An Equality for the Curvature Function of a Simple Closed Curve on the Plane

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Abstract We prove an equality for the curvature function of a simple, closed curve on the plane. This equality leads to another proof of the four-vertex theorem in differential geometry.
AMS Subject Classification 2000 30E20, 35J20, 35J65, 53A04
Keywords conformal, curvature, vertex

Let C be a simple, closed curve on the plane. We assume in the beginning that C is analytic for the convenience of presentation. Let $G(z)$ be a one-to-one conformal mapping from the unit disk centered at the origin to the bounded domain enclosed by C. Standard complex analysis tells us that $G(z)$ is analytic and

$$|G'(z)| \geq m > 0 \tag{1}$$

for a constant m on the closed unit disk (cf. [C] or other textbooks). Let

$$F(z) = G(\frac{z-i}{z+i}) \tag{2}$$

and let

$$\begin{aligned} u(z) &= \mathrm{Re}(\ln(F'(z))) = \ln|F'(z)|, \\ v(z) &= \mathrm{Im}(\ln(F'(z))) = \arg(F'(z)). \end{aligned}$$

Then $F(z)$ maps one-to-one the upper-half plane

$$R_2^+ = \{z = (x_1, x_2) = x_1 + ix_2 \mid x_2 \geq 0\}$$

to the bounded domain enclosed by C, and $u(z), v(z)$ are harmonic functions.

On the boundary $x_2 = 0$, $u(z)$ satisfies

$$u_{x_2}(x_1, 0) = -k(x_1)e^{u(x_1,0)} \tag{3}$$

where $k(x_1)$ is the curvature of C at $F(x_1, 0)$. This boundary condition follows from the following calculation of the curvature of C :

$$\begin{aligned}
k(x_1) &= \frac{\mathrm{Im}(\overline{F'(z)}F''(z))}{|F'(z)|^3} \quad (z = x_1 = (x_1, 0)) \\
&= \mathrm{Im}\frac{\exp(u(z) - iv(z))\exp(u(z) + iv(z))(u_{x_1} + iv_{x_1})}{\exp(3u(z))} \\
&= \mathrm{Im}\frac{1}{\exp(u(z))}(u_{x_1} + iv_{x_1}) \\
&= \frac{1}{\exp(u(z))}(v_{x_1}) = \frac{1}{\exp(u(z))}(-u_{x_2}).
\end{aligned}$$

Our object is to prove the following equality.

Theorem 1 *With $u(x_1, x_2)$ and $k(x_1)$ defined as above,*

$$\int_{-\infty}^{\infty} x_1 k'(x_1)e^{u(x_1,0)}dx_1 = 0. \tag{4}$$

Proof: First we establish the asymptotic behavior of $u(z), v(z), \nabla u(z)$ at infinity. We show that as $z \to \infty$,

$$\begin{aligned}
u(z) &= -2\ln(z) + O(1), \\
v(z) &= -2\arg(z) + \arg(G'(1)) + \frac{\pi}{2} + o(1), \\
\nabla u(z) &= -\frac{2z}{|z|^2} + o(\frac{1}{|z|}).
\end{aligned} \tag{5}$$

Note that

$$F'(z) = G'(\frac{z - i}{z + i})\frac{2i}{(z + i)^2}$$

and that as z approaches infinity,

$$G'(\frac{z - i}{z + i}) \to G'(1),$$

with $G'(1)$ being non-zero. Thus

$$\begin{aligned} u(z) &= \ln |F'(z)| = \ln |2G'(\frac{z-i}{z+i})| - 2\ln|z+i| \\ v(z) &= \arg(F'(z)) = \arg(G'(\frac{z-i}{z+i})) + \frac{\pi}{2} - 2\arg(z+i). \end{aligned}$$

As for $\nabla u(z)$, we have

$$\begin{aligned} &\frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2} \\ &= \frac{\partial u}{\partial x_1} + i\frac{\partial v}{\partial x_1} = (\ln F'(z))' = \frac{1}{F'(z)}F''(z) \\ &= \frac{1}{F'(z)}(G'(\frac{z-i}{z+i})\frac{-4i}{(z+i)^3} + G''(\frac{z-i}{z+i})(\frac{2i}{(z+i)^2})^2) \\ &= \frac{-2}{z+i} + \frac{G''(\frac{z-i}{z+i})}{G'(\frac{z-i}{z+i})}\frac{2i}{(z+i)^2}. \end{aligned}$$

It is elementary to deduce (5) from the calculations above.

Next, we recall that $u(z)$ being a harmonic function implies

$$\frac{\partial}{\partial x_j}(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}) = 0 \quad \text{for each } i = 1, 2. \tag{6}$$

Here and later the convention of summing over a repeated index is assumed. Probably the best way to derive (6) is the following consideration in the calculus of variations. Let ϕ_1, ϕ_2 be two functions that are smooth and have a bounded support on R_2^+ and let $\phi = (\phi_1, \phi_2)$. With $u_t(z) = u(z + t\phi(z))$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}\int |\nabla u_t|^2 \;|_{t=0} = \int 2\nabla u \cdot \nabla(\frac{d}{dt}u_t) \;|_{t=0} \\ &= \int 2\nabla u \cdot \nabla(\frac{\partial u}{\partial x_i}\phi_i) = \int 2\frac{\partial u}{\partial x_j}(\frac{\partial^2 u}{\partial x_i \partial x_j}\phi_i + \frac{\partial u}{\partial x_i}\frac{\partial \phi_i}{\partial x_j}) \\ &= \int (2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}\phi_i - 2\frac{\partial}{\partial x_j}(\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i})\phi_i) \\ &= \int (\frac{\partial}{\partial x_i}|\nabla u|^2 - 2\frac{\partial}{\partial x_j}(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}))\phi_i \\ &= \int \frac{\partial}{\partial x_j}(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})\phi_i. \end{aligned}$$

Then (6) follows. (This argument is of course well known to specialists and appears in the contexts of elasticity, electrostatics, geometry and others.)

The essential step to equality (4) is an integration involving (6). Let B_R^+ be the upper-half disk centered at the origin with radius R. By (6) and Green's Theorem,

$$\begin{aligned} 0 &= \int_{B_R^+} x_i \frac{\partial}{\partial x_j}(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}) \\ &= \oint_{\partial B_R^+} x_i(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})\nu_j - \int_{B_R^+} \delta_{ij}(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}) \\ &= \oint_{\partial B_R^+} x_i(|\nabla u|^2\delta_{ij} - 2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})\nu_j \end{aligned}$$

Above $\nu = (\nu_1, \nu_2)$ is the outward unit normal vector to ∂B_R^+ and the line integral is in the counter-clockwise direction. Let I_1 be the line integral on the line segment $\{(x_1, 0)\,|\,-R \le x_1 \le R\}$ and let I_2 be the line integral on the upper-half circle $\{(R\cos\theta, R\sin\theta)\,|\,0 \le \theta \le \pi\}$. For I_1, we have $x = (x_1, 0)$ and $\nu = (0, -1)$; hence

$$\begin{aligned} I_1 &= \int_{-R}^{R} x_1(|\nabla u|^2\delta_{12} - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2})(-1)dx_1 \\ &= \int_{-R}^{R} x_1 2\frac{\partial u}{\partial x_1}(-k(x_1)e^{u(x_1,0)})dx_1 \qquad \text{(by (3))} \\ &= -2x_1k(x_1)e^{u(x_1,0)}|_{-R}^{R} + \int_{-R}^{R}(2k(x_1) + 2x_1k'(x_1))e^{u(x_1,0)}dx_1 \\ &= -2x_1k(x_1)e^{u(x_1,0)}|_{-R}^{R} - \int_{-R}^{R} 2\frac{\partial u}{\partial x_2}(x_1,0)dx_1 \\ &\quad + \int_{-R}^{R} 2x_1k'(x_1))e^{u(x_1,0)}dx_1 \\ &= (-2Rk(R)e^{u(R,0)} - 2Rk(-R)e^{u(-R,0)}) + 2(v(R,0) - v(-R,0)) \\ &\quad + \int_{-R}^{R} 2x_1k'(x_1))e^{u(x_1,0)}dx_1. \end{aligned}$$

For I_2, we have $x = (R\cos\theta, R\sin\theta)$ and $\nu = x/R$; hence

$$I_2 = \int_0^{\pi}(|\nabla u|^2R^2 - 2\frac{\partial u}{\partial x_i}x_i\frac{\partial u}{\partial x_j}x_j)d\theta.$$

Using the asymptotic expansion (5) and that $k(R)$ and $k(-R)$ are bounded, we conclude that

$$\begin{aligned} I_1 &\to 4\pi + \int_{-\infty}^{\infty} 2x_1 k'(x_1) e^{u(x_1,0)} dx_1, \\ I_2 &\to -4\pi \end{aligned}$$

as $R \to \infty$. Thus we have proved (4).

Let's make a few remarks on (4). Let

$$s = \int_{-\infty}^{x_1} |F'(x_1, 0)| dx_1 = \int_{-\infty}^{x_1} e^{u(x_1,0)} dx_1$$

be a length parameter of C. Then

$$k'(x_1) = \frac{dk}{ds} e^{u(x_1,0)}.$$

Since dk/ds is bounded,

$$x_1 k'(x_1) e^{u(x_1,0)} = x_1 \frac{dk}{ds} e^{2u(x_1,0)} \tag{7}$$

and is absolutely bounded by a constant multiple of $1/(1 + |x_1|)^3$ by (5). Thus the integral in (4) is a proper one.

Also, since we may choose $G(z)$ and $F(z)$ such that $k(0)$ equals the maximum and $k(\pm\infty)$ equal the minimum of the curvature function of C, the equality (4) implies that it is impossible that $x_1 k'(x_1) \le 0$ for all x_1. Thus $k(x_1)$ has at least two more critical points in addition to 0 and $\pm\infty$. That is, C has at least four vertices.

At this point let's examine the regularity assumption of C. Since equality (4) and the four-vertex theorem involve the first derivative of the curvature function of C, the natural assumption is that C is C^3. We shall see that this is also sufficient for (4) and subsequently the four-vertex theorem. With (7) in mind, we recognize that sufficient for all our computations are that $G(z)$ be $C^{2,\alpha}(0 < \alpha < 1)$ and (1) be satisfied on the closed unit disk. Here we need to mention that although (6) involves the third derivative of $G(z)$, we have only used (6) in its integral form. Thus by integrating on a half disk slightly above the real line and then taking a limit, we may proceed with all the computations.

Now since C is assumed to be C^3, it is $C^{2,\alpha}$ for any α in (0,1). We need the case $m = 2$ of the following theorem:

If C is $C^{m,\alpha}$ smooth, where m is a postive integer and $\geq 0 < \alpha < 1$, then $G(z)$ is $C^{m,\alpha}$ and $|G'(z)| > 0$ on the closed unit disk.

This theorem is called Kollogg's theorem in complex analysis. In [Ts] there is a proof on the theorem for the most difficult case of $m = 1$. It is also known to specialists that Kellogg's theorem can be proved by using the Schauder theory and the strong maximum principle for linear elliptic partial differential equations of second order. My colleague Professor N.V. Rao supplied the following proof.

Consider $\ln |G^{-1}(z)|$, which is a Green's function on the domain enclosed by C with a singularity at $G(0)$. For the case of $m \geq 2$, applying the standard Schauder theory and the strong maximum principle as presented in [GT] we conclude that $G^{-1}(z)$ is $C^{m,\alpha}$ smooth and the gradient of $G^{-1}(z)$ does not vanish on the closed domain enclosed by C. Then the same for $G(z)$ on the unit disk by the inverse function theorem. For the case of $m = 1$, the proof goes the same way because both the Schauder theory and the strong maximum priciple are still valid, but one has to look harder in the literature. Gilbarg and Trudinger discussed these subjects in the notes of their book [GT], and they particularly mentioned works of Finn-Gilbarg and Widman. We refer to [CW] and [G] for more modern treatment of the Schauder thoery. We also mention that there are a lot of works on extending the Schauder theory to more general elliptic pde's. There are also related works on the boundary regularity for disk-type minimal surfaces bounded by a simple space curve. In a series papers Professor A. Fridman extended the Schauder thoery to parabolic pde's, and his work was summarized in [F].

From these discussions, we see that the assumption of C being C^3 is natural and sufficient for equality (4).

To conclude, we refer to [CC],[T] for proofs of the four-vertex theorem in differential geometry. The equality we have established is inspired by similar equalities in other contexts (cf. [CL], [P], [W] among many). In a relevant work [O] the author proved a uniqueness theorem for a harmonic function $u(x_1, x_2)$ on the upper-half plane satisfying the boundary condition $u_{x_2}(x_1, 0) = -e^{u(x_1,0)}$ and the constraint $\iint_{R_2^+} e^{2u} < \infty$.

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