Skyrmion in theories with massless Adjoint Quarks

S. Bolognesi and M. Shifman, Phys. Rev. D 75 (2007) 065020 [arXiv:hep-th/0701065]

R. A. and M. Shifman, J. Phys. A 40 (2007) 6221 [arXiv:hep-th/0612211].

R. A., S. Bolognesi and M. Shifman, arXiv:0804.0229 [hep-th].

QCD with Adjont Quarks

$SU(N_c)$ gauge theory with N_f massless adjoint Weyl fermions $\lambda^a_{\alpha\, f}$

$$\langle \lambda^a_{\alpha\, f}\, \lambda^{a\, \alpha}_{g} \rangle \sim \Lambda^3 \delta_{fg}\,.$$

The global symmetry breaking is:

$$\mathrm{SU}(N_f) \times \mathbb{Z}_{2N_cN_f} \to \mathrm{SO}(N_f) \times \mathbb{Z}_2\,,$$

where the discrete factors are the remnants of the anomalous $U(1)$. For $N_f \geq 2$ the low energy effective theory is a sigma model with target space:

$$\mathcal{M}_{N_f} = \mathrm{SU}(N_f)/\mathrm{SO}(N_f).$$

Solitons as Twisted Flux Tubes

For $N_f = 2$, $\mathcal{M}_{N_f} = S^2$

$\pi_2(S^2) = \mathbb{Z}$, which correspond to topologically stable vortices

Then let us take a vortex and twist it once before to form a donut configuration

This corresponds to an Hopf fibration, which is the generator of

$$\pi_3(S^2) = \mathbb{Z}$$

Topological Solitons

For higher N_f the generalization is the following (remember $N_f < 6$ in order to keep asymptotic freedom):

N_f	2	3	4	5
$\pi_3(\mathcal{M}_{N_f})$	$\mathbb{Z}$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Question : Which is the microscopic interpretation of these objects in term of the original $SU(N_c)$ gauge theory ??

A parental theory

A theory with the same continuous global symmetry breaking pattern is:

$SO(N_c)$ gauge theory with N_f Weyl fermions in the vectorial representation

$$\mathcal{M}_{N_f} = \mathrm{SU}(N_f)/\mathrm{SO}(N_f).$$

In this case the skyrmion was matched by Witten (1983) with a baryon:

$$\epsilon_{\alpha_1 \alpha_2 \ldots \alpha_{N_c}} q^{\alpha_1} q^{\alpha_2} \ldots q^{\alpha_{N_c}}$$

Stability in the parental theory

This theory actually has an $\mathrm{O}(N_c)$ symmetry; the quotient $\mathbb{Z}_2 = \mathrm{O}(N_c)/\mathrm{SO}(N_c)$ acts as a global symmetry group. All particles built with the $\epsilon_{\alpha_1\alpha_2\ldots\alpha_{N_c}}$ symbol are odd under this symmetry.

Indeed the skyrmion could be in principle a Pfaffian-Baryon hibrid:

$$\epsilon_{\alpha_1\alpha_2\ldots\alpha_{N_c}} \left(q^{\alpha_1} q^{\alpha_2} \ldots q^{\alpha_r}\right) \left(F^{\alpha_{r+1}\alpha_{r+2}} \ldots F^{\alpha_{N_c-1}\alpha_{N_c}}\right).$$

Note that this is a $\mathbb{Z}_2$ stability

Composite Fermions

In the case of Adjoint QCD, the low energy effective, in addition to the Goldstone Bosons, contains also the following composite fermions:

$$\psi_{\beta\ f} = C\,\mathrm{Tr}\left(\lambda_f^\alpha\, F_{\alpha\beta}\right) \equiv C\,\mathrm{Tr}\left(\lambda_f^\alpha \sigma_{\alpha\beta}^{\mu\nu} F_{\mu\nu}\right),$$

The mass of these objects is $\mathcal{O}(N_c^0)$, while the mass of the Skyrmion scales as $\mathcal{O}(N_c^2)$

Global symmetry quantum numbers

The unbroken global symmetry are $SO(N_f)$ and the $\mathbb{Z}_2$ remnant of the axial $U(1)$, which we call fermion number F. Only $(-1)^F$ is well-defined.

	Q, $N_f = 2$	$\mathrm{SO}(N_f)$, $N_f \geq 3$	F
ψ	1	$\underline{N}_f$	1
π	2	2-Tens Sym, Traceless	0

All the conventional mesons and baryons with mass $\mathcal{O}(N_c^0)$ have:

$$(-1)^Q(-1)^F = 1$$

Skyrmion Stability for $N_f = 2$

The Skyrmion, with mass $O(N_c^2)$, is an exotic state with:

$$(-1)^Q(-1)^F = -1$$

and for this reason is $\mathbb{Z}_2$ stable

The reason is that the fermion number has an anomalous contribution that couples directly to the topological current of the Skyrmion (see S. Bolognesi and M. Shifman, Phys. Rev. D **75** (2007) 065020 [arXiv:hep-th/0701065].)

The generalization to $N_f > 2$ is not so straightforward. For N_f odd we can construct an $SO(N_f)$ singlet with $F = 1$ and mass $\mathcal{O}(N_c)$ from ψ using the antisymmetric tensor $\epsilon^{i_1,\ldots i_k}$.

Generalization to Higher N_f

The generalization to Higher N_f involves an odd relation between statistic and fermion number of the fermion

Skyrmions can be boson with odd fermion number or fermions with even fermion number, and this fact makes them $\mathbb{Z}_2$ stable states

In order to check this issue we need to write the WZNW term for the sigma model with target space $\mathcal{M}_{N_f}$, and then to use it to determine the statistic of the skyrmion. We need also to discuss how the fermion ψ couples to the massless goldstone bosons.

Effective Sigma Model Lagrangian

This is a more concrete way to parameterize the quotient

$$\mathcal{M}_{N_f} = \mathrm{SU}(N_f)/\mathrm{SO}(N_f)$$

Cartan Embedding:

$$U \cdot \mathrm{SO}(N_f) \to W = U \cdot U^t \,,$$

This is a 1-1 map from the quotient space to the submanifold of the symmetric matrices of $SU(N_f)$ The Skyrme model lagrangian reads:

$$\mathcal{L} = \frac{F_\pi^2}{4} \mathrm{Tr} \left(\partial_\mu W \partial^\mu W^\dagger \right) + \frac{1}{e^2} \mathrm{Tr} \left[(\partial_\mu W) W^\dagger, (\partial_\nu W) W^\dagger \right]^2 \,.$$

Parameters

$$U = \exp(i\, V \cdot A \cdot V^\dagger),$$

where A is a traceless real diagonal matrix and V is an $SO(N_f)$ element.

Example: $N_f = 3$

$$A = \frac{1}{2} \begin{pmatrix} \eta/\sqrt{3} + \theta & 0 & 0 \\ 0 & \eta/\sqrt{3} - \theta & 0 \\ 0 & 0 & -2\eta/\sqrt{3} \end{pmatrix},$$

V is a $SO(3)$ rotation with Euler Angles (α, β, γ)

Explicit Metric for $N_f = 3$

$$\mathcal{L}_2 = \frac{1}{4}\Big[2(\partial_\mu\theta)^2 + 2(\partial_\mu\eta)^2 + 2\sin^2\theta(\partial_\mu\alpha)^2$$

$$+(1-\cos\sqrt{3}\eta\cos\theta - \cos\alpha\sin\sqrt{3}\eta\sin\theta)(\partial_\mu\beta)^2$$

$$+\frac{1}{2}(\partial_\mu\gamma)^2\Big[2-(1+\cos\beta)\cos^2\theta - 2\cos\sqrt{3}\eta\cos\theta\sin^2\frac{\beta}{2}$$

$$+2\cos\alpha\sin^2\frac{\beta}{2}\sin\sqrt{3}\eta\sin\theta + \sin^2\theta + \cos\beta\sin^2\theta\Big]$$

$$+\left(4\cos\frac{\beta}{2}\sin^2\theta\right)(\partial_\mu\alpha)(\partial_\mu\gamma) - \left(2\sin\alpha\sin\frac{\beta}{2}\sin\sqrt{3}\eta\sin\theta\right)(\partial_\mu\beta)(\partial_\mu\gamma)\Big].$$

Coupling to Fermion

$$\mathcal{L}_{\rm ferm} = \bar{\psi}_{f\dot{\alpha}} i\partial^{\dot{\alpha}\alpha}\psi_{f\alpha} - \frac{g}{2}\left\{W^{fg}\psi_{\alpha f}\psi^{\alpha}_{g} + \text{H.c.}\right\}.$$

If we expand around the vacuum (identity matrix), we get a mass term for ψ:

$$\bar{\psi}_{f\dot{\alpha}} i\partial^{\dot{\alpha}\alpha}\psi_{f\alpha} - g\left\{\psi^{\alpha}_{f}\psi_{\alpha f} + \text{H.c.}\right\}.$$

The higher order terms give the interactions between the ψ and the Goldstone bosons

WZNW term

$$\Gamma = -\frac{i}{240\pi^2}\int_{B_5} d\Sigma^{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\left[(W^\dagger\partial_\mu W)(W^\dagger\partial_\nu W)(W^\dagger\partial_\rho W)(W^\dagger\partial_\sigma W)(W^\dagger\partial_\lambda W)\right].$$

$$\mathcal{L}_{WZNW} = k\,\Gamma$$

In QCD k has to be an integer. In Adjoint QCD k can be also half-integer.

This difference in the normalization is due to the fact that the minimal S^5 in the subspace of the symmetric matrices in $SU(N_f)$ is **twice** the minimal S^5 in $SU(N_f)$

WZNW term $N_f = 3$

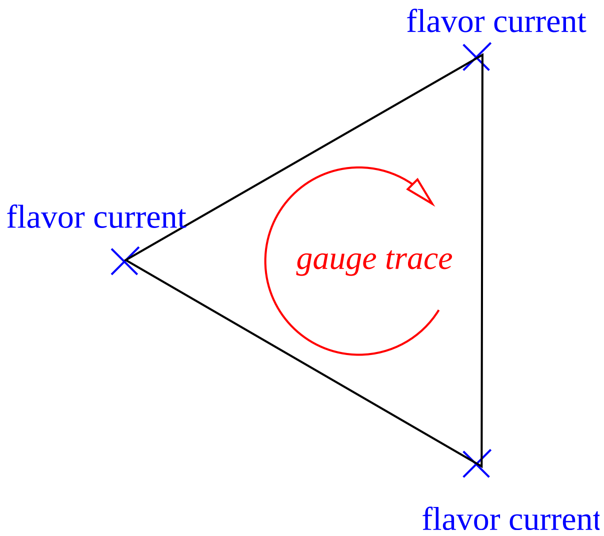
$$S_{\mathrm{WZNW}} = nA\,\frac{i\,60}{64\sqrt{3}} \int_{B_5} d\Sigma^{\mu\nu\rho\sigma\lambda}\ \left(\partial_\mu\theta \cdot \partial_\nu\eta \cdot \partial_\rho\alpha \cdot \partial_\sigma\beta \cdot \partial_\lambda\gamma\right)$$

$$\times \left(\cos\sqrt{3}\eta - \cos\theta\right) \sin\frac{\beta}{2}\,\sin\theta\,.$$

It is simply proportional to the volume form

Anomaly matching

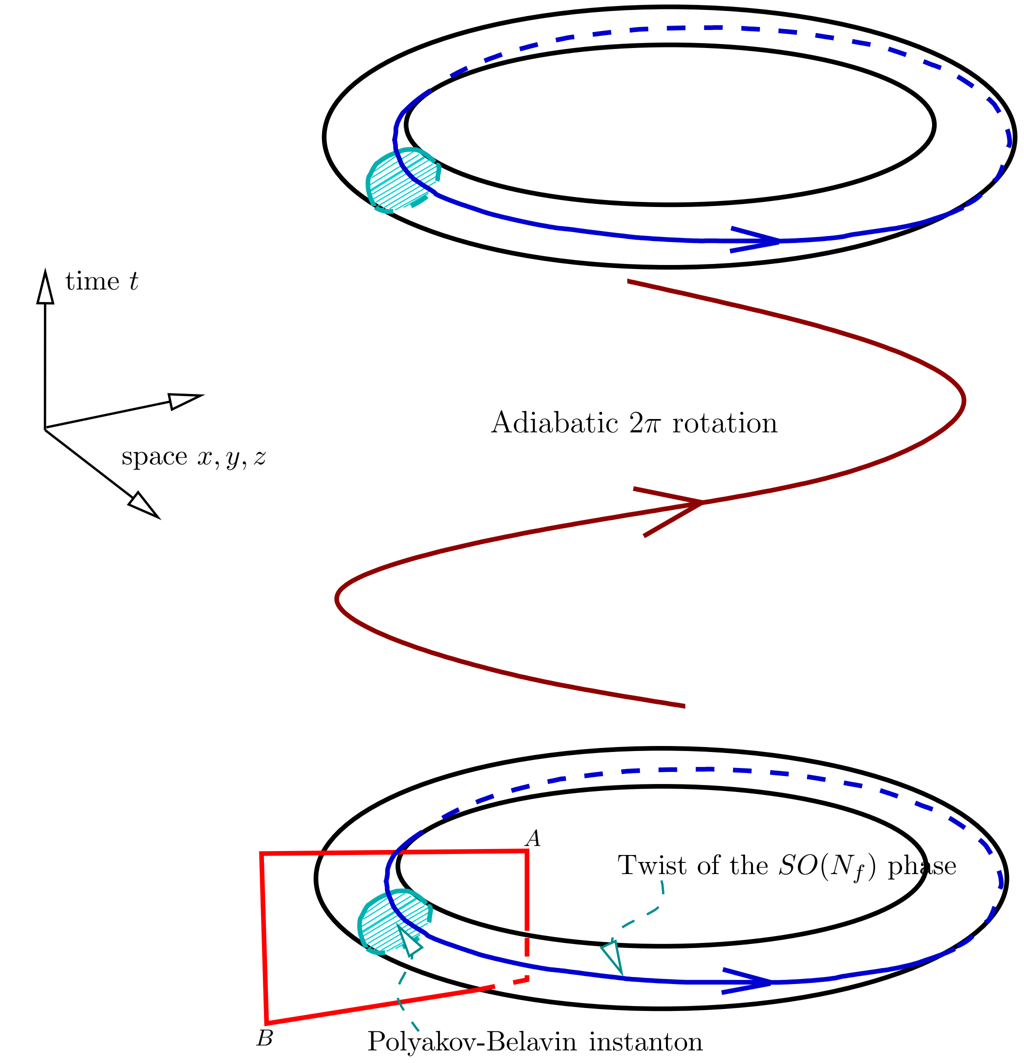
The coefficient k can be determined with an anomaly matching between the ultraviolet and the infrared theory



For Adjoint Qcd we get: $k = \frac{N_c^2 - 1}{2}$.

For the $SO(N_c)$ theory with vectorial matter: $k = \frac{N_c}{2}$.

WZNW term and statistic



WZNW term and statistic

For an adiabatic rotation of the skyrmion, the contribution to the action due to the WZNW is:

$$e^{i2\pi k}$$

So or k half-integer the skyrmion is quantized as a fermion and for k integer as a boson

Extra contribution to the fermion number

Axial anomaly of the fermion current:

$$\partial^\mu J_\mu^{F0} = \frac{1}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{4\pi^2}\partial_\mu(\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma),$$

This expression is matched with the Hopf number of the soliton:

$$s = \frac{1}{4\pi^2}\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \,.$$

The net effect is a shift in the Skyrmion fermion number by one unit, without changing its statistics.

Stability

For N_c odd, the Skyrmion is a boson with an odd fermion number.

For N_c even, it is a fermion with an even fermion number.

In both cases it is a $\mathbb{Z}_2$-stable object.

The reason for the stability is different than in the $SO(N_c)$ theory, where the stability is due to

$$\mathbb{Z}_2 = \mathrm{O}(N_c)/\mathrm{SO}(N_c)$$

Small N_c

$N_c = 2$ is special, because an $SU(2)$ gauge theory with adjoint fermions is the same as an $SO(3)$ theory with vector quarks

The SO(N_c) description in this particular case is better. In this case the fermion ψ coincides with the Pfaffian,

$$\epsilon_{abc} q^a F^{bc},$$

therefore, it does not make sense to introduce it as another independent degree of freedom.

Flux tubes

$$\pi_2(\mathrm{SU}(N_f)/\mathrm{SO}(N_f)) = \mathbb{Z}_2 \text{ at } N_f \geq 3\,,$$

so the sigma model also supports flux tubes

In the $SO(N_c)$ theory they can be identified as flux tubes confining sources in the spinorial representation

For Adjoint QCD the same argument can not be used, because the theory should support $\mathbb{Z}_{N_c}$ flux tube This tells us that the $\mathbb{Z}_2$-strings supported by the chiral low-energy theory are unrelated to the confinement strings of the corresponding microscopic theories.