Robust Strictly Positive Real Synthesis for Polynomial Families of Arbitrary Order [1]

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Abstract For any two *n-th* order polynomials $a(s)$ and $b(s)$, the Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both strictly positive real.

Keywords Robust Stability, Strict Positive Realness, Robust Analysis and Synthesis, Polynomial Segment

1 Introduction

The strict positive realness (SPR) of transfer functions is an important performance specification, and plays a critical role in various fields such as absolute stability/hyperstability theory [15, 20], passivity analysis [11], quadratic optimal control [3] and adaptive system theory [16]. In recent years, stimulated by the parametrization method in robust stability analysis [1, 4, 6], the study of robust strictly positive real systems has received much attention, and great progress has beem made [2, 5] [7]-[10] [12]-[14] [17]-[19] [21]-[35]. However, most results belong to the category of robust SPR analysis. Valuable results in robust SPR synthesis are rare. The following fundamental problem is still open [14, 18, 22, 23] [25]-[28] [30] [31]-[34]:

Suppose $a(s)$ and $b(s)$ are two n-th order Hurwitz polynomials, does there exist, and how to find a (fixed) polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR?

By the definition of SPR, it is easy to know that the Hurwitz stability of the convex combination of $a(s)$ and $b(s)$ is necessary for the existence of polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR. In [13, 14, 19], it was proved that, if $a(s)$ and $b(s)$ have the same even (or odd) parts, such a polynomial $c(s)$ always exists; In [2, 13, 14, 17, 22, 23, 25, 27, 30, 31], it was proved that, if $n \leq 4$ and $a(s), b(s) \in K$ (K is a stable interval polynomial set), such a polynomial $c(s)$ always exists; Recent results show that [22, 23, 25, 27, 28, 30] [32]-[34], if $n \leq 6$ and $a(s)$ and $b(s)$ are the two endpoints of the convex combination of stable polynomials, such a polynomial $c(s)$ always exists. Some sufficient condition for robust SPR synthesis are presented in [2, 5, 10, 17, 22, 23, 27], especially, the design method in [22, 23] is numerically efficient for high-order polynomial segments or interval polynomials, and the derived conditions are necessary and sufficient for low-order polynomial segments or interval polynomials.

This paper presents a constructive proof of the following statement: for any two *n-th* order polynomials $a(s)$ and $b(s)$, the Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR. This also shows that the conditions given in [22, 23] are also necessary and sufficient and the above open problem

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admits a positive answer. Some previously obtained SPR synthesis results for low-order polynomial segments [22, 23, 25, 27, 28, 30] [32]-[34] become special cases of our main result in this paper.

2 Main Results

In this paper, P^n stands for the set of n-th order polynomials of s with real coefficients, R stands for the field of real numbers, $\partial(p)$ stands for the order of polynomial $p(\cdot)$, and $H^n \subset P^n$ stands for the set of n-th order Hurwitz stable polynomials with real coefficients.

In the following definition, $p(\cdot) \in P^m, q(\cdot) \in P^n, f(s) = p(s)/q(s)$ is a rational function.

Definition 1 [29] $f(s)$ is said to be strictly positive real(SPR), if

(i) $\partial(p) = \partial(q)$;

(ii) $f(s)$ is analytic in $\mathrm{Re}[s] \geq 0$, (namely, $q(\cdot) \in H^n$);

(iii) $\mathrm{Re}[f(j\omega)] > 0, \quad \forall \omega \in R$.

If $f(s) = p(s)/q(s)$ is proper, it is easy to get the following property:

Property 1 [9] If $f(s) = p(s)/q(s)$ is a proper rational function, $q(s) \in H^n$, and $\forall \omega \in R, \mathrm{Re}[f(j\omega)] > 0$, then $p(s) \in H^n \cup H^{n-1}$.

The following theorem is the main result of this paper:

Theorem 1 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n$, the necessary and sufficient condition for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both Strict Positive Real, is

$$\lambda b(s) + (1-\lambda) a(s) \in H^n, \lambda \in [0, 1].$$

The statement is obviously true for the cases when $n = 1$ or $n = 2$. We will prove it for the case when $n \geq 3$.

Since SPR transfer functions enjoy convexity property, by Property 1, we get the necessary part of the theorem.

To prove sufficiency, we must first introduce some lemmas.

Lemma 1 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n$, then, for every $k \in \{1, 2, \cdots, n-2\}$, the following quadratic curve is an ellipse in the first quadrant (i.e., $x_i \geq 0, i = 1, 2, \cdots, n-1$) of the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$ [2]:

$$\begin{cases} c_{k+1}^2 - 4c_k c_{k+2} = 0, \\ c_l = 0, \\ l \in \{1, 2, \cdots, n\}, l \neq k, k+1, k+2, \end{cases}$$

[2] When $n = 3$, the ellipse equation is (see [27, 28] for details):

$$(a_2 x_1 - a_1 x_2 - a_3)^2 - 4(a_1 - x_1)(a_3 x_2) = 0.$$

When $n = 4$, the two ellipse equations are (see [25] [30]-[32] for details):

$$\begin{cases} (a_2 x_1 + x_3 - a_1 x_2 - a_3)^2 - 4(a_1 - x_1)(a_3 x_2 - a_2 x_3 - a_4 x_1) = 0, \\ a_4 x_3 = 0, \end{cases}$$
$$\begin{cases} (a_3 x_2 - a_2 x_3 - a_4 x_1)^2 - 4(a_2 x_1 + x_3 - a_1 x_2 - a_3) a_4 x_3 = 0, \\ a_1 - x_1 = 0. \end{cases}$$

When $n = 5$, the three ellipse equations are (see [33] for details):

$$\begin{cases} (a_2 x_1 + x_3 - a_1 x_2 - a_3)^2 - 4(a_1 - x_1)(a_5 + a_3 x_2 + a_1 x_4 - a_2 x_3 - a_4 x_1) = 0, \\ a_4 x_3 - a_3 x_4 - a_5 x_2 = 0, \ a_5 x_4 = 0, \end{cases}$$
$$\begin{cases} (a_5 + a_3 x_2 + a_1 x_4 - a_2 x_3 - a_4 x_1)^2 - 4(a_2 x_1 + x_3 - a_1 x_2 - a_3)(a_4 x_3 - a_3 x_4 - a_5 x_2) = 0, \\ a_1 - x_1 = 0, \ a_5 x_4 = 0, \end{cases}$$
$$\begin{cases} (a_4 x_3 - a_3 x_4 - a_5 x_2)^2 - 4(a_5 + a_3 x_2 + a_1 x_4 - a_2 x_3 - a_4 x_1) a_5 x_4 = 0, \\ a_1 - x_1 = 0, \ a_2 x_1 + x_3 - a_1 x_2 - a_3 = 0. \end{cases}$$

and this ellipse is tangent with the line

$$\begin{cases} c_l = 0, \\ l \in \{1, 2, \cdots, n\}, l \neq k+1, k+2, \end{cases}$$

and the line

$$\begin{cases} c_l = 0, \\ l \in \{1, 2, \cdots, n\}, l \neq k, k+1, \end{cases}$$

respectively, where $c_l := \sum_{j=0}^{n} (-1)^{l+j} a_j x_{2l-j-1}, l = 1, 2, \cdots, n,\ a_0 = 1, x_0 = 1,\ a_i = 0$ if $i < 0$ or $i > n$, and $x_i = 0$ if $i < 0$ or $i > n-1$.

Proof Since $a(s)$ is Hurwitz stable, by using mathematical induction, Lemma 1 is proved by a direct calculation.

For notational simplicity, for $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n$, $\forall k \in \{1, 2, \cdots, n-2\}$, denote

$$\Omega^a_{ek} := \{(x_1, x_2, \cdots, x_{n-1})| \quad \begin{array}{l} c^2_{k+1} - 4c_k c_{k+2} < 0, \\ c_l = 0, l \in \{1, 2, \cdots, n\}, l \neq k, k+1, k+2\}, \end{array}$$

and

$$\Omega^b_{ek} := \{(x_1, x_2, \cdots, x_{n-1})| \quad \begin{array}{l} d^2_{k+1} - 4d_k d_{k+2} < 0, \\ d_l = 0, l \in \{1, 2, \cdots, n\}, l \neq k, k+1, k+2\}, \end{array}$$

where $c_l := \sum_{j=0}^{n} (-1)^{l+j} a_j x_{2l-j-1}, d_l := \sum_{j=0}^{n} (-1)^{l+j} b_j x_{2l-j-1}, l = 1, 2, \cdots, n,\ a_0 = 1, b_0 = 1, x_0 = 1$, $a_i = 0$ and $b_i = 0$ if $i < 0$ or $i > n$, and $x_i = 0$ if $i < 0$ or $i > n-1$.

In what follows, (A, B) stands for the set of points in the line segment connecting the point A and the point B in the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$, not including the endpoints A and B. Denote

$$\Omega^a := \{(x_1, x_2, \cdots, x_{n-1})| \quad \begin{array}{l} (x_1, x_2, \cdots, x_{n-1}) \in \bigcup_{i=1, i<j\leq n-2}^{n-3} (A_i, A_j), \\ \forall A_i \in \Omega^a_{ei}, i \in \{1, 2, \cdots, n-2\}\} \end{array}$$

and

$$\Omega^b := \{(x_1, x_2, \cdots, x_{n-1})| \quad \begin{array}{l} (x_1, x_2, \cdots, x_{n-1}) \in \bigcup_{i=1, i<j\leq n-2}^{n-3} (B_i, B_j), \\ \forall B_i \in \Omega^b_{ei}, i \in \{1, 2, \cdots, n-2\}\}. \end{array}$$

Lemma 2 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n$, if $\Omega^a \cap \Omega^b \neq \phi$, take $(x_1, x_2, \cdots, x_{n-1}) \in \Omega^a \cap \Omega^b$, and let $c(s) := s^{n-1} + (x_1 - \varepsilon)s^{n-2} + x_2 s^{n-3} + \cdots + x_{n-2}s + (x_{n-1} + \varepsilon)$ (ε is a sufficiently small positive number), then for $\frac{c(s)}{a(s)}$ and $\frac{c(s)}{b(s)}$, we have $\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] > 0$ and $\mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}] > 0$.

Proof Suppose $(x_1, x_2, \cdots, x_{n-1}) \in \Omega^a \cap \Omega^b$, let $c(s) := s^{n-1} + (x_1 - \varepsilon)s^{n-2} + x_2 s^{n-3} + \cdots + x_{n-2}s + (x_{n-1} + \varepsilon), \varepsilon > 0, \varepsilon$ sufficiently small.

When $n = 6$, the four ellipse equations are (see [34] for details):

$$\begin{cases} (a_2x_1 + x_3 - a_1x_2 - a_3)^2 - 4(a_1 - x_1)(a_5 + a_3x_2 + a_1x_4 - x_5 - a_2x_3 - a_4x_1) = 0, \\ a_6x_1 + a_4x_3 + a_2x_5 - a_3x_4 - a_5x_2 = 0,\ a_5x_4 - a_4x_5 - a_6x_3 = 0,\ a_6x_5 = 0, \end{cases}$$

$$\begin{cases} (a_5 + a_3x_2 + a_1x_4 - x_5 - a_2x_3 - a_4x_1)^2 - 4(a_2x_1 + x_3 - a_1x_2 - a_3)(a_6x_1 + a_4x_3 + a_2x_5 - a_3x_4 - a_5x_2) = 0, \\ a_1 - x_1 = 0,\ a_5x_4 - a_4x_5 - a_6x_3 = 0,\ a_6x_5 = 0, \end{cases}$$

$$\begin{cases} (a_6x_1 + a_4x_3 + a_2x_5 - a_3x_4 - a_5x_2)^2 - 4(a_5 + a_3x_2 + a_1x_4 - x_5 - a_2x_3 - a_4x_1)(a_5x_4 - a_4x_5 - a_6x_3) = 0 \\ a_1 - x_1 = 0,\ a_2x_1 + x_3 - a_1x_2 - a_3 = 0,\ a_6x_5 = 0, \end{cases}$$

$$\begin{cases} (a_5x_4 - a_4x_5 - a_6x_3)^2 - 4(a_6x_1 + a_4x_3 + a_2x_5 - a_3x_4 - a_5x_2)a_6x_5 = 0, \\ a_1 - x_1 = 0,\ a_2x_1 + x_3 - a_1x_2 - a_3 = 0,\ a_5 + a_3x_2 + a_1x_4 - x_5 - a_2x_3 - a_4x_1 = 0. \end{cases}$$

$\forall\omega\in R$, consider

$$\begin{aligned}\mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] &= \frac{1}{\mid a(j\omega)\mid^2}[c_1\omega^{2(n-1)}+c_2\omega^{2(n-2)}+\cdots+c_{n-1}\omega^2+c_n]\\ &\quad+\mathrm{Re}[\frac{-\varepsilon(j\omega)^{n-2}+\varepsilon}{a(j\omega)}]\\ &= \frac{1}{\mid a(j\omega)\mid^2}[c_1\omega^{2(n-1)}+c_2\omega^{2(n-2)}+\cdots+c_{n-1}\omega^2+c_n]\\ &\quad+\frac{(-\varepsilon)}{|a(j\omega)|^2}(-\omega^{2(n-1)}+\tilde{c}(\omega^2)),\end{aligned}$$

where $c_l := \sum_{j=0}^{n}(-1)^{l+j}a_jx_{2l-j-1}, l=1,2,\cdots,n$, $a_0=1, x_0=1$, $a_i=0$ if $i<0$ or $i>n$, and $x_i=0$ if $i<0$ or $i>n-1$, and $\tilde{c}(\omega^2)$ is a real polynomial with order not greater than $2(n-2)$.

In order to prove that $\forall\omega\in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}]>0$, let $t=\omega^2$, we only need to prove that, for any $\varepsilon>0, \varepsilon$ sufficiently small, the following polynomial $f_1(t)$ satisfies

$$\begin{aligned}f_1(t) :=\ & c_1t^{n-1}+c_2t^{n-2}+\cdots+c_{n-1}t+c_n\\ &+\varepsilon(t^{n-1}-\tilde{c}(t))>0, \quad \forall t\in[0,+\infty).\end{aligned}$$

Since $(x_1,x_2,\cdots,x_{n-1})\in\Omega^a$, by the definition of Ω^a, it is easy to know that

$$g_1(t) := c_1t^{n-1}+c_2t^{n-2}+\cdots+c_{n-1}t+c_n>0, \quad \forall t\in(0,+\infty).$$

Moreover, we obviously have $f_1(0)>0$, and for any $\varepsilon>0$, when t is a sufficiently large or sufficiently small positive number, we have $f_1(t)>0$, namely, there exist $0<t_1<t_2$ such that, for all $\varepsilon>0$, $t\in[0,t_1]\cup[t_2,+\infty)$, we have $f_1(t)>0$.

Denote

$$M_1 = \inf_{t\in[t_1,t_2]} g_1(t),$$

$$N_1 = \sup_{t\in[t_1,t_2]} |t^{n-1}-\tilde{c}(t))|.$$

Then $M_1>0$ and $N_1>0$. Choosing $0<\varepsilon<\frac{M_1}{N_1}$, by a direct calculation, we have

$$\begin{aligned}f_1(t) :=\ & c_1t^{n-1}+c_2t^{n-2}+\cdots+c_{n-1}t+c_n\\ &+\varepsilon(t^{n-1}-\tilde{c}(t))>0, \quad \forall t\in[0,+\infty).\end{aligned}$$

Namely,

$$\forall\omega\in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}]>0.$$

Similarly, since $(x_1,x_2,\cdots,x_{n-1})\in\Omega^b$, there exist $0<t_3<t_4$ such that, for all $\varepsilon>0$, $t\in[0,t_3]\cup[t_4,+\infty)$, we have $f_2(t)>0$,where

$$\begin{aligned}f_2(t) :=\ & d_1t^{n-1}+d_2t^{n-2}+\cdots+d_{n-1}t+d_n\\ &+\varepsilon(t^{n-1}-\tilde{d}(t)),\end{aligned}$$

$$d_l := \sum_{j=0}^{n}(-1)^{l+j}b_jx_{2l-j-1}, \quad l=1,2,\cdots,n,$$

where $b_0=1, x_0=1$, $b_i=0$ if $i<0$ or $i>n$, and $x_i=0$ if $i<0$ or $i>n-1$, and $\tilde{d}(\omega^2)$ is a real polynomial with order not greater than $2(n-2)$ which is determined by the following equation:

$$\mathrm{Re}[\frac{-\varepsilon(j\omega)^{n-2}+\varepsilon}{b(j\omega)}] = \frac{(-\varepsilon)}{|b(j\omega)|^2}(-\omega^{2(n-1)}+\tilde{d}(\omega^2)).$$

Denote

$$g_2(t) := d_1 t^{n-1} + d_2 t^{n-2} + \cdots + d_{n-1} t + d_n,$$

$$M_2 = \inf_{t \in [t_3, t_4]} g_2(t),$$

$$N_2 = \sup_{t \in [t_3, t_4]} |t^{n-1} - \tilde{d}(t))|.$$

Then $M_2 > 0$ and $N_2 > 0$. Choosing $0 < \varepsilon < \frac{M_2}{N_2}$, we have

$$\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}] > 0.$$

Thus, by choosing $0 < \varepsilon < \min\{\frac{M_1}{N_1}, \frac{M_2}{N_2}\}$, Lemma 2 is proved.

Lemma 3 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n$, if $\lambda b(s) + (1-\lambda) a(s) \in H^n, \lambda \in [0,1]$, then $\Omega^a \cap \Omega^b \neq \phi$

Proof If $\forall \lambda \in [0,1], a_\lambda(s) := \lambda b(s) + (1-\lambda) a(s) \in H^n$, by Lemma 1, for any $\lambda \in [0,1]$, $\Omega_{ek}^{a_\lambda}, k = 1, 2, \cdots, n-2$, are $n-2$ ellipses in the first quadrant of the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$.

$\forall \lambda \in [0,1]$, denote

$$\begin{aligned}\Omega^{a_\lambda} := \{(x_1, x_2, \cdots, x_{n-1})| \quad & (x_1, x_2, \cdots, x_{n-1}) \in \textstyle\bigcup_{i=1, i<j\leq n-2}^{n-3} (A_{\lambda i}, A_{\lambda j}), \\ & \forall A_{\lambda i} \in \Omega_{ei}^{a_\lambda}, i \in \{1, 2, \cdots, n-2\}\}\end{aligned}$$

Apparently, when λ changes continuously from 0 to 1, Ω^{a_λ} will change continuously from Ω^a to Ω^b, and $\Omega_{ek}^{a_\lambda}$ will change continuously from Ω_{ek}^a to $\Omega_{ek}^b, k = 1, 2, \cdots, n-2$.

Now assume $\Omega^a \cap \Omega^b = \phi$, by the definitions of Ω^a and Ω^b and Lemma 1, $\exists u_1 > 0$, $u_2 > 0, u_1 \neq a_1, u_1 \neq b_1$, and $\exists \tilde{k} \in \{1, 2, \cdots, n-2\}$, such that the following hyperplane L in the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$

$$L: \quad \frac{x_1}{u_1} + \frac{x_2}{u_2} + \cdots + \frac{x_{n-1}}{u_{n-1}} = 1$$

separates Ω^a and Ω^b, meanwhile, L is tangent with $\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ and $\Omega_{e\tilde{k}}^b$ simultaneously (or tangent with $\Omega_{e1}^b, \Omega_{e2}^b, \cdots, \Omega_{e(n-2)}^b$ and $\Omega_{e\tilde{k}}^a$ simultaneously).

Without loss of generality, suppose that L is tangent with $\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ and $\Omega_{e\tilde{k}}^b$ simultaneously.

In what follows, the notation $[x]$ stands for the largest integer that is smaller than or equal to the real number x, and $\langle y \rangle_z$ stands for the remainder of the nonnegative integer y divided by the positive integer z [3].

Since L is tangent with $\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ and $\Omega_{e\tilde{k}}^b$ simultaneously, note that $a(s)$ is Hurwitz stable and $u_1 > 0, u_1 \neq a_1, u_2 > 0$, using mathematical induction, by a lengthy calculation, we know that the necessary and sufficient condition for L being tangent with $\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ simultaneously

[3] For example, $[1.5] = 1, [0.5] = 0, [-1.5] = -2$, and $\langle 0 \rangle_2 = 0, \langle 1 \rangle_2 = 1, \langle 11 \rangle_3 = 2$.

is [4]

$$\sum_{i=0}^{n}(-1)^{[\frac{i+1}{2}]}a_i u_1^{\langle i+1\rangle_2} u_2^{[\frac{n}{2}]-[\frac{i}{2}]}=0 \tag{1}$$

and

$$u_j=(-1)^{[\frac{j-1}{2}]}u_1^{\langle j\rangle_2}u_2^{[\frac{j}{2}]},\quad j=3,4,\cdots,n-1, \tag{2}$$

where $a_0=1$.

Since $u_j=(-1)^{[\frac{j-1}{2}]}u_1^{\langle j\rangle_2}u_2^{[\frac{j}{2}]}$, $j=3,4,\cdots,n-1$, L is tangent with $\Omega^b_{e\tilde{k}}$, by a direct calculation, we have

$$\sum_{i=0}^{n}(-1)^{[\frac{i+1}{2}]}b_i u_1^{\langle i+1\rangle_2} u_2^{[\frac{n}{2}]-[\frac{i}{2}]}=0 \tag{3}$$

where $b_0=1$.

From (1) and (3), we obviously have $\forall\lambda\in[0,1]$,

$$\sum_{i=0}^{n}(-1)^{[\frac{i+1}{2}]}a_{\lambda i} u_1^{\langle i+1\rangle_2} u_2^{[\frac{n}{2}]-[\frac{i}{2}]}=0 \tag{4}$$

where $a_{\lambda i}:=a_i+\lambda(b_i-a_i), a_0=1, b_0=1, i=0,1,2,\cdots,n$. (4) and (2) show that L is also tangent with $\Omega^{a_\lambda}_{e\tilde{k}}(\forall\lambda\in[0,1])$, but L separates $\Omega^a_{e\tilde{k}}$ and $\Omega^b_{e\tilde{k}}$, and when λ changes continuously from 0 to 1, $\Omega^{a_\lambda}_{e\tilde{k}}$ will change continuously from $\Omega^a_{e\tilde{k}}$ to $\Omega^b_{e\tilde{k}}$, which is obviously impossible. This completes the proof.

Lemma 4 Suppose $a(s)=s^n+a_1s^{n-1}+\cdots+a_n\in H^n, b(s)=s^n+b_1s^{n-1}+\cdots+b_n\in H^n, c(s)=s^{n-1}+x_1s^{n-2}+\cdots+x_{n-1}$, if $\forall\omega\in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}]>0$ and $\mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}]>0$, take

$$\tilde{c}\,(s):=c(s)+\delta\cdot h(s),\quad \delta>0,\ \delta \text{ sufficiently small},$$

(where $h(s)$ is an arbitrarily given monic n-th order polynomial), then $\frac{\tilde{c}\,(s)}{a(s)}$ and $\frac{\tilde{c}\,(s)}{b(s)}$ are both strictly positive real.

Proof Obviously, $\partial(\tilde{c})=\partial(a)=n$, namely, $\tilde{c}(s)$ and $a(s)$ have the same order n. Since $a(s)\in H^n$, there exists $\omega_1>0$ such that, for all $|\,\omega\,|\geq\omega_1$, we have $\mathrm{Re}(\frac{\tilde{c}(j\omega)}{a(j\omega)})>0$.

[4] When $n=3$, we have (see [27, 28] for details):

$$u_1u_2-a_1u_2-a_2u_1+a_3=0.$$

When $n=4$, we have (see [25] [30]-[32] for details):

$$\begin{cases} u_1u_2^2-a_1u_2^2-a_2u_1u_2+a_3u_2+a_4u_1=0,\\ u_3=-u_1u_2.\end{cases}$$

When $n=5$, we have (see [33] for details):

$$\begin{cases} u_1u_2^2-a_1u_2^2-a_2u_1u_2+a_3u_2+a_4u_1-a_5=0,\\ u_3=-u_1u_2,\ u_4=-u_2^2.\end{cases}$$

When $n=6$, we have (see [34] for details):

$$\begin{cases} u_1u_2^3-a_1u_2^3-a_2u_1u_2^2+a_3u_2^2+a_4u_1u_2-a_5u_2-a_6u_1=0,\\ u_3=-u_1u_2,\ u_4=-u_2^2,\ u_5=u_1u_2^2.\end{cases}$$

Denote

$$M_3 = \inf_{|\omega| \le \omega_1} \mathrm{Re}(\frac{c(j\omega)}{a(j\omega)}), \qquad N_3 = \sup_{|\omega| \le \omega_1} | \mathrm{Re}(\frac{h(j\omega)}{a(j\omega)}) |.$$

Then $M_3 > 0$ and $N_3 > 0$. Choosing $0 < \varepsilon < \frac{M_3}{N_3}$, it can be directly verified that

$$\mathrm{Re}(\frac{\tilde{c}(j\omega)}{a(j\omega)}) > 0, \forall \omega \in R.$$

Similarly, $\partial(\tilde{c}) = \partial(b) = n$, and there exists $\omega_2 > 0$ such that, for all $| \omega | \ge \omega_2$, we have $\mathrm{Re}(\frac{\tilde{c}(j\omega)}{b(j\omega)}) > 0$.

Denote

$$M_4 = \inf_{|\omega| \le \omega_2} \mathrm{Re}(\frac{c(j\omega)}{b(j\omega)}), \qquad N_4 = \sup_{|\omega| \le \omega_2} | \mathrm{Re}(\frac{h(j\omega)}{b(j\omega)}) |.$$

Then $M_4 > 0$ and $N_4 > 0$. Choosing $0 < \varepsilon < \frac{M_4}{N_4}$, it can be directly verified that

$$\mathrm{Re}(\frac{\tilde{c}(j\omega)}{b(j\omega)}) > 0, \forall \omega \in R.$$

Thus, by choosing $0 < \varepsilon < \min\{\frac{M_3}{N_3}, \frac{M_4}{N_4}\}$, Lemma 4 is proved.

The sufficiency of Theorem 1 is now proved by simply combining Lemmas 1-4.

Remark 1 From the proof of Theorem 1, we can see that this paper not only proves the existence, but also provides a design method. In fact, based on the main idea of this paper, we have developed a geometric algorithm with order reduction for robust SPR synthesis which is very efficient for high order polynomial segments [26].

Remark 2 The method provided in this paper is constructive, and is insightful and helpful in solving more general robust SPR synthesis problems for polynomial polytopes, multilinear families, etc..

Remark 3 Our main results in this paper can also be extended to discrete-time case. In fact, by applying the bilinear transformation, we can transform the unit circle into the left half plane. Hence, Theorem 1 can be generalized to discrete-time case. Moreover, in discrete-time case, the order of the polynomial obtained by our method is bounded by the order of the given polynomial segment [27, 29].

Remark 4 If $\frac{c(s)}{a(s)}$ and $\frac{c(s)}{b(s)}$ are both SPR, it is easy to know that $\forall \lambda \in [0,1]$, $\frac{c(s)}{\lambda a(s) + (1-\lambda) b(s)}$ is also SPR.

Remark 5 The stability of polynomial segment can be checked by many efficient methods, e.g., eigenvalue method, root locus method, value set method, etc. [1, 4, 6].

3 Conclusions

We have constructively proved that, for any two n-th order polynomials $a(s)$ and $b(s)$, the Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both strictly positive real. By using similar method, we can also constructively prove the existence of SPR synthesis for low order ($n \le 4$) interval polynomials. Namely, when $n \le 4$, the Hurwitz stability of the four Kharitonov vertex polynomials is necessary and sufficient for the existence of a fixed polynomial such that the ratio of this polynomial to any polynomial in the interval polynomial set is SPR invariant [22, 23, 25, 27, 30, 31]. The SPR synthesis problem for high order interval polynomials is currently under investigation.

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