Backward uniqueness for the heat operator in half space

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Abstract We prove a backward uniqueness result for the heat operator with variable lower order terms in a half space. The main point of the result is that the boundary conditions are not controlled by the assumptions.

1 Introduction

In this paper, which can be thought of as a continuation of [3] and [4], we deal with the following backward uniqueness problem for the heat operator. Let $\mathbb{R}^n_+ = \{x = (x_i) \in \mathbb{R}^n \parallel x_n > 0\}$ and $Q_+ = \mathbb{R}^n_+ \times]0,1[$. We consider a vector-valued function $u : Q_+ \to \mathbb{R}^n$, which is "sufficiently regular" and satisfies

$$|\partial_t u + \Delta u| \leq c_1(|\nabla u| + |u|) \quad \text{in} \quad Q_+ \tag{1.1}$$

for some $c_1 > 0$ and

$$u(\cdot, 0) = 0 \quad \text{in} \quad \mathbb{R}^n_+. \tag{1.2}$$

Do (1.1) and (1.2) imply $u \equiv 0$ in Q_+? We prove that the answer is positive if we impose natural restrictions on the growth of the function u at infinity. For example, we can consider

$$|u(x,t)| \leq e^{M|x|^2} \tag{1.3}$$

for all $(x,t) \in Q_+$ and for some $M > 0$. Natural regularity assumptions, under which (1.1)–(1.3) can be considered are, for example, as follows:

$$\left.\begin{array}{c} u \text{ and distributional derivatives } \partial_t u,\ \nabla^2 u \text{ are square} \\ \text{integrable over bounded subdomains of } Q_+. \end{array}\right\} \tag{1.4}$$

We can formulate our main result.

Theorem 1.1 *Using the notation introduced above, assume that u satisfies conditions (1.1)–(1.4). Then $u \equiv 0$ in Q_+.*

This extends the main result of [3] and [4], where an analogue of Theorem 1.1 was proved for Q_+ replaced with $(\mathbb{R}^n \setminus B(R))\times]0,T[$. Here, as usual, $B(R)$ denotes the n-dimensional ball of radius R with the center at the origin.

Such results are of interest in control theory, see for example [8]. Also, as it was explained in [10], results of this type are helpful in regularity theory for the Navier-Stokes equations. Combining methods developed in [10] and the boundary regularity results in [9], one can show that the so-called Leray-Hopf solutions to the Navier-Stokes equations in a regular domain $\Omega \in \mathbb{R}^3$, which satisfy the homogeneous Dirichlet boundary condition and belong to the space $L_\infty(0,1;L_3(\Omega;\mathbb{R}^3))$, are regular up to the boundary of Ω. A detailed proof of this result will be explained elsewhere.

Similarly to papers [3] and [4], the proof of Theorem 1.1 is based on two Carleman-type inequalities. The first one is essentially the same as the one used in [3] and [4] (see also [1], [5], and [11]) and has the form

$$\begin{aligned}&\int\limits_{\mathbb{R}^n\times]0,2[} h^{-2a}(t)e^{-\frac{|x|^2}{4t}}\big(a\tfrac{|w|^2}{t}+|\nabla w|^2\big)\,dxdt\\ &\leq c_2\int\limits_{\mathbb{R}^n\times]0,2[} h^{-2a}(t)e^{-\frac{|x|^2}{4t}}|\partial_t w+\Delta w|^2\,dxdt.\end{aligned}\tag{1.5}$$

Here, $h(t)=te^{\frac{1-t}{3}}$ and c_2 is an absolute positive constant. Inequality (1.5) holds for any functions $w\in C_0^\infty(\mathbb{R}^n\times]0,2[;\mathbb{R}^n)$ and for any positive number a.

The second inequality is, in a sense, an anisotropic one:

$$\begin{aligned}&\int\limits_{(\mathbb{R}^n_+ + e_n)\times]0,1[} t^2e^{2\phi(x,t)}\Big(a\frac{|w|^2}{t^2}+\frac{|\nabla w|^2}{t}\Big)\,dxdt\\ &\leq c_3\int\limits_{(\mathbb{R}^n_+ + e_n)\times]0,1[} t^2e^{2\phi(x,t)}|\partial_t w+\Delta w|^2\,dxdt.\end{aligned}\tag{1.6}$$

Here, $\phi=\phi^{(1)}+\phi^{(2)}$, $\phi^{(1)}(x,t)=-\frac{|x'|^2}{8t}$, $\phi^{(2)}(x,t)=a(1-t)\frac{x_n^{2\alpha}}{t^\alpha}$, $\alpha\in]1/2,1[$ is fixed, $x'=(x_1,x_2,...,x_{n-1})$ so that $x=(x',x_n)$, $c_3(\alpha)>0$, and $e_n=(0,0,...,0,1)$. Inequality (1.6) is fulfilled for any function $w\in C_0^\infty((\mathbb{R}^n_+ + e_n)\times]0,1[);\mathbb{R}^n)$ and for any number $a>a_0(\alpha)$.

Our concluding remark is that Theorem 1.1 is true for functions $u: Q_+\to\mathbb{R}^m$ with $1\leq m<+\infty$. This is an easy exercise for the reader.

Our paper is organized as follows. In the second section, we prove Theorem 1.1, assuming validity of Carleman's inequalities. The proof of Carleman's inequalities are given in the third and fourth sections, respectively.

2 Proof of Theorem 1.1

In what follows, we always assume that function u is extended by zero to negative values of t.

We start with proofs of several lemmas. The first of them plays the crucial role in our approach. It enables us to apply powerful technique of Carleman's inequalities.

Lemma 2.1 *Suppose that conditions (1.1), (1.2), and (1.4) are fulfilled. There exists an absolute positive constant $A_0 < 1/32$ with the following properties. If*

$$|u(x,t)| \leq e^{A|x|^2} \tag{2.1}$$

for all $(x,t) \in Q_+$ and for some $A \in [0, A_0]$, then there are constants $\beta(A) > 0$, $\gamma(c_1) \in]0, 1/12[$, and $c_4(c_1, A) > 0$ such that

$$|u(x,t)| \leq c_4 e^{4A|x'|^2} e^{-\beta \frac{x_n^2}{t}} \tag{2.2}$$

for all $(x,t) \in (\mathbb{R}^n_+ + 2e_n) \times]0, \gamma[$.

PROOF According to the regularity theory of solutions to parabolic equations (see [7]), we may assume

$$|u(x,t)| + |\nabla u(x,t)| \leq c_5 e^{2A|x|^2} \tag{2.3}$$

for all $(x,t) \in (\mathbb{R}^n_+ + e_n) \times]0, 1/2[$.

We fix $x_n > 2$ and $t \in]0, \gamma[$ and introduce the new function v by usual parabolic scaling

$$v(y,s) = u(x + \lambda y, \lambda^2 s - t/2).$$

The function v is well defined on the set $Q_\rho = B(\rho) \times]0, 2[$, where $\rho = (x_n - 1)/\lambda$ and $\lambda = \sqrt{3t} \in]0, 1/2[$. Then, relations (1.1), (1.2), and (2.3) take the form:

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|) \quad \text{a.e. in} \quad Q_\rho; \tag{2.4}$$

$$|v(y,s)| + |\nabla v(y,s)| \leq c_5 e^{4A|x|^2} e^{4A\lambda^2 |y|^2} \tag{2.5}$$

for $(y,s) \in Q_\rho$;

$$v(y,s) = 0 \tag{2.6}$$

for $y \in B(\rho)$ and for $s \in]0, 1/6]$.

In order to apply inequality (1.5), we choose two smooth cut-off functions:

$$\phi_\rho(y) = \begin{cases} 0 & |y| > \rho - 1/2 \\ 1 & |y| < \rho - 1 \end{cases},$$

$$\phi_t(s) = \begin{cases} 0 & 7/4 < s < 2 \\ 1 & 0 < s < 3/2 \end{cases}.$$

All functions take values in $[0,1]$. In addition, function ϕ_ρ satisfies the conditions: $|\nabla^k \phi_\rho| < C_k$, $k = 1, 2$. We let $\eta(y,s) = \phi_\rho(y)\phi_t(s)$ and $w = \eta v$. It follows from (2.4) that

$$|\partial_s w + \Delta w| \leq c_1 \lambda (|\nabla w| + |w|) + \chi c_6 (|\nabla v| + |v|). \tag{2.7}$$

Here, c_6 is a positive constant depending on c_1 and C_k only, $\chi(y,s) = 1$ if $(y,s) \in \omega = \{\rho - 1 < |y| < \rho\} \times]3/2, 2[$ and $\chi(y,s) = 0$ if $(y,s) \notin \omega$. Obviously, function w has the compact support in $\mathbb{R}^n \times]0,2[$ and we may use inequality (1.5). As a result, we have

$$I \equiv \int\limits_{Q_\rho} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2)\, dyds \leq c_2 10 (c_1^2 \lambda^2 I + c_6^2 I_1), \tag{2.8}$$

where

$$I_1 = \int\limits_{Q_\rho} \chi(y,s) h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|v|^2 + |\nabla v|^2)\, dyds.$$

Choosing $\gamma = \gamma(c_1)$ sufficiently small, we can assume that the inequality $c_2 10 c_1^2 \lambda^2 \leq 1/2$ holds and then (2.8) implies

$$I \leq c_7(c_1) I_1. \tag{2.9}$$

On the other hand, if $A < 1/32$, then

$$8A\lambda^2 - \frac{1}{4s} < -\frac{1}{8s} \tag{2.10}$$

for $s \in]0,2]$. By (2.5) and (2.10), we have

$$\begin{aligned} I_1 &\leq c_5^2 e^{8A|x|^2} \int\limits_0^2 \int\limits_{B\rho} \chi(y,s) h^{-2a}(s) e^{-\frac{|y|^2}{8s}}\, dyds \\ &\leq c_8 e^{8A|x|^2} \Big[h^{-2a}(3/2) + \int\limits_0^2 h^{-2a}(s) e^{-\frac{(\rho-1)^2}{8s}}\, ds \Big]. \end{aligned} \tag{2.11}$$

Now, taking into account (2.11), we deduce the bound

$$D \equiv \int\limits_{B(1)} \int\limits_{\frac{1}{2}}^{1} |w|^2 \, dyds = \int\limits_{B(1)} \int\limits_{\frac{1}{2}}^{1} |v|^2 \, dyds$$

$$\leq c_9 \int\limits_{Q_\rho} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2) \, dyds$$

$$\leq c_{10}(c_1) e^{8A|x|^2} \Big[h^{-2a}(3/2) + \int\limits_0^2 h^{-2a}(s) e^{-\frac{\rho^2}{32s}} \, ds \Big]$$

$$= c_{10} e^{8A|x|^2 - 2\beta\rho^2} \Big[h^{-2a}(3/2) e^{2\beta\rho^2} + \int\limits_0^2 h^{-2a}(s) e^{2\beta\rho^2 - \frac{\rho^2}{32s}} \, ds \Big].$$

We can take $\beta = 8A < 1/256$ and then choose

$$a = \beta\rho^2 / \ln h(3/2).$$

Since $\rho \geq x_n$, such a choice leads to the estimate

$$D \leq c_{10} e^{8A|x'|^2} e^{-\beta\rho^2} \Big[1 + \int\limits_0^2 g(s) \, ds \Big],$$

where $g(s) = h^{-2a}(s) e^{-\frac{\rho^2}{64s}}$. It is easy to check that $g'(s) \geq 0$ for $s \in]0,2[$ if $\beta < \frac{1}{96} \ln h(3/2)$. So, we have

$$D \leq 2c_{10} e^{8A|x'|^2} e^{-\beta\rho^2} \leq 2c_{10} e^{8A|x'|^2} e^{-\frac{\beta x_n^2}{12t}}. \tag{2.12}$$

On the other hand, the regularity theory implies

$$|v(0,1/2)|^2 = |u(x,t)|^2 \leq c'_{10} D. \tag{2.13}$$

Combining (2.12) and (2.13), we complete the proof of Lemma 2.1. Lemma 2.1 is proved.

Next lemma will be a consequence of Lemma 2.1 and the second Carleman inequality (see (1.6)).

Lemma 2.2 *Suppose that the function u satisfies conditions (1.1), (1.2), (1.4), and (2.1). There exists a number $\gamma_1(c_1, c_3) \in]0, \gamma/2]$ such that*

$$u(x,t) = 0 \tag{2.14}$$

for all $x \in \mathbb{R}^n_+$ and for all $t \in]0, \gamma_1[$.

PROOF As usual, by Lemma 2.1 and by the regularity theory, we may assume

$$|u(x,t)| + |\nabla u(x,t)| \leq c_{11}(c_1, A) e^{8A|x'|^2} e^{-\beta \frac{x_n^2}{2t}} \tag{2.15}$$

for all $x \in \mathbb{R}^n_+ + 3e_n$ and for all $t \in]0, \gamma/2]$.

By scaling, we define function $v(y,s) = u(\lambda y, \lambda^2 s - \gamma_1)$ for $(y,s) \in Q_+$ with $\lambda = \sqrt{2\gamma_1}$. This function satisfies the relations:

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|) \quad \text{a.e. in} \quad Q_+; \tag{2.16}$$

$$v(y,s) = 0 \tag{2.17}$$

for all $x \in \mathbb{R}^n_+$ and for all $s \in]0, 1/2[$;

$$|\nabla v(y,s)| + |v(y,s)| \leq c_{11} e^{8A\lambda^2 |y'|^2} e^{-\frac{\beta \lambda^2 y_n^2}{2(\lambda^2 s - \gamma_1)}} \leq c_{11} e^{8A\lambda^2|y'|^2} e^{-\beta \frac{y_n^2}{2s}} \tag{2.18}$$

for all $s > 1/2$ and for all $y \in \mathbb{R}^n_+ + \frac{3}{\lambda} e_n$. Since $A < 1/32$ and $\lambda \leq \sqrt{\gamma} \leq 1/\sqrt{12}$, (2.18) can be reduced to the form

$$|\nabla v(y,s)| + |v(y,s)| \leq c_{11} e^{\frac{|y'|^2}{48}} e^{-\beta \frac{y_n^2}{2s}} \tag{2.19}$$

for the same y and s as in (2.18).

Let us fix two smooth cut-off functions:

$$\psi_1(y_n) = \begin{cases} 0 & y_n < \frac{3}{\lambda} + 1 \\ 1 & y_n > \frac{3}{\lambda} + 2 \end{cases},$$

and

$$\psi_2(r) = \begin{cases} 0 & r > -1/2 \\ 1 & r < -3/4 \end{cases}.$$

We set (see (1.6) for the definition of $\phi^{(1)}$ and $\phi^{(2)}$)

$$\phi_B(y_n, s) = \frac{1}{a} \phi^{(2)}(y_n, s) - B = (1-s) \frac{y_n^{2\alpha}}{s^\alpha} - B,$$

where $\alpha \in]1/2, 1[$ is fixed, $B = \frac{2}{a}\phi^{(2)}(\frac{3}{\lambda} + 2, 1/2)$, and

$$\eta(y_n, s) = \psi_1(y_n)\psi_2(\phi_B(y_n, s)/B), \qquad w(y, s) = \eta(y_n, s)v(y, s).$$

Although function w is not compactly supported in Q^1_+, but, by the statement of Lemma 2.1 and by the special structure of the weight in (1.6), we can claim validity of (1.6) for w. As a result, we have

$$\int\limits_{Q^1_+} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} (|w|^2 + |\nabla w|^2)\, dyds$$

$$\leq c_3 \int\limits_{Q^1_+} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} |\partial_s w + \Delta w|^2\, dyds.$$

Arguing as in the proof of Lemma 2.1, we can select $\gamma_1(c_1, c_3)$ so small that

$$I \equiv \int\limits_{Q^1_+} s^2 e^{2a\phi_B} (|w|^2 + |\nabla w|^2) e^{-\frac{|y'|^2}{4s}}\, dyds$$

$$\leq c_{12}(c_1, c_3) \int\limits_{(\mathbb{R}^n_+ + (\frac{3}{\lambda}+1)e_n)\times]1/2,1[} \chi(y_n, s)(sy_n)^2 e^{2a\phi_B} (|v|^2 + |\nabla v|^2) e^{-\frac{|y'|^2}{4s}}\, dyds,$$

where $\chi(y_n, s) = 1$ if $(y_n, s) \in \omega$, $\chi(y_n, s) = 0$ if $(y_n, s) \notin \omega$, and

$$\omega \equiv \{(y_n, s) \parallel y_n > 1, \quad 1/2 < s < 1, \quad \phi_B(y_n, s) < -B/2\}.$$

Now, we wish to estimate the right hand side of the last inequality with the help of (2.19). We find

$$I \leq c_{13} e^{-Ba} \int\limits_{\frac{3}{\lambda}+1}^{+\infty} \int\limits_{1/2}^{1} (y_n s)^2 e^{-\beta \frac{y_n^2}{s}}\, dy_n ds \int\limits_{\mathbb{R}^{n-1}} e^{(\frac{1}{24} - \frac{1}{4s})|y'|^2}\, dy'.$$

Passing to the limit as $a \to +\infty$, we see that $v(y, s) = 0$ if $1/2 \leq s < 1$ and $\phi_B(y_n, s) > 0$. Using unique continuation through spatial boundaries (see [2]), we show that $v(y, s) = 0$ if $\in \mathbb{R}^n_+$ and $0 < s < 1$. Lemma 2.2 is proved.

Now, Theorem 1.1 follows from Lemmas 2.1 and 2.2 with the help of more or less standard arguments. We shall demonstrate them just for completeness.

Lemma 2.3 *Suppose that the function u meets all conditions of Lemma 2.2. Then $u \equiv 0$ in Q_+.*

PROOF By Lemma 2.2, $u(x,t) = 0$ for $x \in \mathbb{R}^n_+$ and for $t \in]0, \gamma_1[$. By scaling, we introduce the function $u^{(1)}(y,s) = u(\sqrt{1-\gamma_1}y, (1-\gamma_1)s + \gamma_1)$. It easy to check that function $u^{(1)}$ is well-defined in Q_+ and satisfies all conditions of Lemma 2.2 with the same constants c_1 and A. Therefore, $u^{(1)}(y,s) = 0$ for $y_n > 0$ and for $0 < s < \gamma_1$. The latter means that $u(x,t) = 0$ for $x_n > 0$ and for $0 < t < \gamma_2 = \gamma_1 + (1-\gamma_1)\gamma_1$. Then, we introduce the function

$$u^{(2)}(y,s) = u(\sqrt{1-\gamma_2}y, (1-\gamma_2)s + \gamma_2), \qquad (y,s) \in Q_+,$$

and apply Lemma 2.2. After k steps we shall see that $u(x,t) = 0$ for $x_n > 0$ and for $0 < t < \gamma_{k+1}$, where $\gamma_{k+1} = \gamma_k + (1-\gamma_k)\gamma_1 \to 1$. Lemma 2.3 is proved.

PROOF OF THEOREM 1.1 Assume that $A_0 < M$. Then $\lambda^2 \equiv \frac{A_0}{2M} < \frac{1}{2}$. Introducing function $v(y,s) = u(\lambda y, \lambda^2 s)$, $(y,s) \in Q_+$, we see that this function satisfies all conditions of Lemma 2.3 with constants c_1 and $A = \frac{1}{2}A_0$. Therefore, $u(x,t) = 0$ for $x_n > 0$ and for $0 < t < \frac{A_0}{2M}$. Now, we repeat arguments of Lemma 2.3, replacing γ_1 to $\frac{A_0}{2M}$ and A to M and end up with the proof of the theorem. Theorem 1.1 is proved.

3 Proof of the first Carleman inequality

Our proof follows standard techniques used in the L_2-theory of Carleman inequalities, see for example [6] and [11].

Let u be an arbitrary function from $C_0^\infty(\mathbb{R}^N \times]0,2[; \mathbb{R}^n)$. We set $\phi(x,t) = -\frac{|y|^2}{8t} - (a+1)\ln h(t)$ and $v = e^\phi u$. Then, we have

$$Lv \equiv e^\phi(\partial_t + \Delta u) = \partial_t v - \operatorname{div}(v \otimes \nabla\phi) - \nabla v \nabla\phi + \Delta v + (|\nabla\phi|^2 - \partial_t\phi)v.$$

The main trick in the above approach is the decomposition of operator tL into symmetric and skew symmetric parts, i.e.,

$$tL = S + A, \tag{3.1}$$

where

$$Sv \equiv t(\Delta v + (|\nabla\phi|^2 - \partial_t\phi)v) - \frac{1}{2}v \tag{3.2}$$

and

$$Av \equiv \frac{1}{2}(\partial_t(tv) + t\partial_t v) - t(\mathrm{div}(v \otimes \nabla\phi) + \nabla v \nabla\phi). \tag{3.3}$$

Obviously,

$$\begin{aligned} &\int t^2 e^2 \phi |\partial_t + \Delta u|^2 \, dxdt = \int t^2 |Lv|^2 \, dxdt \\ &= \int |Sv|^2 \, dxdt + \int |Av|^2 \, dxdt + \int [S, A] v \cdot v \, dxdt, \end{aligned} \tag{3.4}$$

where $[S, A] = SA - AS$ is the commutator of S and A. Simple calculations show that

$$\begin{aligned} I \equiv &\int [S, A] v \cdot v \, dxdt = \\ &= 4 \int t^2 \Big[\phi_{,ij} v_{,i} \cdot v_{,j} + \phi_{,ij} \phi_{,i} \phi_{,j} |v|^2 \Big] \, dxdt \\ &+ \int t^2 |v|^2 (\partial_t^2 \phi - 2\partial_t |\nabla\phi|^2 - \Delta^2 \phi) \, dxdt \\ &+ \int t |\nabla v|^2 \, dxdt - \int t |v|^2 (|\nabla\phi|^2 - \partial_t \phi) \, dxdt. \end{aligned} \tag{3.5}$$

Here and in what follows, we adopt the convention on summation over repeated Latin indices, running from 1 to n, Partial derivatives in spatial variables are denoted by comma in lower indices, i.e., $v_{,i} = \frac{\partial v}{\partial x_i}$, $\nabla v = (v_{i,j})$, etc. Given choice of function ϕ, we have

$$I = (a+1) \int t^2 \Big[-\Big(\frac{h'(t)}{h(t)}\Big)' - \frac{h'(t))}{th(t)} \Big] |v|^2 \, dxdt = \frac{a+1}{2} \int t |v|^2 \, dxdt. \tag{3.6}$$

By the simple identity

$$|\nabla v|^2 = \frac{1}{2}(\partial_t + \Delta)|v|^2 - v \cdot (\partial_t v + \Delta v), \tag{3.7}$$

we find

$$\begin{aligned} \int t^2 |\nabla v|^2 \, dxdt = &- \int t |v|^2 \, dxdt - \int t^2 v \cdot Lv \, dxdt \\ &+ \int t^2 |v|^2 (|\nabla\phi|^2 - \partial_t \phi) \, dxdt. \end{aligned} \tag{3.8}$$

In our case,

$$|\nabla\phi|^2 - \partial_t \phi = -|\nabla\phi|^2 + (a+1) t^2 \frac{h'(t)}{h(t)}.$$

The latter relation (together with (3.6)) implies the bound

$$\begin{gathered}\int t^2(|\nabla v|^2+|v|^2|\nabla\phi|^2)\,dxdt\\ \le 3I-\int t^2 v\cdot Lv\,dxdt\le b_1\int t^2|Lv|^2\,dxdt,\end{gathered}\tag{3.9}$$

where b_1 is an absolute positive constant. Since

$$e^{\phi}|\nabla u|\le|\nabla v|+|v||\nabla\phi|,\tag{3.10}$$

it follows from (3.4)–(3.10) that

$$\begin{gathered}\int h^{-2a}(t)(th^{-1}(t))^2\Big((a+1)\frac{|u|^2}{t}+|\nabla u|^2\Big)e^{2\phi}\,dxdt\\ \le b_2\int h^{-2a}(t)(th^{-1}(t))^2|\partial_t u+\Delta u|^2e^{2\phi}\,dxdt.\end{gathered}$$

Here, b_2 is an absolute positive constant. Inequality (1.5) is proved.

4 Proof of the second Carleman inequality

Let $u\in C_0^\infty(Q_+^1;\mathbb{R}^n)$ and let

$$\phi=\phi^{(1)}+\phi^{(2)},\quad \phi^{(1)}(x,t)=-\frac{|x'|^2}{8t},\quad \phi^{(2)}(x,t)=a(1-t)\frac{x_n^{2\alpha}}{t^\alpha},\tag{4.1}$$

where $Q_+^1=(\mathbb{R}^n_++e_n)\times]0,1[$, $x=(x',x_n)$, $\alpha\in]1/2,1]$ is fixed, and a is a positive parameter.

We are going to use formulas (3.1)–(3.5) for new functions u, v, and ϕ. All integrals in those formulas are taken now over Q_+^1.

First, we observe that

$$\begin{gathered}\nabla\phi=\nabla\phi^{(1)}+\nabla\phi^{(2)}\\ \nabla\phi^{(1)}(x,t)=-\tfrac{x'}{4t},\qquad \nabla\phi^{(2)}(x,t)=2\alpha a\tfrac{1-t}{t^\alpha}x_n^{2\alpha-1}e_n\end{gathered}\tag{4.2}$$

Therefore,

$$\nabla\phi^{(1)}\cdot\nabla\phi^{(2)}=0,\qquad |\nabla\phi|^2=|\nabla\phi^{(1)}|^2+|\nabla\phi^{(2)}|^2.\tag{4.3}$$

Moreover,

$$\nabla^2\phi = \nabla^2\phi^{(1)} + \nabla^2\phi^{(2)},$$

$$\phi^{(1)}_{,ij} = \begin{cases} -\frac{\delta_{ij}}{4t} & \text{if} \quad 1 \le i, j \le n-1 \\ 0 & \text{if} \quad i = n \quad \text{or} \quad j = n \end{cases}, \tag{4.4}$$

$$\phi^{(2)}_{,ij} = \begin{cases} 0 & \text{if} \quad i \neq n \quad \text{or} \quad j \neq n \\ 2\alpha(2\alpha-1)a\frac{1-t}{t^\alpha}x_n^{2\alpha-2} & \text{if} \quad i = n \quad \text{and} \quad j = n \end{cases}.$$

In particular, (4.4) implies

$$\phi_{,ij}\phi_{,i}\phi_{,j} = -\frac{1}{4t}|\nabla\phi^{(1)}|^2 + 2\alpha(2\alpha-1)a\frac{1-t}{t^\alpha}x_n^{2\alpha-2}|\nabla\phi^{(2)}|^2 \ge -\frac{1}{4t}\frac{|x'|^2}{16t^3}. \tag{4.5}$$

Using (4.3)–(4.5), we present integral I in (3.5) in the following way

$$I = I_1 + I_2 + \int t|\nabla v|^2\,dxdt, \tag{4.6}$$

where

$$I_s = 4\int t^2\left[\phi^{(s)}_{,ij}v_{,i}\cdot v_{,j} + \phi^{(s)}_{,ij}\phi^{(s)}_{,i}\phi^{(s)}_{,j}|v|^2\right]dxdt$$

$$+\int t^2|v|^2\Big(\partial_t^2\phi^{(s)} - 2\partial_t|\nabla\phi^{(s)}|^2 - \Delta^2\phi^{(s)}$$

$$-\frac{1}{t}|\nabla\phi^{(s)}|^2 + \frac{1}{t}\partial_t\phi^{(s)}\Big)\,dxdt, \qquad s = 1, 2.$$

Direct calculations give us

$$I_1 = -\int t(|\nabla v|^2 - |v_{,n}|^2)\,dxdt$$

and, therefore,

$$I = \int t|v_{,n}|^2\,dxdt + I_2. \tag{4.7}$$

Now, our aim is to estimate I_2 from below. Since $\alpha \in]1/2, 1[$, we can skip the fist integral in the expression for I_2. As a result, we have

$$I_2 \ge \int t^2|v|^2(A_1 + A_2 + A_3)\,dxdt, \tag{4.8}$$

where

$$A_1 = -\partial_t |\nabla \phi^{(2)}|^2,$$
$$A_2 = A_1 - \Delta^2 \phi^{(2)} - \frac{1}{t} |\nabla \phi^{(2)}|^2,$$
$$A_3 = \partial_t^2 \phi^{(2)} + \frac{1}{t} \partial_t \phi^{(2)}.$$

For A_2, we find

$$A_2 \geq \frac{1-t}{t^\alpha} x_n^{2\alpha-4} a(2\alpha-1) \Big[\frac{4\alpha^2 a x_n^{2\alpha+2}}{t^{\alpha+1}} - 2\alpha(2\alpha-2)(2\alpha-3) \Big].$$

Since $x_n \geq 1$ and $0 < t < 1$, we see that $A_2 > 0$ for all $a \geq a_0(\alpha)$. Hence, it follows from (4.7) and (4.8) that

$$I \geq \int t^2 |v|^2 (A_1 + A_3)\, dxdt. \tag{4.9}$$

It is not difficult to check the following inequality

$$A_3 \geq a(2\alpha - 1) \frac{x_n^{2\alpha}}{t^{\alpha+2}}. \tag{4.10}$$

On the other hand,

$$-\partial_t |\nabla \phi^{(2)}|^2 - \frac{1}{t} |\nabla \phi^{(2)}|^2 \geq (2\alpha - 1) \frac{1-t}{t^{2\alpha+1}} 4\alpha^2 a^2 x_n^{2(2\alpha-1)} \geq 0$$

and thus

$$A_1 \geq \frac{1}{t} |\nabla \phi^{(2)}|^2. \tag{4.11}$$

Combining (4.9)–(4.11), we deduce from (3.4) the estimate

$$\begin{aligned} \textstyle\int t^2 |Lv|^2\, dxdt &\geq I \\ &\geq a(2\alpha-1) \textstyle\int \frac{x_n^{2\alpha}}{t^\alpha} |v|^2\, dxdt + \int t|v|^2 |\nabla \phi^{(2)}|^2\, dxdt \\ &\geq a(2\alpha-1) \textstyle\int |v|^2\, dxdt + \int t|v|^2 |\nabla \phi^{(2)}|^2\, dxdt. \end{aligned} \tag{4.12}$$

Using (3.7), we can find the following analog of (3.8)

$$\begin{aligned} \textstyle\int t|\nabla v|^2\, dxdt = -\frac{1}{2} \int |v|^2\, dxdt - \int tv \cdot Lv\, dxdt \\ + \textstyle\int t|v|^2 (|\nabla \phi|^2 - \partial_t \phi)\, dxdt. \end{aligned} \tag{4.13}$$

Due to special structure of ϕ, we have

$$|\nabla\phi|^2 - \partial_t\phi = |\nabla\phi^{(1)}|^2 - \partial_t\phi^{(1)} + |\nabla\phi^{(2)}|^2 - \partial_t\phi^{(2)}$$
$$= -|\nabla\phi^{(1)}|^2 + |\nabla\phi^{(2)}|^2 - \partial_t\phi^{(2)}$$

and, therefore, (4.13) can be reduced to the form

$$\int \Big(t|\nabla v|^2 + t|v|^2(|\nabla\phi^{(1)}|^2 + |\nabla\phi^{(2)}|^2)\Big)\,dxdt$$
$$= \int t\Big(|\nabla v|^2 + |v|^2|\nabla\phi|^2\Big)\,dxdt = -\tfrac{1}{2}\int |v|^2\,dxdt \tag{4.14}$$
$$-\int tv\cdot Lv\,dxdt + 2\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt - \int t|v|^2\partial_t\phi^{(2)}\,dxdt.$$

But

$$-t\partial_t\phi^{(2)} \leq a\frac{x_n^{2\alpha}}{t^\alpha}$$

and, by (3.10) and (4.14),

$$\tfrac{1}{2}\int te^{2\phi}|\nabla u|^2 \leq -\int v\cdot(tLv)\,dxdt$$
$$+2\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt + a\int \tfrac{x_n^{2\alpha}}{t^\alpha}|v|^2\,dxdt. \tag{4.15}$$

The Cauchy-Scwartz inequality, (4.12), and (4.12) imply required inequality (1.6).

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