Strong Coupling Expansion of Cusp Anomalous Dimension in Planar $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

Benjamin BASSO

Laboratoire de Physique Théorique d'Orsay
Université de Paris XI

May 16, 2008
CAQCD08, Theoretical Physics Institute, University of Minnesota

Based on 0708.3933,
in collaboration with Gregory KORCHEMSKY and Jan KOTAŃSKI

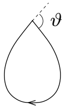
Outline

- What is the cusp anomalous dimension and where does it appear?
- Integrable spin chain and Bethe ansatz approach to calculate the cusp anomalous dimension
- Strong coupling expansion from BES equation
- Test of AdS/CFT correspondence

Cusp Anomalous Dimension

Definition : Cusp anomalous dimension governs the renormalization of Wilson loops evaluated over a closed euclidean contour with a cusp [Polyakov'80]

$$\left\langle \mathrm{Tr}\ \mathrm{P}\exp\left(\imath \oint_{\mathrm{C}} dx \cdot A(x) \right) \right\rangle \sim (\Lambda_{\mathrm{UV}})^{\Gamma_{\mathrm{cusp}}(g,\vartheta)}$$

$\mathrm{C} =$ 

Controls infrared asymptotics of scattering amplitudes in gauge theories [Korchemsky,Radyushkin'86]

- An integration contour C is defined by the particle momenta
- The cusp angle ϑ is related to the scattering angles in Minkowski space

$$\Gamma_{\mathrm{cusp}}(g,\vartheta) = \vartheta\ \Gamma_{\mathrm{cusp}}(g) + \mathcal{O}(\vartheta^0) \qquad |\vartheta| \gg 1$$

Ubiquitous observable of gauge theories :

- IR singularities of on-shell gluon scattering amplitudes
- Logarithmic scaling of anomalous dimensions of high-spin Wilson operators
- ...

Weak and Strong Coupling Expansion of Cusp in Planar $\mathcal{N} = 4$ SYM

Cusp anomalous dimension in the 't Hooft planar limit

$$N_c \to \infty \qquad \text{with the 't Hooft coupling} \qquad g^2 \equiv g^2_{\mathrm{YM}} N_c / 16\pi^2 \qquad \text{fixed}$$

- Weak coupling expansion [Kotikov,Lipatov,Onishchenko,Velizhanin'04],[Bern,Czakon,Dixon,Kosower,Smirnov'06], [Cachazo,Spradlin,Volovich'06]

$$\Gamma_{\mathrm{cusp}}(g) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right) g^8 + \mathcal{O}(g^{10})$$

Fulfills the Kotikov-Lipatov maximal transcendentality principle

- Strong coupling expansion from AdS/CFT correspondence [Gubser,Klebanov,Polyakov'02],[Kruczenski'02], [Frolov,Tseytlin'02],[Casteil,Kristjansen'07]

$$\Gamma_{\mathrm{cusp}}(g) = 2g - \frac{3\ln 2}{2\pi} + \mathcal{O}(1/g)$$

Semiclassical expansion of the energy of a folded string spinning in $AdS_3 \times S^1$

- A conjecture was put forward (Beisert-Eden-Staudacher equation) about the form of the all-loop cusp anomalous dimension derived from Bethe Ansatz equations

Anomalous Dimensions of High-Spin Operators

Wilson operators : Single-trace operators built from L complex scalars $\mathcal{Z}(0)$ and N lightcone derivatives D_+

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)...D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, ..., n_L) \in \mathbb{N}^L$$

Quantum numbers : Twist L and Lorentz spin $N = n_1 + ... + n_L$

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \left[(x-y)^2\right]^{-\Delta_L(N)}$$

Scaling dimensions of Wilson operators = spectrum of the dilatation operator

$$\Delta_L(N) = \Delta_L^{(0)}(N) + \delta_L(N) \qquad \delta_L(N) = \mathcal{O}(g^2)$$

Feature : Anomalous dimensions of Wilson operators $\delta_L(N)$ with large spin $N \gg 1$ occupy the band

$$2\ \Gamma_{\mathrm{cusp}}(g) \ln N \leqslant \delta_L(N) \leqslant L\ \Gamma_{\mathrm{cusp}}(g) \ln N$$

Minimal anomalous dimension has universal scaling behavior in a generic Yang-Mills theory

[Korchemsky'95],[Belitsky,Gorsky,Korchemsky'03],

$$\delta_{\min}(N) = 2\ \Gamma_{\mathrm{cusp}}(g) \ln N + \mathcal{O}(N^0)$$

Spin Chain Representation

Kinematics

Single-trace Wilson composite operators built from L complex scalar fields $\mathcal{Z}(0)$ and N lightcone derivatives D_+

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)\dots D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, \dots, n_L) \in \mathbb{N}^L$$

- $\mathrm{Tr}\{\mathcal{Z}(0)\dots\mathcal{Z}(0)\dots\mathcal{Z}(0)\} \to$ vacuum state of the spin chain
- $\mathrm{Tr}\{\mathcal{Z}(0)\dots D_+\mathcal{Z}(0)\dots\mathcal{Z}(0)\} \to$ one-particle state of the spin chain (magnon)

Quantum numbers :

- Twist $L \to$ spin chain length
- Lorentz spin $N = n_1 + \dots + n_L \to$ number of excitations (magnons) over the vacuum

Dynamics

Wilson operators with same quantum numbers mix under a change of the renormalization scale according to the Callan-Symanzik equation

$$\mu\frac{\partial}{\partial\mu}\mathcal{O}_{\mathbf{n}}(0) = \left(\mathbb{H}\cdot\mathcal{O}\right)_{\mathbf{n}}(0) \qquad \mu\,:\ \text{renormalization scale}$$

- $\mathbb{H}$ dilatation operator of the conformal $\mathcal{N} = 4$ gauge theory $\to$Hamiltonian of the spin chain
- Spectrum of scaling dimensions $\Delta_L(N) \to$ spectrum of energies of the spin chain

Integrability : Cusp Anomaly from Bethe Ansatz

$\mathbb{H}$ is integrable and can be solved by means of Bethe ansatz

[Lipatov'97],[Braun,Derkachov,Manashov'98],[Belitsky'98],[Korchemsky'98],
[Minahan,Zarembo'02],[Beisert,Staudacher'03],[Beisert,Kristjansen,Staudacher'03],[Beisert'04]

Bethe ansatz for (planar) dilatation operator (one-loop example)

- Bethe equations for integrable $SL(2)$ Heisenberg spin chain

$$\left(\frac{u_k + \imath/2}{u_k - \imath/2}\right)^L = \prod_{j\neq k}^{N} \frac{u_k - u_j - \imath}{u_k - u_j + \imath} \text{ (quantization conditions)} \qquad \delta_L(N) = g^2 \sum_{k=1}^{N} \frac{1}{u_k^2 + 1/4} \text{ (energy)}$$

Large spin limit $N \gg 1$: condensation of the Bethe roots $\{u_k\} \rightarrow$ continuum limit of the Bethe equations [Korchemsky'95]

- Distribution density of Bethe roots

  Minimal anomalous dimensions

$$\rho(z) = \frac{1}{N} \sum_{k=1}^{N} \delta\left(z - \frac{u_k}{N}\right) \qquad \text{solution to} \qquad \int_{-1/2}^{+1/2} dx \frac{\rho(x)}{x^2 - z^2} = 2\pi\delta(z)$$

- Bethe roots condense at the origin $\rightarrow$ logarithmic scaling $\sim 2\,\Gamma_{\mathrm{cusp}}(g) \ln N$

$$\rho(z) = \frac{1}{\pi} \ln \frac{1+\sqrt{1-4z^2}}{1-\sqrt{1-4z^2}} \quad \rightarrow \quad \delta_{\min}(N) = g^2 \int_{-1/2}^{+1/2} dz \frac{\rho(z)/N}{z^2 + (1/2N)^2} \sim 8g^2 \ln N$$

All-Loop Asymptotic Bethe Ansatz

Ingredients

- Exploration and understanding of integrable structures in planar $\mathcal{N} = 4$ SYM at higher loop
  [Serban,Staudacher'04],[Eden,Jarczak,Sokatchev'04],[Beisert,Dippel,Staudacher'04],[Staudacher'04]
- Comparison with integrable structures on the stringy side of AdS/CFT correspondence
  [Bena,Polchinski,Roiban'03],[Kazakov,Marshakov,Minahan,Zarembo'04]
- Necessity and presence of a dressing phase [Arutyunov,Frolov,Staudacher'04],[Beisert,Tseytlin'05],[Hernández,López'06]
- Crossing-symmetry of the dressing phase [Janik'06],[Arutyunov,Frolov'06]

Proposal for $SL(2)$ sector

- All-loop asymptotic Bethe ansatz [Beisert,Staudacher'05],[Beisert'05]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{N} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp\left(2\imath\theta(u_k, u_j)\right) \qquad u_k \pm \imath/2 = x_k^\pm + g^2/x_k^\pm$$

  with the dressing phase $\theta(u_k, u_j)$ $\left(= \mathcal{O}\left(g^6\right)\right)$ [Beisert,Hernández,López'06],[Beisert,Eden,Staudacher'06]
- All-loop 'asymptotic' anomalous dimensions

$$\delta_L(N) = g^2 \sum_{j=1}^{N} \left[\frac{\imath}{x_j^+} - \frac{\imath}{x_j^-}\right]$$

- Wrapping effect $\rightarrow$ accuracy $= \mathcal{O}(g^{2L})$ when compared with gauge perturbation theory

Large Spin Continuum Limit : BES equation

The large spin limit of the all-loop asymptotic Bethe ansatz is consistent with the logarithmic scaling and leads to the...

BES equation for the distribution density of roots [Eden,Staudacher'06],[Beisert,Eden,Staudacher'06]

$$\sigma(t) = \frac{t}{e^t - 1}\left(K(2gt,0) - 4g^2\int_0^\infty dt' K(2gt,2gt')\sigma(t')\right)$$

with the kernel

$$K(t,t') = K^{(m)}(t,t') + 2\ K^{(d)}(t,t')$$

sum of the main scattering kernel and of the dressing kernel

$$K^{(m)}(t,t') = K_0(t,t') + K_1(t,t')$$

$$K^{(d)}(t,t') = 4g^2\int_0^\infty dt'' K_1(t,2gt'')\frac{t''}{e^{t''}-1}K_0(2gt'',t')$$

where

$$K_0(t,t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n\geqslant 1}(2n-1)J_{2n-1}(t)J_{2n-1}(t')$$

$$K_1(t,t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n\geqslant 1}(2n)J_{2n}(t)J_{2n}(t')$$

Relation to the cusp anomaly : $\Gamma_{\mathrm{cusp}}(g) = 8g^2\sigma(0)$

Weak Coupling Expansion of Cusp from BES Equation

BES equation

$$\sigma(t) = \frac{t}{e^t - 1}\left(K(2gt,0) - 4g^2\int_0^{+\infty} dt' K(2gt,2gt')\sigma(t')\right)$$

Solution at weak coupling

[Eden,Staudacher'06],[Beisert,Eden,Staudacher'06]
[Belitsky'06]

$$\sigma(t) = \frac{t}{e^t - 1}\left[K(2gt,0) - 4g^2\int_0^{\infty} dt' K(2gt,2gt')\frac{t'}{e^{t'}-1}K(2gt',0) + O(g^4)\right]$$

Weak coupling expansion of the cusp anomaly

$$\Gamma_{\text{cusp}}(g) = 8g^2\sigma(0) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right)g^8$$

$$+32\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\zeta_3^2 + 40\zeta_3\zeta_5\right)g^{10} + \mathcal{O}(g^{12})$$

- Reproduces the known four-loop result
- Verifies the KL maximal transcendentality principle to all loop [Kotikov,Lipatov'06]
- Numerical analysis indicates that the weak coupling expansion is convergent

Strong Coupling Expansion : Numerical Approach

Strategy

[Benna,Benvenuti,Klebanov,Scardicchio'06]

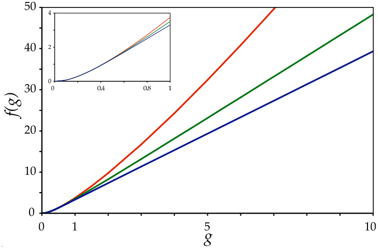
- Expand the solution $s(t) = (e^t - 1)\sigma(t)/t$ over the Bessel functions
- Truncate the series at sufficiently large number of terms $M \sim g$

$$s(t) = \sum_{n=1}^{M} s_n(g)\, \frac{J_n(2gt)}{2gt} \qquad s_{n>M}(g) = 0$$

The integral equation becomes a finite-dimensional matrix equation for the coefficients $s_n(g)$

- Solve numerically the matrix equation and extract the cusp anomaly $\Gamma_{\mathrm{cusp}}(g) = 4g^2 s_1(g)$

Result



$$f(g) = 2\Gamma_{\mathrm{cusp}}(g) = (4.000000 \pm 0.000001)g$$
$$-(0.661907 \pm 0.000002) - \frac{0.0232 \pm 0.0001}{g} + \ldots$$

The first two terms are in remarkable agreement with the string theory result!

$$0.661907 = \frac{3\ln 2}{\pi}, \qquad 0.0232 = ?$$

Strong Coupling Expansion of Cusp from BES Equation I

Analytically, the strong coupling solution was first analyzed at leading order

[Kotikov,Lipatov'06],[Benna,Benvenuti,Klebanov,Scardicchio'07], [Kostov,Serban,Volin'07], [Alday,Arutyunov,Benna,Eden,Klebanov'07],[Beccaria,De Angelis,Forini'07]

and then in a more systematic approach [B.,Korchemsky,Kotański'07],[Belitsky'07],[Kostov,Serban,Volin'08]

Result

$$\Gamma_{\rm cusp}\left(g+c_1\right) = 2g\Big[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2\,{c_2}^2\right) g^{-4} - \left(c_5 + 23\,c_2 c_3\right) g^{-5} - \left(c_6 + \frac{166}{7}\,c_2 c_4 + 54\,{c_3}^2 + 25\,c_2^3\right) g^{-6} + O\left(g^{-7}\right)\Big]$$

where the expansion coefficients are given by

$$c_1 = \frac{3\ln 2}{4\pi} \qquad c_2 = \frac{1}{16\pi^2}\mathrm{K} \qquad c_3 = \frac{27}{2^{11}\pi^3}\zeta(3)$$

$$c_4 = \frac{21}{2^{10}\pi^4}\beta(4) \qquad c_5 = \frac{43065}{2^{21}\pi^5}\zeta(5) \qquad c_6 = \frac{1605}{2^{15}\pi^6}\beta(6)$$

with the special functions

$$\zeta(x) = \sum_{n\geq 1} n^{-x} = \text{Riemann zeta function}$$

$$\beta(x) = \sum_{n\geq 0} (-1)^n (2n+1)^{-x} = \text{Dirichlet zeta function}$$

$$\mathrm{K} = \beta(2) = \text{Catalan's constant}$$

Strong Coupling Expansion of Cusp from BES Equation II

Result

$$\Gamma_{\mathrm{cusp}}(g + c_1) = 2g\Big[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2\, {c_2}^2\right) g^{-4}$$
$$- \left(c_5 + 23\, c_2 c_3\right) g^{-5} - \left(c_6 + \frac{166}{7}\, c_2 c_4 + 54\, {c_3}^2 + 25\, c_2^3\right) g^{-6} + O\left(g^{-7}\right)\Big]$$

Feature :

- Agreement with numerical values obtained within [Benna,Benvenuti,Klebanov,Scardicchio'07] approach
- Maximal transcendentality principle at strong coupling :

$$\begin{array}{rcl} \text{weak coupling} & \rightarrow & \text{strong coupling} \\ \zeta(2n) & \rightarrow & \beta(2n) \\ \zeta(2n-1) & \rightarrow & \zeta(2n-1) \end{array}$$

- c_1-dependent terms inside $\Gamma_{\mathrm{cusp}}(g)$ can be resummed by shifting $g \rightarrow g + c_1$

AdS/CFT correspondence

$$\Gamma_{\rm cusp}(g) = \begin{cases} \text{semiclassical energy of string spinning on AdS3} \\ \text{v.e.v. of the Wilson loop expectation value with a cusp} \end{cases}$$

[Gubser,Klebanov,Polyakov'02],[Frolov,Tseytlin'02]

[Kruczenski'02]

Perturbative string theory prediction

$$\Gamma_{\rm cusp}(g) = 2g\left[1 - c_1 g^{-1} - c_2 g^{-2} - c_3 g^{-3} + \mathcal{O}\left(g^{-4}\right)\right]$$

with '1'= classical solution, c_1 = 1-loop correction, c_2 = 2-loop correction, ...

$c_2 = \mathrm{K}/(4\pi)^2$ was recently confirmed by superstring computation [Roiban,Tirziu,Tseytlin'07],[Roiban,Tseytlin'07]

AdS/CFT correspondence works fine to two-loop accuracy!

Verification of the prediction for higher order coefficients c_k (with $k \leqslant 40$) remains a challenge for the string theory:

- All expansion coefficients except the first one are negative
- The expansion coefficients grow factorially at large orders

$$c_k \propto \frac{\Gamma(k-1/2)}{(2\pi)^k} \qquad \text{for } k \gg 1$$

Strong coupling expansion of the cusp anomalous dimension is only asymptotic and is not Borel summable!

What is the physical meaning of these properties on the string theory side?

Does perturbative string theory make sense?

- Borel improved expansion of the cusp anomalous dimension

$$\Gamma_{\rm cusp}(g) \sim -g \sum_k \frac{\Gamma(k-\frac{1}{2})}{(2\pi g)^k} = g \int_0^\infty \frac{du\, u^{-1/2} e^{-u}}{u - 2\pi g}$$

... but it is not well-defined due to a pole at $u = 2\pi g$

- The cusp anomaly (= energy of quantum spinning folded string) receives 'nonperturbative' contribution at large g

$$\Delta\Gamma_{\rm cusp}(g) \sim g^{1/2} e^{-2\pi g}$$

perturbative expansion in string theory is not well-defined

What could be an origin of such corrections?

- Alday-Maldacena proposal:
  - Massless excitations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma-model are described by noncritical $O(6)$ sigma-model with a UV cut-off set by the masses of massive (fermions + bosons) excitations
  - The $O(6)$ sigma-model develops a mass gap which affects the cusp anomalous dimension

$$\Delta\Gamma_{\rm cusp}(g) \sim m^2 = g^{-2\beta_2/\beta_1^2} e^{4g/\beta_1}$$

Two-loop beta function for the $O(6)$ sigma-model $\beta_1 = -2/\pi$, $\beta_2 = -1/\pi^2$

Does perturbative string theory make sense?

- Borel improved expansion of the cusp anomalous dimension

$$\Gamma_{\text{cusp}}(g) \sim -g \sum_k \frac{\Gamma(k-\frac{1}{2})}{(2\pi g)^k} = g \int_0^\infty \frac{du\, u^{-1/2} e^{-u}}{u - 2\pi g}$$

... but it is not well-defined due to a pole at $u = 2\pi g$

- The cusp anomaly (= energy of quantum spinning folded string) receives 'nonperturbative' contribution at large g

$$\Delta\Gamma_{\text{cusp}}(g) \sim g^{1/2} e^{-2\pi g}$$

perturbative expansion in string theory is not well-defined

What could be an origin of such corrections?

- Alday-Maldacena proposal:
  - Massless excitations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma-model are described by noncritical $O(6)$ sigma-model with a UV cut-off set by the masses of massive (fermions + bosons) excitations
  - The $O(6)$ sigma-model develops a mass gap which affects the cusp anomalous dimension

$$\Delta\Gamma_{\text{cusp}}(g) \sim m^2 = g^{-2\beta_2/\beta_1^2} e^{4g/\beta_1}$$

perfect agreement with

Two-loop beta function for the $O(6)$ sigma-model $\beta_1 = -2/\pi$, $\beta_2 = -1/\pi^2$

Conclusions

- BES equation can be solved analytically both at weak and strong coupling :
  - At weak coupling the cusp anomalous dimension is given by a convergent series in g^2
  - At strong coupling the cusp anomalous dimension is given by an asymptotic series in $1/g$
- Both at weak and strong coupling the BES interpolation agrees with the known gauge and string results
- The strong coupling expansion is non Borel-summable and suffers from non-perturbative ambiguities
- Non-perturbative corrections to the cusp anomalous dimension are governed by the dynamical infrared scale of the $O(6)$ sigma-model
- Quantitative description of these non-perturbative corrections is still missing