm, and suboptimal in $L^\infty((0, T); L^2(\Omega))$ -norm. It might be possible to obtain optimal order error estimate of the form $O(k + h^2)$ in $L^\infty((0, T); L^2(\Omega))$ -norm by replacing the standard elliptic projection operator P_h by some nonlinear projection operator R_h associated with the underlying differential problem (1.5)-(1.8). If such an optimal order estimate can be obtained, it is easy to see that the conclusion of Theorem 3.8 also holds for $N = 3$.*

(b). *The proof does not use the discrete stability estimates derived in Lemmas 3.2 and 3.3.*

(c). *As expected, the fully implicit scheme (i.e., $(\mu, \nu) = (1, 1)$) produces a smaller error than its semi-implicit (i.e., $(\mu, \nu) = (0, 1)$ and $(\mu, \nu) = (1, 0)$) counterparts do, although they all have same asymptotic rate of convergence, since four terms on the right-hand side of (3.72) vanish when $(\mu, \nu) = (1, 1)$.*

(d). *Both $(u_h^0, \varphi_h^0) = (Q_h u_0, Q_h \varphi_0)$ and $(u_h^0, \varphi_h^0) = (I_h u_0, I_h \varphi_0)$ are valid choices of starting values for the schemes in (3.1)-(3.2) (cf. Remark 3.3).*

(e). *The growth condition (3.54) is used to ensure positivity of the coefficient for $\|E_\varphi^m\|_{L^2}^2$ on the left-hand side of (3.72); as a result, the constant \hat{C} in the error estimate (3.58) depends on $\frac{1}{\varepsilon}$ and $\frac{1}{k_\varepsilon}$ only in some low polynomial order. In Section 2, we have proved that this growth condition (3.54) holds at least locally in time. So a natural question is whether Theorem 3.8 still holds without assuming the growth condition (3.54). The answer to this question is yes. However, in this case since the coefficient for $\|E_\varphi^m\|_{L^2}^2$ on the left-hand side of (3.72) may not be positive, so we have to move this term to the right-hand side and treat it as a 'bad' term. As a result, the constant \hat{C} in the error estimate (3.58) now depends on $\|\nabla u\|_{L^\infty(L^\infty)}^2$ exponentially, hence, it also depends on $\frac{1}{\varepsilon}$ and $\frac{1}{k_\varepsilon}$ exponentially. In this case, using a different interpolation of nonlinear terms that arise in the error analysis, it can be shown that the estimate also holds for $(u, \varphi) \in [L^2((0, T); H^2(\Omega))]^2$.*

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