ON EXTON'S GENERALIZED QUADRUPLE HYPERGEOMETRIC FUNCTIONS AND CHANDEL'S FUNCTION OF MATRIX ARGUMENTS

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ABSTRACT

In continuation of our previous studies [6,7] we have established five results in this paper- one for the Chandel's ${}^{(k)}_{(1)}E^{(n)}_{C}$ function, one for the Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ function and three for the Exton's ${}^{(k)}_{(2)}E^{(n)}_{D}$ function of matrix arguments.

INTRODUCTION

Exton [2,3] has given two functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ which, according to him, are generalizations of certain of the quadruple hypergeometric functions discussed by him in [3].Chandel [1] has also given a similar function ${}^{(k)}_{(1)}E^{(n)}_{C}$. We have already defined the functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(1)}E^{(n)}_{C}$ for the matrix arguments case in our previous article [7].

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In this paper we have defined the function ${}^{(k)}_{(2)}E^{(n)}_{D}$ with matrix arguments and have proved some results for these functions. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [5, 6].

1. The Chandel's ${}^{(k)}_{(1)}E^{(n)}_{C}$ Function

THEOREM 1.1:

$$
{}^{(k)}_{(1)}E^{(n)}_{C}(a,a',b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)
$$

$$
=\frac{1}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b)}\int_{U>0}\int_{V>0}\int_{W>0} e^{-tr(U+V+W)}|U|^{a-(p+1)/2}\times
$$

$$
|V|^{a'-(p+1)/2}|W|^{b-(p+1)/2}\,{}_0F_1(\ ;c_1;-W^{1/2}U^{1/2}X_1U^{1/2}W^{1/2})\cdots\times
$$

$$
{}_0F_1(\ ;c_k;-W^{1/2}U^{1/2}X_kU^{1/2}W^{1/2})\,{}_0F_1(\ ;c_{k+1};-W^{1/2}V^{1/2}X_{k+1}V^{1/2}W^{1/2})\times
$$

$$
\cdots{}_0F_1(\ ;c_n;-W^{1/2}V^{1/2}X_nV^{1/2}W^{1/2})dUdVdW \qquad \cdots\cdots(1.1)
$$

for $Re(a,a',b)>(p-1)/2$.

PROOF: Taking the M-transform (matrix transform) of the right side of eq.(1.1) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$
\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\times
$$

$$
\cdots|X_n|^{\rho_n-(p+1)/2}\,{}_0F_1(\ ;c_1;-W^{1/2}U^{1/2}X_1U^{1/2}W^{1/2})\cdots\times
$$

$$
{}_0F_1(\ ;c_k;-W^{1/2}U^{1/2}X_kU^{1/2}W^{1/2})\,{}_0F_1(\ ;c_{k+1};-W^{1/2}V^{1/2}X_{k+1}V^{1/2}W^{1/2})\times
$$

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$$\cdots\ {}_0F_1(\ ;c_n;-W^{1/2}V^{1/2}X_nV^{1/2}W^{1/2})dX_1\cdots dX_k dX_{k+1}\cdots dX_n \qquad \cdots\cdots(1.2)$$

Applying the following transformations to the expression (1.2),

$Y_i = W^{1/2}U^{1/2}X_iU^{1/2}W^{1/2}$, $Y_j = W^{1/2}V^{1/2}X_jV^{1/2}W^{1/2}$ with $dY_i = |W|^{(p+1)/2}\times |U|^{(p+1)/2}dX_i; dY_j = |W|^{(p+1)/2}|V|^{(p+1)/2}dX_j$; and $|Y_i| = |W||U||X_i|$; $|Y_j| = |W||V||X_j|$; for $i = 1,\cdots,k$ and $j = k+1,\cdots,n$

and then writing the M-transforms of the ${}_0F_1$ functions, we are led to,

$$|U|^{-\rho_1-\cdots-\rho_k}|V|^{-\rho_{k+1}-\cdots-\rho_n}|W|^{-\rho_1-\cdots-\rho_n}\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)}{\Gamma_p(c_1-\rho_1)}\cdots\times$$

$$\frac{\Gamma_p(c_n)\Gamma_p(\rho_n)}{\Gamma_p(c_n-\rho_n)} \qquad \cdots\cdots(1.3)$$

On substituting this expression on the right side of eq.(1.1) and then integrating out the variables U,V,W in the resulting expression by using a Gamma integral produces $M[{}^{(k)}_{(1)}E^{(n)}_C]$ as given by eq.(3.2) of the authors' paper [7].

2. The Exton's ${}^{(k)}_{(1)}E^{(n)}_D$ Function

THEOREM 2.1:

$${}^{(k)}_{(1)}E^{(n)}_D(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$

$$=\frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\int_{U_1>0}\cdots\int_{U_n>0}e^{-tr(U_1+\cdots+U_n)}|U_1|^{b_1-(p+1)/2}\cdots\times$$

$$|U_n|^{b_n-(p+1)/2}\Psi_2(a;c,c';-U_1^{1/2}X_1U_1^{1/2}-\cdots-U_k^{1/2}X_kU_k^{1/2},-U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}$$

$$-\cdots-U_n^{1/2}X_nU_n^{1/2})dU_1\cdots dU_n \qquad \cdots\cdots(2.1)$$

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for $\mathrm{Re}(b_1,\cdots,b_n) > (p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.1) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$\int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\times$$

$$\cdots|X_n|^{\rho_n-(p+1)/2}\Psi_2(a;c,c';-U_1^{1/2}X_1U_1^{1/2}-\cdots-U_k^{1/2}X_kU_k^{1/2},$$

$$-U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}-\cdots-U_n^{1/2}X_nU_n^{1/2})dX_1\cdots dX_k dX_{k+1}\cdots dX_n \qquad \cdots\cdots(2.2)$$

Applying the transformations

$Y_j = U_j^{1/2}X_jU_j^{1/2}$ with $dY_j = |U_j|^{(p+1)/2}dX_j$ and $|Y_j| = |U_j||X_j|$ for $j=1,\cdots,n$

to the above expression generates,

$$|U_1|^{-\rho_1}\cdots|U_n|^{-\rho_n}\int_{Y_1>0}\cdots\int_{Y_n>0}|Y_1|^{\rho_1-(p+1)/2}\cdots|Y_k|^{\rho_k-(p+1)/2}\times$$

$$|Y_{k+1}|^{\rho_{k+1}-(p+1)/2}\cdots|Y_n|^{\rho_n-(p+1)/2}\Psi_2(a;c,c';-Y_1-\cdots-Y_k,$$

$$-Y_{k+1}-\cdots-Y_n)dY_1\cdots dY_k dY_{k+1}\cdots dY_n \qquad \cdots\cdots(2.3)$$

Now, we apply the following transformations to the expression (2.3),

$Z_1 = Y_1, Z_2 = Y_1+Y_2,\cdots,Z_k = Y_1+\cdots+Y_k; Z_{k+1} = Y_{k+1},$

$Z_{k+2} = Y_{k+1}+Y_{k+2},\cdots,Z_n = Y_{k+1}+\cdots+Y_n$; with $dZ_1\cdots dZ_k = dY_1\cdots dY_k$ and $dZ_{k+1}\cdots dZ_n = dY_{k+1}\cdots dY_n$ (from eq.(6.7) page 95 of Mathai [4]), where, $0<Z_1<\cdots<Z_k$ and $0<Z_{k+1}<\cdots<Z_n$.

Then integrating out the variables $Z_1,\cdots,Z_{k-1}$ and $Z_{k+1},\cdots,Z_{n-1}$ one-by-one and in order by using a type-1 Beta integral and on writing the M-transform of a Ψ_2- function, we obtain,

$$|U_1|^{-\rho_1}\cdots|U_n|^{-\rho_n}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\Gamma_p(c)\Gamma_p(c')\Gamma_p(a-\rho_1-\cdots-\rho_n)}{\Gamma_p(a)\Gamma_p(c-\rho_1-\cdots-\rho_k)\Gamma_p(c'-\rho_{k+1}-\cdots-\rho_n)} \qquad \cdots\cdots(2.4)$$

Substituting this expression on the right side of eq.(2.1) and then integrating out the variables $U_1,\cdots,U_n$ in the resulting expression by using a Gamma integral, the outcome is $M[{}^{(k)}_{(1)}E^{(n)}_{D}]$ as given by eq.(3.1) of the authors' article [7].

3. The Exton's ${}^{(k)}_{(2)}E^{(n)}_{D}$ Function

DEFINITION 3.1: The ${}^{(k)}_{(2)}E^{(n)}_{D}$- function of matrix arguments

$${}^{(k)}_{(2)}E^{(n)}_{D} = {}^{(k)}_{(2)}E^{(n)}_{D}(a,a',b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$$

is defined as that class of functions for which the matrix transform (M-transform) is the following:

$$M[{}^{(k)}_{(2)}E^{(n)}_{D}] = [\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$${}^{(k)}_{(2)}E^{(n)}_{D}(a,a',b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)dX_1\cdots dX_n]$$

$$=\frac{\Gamma_p(a-\rho_1-\cdots-\rho_k)}{\Gamma_p(a)}\frac{\Gamma_p(a'-\rho_{k+1}-\cdots-\rho_n)}{\Gamma_p(a')}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\cdots\frac{\Gamma_p(b_n-\rho_n)}{\Gamma_p(b_n)}\times$$

$$\frac{\Gamma_p(c)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \qquad \cdots\cdots(3.1)$$

for $\mathrm{Re}(a-\rho_1-\cdots-\rho_k, a'-\rho_{k+1}-\cdots-\rho_n, b_j-\rho_j, c-\rho_1-\cdots-\rho_n, \rho_j)$
$>(p-1)/2;\ j=1,\cdots,n.$

THEOREM 3.1:

$${}^{(k)}_{(2)}E^{(n)}_{D}(a,a',b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$$

$$=\frac{\Gamma_p(c)}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_n)}\int\cdots(n)\cdots\int|U_1|^{b_1-(p+1)/2}\cdots\times$$

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$$|U_n|^{b_n-(p+1)/2}|I-U_1-\cdots-U_n|^{c-b_1-\cdots-b_n-(p+1)/2}\times$$

$$\left|I+U_1^{1/2}X_1U_1^{1/2}+\cdots+U_k^{1/2}X_kU_k^{1/2}\right|^{-a}\times$$

$$\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a'}dU_1\cdots dU_n \qquad \cdots\cdots(3.2)$$

where $U_i' = U_i > 0$ and $0 < U_1+\cdots+U_n < I$ and for $\mathrm{Re}(b_i, c-b_1-\cdots-b_n)$ $> (p-1)/2;\ i=1,\cdots,n.$

PROOF: Taking the M-transform of the right side of eq.(3.2) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\times$$

$$\cdots|X_n|^{\rho_n-(p+1)/2}\left|I+U_1^{1/2}X_1U_1^{1/2}+\cdots+U_k^{1/2}X_kU_k^{1/2}\right|^{-a}\times$$

$$\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a'}dX_1\cdots dX_k dX_{k+1}\cdots dX_n \quad \cdots\cdots(3.3)$$

On using the transformations

$Y_j = U_j^{1/2}X_jU_j^{1/2}$, with $dY_j = |U_j|^{(p+1)/2}dX_j$; and $|Y_j| = |U_j||X_j|$ for $j=1,\cdots,n$ in the above expression and then integrating out the variables $Y_1,\cdots,Y_k$ and $Y_{k+1},\cdots,Y_n$ by using a type-2 Dirichlet integral, we are led to,

$$|U_1|^{-\rho_1}\cdots|U_n|^{-\rho_n}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\Gamma_p(a-\rho_1-\cdots-\rho_k)}{\Gamma_p(a)\Gamma_p(a')}\times$$

$$\Gamma_p(a'-\rho_{k+1}-\cdots-\rho_n) \qquad \cdots\cdots(3.4)$$

Substituting this expression on the right side of eq.(3.2) and then integrating out $U_1, \cdots, U_n$ in the resulting expression by using a type-1 Dirichlet integral we have $M[{}^{(k)}_{(2)}E^{(n)}_{D}]$ as given by eq.(3.1).

It may be easily seen that a limiting form of eq.(3.2) has the following form:

$$\lim_{\alpha\to\infty} {}^{(k)}_{(2)}E^{(n)}_{D}(\alpha, \alpha, b_1, \cdots, b_n; c; \frac{-X_1}{\alpha}, \cdots, \frac{-X_n}{\alpha})$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c - b_1 - \cdots - b_n)} \int \cdots (n) \cdots \int |U_1|^{b_1-(p+1)/2} \cdots \times$$

$$|U_n|^{b_n-(p+1)/2} |I - U_1 - \cdots - U_n|^{c-b_1-\cdots-b_n-(p+1)/2} \times$$

$$e^{-tr(U_1X_1+\cdots+U_nX_n)} dU_1 \cdots dU_n \qquad \cdots\cdots(3.5)$$

where $U_i' = U_i > 0$ and $0 < U_1 + \cdots + U_n < I$ and for $Re(b_i, c - b_1 - \cdots - b_n) > (p-1)/2;\ i = 1, \cdots, n.$

THEOREM 3.2: A case of reducibility:-

$$\lim_{\alpha\to\infty} {}^{(k)}_{(2)}E^{(n)}_{D}(\alpha, \alpha, b_1, \cdots, b_n; c; \frac{-X}{\alpha}, \cdots (n) \cdots, \frac{-X}{\alpha})$$

$$= {}_1F_1(b_1 + \cdots + b_n; c; -X) \qquad \cdots\cdots(3.6)$$

PROOF: To prove this theorem we put $X_1 = X_2 = \cdots = X_n = X$ in eq.(3.5) and apply the following transformations to the resulting expression $V_1 = U_1, V_2 = U_1 + U_2, \cdots, V_n = U_1 + \cdots + U_n$; with $dU_1 \cdots dU_n = dV_1 \cdots dV_n$ (from eq.(6.7) page 95 of Mathai [4]), where, $0 < V_1 < \cdots < V_n < I$.

Then integrating out the variables $V_1, \cdots, V_{n-1}$ one-by-one and in order by using a type-1 Beta integral and finally using the theorem 2.3.4 page 42 of Mathai [5], the desired result follows.

THEOREM 3.3:

$$ {}^{(k)}_{(2)}E_D^{(n)}(a,a',b_1,\cdots,b_n;c;-X_1,\cdots,-X_n) $$

$$ = \frac{1}{\Gamma_p(a)\Gamma_p(a')} \int_{U>0}\int_{V>0} e^{-tr(U+V)} |U|^{a-(p+1)/2} |V|^{a'-(p+1)/2} \times $$

$$ \Phi_2^{(n)}(b_1,\cdots,b_n;c;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2},-V^{1/2}X_{k+1}V^{1/2},\cdots, $$

$$ -V^{1/2}X_nV^{1/2})dUdV \qquad \cdots\cdots(3.7) $$

for $Re(a,a') > (p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(3.7) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$ \int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\times $$

$$ \cdots|X_n|^{\rho_n-(p+1)/2}\,\Phi_2^{(n)}(b_1,\cdots,b_n;c;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2}, $$

$$ -V^{1/2}X_{k+1}V^{1/2},\cdots,-V^{1/2}X_nV^{1/2})dX_1\cdots dX_k dX_{k+1}\cdots dX_n \qquad \cdots\cdots(3.8) $$

Making use of the transformations

$Y_i = U^{1/2}X_iU^{1/2};\ Y_j = V^{1/2}X_jV^{1/2}$ with $dY_i = |U|^{(p+1)/2}dX_i$;

$dY_j = |V|^{(p+1)/2}dX_j$; and $|Y_i| = |U||X_i|;\ |Y_j| = |V||X_j|$ for $i=1,\cdots,k;\ j=k+1,\cdots,n$

to the expression (3.8) and then using eq.(1.4) of the authors' paper [7], we obtain,

$$ |U|^{-\rho_1-\cdots-\rho_k}|V|^{-\rho_{k+1}-\cdots-\rho_n}\frac{\Gamma_p(b_1-\rho_1)\Gamma_p(\rho_1)}{\Gamma_p(b_1)}\cdots\frac{\Gamma_p(b_n-\rho_n)\Gamma_p(\rho_n)}{\Gamma_p(b_n)}\times $$

$$ \frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \qquad \cdots\cdots(3.9) $$

Substituting the above expression on the right side of eq.(3.7) and then integrating out the variables U and V in the resulting expression by using a Gamma integral produces $M[{}_{(2)}^{(k)}E_D^{(n)}]$ as given by eq.(3.1) above.

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