H^∞ Performance of Interval Systems

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Abstract

In this paper, we study H^∞ performance of interval systems. We prove that, for an interval system, the maximal H^∞ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices.

Keywords: H^∞ Control Theory, Uncertain Systems, Performance Analysis, Robustness, Interval Model, Sensitivity Functions.

1   Introduction

Motivated by the seminal theorem of Kharitonov on robust stability of interval polynomials[1, 2], a number of papers on robustness analysis of uncertain systems have been published in the past few years[3, 4, 5, 6, 7, 8, 9, 10]. Kharitonov's theorem states that the Hurwitz stability of the real (or complex) interval polynomial family can be guaranteed by the Hurwitz stability of four (or eight) prescribed critical vertex polynomials in this family. This result is significant since it reduces checking stability of infinitely many polynomials to checking stability of finitely many polynomials, and the number of critical vertex polynomials need to be checked is independent of the order of the polynomial family. An important extension of Kharitonov's theorem is the edge theorem discovered by Bartlett, Hollot and Huang[4]. The edge theorem states that the stability of a polytope of polynomials can be guaranteed by the stability of its one-dimensional exposed edge polynomials. The significance of the edge theorem is that it allows some (affine) dependency among polynomial coefficients, and applies to more general stability regions, e.g., unit circle, left sector, shifted half plane, hyperbola region, etc. When the dependency among polynomial coefficients is nonlinear, however, Ackermann shows that checking a subset of a polynomial family generally can not guarantee the stability of the entire family[11, 12].

In this paper, we prove that, for an interval system, the maximal H_∞ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices. This

result is useful in robust performance analysis and H_∞ control design for dynamic systems under parametric perturbations.

2 Main Results

Denote the m-th, n-th ($m < n$) order real interval polynomial families $K_g(s)$, $K_f(s)$ as

$$K_g(s) = \{g(s)|g(s) = \sum_{i=0}^{m} b_i s^i, b_i \in [\underline{b_i}, \overline{b_i}], i = 0, 1, ..., m\}, \tag{1}$$

$$K_f(s) = \{f(s)|f(s) = \sum_{i=0}^{n} a_i s^i, a_i \in [\underline{a_i}, \overline{a_i}], i = 0, 1, ..., n\}. \tag{2}$$

For any $f(s) \in K_f(s)$, it can be expressed as

$$f(s) = \alpha_f(s^2) + s\beta_f(s^2), \tag{3}$$

where

$$\alpha_f(s^2) = a_0 + a_2 s^2 + a_4 s^4 + a_6 s^6 + ..., \tag{4}$$

$$\beta_f(s^2) = a_1 + a_3 s^2 + a_5 s^4 + a_7 s^6 + ... \tag{5}$$

Obviously, for any fixed $\omega \in R$, $\alpha_f(-\omega^2)$ and $\omega\beta_f(-\omega^2)$ are the real and imaginary parts of $f(j\omega) \in C$ respectively.

For the interval polynomial family $K_f(s)$, define

$$\alpha_f^{(1)}(s^2) = \underline{a_0} + \overline{a_2} s^2 + \underline{a_4} s^4 + \overline{a_6} s^6 + ..., \tag{6}$$

$$\alpha_f^{(2)}(s^2) = \overline{a_0} + \underline{a_2} s^2 + \overline{a_4} s^4 + \underline{a_6} s^6 + ..., \tag{7}$$

$$\beta_f^{(1)}(s^2) = \underline{a_1} + \overline{a_3} s^2 + \underline{a_5} s^4 + \overline{a_7} s^6 + ..., \tag{8}$$

$$\beta_f^{(2)}(s^2) = \overline{a_1} + \underline{a_3} s^2 + \overline{a_5} s^4 + \underline{a_7} s^6 + ..., \tag{9}$$

and denote the four Kharitonov vertex polynomials of $K_f(s)$ as

$$f_{ij}(s) = \alpha_f^{(i)}(s^2) + s\beta_f^{(j)}(s^2), \qquad i, j = 1, 2 \tag{10}$$

For the interval polynomial family $K_g(s)$, the corresponding $\alpha_g^{(i)}(s), \beta_g^{(j)}(s)$ and $g_{ij}(s) \in K_g(s)$ can be defined analogously.

Denote by H the set of all Hurwitz stable polynomials (i.e. all of their roots lie within the open left half of the complex plane).

For the proper stable rational function $\frac{p(s)}{q(s)}$, the H_∞ norm is defined as

$$\left\|\frac{p(s)}{q(s)}\right\|_{\infty} = \sup\left\{\left|\frac{p(j\omega)}{q(j\omega)}\right| \ \middle| \ \omega \in (-\infty, +\infty)\right\} \tag{11}$$

The proper complex rational function $\frac{p(s)}{q(s)}$ is said to be strictly positive real, if 1) $q(s) \in H$; and 2) for any $\omega \in R$, $\Re\frac{p(j\omega)}{q(j\omega)} > 0$.

Denote by SPR the set of all strictly positive real rational functions.

Lemma 2.1[14]

For any fixed $\omega \in R$, $f(s) \in K_f(s)$, we have

$$\alpha_f^{(1)}(-\omega^2) \le \alpha_f(-\omega^2) \le \alpha_f^{(2)}(-\omega^2), \tag{12}$$

$$\beta_f^{(1)}(-\omega^2) \le \beta_f(-\omega^2) \le \beta_f^{(2)}(-\omega^2). \tag{13}$$

Lemma 2.2[11] (Zero Exclusion Principle)

For the n-th order polynomial family

$$f(s,T) =: \{f(s,t)|t \in T\}, \tag{14}$$

where T is a bounded connected closed set, and the coefficients of $f(s,t)$ are continuous functions of t, then $f(s,T) \in H$ if and only if

1) there exists $t^* \in T$, such that $f(s,t^*) \in H$;
2) $0 \notin f(j\omega,T)$, $\forall \omega \in R$.

Consider the strictly proper open-loop transfer function

$$P = \frac{g(s)}{f(s)} \tag{15}$$

and suppose the closed-loop system is stable under negative unity feedback. Denote its sensitivity function as

$$S = \frac{1}{1+P} = \frac{f(s)}{f(s) + g(s)} \tag{16}$$

Apparently, we have

$$\|S\|_{\infty} \ge 1 \tag{17}$$

For notational simplicity, define

$$J_{i_1 j_1 i_2 j_2}(s) = g_{i_1 j_1}(s) + (1 + \delta e^{j\theta})f_{i_2 j_2}(s), \qquad \delta \in (0,1), \ \ i_1, j_1, i_2, j_2 = 1, 2, \ \ \theta \in [-\pi, \pi]. \tag{18}$$

Lemma 2.3

Suppose $g(s) + f(s) \in H$. Then, for any $\gamma > 1$, we have

$$\|S\|_{\infty} < \gamma \iff g(s) + (1 + \frac{1}{\gamma}e^{j\theta})f(s) \in H, \quad \forall \theta \in [-\pi, \pi]. \tag{19}$$

Proof: Necessity: Since $g(s)+f(s) \in H$ and $||\frac{\frac{1}{\gamma}f(s)}{f(s)+g(s)}||_\infty < 1$, by Rouche's Theorem, we know that

$$[g(s) + f(s)] + \frac{1}{\gamma}e^{j\theta}f(s) \in H, \quad \forall \theta \in [-\pi, \pi] \tag{20}$$

Sufficiency: Now suppose on the contrary that $||S||_\infty \geq \gamma$, namely, $||\frac{\frac{1}{\gamma}f(s)}{f(s)+g(s)}||_\infty \geq 1$. Since $|\frac{\frac{1}{\gamma}f(s)}{f(s)+g(s)}|_{s=j\omega}|$ is a contiunous function of ω, and since

$$\lim_{\omega \to \infty} |\frac{\frac{1}{\gamma}f(s)}{f(s)+g(s)}|_{s=j\omega}| = \frac{1}{\gamma} < 1 \tag{21}$$

there must exist ω_0 such that

$$|\frac{\frac{1}{\gamma}f(s)}{f(s)+g(s)}|_{s=j\omega_0}| = 1 \tag{22}$$

Therefore, there exists $\theta_0 \in [-\pi, \pi]$ such that

$$\{g(s) + f(s) + \frac{1}{\gamma}e^{j\theta_0}f(s)\}|_{s=j\omega_0} = 0 \tag{23}$$

which contradicts the original hypothesis. This completes the proof.

Lemma 2.4
For any $\delta \in (0, 1), \theta \in [-\pi, \pi]$, we have

$$W(s) =: \{g(s) + (1+\delta e^{j\theta})f(s)|g(s) \in K_g(s), f(s) \in K_f(s)\} \subset H \iff \tag{24}$$

$$J_{1111}, J_{1212}, J_{2222}, J_{2121}, J_{1112}, J_{1222}, J_{2221}, J_{2111}, J_{1211}, J_{2212}, J_{2122}, J_{1121} \in H \tag{25}$$

Proof: Necessity is obvious. To prove sufficiency, note that $W(s)$ is a set of polynomials with complex coefficients, and with constant order n. By Lemma 2.2, it suffices to show that

$$0 \notin W(j\omega), \quad \forall \omega \in R \tag{26}$$

Since $0 \notin W(j\omega_\infty)$ for sufficiently large ω_∞, we only need to show that

$$0 \notin \partial W(j\omega), \quad \forall \omega \in R \tag{27}$$

where $\partial W(j\omega)$ stands for the boundary of $W(j\omega)$ in the complex plane.

To construct $\partial W(j\omega)$, note that $\arg(1+\delta e^{j\theta}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Suppose now $\omega \geq 0$ and $\arg(1+\delta e^{j\theta}) \in [0, \frac{\pi}{2})$. Then by Lemma 2.1, we know that $K_g(j\omega), K_f(j\omega)$ are rectangles with edges parallel to the coordinate axes. The four vertices of $K_g(j\omega)$ are $g_{11}(j\omega), g_{12}(j\omega), g_{21}(j\omega), g_{22}(j\omega)$, respectively; and the four vertices of $K_f(j\omega)$ are $f_{11}(j\omega), f_{12}(j\omega), f_{21}(j\omega), f_{22}(j\omega)$, respectively. $(1+\delta e^{j\theta})K_f(j\omega)$ is generated by rotating $K_f(j\omega)$ by $\arg(1+\delta e^{j\theta})$ counterclockwisely, and then scaling by

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$|1 + \delta e^{j\theta}|$. Thus, $W(j\omega) = K_g(j\omega) + (1 + \delta e^{j\theta})K_f(j\omega)$ is a convex polygon with eight edges. These edges are parallel to either the edges of $K_g(j\omega)$ or the edges of $(1 + \delta e^{j\theta})K_f(j\omega)$. Therefore, their orientations are fixed (independent of ω). The eight vertices of $W(j\omega)$ are (clockwisely) $J_{1111}(j\omega)$, $J_{1112}(j\omega)$, $J_{1212}(j\omega)$, $J_{1222}(j\omega)$, $J_{2222}(j\omega)$, $J_{2221}(j\omega)$, $J_{2121}(j\omega)$, $J_{2111}(j\omega)$, respectively.

Now suppose on the contrary that there exists $\omega_0 \geq 0$ such that

$$0 \in \partial W(j\omega_0) \tag{28}$$

Without loss of generality, suppose

$$0 \in \{\lambda J_{1111}(j\omega_0) + (1 - \lambda)J_{1112}(j\omega_0) | \lambda \in [0, 1]\} \tag{29}$$

Namely, there exists $\lambda_0 \in (0, 1)$ such that

$$\lambda_0 J_{1111}(j\omega_0) + (1 - \lambda_0)J_{1112}(j\omega_0) = 0 \tag{30}$$

Since $J_{1111}(s), J_{1112}(s) \in H$, we have

$$\frac{d}{d\omega} \arg J_{1111}(j\omega) > 0, \qquad \frac{d}{d\omega} \arg J_{1112}(j\omega) > 0 \tag{31}$$

Thus[15]

$$\frac{d}{d\omega} \arg[J_{1112}(j\omega) - J_{1111}(j\omega)]|_{\omega=\omega_0} = \tag{32}$$

$$(1 - \lambda_0)\frac{d}{d\omega} \arg J_{1111}(j\omega)|_{\omega=\omega_0} + \lambda_0 \frac{d}{d\omega} \arg J_{1112}(j\omega)|_{\omega=\omega_0} > 0 \tag{33}$$

This contradicts the fact that the edges of $W(j\omega)$ have fixed orientations. Thus

$$0 \notin \partial W(j\omega) \tag{34}$$

Suppose now $\omega \leq 0$ and $\arg(1 + \delta e^{j\theta}) \in (-\frac{\pi}{2}, 0]$. Then $K_g(j\omega), (1 + \delta e^{j\theta})K_f(j\omega)$ are the mirror images (with respect to the real axis) of the corresponding sets in the case of $\omega \geq 0$ and $\arg(1 + \delta e^{j\theta}) \in [0, \frac{\pi}{2})$. Therefore, following an identical line of arguments, we have

$$0 \notin \partial W(j\omega) \tag{35}$$

The cases when $\omega \geq 0$ and $\arg(1+\delta e^{j\theta}) \in (-\frac{\pi}{2}, 0]$ and when $\omega \leq 0$ and $\arg(1+\delta e^{j\theta}) \in [0, \frac{\pi}{2})$ are also symmetric with respect to the real axis. Hence, we only need to consider the former case. In this case, $K_g(j\omega), K_f(j\omega)$ are rectangles with edges parallel to the coordinate axes. $(1 + \delta e^{j\theta})K_f(j\omega)$ is generated by rotating $K_f(j\omega)$ by $|\arg(1 + \delta e^{j\theta})|$ clockwisely, and then scaling by $|1 + \delta e^{j\theta}|$. Thus, $W(j\omega) = K_g(j\omega)+(1+\delta e^{j\theta})K_f(j\omega)$ is a convex polygon with eight edges. These edges are parallel to either the edges of $K_g(j\omega)$ or the edges of $(1 + \delta e^{j\theta})K_f(j\omega)$. Therefore, their orientations are fixed (independent of ω). The eight vertices of $W(j\omega)$ are (clockwisely) $J_{1111}(j\omega)$, $J_{1211}(j\omega)$, $J_{1212}(j\omega)$, $J_{2212}(j\omega)$, $J_{2222}(j\omega)$, $J_{2122}(j\omega)$, $J_{2121}(j\omega)$, $J_{1121}(j\omega)$, respectively. Thus, following a similar argument, we have

$$0 \notin \partial W(j\omega) \tag{36}$$

This completes the proof.

The following theorem shows that, for an interval system, the maximal H_∞ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices.

Theorem 2.1

Suppose $g_{ij}(s) + f_{ij}(s) \in H, \ i, j = 1, 2$. Then

$$\max\{\|\frac{f(s)}{f(s) + g(s)}\|_\infty | g(s) \in K_g(s), f(s) \in K_f(s)\} = \tag{37}$$

$$\max\{\|\frac{f_{i_2 j_2}(s)}{f_{i_2 j_2}(s) + g_{i_1 j_1}(s)}\|_\infty | (i_1 j_1 i_2 j_2) = (1111), (1212), \tag{38}$$

$$(2222), (2121), (1112), (1222), (2221), (2111), (1211), (2212), (2122), (1121)\} \tag{39}$$

Proof: Since $g_{ij}(s) + f_{ij}(s) \in H, \ i, j = 1, 2$, by Kharitonov's Theorem[1], we know that $K_g(s) + K_f(s) \subset H$. Let

$$\gamma_1 = \max\{\|\frac{f(s)}{f(s) + g(s)}\|_\infty | g(s) \in K_g(s), f(s) \in K_f(s)\} \tag{40}$$

$$\gamma_2 = \max\{\|\frac{f_{i_2 j_2}(s)}{f_{i_2 j_2}(s) + g_{i_1 j_1}(s)}\|_\infty | (i_1 j_1 i_2 j_2) = (1111), (1212), \tag{41}$$

$$(2222), (2121), (1112), (1222), (2221), (2111), (1211), (2212), (2122), (1121)\} \tag{42}$$

Then apparently

$$\gamma_1 \geq \gamma_2 \geq 1 \tag{43}$$

Now suppose $\gamma_1 \neq \gamma_2$, namely, $\gamma_1 > \gamma_2$. Then there exists γ_0 such that $\gamma_1 > \gamma_0 > \gamma_2$. Thus, for any $(i_1 j_1 i_2 j_2) \in \{(1111), (1212), (2222), (2121), (1112), (1222), (2221), (2111), (1211), (2212), (2122), (1121)\}$, we have

$$\|\frac{f_{i_2 j_2}(s)}{f_{i_2 j_2}(s) + g_{i_1 j_1}(s)}\|_\infty < \gamma_0 \tag{44}$$

Hence, by Lemma 2.3, we have

$$g_{i_1 j_1}(s) + (1 + \frac{1}{\gamma_0} e^{j\theta}) f_{i_2 j_2}(s) \in H, \quad \forall \theta \in [-\pi, \pi] \tag{45}$$

By Lemma 2.4, we know that

$$\{g(s) + (1 + \frac{1}{\gamma_0} e^{j\theta}) f(s) | g(s) \in K_g(s), f(s) \in K_f(s)\} \subset H, \quad \forall \theta \in [-\pi, \pi] \tag{46}$$

Therefore, by Lemma 2.3, for any $g(s) \in K_g(s), f(s) \in K_f(s)$, we have

$$\|\frac{f(s)}{f(s) + g(s)}\|_\infty < \gamma_0 \tag{47}$$

Namely

$$\max\{\|\frac{f(s)}{f(s) + g(s)}\|_\infty | g(s) \in K_g(s), f(s) \in K_f(s)\} < \gamma_0 \tag{48}$$

That is, $\gamma_1 < \gamma_0$, which contradicts $\gamma_1 > \gamma_0 > \gamma_2$. This completes the proof.

3 Conclusions

We have proved that, for an interval system, the maximal H^∞ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices. This result is useful in robust performance analysis and H_∞ control design for dynamic systems under parametric perturbations.

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