

HARNACK INEQUALITY FOR NONDIVERGENT ELLIPTIC OPERATORS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We consider second-order linear elliptic operators of nondivergence type which is intrinsically defined on Riemannian manifolds. Cabré proved a global Krylov-Safonov Harnack inequality under the assumption that the sectional curvature is nonnegative. We improve Cabré's result and, as a consequence, we give another proof to Harnack inequality of Yau for positive harmonic functions on Riemannian manifolds with nonnegative Ricci curvature using the nondivergence structure of the Laplace operator.

1. INTRODUCTION AND MAIN RESULTS

In this article we study Harnack inequality for solutions of second-order elliptic equations on Riemannian manifolds. Harnack inequality is well understood for divergence form operators on manifolds satisfying certain properties, namely the volume doubling property and the weak Poincaré inequality; see [6, 10, 11]. For example, the above two conditions hold for manifolds with nonnegative Ricci curvature. Recently, Cabré [1] considered the nondivergence type operators and proved the Harnack inequality on manifolds with nonnegative sectional curvature; see below for the definition of nondivergent operators. His result is an extension of the Euclidean Krylov-Safonov Harnack inequality [7, 9].

However, compared to the Harnack inequality for the divergent operators, Cabré's Harnack inequality doesn't seem to be quite optimal. For example, Laplace operator has both divergent and nondivergent structure and it is well known that positive harmonic functions on Riemannian manifolds with nonnegative Ricci curvature satisfy the Harnack inequality, which was earlier proved by Yau [15]. Since Cabré's Harnack inequality requires the underlying manifold to have nonnegative sectional curvature, it does not directly imply the Harnack inequality of Yau.

The motivation to our work was to give another proof to Yau's Harnack inequality using the nondivergent structure of Laplace operator and by doing so, we expected to generalize Cabré's result to Riemannian manifolds with nonnegative Ricci curvature. The attempt was partly successful. We found a sufficient condition in terms of the distance function of the underlying Riemannian manifold that implies a global Krylov-Safonov Harnack inequality for solutions of uniformly elliptic equations of nondivergence type. It is not quite clear to us whether or not Riemannian manifolds with nonnegative Ricci curvature satisfy the condition of ours, but we shall show that, at least, our result implies the Harnack inequality of Yau. Also, we shall provide the examples of such Riemannian manifolds using the curvature assumption. Our examples strictly include the manifolds with

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nonnegative sectional curvature, which means that our result generalizes that of Cabré's.

Let us begin with the definition of the nondivergent elliptic operators on Riemannian manifolds. Let (M, g) be a smooth, complete, non-compact Riemannian manifold of dimension n . For any $x \in M$, we denote by $T_x M$ the tangent space of M at x . For any $x \in M$, let A_x be a positive definite symmetric endomorphism of $T_x M$. We assume that

$$(1) \quad \lambda |X|^2 \leq \langle A_x X, X \rangle \leq \Lambda |X|^2 \quad \forall x \in M, \quad \forall X \in T_x M,$$

for some positive constants λ and Λ . Here $\langle X, Y \rangle = g(X, Y)$ and $|X|^2 = \langle X, X \rangle$. We consider the second-order, linear, uniformly elliptic operators L defined by

$$Lu = \text{tr}(A_x \circ D^2 u) = \text{tr}\{X \mapsto A_x \nabla_X \nabla u\},$$

where \circ denotes composition of endomorphisms, tr is the trace, and $D^2 u$ denote the Hessian of the function u . We recall that the Hessian of u at $x \in M$ is the endomorphism of $T_x M$ defined by

$$D^2 u \cdot X = \nabla_X \nabla u,$$

where $\nabla u(x) \in T_x M$ is the gradient of u at x . Our definition of nondivergent elliptic operator is the same as appears in [14], and since

$$\begin{aligned} Lu &= \text{tr}(A_x \circ D^2 u) = \text{tr}\{X \mapsto A_x \nabla_X \nabla u\} \\ &= \text{tr}(D^2 u \circ A_x) = \text{tr}\{X \mapsto \nabla_{A_x X} \nabla u\}, \end{aligned}$$

our definition of Lu coincides with that of Cabré's. In contrast, divergent operators in manifolds are of the form

$$\mathcal{L}u = \text{div}(A_x \nabla u),$$

where div denotes the divergence. The Laplace operator has both divergent and nondivergent structure in the sense that Δu can be viewed as either $\Delta u = \text{div}(\nabla u)$ or $\Delta u = \text{tr}(D^2 u)$.

Let p be a fixed point in M and let $d_p(x)$ be the distance function defined by $d_p(x) = d(p, x)$, where $d(p, x)$ is the geodesic distance between p and x . Recall that d_p is smooth on $M \setminus (\text{Cut}(p) \cup \{p\})$, where $\text{Cut}(p)$ is the cut locus of p ; see e.g. [12, 13]. We assume that M satisfies the following conditions:

$$(2) \quad \Delta d_p(x) \leq \frac{n-1}{d_p(x)} \quad \text{for } x \notin \text{Cut}(p) \cup \{p\} \quad \forall p \in M \quad \text{and}$$

$$(3) \quad Ld_p(x) \leq \frac{a_L}{d_p(x)} \quad \text{for } x \notin \text{Cut}(p) \cup \{p\} \quad \forall p \in M,$$

where a_L is a constant which may depend on the given operator L .

By the Laplace comparison theorem (see e.g. [12, 13]), any manifold with nonnegative Ricci curvature satisfies the condition (2). We shall give the examples of the manifolds satisfying the condition (3) shortly.

Let us state our main theorem.

Theorem 1.1. *Assume that M satisfies the conditions (2) and (3). Let u be a smooth function in a ball B_{2R} satisfying $u \geq 0$ in B_{2R} . Then*

$$\sup_{B_R} u \leq C \left\{ \inf_{B_R} u + \frac{R^2}{|B_{2R}|^{1/n}} \|Lu\|_{L^n(B_{2R})} \right\},$$

where C is a constant depending only on λ, Λ, n and a_L .

Here, B_{2R} is any geodesic ball of radius $2R$, B_R is the geodesic ball of radius R concentric with B_{2R} , and $|B_{2R}|$ is the volume of B_{2R} . As an immediate consequence, we obtain the following Liouville property for solutions of $Lu = 0$ in M :

Corollary 1.2. *Let u be a smooth solution of $Lu = 0$ in M that is bounded from below. Then u is constant.*

We now give the examples of manifolds satisfying (2) and (3). This can be done in a most transparent way by stating it in terms of the *Pucci's extremal operator*. For a symmetric endomorphism S on T_xM , we define

$$\mathcal{M}^-[S, \lambda, \Lambda] = \mathcal{M}^-[S] = \lambda \sum_{\kappa_i > 0} \kappa_i + \Lambda \sum_{\kappa_i < 0} \kappa_i,$$

where $\kappa_i = \kappa_i(S)$ are the eigenvalues of S . Suppose M satisfies the following curvature assumption for any unit vector e in T_xM and for all $x \in M$:

$$(4) \quad \mathcal{M}^-[R(e)] \geq 0,$$

where $R(e)$ is the Ricci transformation on T_xM ; see Sec. 2 for its definition. Note that the inequality $\mathcal{M}^-[R(e)] \leq \text{tr}(A_x \circ R(e))$ holds provided that A_x satisfies the ellipticity condition (1). Thus, by taking $A_x = \lambda \text{Id}$, we find that the condition (4) implies the nonnegativity of Ricci curvature of M and hence, by the Laplace comparison theorem (see [12, 13]), it implies (2). We shall see in the next section that (4) also implies the condition (3) with $a_L = (n-1)\Lambda$; see Lemma 2.1 below. Then, the next corollary will follow immediately.

Corollary 1.3. *Assume that M satisfies the curvature condition (4). Let u be a smooth function in a ball B_{2R} satisfying $u \geq 0$ in B_{2R} . Then*

$$\sup_{B_R} u \leq C \left\{ \inf_{B_R} u + \frac{R^2}{|B_{2R}|^{1/n}} \|Lu\|_{L^n(B_{2R})} \right\},$$

where C is a universal constant; i.e., a constant depending on the ellipticity constants λ, Λ and the dimension n only.

Note that in the case when $L = \Delta$, the condition (4) is just to say that the Ricci curvature is nonnegative. Hence, Theorem 1.1 and Corollary 1.2, respectively, imply the Harnack inequality and the Liouville property by Yau [15]. Moreover, since the condition (4) holds trivially for the manifolds with nonnegative sectional curvature, our result also generalizes the Harnack inequality of Cabré [1].

2. PRELIMINARIES

Recall, for $x \in M$, the exponential map $\exp_x : T_xM \rightarrow M$. For $X \in T_xM$ with $|X| = 1$, $\gamma(t) = \exp_x(tX)$ is the unique unit speed geodesic that starts from x and goes in the direction of X . Let

$$t_0 = \sup \{t > 0 : \gamma \text{ is the unique minimal geodesic joining } x \text{ and } \gamma(t)\}.$$

If $t_0 < \infty$, then $\gamma(t_0)$ is called a *cut point* of x . We denote

$$\text{Cut}(x) = \text{set of all cut points of } x.$$

If we denote $S_x = \{X \in T_xM : |X| = 1\}$, it is clear that for any $X \in S_x$, there can be at most one cut point on the geodesic $\exp_x(tX)$, $t > 0$. If $\exp_x(t_0X) = y$ is a

cut point of x then we set $\mu(X) = d(x, y)$, the geodesic distance between x and y . If there is no cut point, we set $\mu(X) = \infty$. Define

$$E_x := \{tX : 0 \leq t < \mu(X), X \in S_x\}.$$

Then, it can be shown that $\exp_x : E_x \rightarrow \exp_x(E_x)$ is a diffeomorphism and, also that $\text{Cut}(x)$ has n -dimensional measure zero; see [12, 13].

Let the Riemannian curvature tensor be defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where ∇ is the Levi-Civita connection. For a unit vector $e \in T_x M$, $R(e)$ will denote the Ricci transformation of $T_x M$ into itself given by $R(e)X := R(X, e)e$. The Morse index form $I(\cdot, \cdot)$ is defined by

$$I(X, Y) = I_0^\ell(X, Y) = \int_0^\ell \{ \langle \nabla_{\gamma'} X, \nabla_{\gamma'} Y \rangle - \langle R(\gamma', X)Y, \gamma' \rangle \} dt,$$

where $\gamma : [0, \ell] \rightarrow M$ is a geodesic parametrized by arc length and X, Y are piecewise smooth vector fields along γ .

Now, we show that the condition (4) implies (3) with $a_L = (n-1)\Lambda$.

Lemma 2.1. *Let M be an n -dimensional complete Riemannian manifold satisfying the curvature condition (4). Let d_p be the distance function from a fixed point $p \in M$. If $x \notin \text{Cut}(p) \cup \{p\}$, then*

$$Ld_p(x) \leq \frac{a_L}{d_p(x)}, \quad \text{where } a_L = (n-1)\Lambda.$$

Proof. Let $\gamma : [0, \rho] \rightarrow M$ be the minimal geodesic parametrized by arc length joining $\gamma(0) = p$ and $\gamma(\rho) = x$. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ on $T_x M$ such that $e_1 = \gamma'(\rho)$ and $\{e_i\}_{i=1}^n$ are eigenvectors of $D^2 d_p$ on $T_x M$. We extend $\{e_i\}_{i=1}^n$ to $\{e_i(t)\}_{i=1}^n$ on $t \in [0, \rho]$ by parallel translation. Let $X_i, i = 2, \dots, n$, be the Jacobi fields along γ such that $X_i(0) = 0, X_i(\rho) = e_i$ and $[X_i, \gamma'] = 0$. Then $\langle D^2 d_p(e_i), e_i \rangle = \langle \nabla_{\gamma'} X_i, X_i \rangle(\rho) = I(X_i, X_i)$. Let $Y_i = f(t)e_i(t)$, where $f(t) = \frac{t}{\rho}$.

Since a Jacobi field minimizes the index form among all vector fields along the same geodesic with the same boundary data, we have $I(X_i, X_i) \leq I(Y_i, Y_i)$. Denote $a_{ij} := \langle A_x e_i, e_j \rangle$. Then

$$Ld_p(x) = \sum_{i=2}^n a_{ii} \langle D^2 d_p(e_i), e_i \rangle = \sum_{i=2}^n a_{ii} I(X_i, X_i) \leq \sum_{i=2}^n a_{ii} I(Y_i, Y_i).$$

On the other hand,

$$\begin{aligned} \sum_{i=2}^n a_{ii} I(Y_i, Y_i) &= \sum_{i=2}^n a_{ii} \int_0^\rho |f'|^2 - \int_0^\rho f^2 \sum_{i=2}^n a_{ii} \langle R(\gamma', e_i)e_i, \gamma' \rangle \\ &\leq \sum_{i=2}^n a_{ii} \int_0^\rho |f'|^2 - \int_0^\rho f^2 \mathcal{M}^- [R(\gamma')] \\ &\leq \frac{(n-1)\Lambda}{\rho}. \end{aligned}$$

This completes the proof. \square

Observe that in the case when $\lambda = \Lambda = 1$ (i.e., $L = \Delta$), condition (4) reduces to the nonnegative Ricci curvature assumption on M . Also, note that if the sectional curvature of M is nonnegative, then (4) holds trivially.

Let us recall that, if ϕ is a smooth map from M to another Riemannian manifold N , the Jacobian of ϕ is the absolute value of the determinant of the differential of ϕ , that is, $\text{Jac } \phi(x) := |\det d\phi(x)|$. This determinant is computed when expressing $d\phi(x)$ in an orthonormal basis of $T_x M$ and an orthonormal basis of $T_{\phi(x)} N$, and hence it is defined up to a sign. Its absolute value, $\text{Jac } \phi(x)$, is therefore well-defined. The following is the area formula on M , which follows easily from the area formula in Euclidean space using a partition of unity; see [1]: For any smooth map ϕ from M to M and any measurable subset E of M , we have

$$(5) \quad \int_E \text{Jac } \phi(x) dV(x) = \int_M \mathcal{H}^0[E \cap \phi^{-1}(y)] dV(y),$$

where \mathcal{H}^0 is the counting measure.

The following lemma is quoted from [1, Lemma 3.2].

Lemma 2.2. *Let v be a smooth function in an open set Ω of M . Consider the map ϕ from Ω to M defined by*

$$\phi(p) = \exp_p \nabla v(p).$$

Let $x \in \Omega$ and suppose that $\nabla v(x) \in E_x$. Set $y = \phi(x)$. Then

$$\text{Jac } \phi(x) = \text{Jac } \exp_x(\nabla v(x)) \cdot |\det D^2(v + d_y^2/2)(x)|,$$

where $\text{Jac } \exp_x(\nabla v(x))$ denotes the Jacobian of \exp_x , a map from $T_x M$ to M at the point $\nabla v(x) \in T_x M$.

In the normal polar coordinates (r, θ) , the area element $J(r, \theta)d\theta$ of the geodesic sphere $\partial B_r(x)$ of radius r centered at x is given by $r^{n-1}A(r, \theta)d\theta$, where $A(r, \theta)$ is the Jacobian of the map \exp_x at $r\theta \in T_x M$. Assume that M satisfies the condition (2). For $y = (r, \theta)$ not in the cut locus of x , we have the inequality, $\frac{\partial}{\partial r} \ln J(r, \theta) = \Delta r \leq \frac{n-1}{r}$; see [8]. By integrating it, we find $A(r, \theta) = r^{1-n}J(r, \theta)$ is a nonincreasing function of r and, in particular, $A(r, \theta) \leq 1$ since $\lim_{r \rightarrow 0} A(r, \theta) = 1$. Hence, Lemma 2.2 implies

$$(6) \quad \text{Jac } \phi(x) \leq |\det D^2(v + d_y^2/2)(x)|.$$

Also, from the above observation, it follows that $r^{1-n}|B_r|$ is nonincreasing in r ; see e.g. [8, 13]. This is generally known as Bishop's volume comparison theorem. We state it as a lemma.

Lemma 2.3 (Bishop). *Let M be an n -dimensional Riemannian manifold satisfying (2). For any $x \in M$, $|B_R(x)|/R^n$ is nonincreasing with respect to R . In particular,*

$$(7) \quad |B_R(x)|/|B_r(x)| \leq R^n/r^n \quad \text{if } 0 < r < R,$$

and M satisfies the volume doubling property; i.e. $|B_{2R}(x)| \leq 2^n |B_R(x)|$.

3. PROOF OF HARNACK INEQUALITY

Throughout the entire section, we shall assume that M satisfies (2) and (3). We will closely follow the outline of Cabré's proof of Harnack inequality in [1], which in turn carried over the basic scheme in the book by Caffarelli and Cabré [3]; see also [2]. Our goal is to establish Lemma 3.1 below. It corresponds to [1, Lemma 5.1], in the proof of which Cabré relied on the assumption that the underlying manifold

M has nonnegative sectional curvature. If one investigates his arguments carefully, it will turn out that, besides [1, Lemma 5.1], the only geometrical property of M he used in proving the subsequent lemmas in Sec. 6 and Sec. 7 of [1] is the volume growth condition (7). Therefore, once we prove Lemma 3.1, the proof of Theorem 1.1 follows from the one in [1] verbatim since, by Lemma 2.3 of the previous section, the property (7) holds for any manifold satisfying the condition (2).

Lemma 3.1. *Let u be a smooth function of in a ball B_{7R} satisfying*

$$Lu \leq f \text{ in } B_{7R}, \quad u \geq 0 \text{ in } B_{7R}, \quad \inf_{B_{2R}} u \leq 1,$$

and

$$\frac{R^2}{|B_{7R}|^{1/n}} \|f\|_{L^n(B_{7R})} \leq \varepsilon_\delta.$$

Then, for any $0 < \delta < 1$,

$$(8) \quad \frac{|\{u \leq M_\delta\} \cap B_{\delta R}|}{|B_{7R}|} \geq \mu_\delta,$$

where ε_δ , $0 < \mu_\delta < 1$ and $M_\delta > 1$ are positive constants depending on δ .

We need a series of lemmas to prove Lemma 3.1 above. The next lemma is a modification of [1, Lemma 4.1], where Cabré assumed the nonnegativity of the sectional curvature to get the same conclusion.

We denote by f^+ the positive part of a function, that is, $f^+ = \max(f, 0)$.

Lemma 3.2. *Let u be a smooth function in a ball $B_{7R} := B_{7R}(p)$ satisfying $u \geq 0$ in $B_{7R} \setminus B_{5R}$ and $\inf_{B_{2R}} u \leq 1$. Then*

$$(9) \quad |B_R| \leq \frac{1}{(n\lambda)^n} \int_{\{u \leq 6\} \cap B_{5R}} \left\{ (R^2 Lu + a_L + \Lambda)^+ \right\}^n dV.$$

Proof. For any $y \in B_R$ we consider the continuous function

$$w_y := R^2 u + \frac{1}{2} d_y^2.$$

We have that $\inf_{B_{2R}} w_y \leq R^2 + (3R)^2/2 = 11R^2/2$. In $B_{7R} \setminus B_{5R}$ we have, since $u \geq 0$ here, $w_y \geq (4R)^2/2 > 11R^2/2$. We conclude that the minimum of w_y in $\overline{B_{5R}}$ is achieved at some point of B_{5R} , which is also a minimum of w_y in B_{7R} . That is,

$$\inf_{B_{7R}} w_y = \inf_{B_{5R}} w_y = w_y(x),$$

for some $x \in B_{5R}$. It is not hard to see that $y = \exp_x \nabla(R^2 u)(x)$; see [1, pp. 637–638]. We are therefore led to consider the smooth map from B_{7R} to M

$$\phi(z) = \exp_z \nabla(R^2 u)(z),$$

and the measurable set

$$E := \left\{ x \in B_{5R} : \exists y \in B_R \text{ such that } w_y(x) = \inf_{B_{7R}} w_y \right\}.$$

We have proved that for any $y \in B_R$ there is at least one $x \in E$ such that $\phi(x) = y$. The area formula (5) gives

$$(10) \quad |B_R| \leq \int_M \mathcal{H}^0[E \cap \phi^{-1}(y)] dV(y) = \int_E (\text{Jac } \phi) dV.$$

We claim that $\text{Jac } \phi(x) \leq (n\lambda)^{-n} \{R^2Lu(x) + a_L + \Lambda\}^n$ for any $x \in E$.

Let $x \in E$ and take $y \in B_R$ such that $w_y(x) = \inf_{B_{7R}} w_y$. If x is not a cut point of y , then by (6) we find

$$\text{Jac } \phi(x) \leq |\det D^2(R^2u + d_y^2/2)(x)| = |\det D^2w_y(x)|.$$

Since w_y achieves its minimum at x , $D^2w_y(x) \geq 0$. Here we used \leq to denote the usual order between symmetric endomorphisms. Therefore, by using the well-known inequality,

$$(11) \quad \det A \cdot \det B \leq \{n^{-1} \text{tr}(A \circ B)\}^n, \quad A, B \text{ symmetric } \geq 0,$$

we conclude

$$\begin{aligned} \text{Jac } \phi(x) &\leq \det D^2w_y(x) \leq \frac{1}{\lambda^n} \det A_x \cdot \det D^2w_y(x) \\ &\leq \frac{1}{(n\lambda)^n} \{\text{tr}(A_x \circ D^2w_y(x))\}^n \\ &= \frac{1}{(n\lambda)^n} \{Lw_y(x)\}^n \\ &= \frac{1}{(n\lambda)^n} \{R^2Lu(x) + L(d_y^2/2)(x)\}^n \\ &\leq \frac{1}{(n\lambda)^n} \{R^2Lu(x) + a_L + \Lambda\}^n, \end{aligned}$$

where we used

$$L(d_p^2/2) = d_p L d_p + \langle A \nabla d_p, \nabla d_p \rangle \leq a_L + \Lambda |\nabla d_p|^2$$

in the last step to get $L(d_y^2/2) \leq a_L + \Lambda$.

We also have to consider the other case, namely the case when x is a cut point of y . In general, this kind of situation is easily overcome. Indeed, we will make use of *upper barrier* technique due to Calabi [4]; see also [5]. Since $y = \exp_x \nabla(R^2u)(x)$, x is not a cut point of $y_s = \phi_s(x) := \exp_x \nabla(sR^2u)(x)$, for all $0 \leq s < 1$. By continuity, $\text{Jac } \phi(x) = \lim_{s \rightarrow 1} \text{Jac } \phi_s(x)$. As before,

$$\text{Jac } \phi_s(x) \leq |\det D^2(sR^2u + d_{y_s}^2/2)(x)|.$$

Also,

$$\begin{aligned} \lim_{s \rightarrow 1} |\det D^2(sR^2u + d_{y_s}^2/2)(x)| &= \lim_{s \rightarrow 1} |\det D^2(R^2u + d_{y_s}^2/2)(x)| \\ &= \lim_{s \rightarrow 1} |\det D^2w_{y_s}(x)|. \end{aligned}$$

By the triangle inequality, we have

$$R^2u + \frac{1}{2}[d_{y_s} + d(y_s, y)]^2 \geq w_y.$$

Note that the equality holds at x . Since the function

$$R^2u + \frac{1}{2}[d_{y_s} + d(y_s, y)]^2 = w_{y_s} + d(y_s, y)d_{y_s} + \frac{1}{2}d(y_s, y)^2$$

is smooth near x and has a local minimum at x (recall that w_y has a minimum at x), its Hessian at x is nonnegative definite. Let $-k^2$ ($k > 0$) be a lower bound

of sectional curvature along the minimal geodesic joining x and y . By the Hessian comparison theorem (see, e.g. [12], [13]),

$$D^2 d_{y_s}(x) \leq k \coth(kd(x, y_s)) \text{Id} \leq N \text{Id},$$

uniformly in $s \in [\frac{1}{2}, 1)$ for some number N . Therefore, for all $s \in [\frac{1}{2}, 1)$,

$$0 \leq D^2 w_{y_s}(x) + d(y_s, y) D^2 d_{y_s}(x) \leq D^2 w_{y_s}(x) + Nd(y_s, y) \text{Id}.$$

In particular, $D^2 w_{y_s}(x) + Nd(y_s, y) \text{Id}$ is nonnegative definite. As before, using inequality (11), we obtain $\forall s \in [\frac{1}{2}, 1)$

$$\begin{aligned} \det(D^2 w_{y_s}(x) + Nd(y_s, y) \text{Id}) &\leq \frac{1}{\lambda^n} \{n^{-1} Lw_{y_s}(x) + Nd(y_s, y) \Lambda\}^n \\ &\leq \frac{1}{\lambda^n} \{n^{-1} (R^2 Lu(x) + a_L + \Lambda) + Nd(y_s, y) \Lambda\}^n. \end{aligned}$$

Since $d(y_s, y) \rightarrow 0$ as $s \rightarrow 1$, we observe

$$\begin{aligned} 0 &\leq \liminf_{s \rightarrow 1} |\det D^2 w_{y_s}(x)| = \liminf_{s \rightarrow 1} \det(D^2 w_{y_s}(x) + Nd(y_s, y) \text{Id}) \\ &\leq \frac{1}{\lambda^n} \liminf_{s \rightarrow 1} \{n^{-1} (R^2 Lu(x) + a_L + \Lambda) + Nd(y_s, y) \Lambda\}^n \\ &= \frac{1}{(n\lambda)^n} \{R^2 Lu(x) + a_L + \Lambda\}^n. \end{aligned}$$

Putting together, we finally obtain

$$\text{Jac } \phi(x) \leq \frac{1}{(n\lambda)^n} \{R^2 Lu(x) + a_L + \Lambda\}^n \quad \forall x \in E,$$

which proves the claim. Obviously,

$$(12) \quad \int_E (\text{Jac } \phi) dV \leq \frac{1}{(n\lambda)^n} \int_E \{(R^2 Lu(x) + a_L + \Lambda)^+\}^n dV.$$

At the beginning of the proof we have shown

$$E \subset \{u \leq 11/2\} \cap B_{5R} \subset \{u \leq 6\} \cap B_{5R}.$$

Now, the lemma follows from (10) and (12). \square

The following technical lemma, which consists of the construction of a barrier, will be used in the proof of Lemma 3.1. It corresponds to [1, Lemma 5.5] and [3, Lemma 4.1].

Lemma 3.3 (A barrier function). *Let $p \in M$, $R > 0$, and $0 < \delta < 1$. There exists a continuous function v_δ in $B_{7R} = B_{7R}(p)$, smooth in $M \setminus \text{Cut}(p)$ and such that*

- (a) $v_\delta \geq 0$ in $B_{7R} \setminus B_{5R}$,
- (b) $v_\delta \leq 0$ in B_{2R} ,
- (c) $R^2 L v_\delta + a_L + \Lambda \leq 0$ a.e. in $(B_{5R} \setminus B_{\delta R})$,
- (d) $R^2 L v_\delta \leq C_\delta$ a.e. in B_{5R} and
- (e) $v_\delta \geq -C_\delta$ in B_{7R} ,

for some positive constant C_δ depending on δ .

Proof. We take $v_\delta = \psi_\delta(d_p/R)$, where ψ_δ is a smooth increasing function on \mathbb{R}^+ such that $\psi'_\delta(0) = 0$ and that $\psi_\delta(t) = (\frac{3}{5})^{-\alpha} - (\frac{t}{5})^{-\alpha}$ if $t \geq \delta$. The number $\alpha > 1$ will be chosen later. Clearly, v_δ is continuous in M . Also, since d_p is smooth outside

$\text{Cut}(p) \cup \{p\}$ and $\psi'_\delta(0) = 0$, v_δ is smooth in $M \setminus \text{Cut}(p)$. Recall that $\text{Cut}(p)$ is a closed set of measure zero.

Properties (a) and (b) are clear. For the rest of proof, we will denote $\rho := d_p/R$. From the identity, $L[\varphi(u)] = \varphi'(u)Lu + \varphi''(u)\langle A\nabla u, \nabla u \rangle$, we get

$$(13) \quad \begin{aligned} Lv_\delta &= \frac{1}{R}\psi'_\delta(\rho)Ld_p + \frac{1}{R^2}\psi''_\delta(\rho)\langle A\nabla d_p, \nabla d_p \rangle \\ &= \frac{1}{R^2}\frac{\psi'_\delta(\rho)}{\rho}d_pLd_p + \frac{1}{R^2}\psi''_\delta(\rho)\langle A\nabla d_p, \nabla d_p \rangle \quad \text{in } M \setminus \text{Cut}(p). \end{aligned}$$

For $\delta \leq \rho < 5$, we have

$$\psi'_\delta(\rho) = \frac{\alpha}{5}\left(\frac{\rho}{5}\right)^{-\alpha-1} \quad \text{and} \quad \psi''_\delta(\rho) = -\frac{\alpha(\alpha+1)}{5^2}\left(\frac{\rho}{5}\right)^{-\alpha-2}.$$

We recall that

$$(14) \quad d_pLd_p \leq a_L \quad \text{and} \quad \lambda \leq \langle A\nabla d_p, \nabla d_p \rangle \leq \Lambda \quad \text{in } M \setminus \text{Cut}(p).$$

Hence, in $(B_{5R} \setminus B_{\delta R}) \setminus \text{Cut}(p)$

$$R^2Lv_\delta \leq \frac{\alpha}{25}\left(\frac{\rho}{5}\right)^{-\alpha-2}(a_L - (\alpha+1)\lambda)$$

which can be made less than $-(a_L + \Lambda)$ if α is chosen large enough. This proves property (c).

Also, (13), (14), and the assumptions on ψ_δ imply that in $B_{5R} \setminus \text{Cut}(p)$

$$R^2Lv_\delta \leq a_L \sup_{0 < \rho < 5} \frac{\psi'_\delta(\rho)}{\rho} + \Lambda \sup_{0 < \rho < 5} |\psi''_\delta(\rho)| < +\infty,$$

which proves property (d). Property (e) is clear. \square

In the proof of Lemma 3.1, we will apply Lemma 3.2 to $u+v_\delta$ with v_δ constructed in the proof of Lemma 3.3. The possible existence of cut locus of p may present some technical issue here; $u+v_\delta$ is not necessarily smooth in B_{7R} . There are various ways to overcome this difficulty but we will again stick to Cabré's approach in [1]. It is not hard to verify the following lemma, which asserts that one can approximate v_δ by a sequence of smooth functions; see [1, pp. 641–645].

Lemma 3.4. *Let $p \in M$, $R > 0$, and $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a smooth increasing function such that $\psi'(0) = 0$. Let $v = \psi \circ d_p$. Then there exist a smooth function $0 \leq \xi \leq 1$ in M with $\xi \equiv 1$ in $B_{5R} := B_{5R}(p)$ and $\text{supp } \xi \subset B_{7R}$ and a sequence $\{w_k\}_{k=1}^\infty$ of smooth functions in M such that*

$$\begin{cases} w_k \rightarrow \xi v & \text{uniformly in } M, \\ Lw_k \rightarrow Lv & \text{a.e. in } B_{5R}, \text{ and} \\ D^2w_k \leq C \text{Id} & \text{in } M \text{ for some constant } C \text{ independent of } k. \end{cases}$$

Now, we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let v_δ be as in Lemma 3.3. Let $\{w_k\}$ be a sequence of smooth functions approximating v_δ in the way as in Lemma 3.4. Note that $u + v_\delta \geq 0$ in $B_{7R} \setminus B_{5R}$ and $\inf_{B_{2R}}(u + v_\delta) \leq 1$. Replacing $u + w_k$ by $(u + w_k + \varepsilon_k)/(1 + 2\varepsilon_k)$ for some sequence $0 < \varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, we may assume that $u + w_k$ satisfies the hypothesis of Lemma 3.2, and hence (9) with u replaced by $u + w_k$. Let $\varepsilon > 0$ and note that $\{u + w_k \leq 6\} \cap B_{5R} \subset \{u + v_\delta \leq 6 + \varepsilon\} \cap B_{5R}$ if k is large enough, and that $\{(R^2L(u + w_k) + a_L + \Lambda)^+\}^n$ is uniformly bounded in M independently of k . (since $D^2w_k \leq C \text{Id}$). Letting k tend to infinity and applying the dominated

convergence theorem, we get that $u + v_\delta$ satisfies (9) with $\{u \leq 6\}$ replaced by $\{u + v_\delta \leq 6 + \varepsilon\}$. Now letting ε tend to zero, we obtain, since $Lu \leq f$,

$$\begin{aligned} |B_R| &\leq \frac{1}{(n\lambda)^n} \int_{\{u+v_\delta \leq 6\} \cap B_{5R}} \left\{ (R^2(f + Lv_\delta) + a_L + \Lambda)^+ \right\}^n dV \\ &= \frac{1}{(n\lambda)^n} \left[\int_{E_1} + \int_{E_2} \left\{ (R^2(f + Lv_\delta) + a_L + \Lambda)^+ \right\}^n dV \right], \end{aligned}$$

where $E_1 = \{u + v_\delta \leq 6\} \cap (B_{5R} \setminus B_{\delta R})$ and $E_2 = \{u + v_\delta \leq 6\} \cap B_{\delta R}$.

Using (c), (d) and (e) in Lemma 3.3 and (7), we get

$$|B_{7R}| \leq 7^n |B_R| \leq C'_\delta \left\{ R^{2n} \|f\|_{L^n(B_{7R})}^n + |\{u \leq M_\delta\} \cap B_{\delta R}| \right\},$$

for some constants C'_δ and M_δ depending on δ . We easily conclude (8). \square

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