ON THE REGULARITY OF SOLUTIONS TO A PARABOLIC SYSTEM RELATED TO MAXWELL'S EQUATIONS

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Abstract. The goal of this paper is to establish Hölder estimates for the solutions of a certain parabolic system related to Maxwell's equations. Such an estimate is employed to get the local Hölder continuity of the magnetic field arising from Maxwell's equations in a quasi-stationary electromagnetic field, provided the resistivity of the material is continuous in time.

1. Introduction

Let Ω be a domain in $\mathbb{R}^3$ and let $Q = \Omega \times (0,T)$ be a cylinder in $\mathbb{R}^4$. Let $A(x,t)$ be a 3×3 symmetric matrix such that there exists a number $\nu \in (0,1)$ satisfying

$$\nu\,|\xi|^2 \le \langle A(x,t)\xi,\xi\rangle \quad \text{and} \quad |A(x,t)| \le \nu^{-1} \quad \forall (x,t) \in Q. \tag{1.1}$$

Here $\langle \xi,\eta\rangle$ denotes the usual inner product of vectors $\xi,\eta \in \mathbb{R}^3$ and $|X|$ is the Euclidean length of $X \in \mathbb{R}^N$, i.e., $|X|^2$ is the sum of squares of each entry of X.

In this paper we study the regularity of the solutions of the following system:

$$\left.\begin{array}{c} \boldsymbol{u}_t + \nabla\times[A(x,t)\nabla\times\boldsymbol{u}] = 0 \\ \nabla\cdot\boldsymbol{u} = 0 \end{array}\right\} \quad \text{in} \quad Q. \tag{1.2}$$

Here, we denote $\nabla\times\boldsymbol{u} = \operatorname{curl}\boldsymbol{u}$ and $\nabla\cdot\boldsymbol{u} = \operatorname{div}\boldsymbol{u}$, where $\boldsymbol{u} = (u^1,u^2,u^3) \in \mathbb{R}^3$.

Recently, in [3], the elliptic case of this system has been studied by the first two authors. Without imposing any assumptions other than (1.1), they derived a priori Hölder estimate of its weak solutions. Although a special type of $A(x)$ was considered in the article [3], the main result [3, Theorem 2.1] remains valid as long as $A(x)$ satisfies (1.1). We would like to mention that independently, Yin obtained a similar result in [12]. In this context, it is an interesting question to ask whether or not the weak solutions of the system (1.2) are locally Hölder continuous when the coefficients $A(x,t)$ are assumed to be only measurable.

Our main result states that weak solutions of the system (1.2) are locally Hölder continuous in the case that $A(x,t)$ is uniformly continuous in time or $A(x,t)$ is of the form $a(t)B(x)$ where $a(t)$ is a scalar function and $B(x)$ is a symmetric matrix (see Theorem 3.2 and Theorem 3.3 below). The main idea is to use the elliptic result of [3], as well as standard perturbation techniques and the structure of the system (1.2).

As an application of the main result, we consider the following system arising from Maxwell's equations in a quasi-stationary electro-magnetic field where the

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displacement of the electric current is assumed to be time independent:

$$\left.\begin{array}{c}\boldsymbol{H}_t + \nabla\times[\varrho(x,t)\nabla\times\boldsymbol{H}] = 0\\ \nabla\cdot\boldsymbol{H} = 0\end{array}\right\} \quad \text{in } Q.$$

Here, the vector $\boldsymbol{H}$ represents the strength of the magnetic field and ϱ the electrical resistivity of the material (see [5], [10]). Using our result from Section 3, one finds that if ϱ is assumed to be continuous in time, then $\boldsymbol{H}$ is locally Hölder continuous.

Another application is the following coupled system introduced in [9]:

$$\left.\begin{array}{c}\boldsymbol{H}_t + \nabla\times[\varrho(u)\nabla\times\boldsymbol{H}] = 0\\ \nabla\cdot\boldsymbol{H} = 0\\ u_t - \Delta u = \varrho(u)\left|\nabla\times\boldsymbol{H}\right|^2\end{array}\right\} \quad \text{in} \quad Q = \Omega\times(0,T), \tag{1.3}$$

where $\boldsymbol{H}$ and u are the magnetic field and the temperature of the material, respectively, and $\varrho(u)$ is the electrical resistivity of the material. In [10] Yin proved the Hölder continuity of $(\boldsymbol{H}, u)$ under the assumption that $\varrho(u)$ is continuous. Using our result of Section 3, the same conclusion holds, only assuming the continuity of $\varrho(u)$ in time. Finally we consider the Stokes system with measurable coefficients:

$$\left.\begin{array}{c}\boldsymbol{u}_t - A(x,t)\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}\\ \operatorname{div}\boldsymbol{u} = 0\end{array}\right\} \quad \text{in} \quad Q = \mathbb{R}^3\times(0,\infty).$$

With the same assumption on $A(x,t)$ as in Section 3, we prove that u is locally Hölder continuous.

This paper is organized as follows: In Section 2, we introduce the notations and recall some known results used in our proofs. In Section 3, we study the system (1.2) and prove our main result. In Section 4, we study some parabolic systems related to Maxwell's equations and the Stokes system with measurable coefficients, as applications.

2. Notations and Preliminaries

In this section, we introduce the notations which will be used throughout this article and also recall some well-known facts. Let us begin with the notations.

- $z_0 = (x_0, t_0)$ denotes an arbitrary point in $\mathbb{R}^{n+1}$, where $x_0 \in \mathbb{R}^n$ and $t_0 \in (-\infty, \infty)$.
- $B_r = B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.
- $Q_r = Q_r(z_0) = \left\{(x,t) \in \mathbb{R}^{n+1} : |x - x_0| < r, -r^2 < t - t_0 < 0\right\}$.
- $Q_{r,t} = Q_{r,t}(z_0) = \{(x,t) \in Q_r(z_0)\}$; i.e., $Q_{r,t}(z_0) = B_r(x_0)\times\{t\}$ if $t \in (t_0 - r^2, t_0)$ and $Q_{r,t}(z_0) = \emptyset$ otherwise.
- For $Q' \subset Q$, $\partial_p Q'$ is the parabolic boundary of Q'.
- $Q' \Subset Q$ means $\overline{Q'}$ is compact and $Q' \cup \partial_p Q' \subset Q$.
- We denote by $\fint_S f$ the average of f on S; i.e., $\fint_S f = \int_S f / \int_S 1$ and we denote $f_r = f_{z_0,r} = \fint_{Q_r(z_0)} f$.
- For $\Omega \subset \mathbb{R}^n$ and $q > 1$, $L^q(\Omega)$ denotes the Banach space of measurable functions with the following norms:

$$\|u\|_{L^q(\Omega)} = \left(\int_\Omega |u(x)|^q\, dx\right)^{1/q} \quad \text{and} \quad \|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |u|.$$

- Let $Q = \Omega \times (a,b)$. For $1 \le q, r \le \infty$, $L^{q,r}(Q)$ denotes the Banach space of all measurable functions with the finite norm
$$\|u\|_{q,r;\,Q} = \|u\|_{L^{q,r}(Q)} = \left(\int_a^b \left(\int_\Omega |u(x,t)|^q \, dx \right)^{r/q} dt \right)^{1/r}.$$
- $L^{q,q}(Q)$ will be denoted by $L^q(Q)$ and the norm $\|\cdot\|_{L^{q,q}(Q)}$ by $\|\cdot\|_{L^q(Q)}$.
- For $1 \le q \le \infty$, $W^{k,q}(\Omega)$ denotes the usual Sobolev space; i.e., $W^{k,q}(\Omega) = \{u : D^\alpha u \in L^q(\Omega), 0 \le |\alpha| \le k\}$, and $W_0^{k,q}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $W^{k,q}(\Omega)$.
- For $1 \le q, r \le \infty$, $L^r\left((a,b); W^{k,q}(\Omega)\right)$ denotes the Banach space of all measurable functions with the finite norm
$$\|u\|_{L^r((a,b);W^{k,q}(\Omega))} = \left(\int_a^b \|u(\cdot,t)\|^r_{W^{k,q}(\Omega)} \, dt \right)^{1/r}.$$
- $C^{\alpha,\alpha/2}(Q)$ denotes the Banach space of functions that are Hölder continuous with the exponent $\alpha \in (0,1)$, and
$$[u]_{\alpha,\alpha/2;\,Q} = [u]_{C^{\alpha,\alpha/2}(Q)} = \sup_{z \ne z' \in Q} \frac{|u(z) - u(z')|}{d(z,z')^\alpha},$$
where $d(\cdot,\cdot)$ is the parabolic metric; i.e., $d(z,z') = |x - x'| + |t - t'|^{1/2}$.
- $u \in L^{p,q}_{\mathrm{loc}}$ (resp., $u \in C^{\alpha,\alpha/2}_{\mathrm{loc}}$) means $u \in L^{p,q}(Q')$ (resp., $u \in C^{\alpha,\alpha/2}(Q')$) for all $Q' = \Omega' \times (a,b) \Subset Q$.
- The Morrey space $M^{2,\mu}(Q)$ is defined to be the set of all functions $u \in L^2_{\mathrm{loc}}(Q)$ with the finite norm
$$\|u\|_{M^{2,\mu}(Q)} = \sup_{Q_\rho(z) \subset Q} \left(\rho^{-\mu} \int_{Q_\rho(z)} |u|^2 \right)^{1/2}.$$
- We denote by $N = N(\alpha, \beta, \ldots)$ a constant depending on the prescribed quantities $\alpha, \beta, \ldots$.

The following lemma is Campanato's integral characterization of Hölder continuous functions (see e.g. [6, Lemma 4.3, page 50]).

Lemma 2.1. *Let $f \in L^2(Q_{2R}(z_0))$ and suppose there are positive constants $\alpha \le 1$ and H such that $\int_{Q_r(z)} |f - f_{z,r}|^2 \le H^2 r^{n+2+2\alpha}$ for any $z \in Q_R(z_0)$ and any $r \in (0,R)$. Then f is Hölder continuous with the exponent α in $Q_R(z_0)$ and $[f]_{\alpha,\alpha/2;Q_R(z_0)} \le N(n,\alpha)H$.*

The following lemma can be found in Giaquinta [1, Lemma 2.1, page 86].

Lemma 2.2. *Let $\phi(t)$ be a nonnegative and nondecreasing function. Suppose that*
$$\phi(\rho) \le A\left[\left(\frac{\rho}{r}\right)^\alpha + \varepsilon \right] \phi(r) + Br^\beta$$
for all $\rho < r \le R$, with A, α, β nonnegative constants, $\beta < \alpha$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A,\alpha,\beta)$ such that if $\varepsilon < \varepsilon_0$, for all $\rho < r \le R$ we have
$$\phi(\rho) \le c\left[\left(\frac{\rho}{r}\right)^\beta \phi(r) + B\rho^\beta \right]$$
where c is a constant depending on α, β, A.

3. Main result

This section deals with the linear theory and a priori estimates. To avoid the technicalities, the coefficient $A(x,t)$ will be assumed to be smooth as in [3]. Nonetheless, the constant appearing in the a priori estimates will not depend on the extra smoothness of $A(x,t)$. Also, by a solution, we always mean a smooth solution unless otherwise stated. Indeed, the energy inequality below indicates that our system (1.2) is strongly parabolic. Hence, the general theory on the parabolic systems of divergence type can be applied, and by requiring $A(x,t)$ to be smooth, we easily see that the weak solutions of (1.2) are actually smooth (see, e.g. [4]).

Lemma 3.1 (Energy inequality)**.** *Let $\boldsymbol{u}$ be a solution of* (1.2)*. Let $R>0$ be such that $Q_{\lambda R}:=Q_{\lambda R}(z_0)\subset Q$ for some $\lambda>1$. Then the following estimate holds:*

$$\sup_{t_0-R^2\le s\le t_0}\int_{B_R}|\boldsymbol{u}(\cdot,s)|^2+\int_{Q_R}|\nabla\boldsymbol{u}|^2\le\frac{N(\nu,\lambda)}{R^2}\int_{Q_{\lambda R}}|\boldsymbol{u}|^2.$$

Proof. Assume, for simplicity, that $z_0=(0,0)$. Let η be a cut-off function which vanishes near $\partial_p Q_{\lambda R}$. Using $\eta^2\boldsymbol{u}$ as a test function, we find (see e.g. [4] or [6])

$$\sup_{-R^2\le s\le 0}\int_\Omega\eta^2|\boldsymbol{u}(\cdot,s)|^2+\int_Q\eta^2|\nabla\times\boldsymbol{u}|^2\le N(\nu)\int_Q\left(|\eta_t|+|\nabla\eta|^2\right)|\boldsymbol{u}|^2.$$

On the other hand, from the vector identity

$$\nabla\times\nabla\times\boldsymbol{u}=-\Delta\boldsymbol{u}+\nabla(\nabla\cdot\boldsymbol{u}) \tag{3.1}$$

and the equation $\nabla\cdot\boldsymbol{u}=0$, we find

$$\int_Q\eta^2|\nabla\boldsymbol{u}|^2\le N\left[\int_Q\eta^2|\nabla\times\boldsymbol{u}|^2+\int_Q|\nabla\eta|^2|\boldsymbol{u}|^2\right].$$

Combining the above inequalities and choosing a suitable η, we finish the proof. □

Next lemma states that if $A(x,t)$ does not depend on time; i.e., $A=A(x)$, then any solution $\boldsymbol{u}$ of (1.2) satisfies $\boldsymbol{u}_t\in L^2_{\mathrm{loc}}(Q)$ and $\nabla\boldsymbol{u}\in L^{2,\infty}_{\mathrm{loc}}(Q)$.

Lemma 3.2. *Let $\boldsymbol{u}$ be a solution of* (1.2) *with $A=A(x)$. Let $R>0$ be such that $Q_{\lambda R}:=Q_{\lambda R}(z_0)\subset Q$ for some $\lambda>1$. Then, we have*

$$\int_{Q_R}|\boldsymbol{u}_t|^2+\sup_{t_0-R^2\le s\le t_0}\int_{B_R}|\nabla\boldsymbol{u}(\cdot,s)|^2\le\frac{N(\nu,\lambda)}{R^4}\int_{Q_{\lambda R}}|\boldsymbol{u}|^2.$$

Proof. As before, we assume $z_0=(0,0)$. Let η be a cut-off function vanishing near $\partial_p Q_{\sqrt{\lambda}R}$. Using $\eta^2\boldsymbol{u}_t$ as a test function, we find (see, e.g. [4])

$$\int_Q\eta^2|\boldsymbol{u}_t|^2+\sup_{-R^2\le s\le 0}\int_\Omega\eta^2|\nabla\times\boldsymbol{u}(\cdot,s)|^2\le N(\nu)\int_Q(|\eta_t|+|\nabla\eta|^2)|\nabla\boldsymbol{u}|^2.$$

Also, from (3.1) we find that

$$\int_\Omega\eta^2|\nabla\boldsymbol{u}(\cdot,s)|^2\le N\left[\int_\Omega\eta^2|\nabla\times\boldsymbol{u}(\cdot,s)|^2+\int_\Omega|\nabla\eta|^2|\boldsymbol{u}(\cdot,s)|^2\right].$$

Combining the above inequalities and applying Lemma 3.1 with Q_R and λ replaced by $Q_{\sqrt{\lambda}R}$ and $\sqrt{\lambda}$, respectively, we complete the proof. □

Next lemma says that if $A=A(x)$, then $\boldsymbol{u}_t$ belongs to $L^{2,\infty}_{\mathrm{loc}}(Q)$. This is where the result of the elliptic case proved in [3] is applied.

Lemma 3.3. *Let $\boldsymbol{u}$ be a solution of* (1.2) *with $A = A(x)$. Let $R > 0$ be such that $Q_{\lambda R} := Q_{\lambda R}(z_0) \subset Q$ for some $\lambda > 1$. Then, the following estimate holds:*

$$\sup_{t_0 - R^2 \le s \le t_0} \int_{B_R} |\boldsymbol{u}_t(\cdot, s)|^2 + \int_{Q_R} |\nabla \boldsymbol{u}_t|^2 \le \frac{N(\nu, \lambda)}{R^6} \int_Q |\boldsymbol{u}|^2 .$$

Proof. As before, we assume $z_0 = (0, 0)$. Taking the derivatives with respect to time in (1.2), we get

$$\boldsymbol{u}_{tt} + \nabla \times (A \nabla \times \boldsymbol{u}_t) = 0, \qquad \nabla \cdot \boldsymbol{u}_t = 0.$$

Applying Lemma 3.1 to $\boldsymbol{u}_t$, and then using Lemma 3.2 we find

$$\sup_{-R^2 \le s \le 0} \int_{B_R} |\boldsymbol{u}_t(\cdot, s)|^2 + \int_{Q_R} |\nabla \boldsymbol{u}_t|^2 \le \frac{N(\nu, \lambda)}{R^2} \int_{Q_{\sqrt{\lambda} R}} |\boldsymbol{u}_t|^2 \le \frac{N(\nu, \lambda)}{R^6} \int_{Q_{\lambda R}} |\boldsymbol{u}|^2 .$$

This completes the proof. □

The following lemma is adapted from Struwe [8]. One can easily verify it by following the proof of Lemma 3 in [8] line by line.

Lemma 3.4. *Let $\boldsymbol{u}$ be a solution of* (1.2). *Let $R > 0$ be such that $Q_{\lambda R} = Q_{\lambda R}(z_0) \subset Q$ for some $\lambda > 1$. Then, there exist a constant $N = N(\nu, \lambda)$ such that*

$$\int_{Q_R} |\boldsymbol{u} - \boldsymbol{u}_R|^2 \le N R^2 \int_{Q_{\lambda R}} |\nabla \boldsymbol{u}|^2 . \tag{3.2}$$

The next theorem is our first main result. The idea is to treat $\boldsymbol{u}_t$ as an inhomogeneous term and apply the elliptic result of [3]. This kind of approach is found, for example, in [7].

Theorem 3.1. *Let $\boldsymbol{u}$ be a solution of* (1.2) *with $A = A(x)$. Then, $\boldsymbol{u}$ is locally Hölder continuous in Q. Moreover, the following estimate holds: for any $R > 0$ such that $Q_{6R} = Q_{6R}(z_0) \subset Q$,*

$$[\boldsymbol{u}]_{C^{\alpha, \alpha/2}(Q_R)} \le N(\nu) R^{-(5+2\alpha)} \|\boldsymbol{u}\|_{L^2(Q_{6R})} ,$$

where $\alpha = \alpha(\nu) \in (0, 1/2]$.

Proof. We assume that $R = 1$ and $z_0 = (0, 0)$. The general case is recovered by a simple coordinate change $(x, t) \mapsto ((x - x_0)/R, (t - t_0)/R^2)$. From Lemma 3.3, we know that $\boldsymbol{u}_t \in L^{2,\infty}(Q_5)$. Also, we find that $\nabla \cdot \boldsymbol{u}_t = 0$ from the vector identity $\nabla \cdot \nabla \times \boldsymbol{F} = 0$. Hence, we can rewrite (1.2) as

$$\nabla \times (A(x) \nabla \times \boldsymbol{u}) = -\boldsymbol{u}_t \tag{3.3}$$

and apply the elliptic result [3, Theorem 2.1] to find that for any $t \in (-4^2, 0)$, we have

$$[\boldsymbol{u}(\cdot, t)]_{C^\alpha(Q_{4,t})} \le N(\nu) \left(\|\boldsymbol{u}(\cdot, t)\|_{L^2(Q_{5,t})} + \|\boldsymbol{u}_t(\cdot, t)\|_{L^2(Q_{5,t})} \right), \tag{3.4}$$

where $\alpha = \alpha(\nu) \in (0, 1/2]$ (see Remark 3.1 below).

From Lemma 3.1 and Lemma 3.3, the right hand side of (3.4) is uniformly bounded for $t \in (-4^2, 0)$ hence

$$[\boldsymbol{u}(\cdot, t)]_{C^\alpha(Q_{4,t})} \le N(\nu) \|\boldsymbol{u}\|_{L^2(Q_6)} \quad \forall t \in (-4^2, 0). \tag{3.5}$$

Fix $z = (x, t) \in Q_1$ and $r \le 1$. From Lemma 3.4, we have

$$\int_{Q_r(z)} |\boldsymbol{u} - \boldsymbol{u}_{z,r}|^2 \le N r^2 \int_{Q_{2r}(z)} |\nabla \boldsymbol{u}|^2 . \tag{3.6}$$

From (3.3), the following Caccioppoli type inequality holds (see [3, Lemma 4.4]).

$$\int_{Q_{2r,t}(z)} |\nabla \boldsymbol{u}|^2 \le N \left(\frac{1}{r^2} \int_{Q_{3r,t}(z)} |\boldsymbol{u} - \overline{\boldsymbol{u}}(t)|^2 + \|\boldsymbol{u}_t(\cdot,t)\|^2_{L^{6/5}(Q_{3r,t}(z))} \right),$$

where $\overline{\boldsymbol{u}}(t) = \fint_{Q_{3r,t}(z)} \boldsymbol{u}(\cdot,t)$. Note that, from (3.5),

$$\frac{1}{r^2} \int_{Q_{3r,t}(z)} |\boldsymbol{u} - \overline{\boldsymbol{u}}(t)|^2 \le N r^{1+2\alpha} [\boldsymbol{u}(\cdot,t)]^2_{C^\alpha(Q_{4,t})} \le N r^{1+2\alpha} \|\boldsymbol{u}\|^2_{L^2(Q_6)}.$$

Also, from Hölder's inequality and Lemma 3.3

$$\|\boldsymbol{u}_t(\cdot,t)\|^2_{L^{6/5}(Q_{3r,t}(z))} \le N r^2 \|\boldsymbol{u}_t(\cdot,t)\|^2_{L^2(Q_{3r,t}(z))} \le N r^2 \|\boldsymbol{u}\|^2_{L^2(Q_6)}.$$

Therefore, we have

$$\begin{aligned} \int_{Q_{2r}(z)} |\nabla \boldsymbol{u}|^2 &= \int_{t-4r^2}^{t} \int_{Q_{2r,t}(z)} |\nabla \boldsymbol{u}(y,s)|^2 \, dyds \\ &\le N r^2 \left(r^{1+2\alpha} + r^2 \right) \|\boldsymbol{u}\|^2_{L^2(Q_6)} \le N r^{3+2\alpha} \|\boldsymbol{u}\|^2_{L^2(Q_6)}, \end{aligned}$$

where we used $\alpha \le 1/2$ in the last step. Hence, using (3.6) we conclude

$$\fint_{Q_r(z)} |\boldsymbol{u} - \boldsymbol{u}_{z,r}|^2 \le N(\nu) r^{2\alpha} \|\boldsymbol{u}\|^2_{L^2(Q_6)}, \quad \forall r \le 1, \ \forall z \in Q_1.$$

Finally, from Lemma 2.1, we get $[\boldsymbol{u}]_{\alpha,\alpha/2,Q_1} \le N(\nu) \|\boldsymbol{u}\|_{L^2(Q_6)}$. This completes the proof. □

Remark 3.1. In [3, Theorem 2.1], $A(x)$ is assumed to be of the form $a(x)I$. However, the proof works for general $A(x)$ as long as $A(x)$ satisfies (1.1). Also, one can find in the proof that $\alpha \le \gamma$, where $\gamma = 2 - 3/2 = 1/2$ in our case.

Definition 3.1. Let $f : Q \to \mathbb{R}^N$ be measurable. For all $r > 0$ and $z = (x,t)$ such that $Q_r(z) \subset Q$, we define $\omega_f(r;z) := \operatorname{ess\,sup}_{(y,s)\in Q_r(z)} |f(y,t) - f(y,s)|$ and

$$\omega_f(r) := \sup \{ \omega_f(r;z) : \forall z \in Q \text{ such that } Q_r(z) \subset Q \}. \tag{3.7}$$

Lemma 3.5. *Let $\boldsymbol{u}$ be a solution of* (1.2). *Suppose there is a fixed $\tau \in (0,1)$ such that for all $R > 0$ satisfying $Q_R \subset Q$,*

$$[\boldsymbol{u}]_{C^{\alpha,\alpha/2}(Q_{\tau R})} \le N R^{-(5+2\alpha)/2} \|\boldsymbol{u}\|_{L^2(Q_R)}. \tag{3.8}$$

Then for all $0 < \rho \le r \le R$, we have the following estimate for $\nabla \boldsymbol{u}$:

$$\int_{Q_\rho} |\nabla \boldsymbol{u}|^2 \le N(\tau) \left(\frac{\rho}{r} \right)^{3+2\alpha} \int_{Q_r} |\nabla \boldsymbol{u}|^2. \tag{3.9}$$

Proof. We may assume $\rho < (\tau/4)r$, for (3.9) is obvious if $\rho \ge (\tau/4)r$. We denote $[\boldsymbol{u}]_{\alpha,r} := [\boldsymbol{u}]_{C^{\alpha,\alpha/2}(Q_r)}$. Applying the energy inequality to $\boldsymbol{u} - \boldsymbol{u}_{2\rho}$, which is also a solution of (1.2), we find (recall $2\rho \le (\tau/2)r$)

$$\int_{Q_\rho} |\nabla \boldsymbol{u}|^2 \le N \rho^{-2} \int_{Q_{2\rho}} |\boldsymbol{u} - \boldsymbol{u}_{2\rho}|^2 \le N \rho^{3+2\alpha} [\boldsymbol{u}]^2_{\alpha,(\tau/2)r}. \tag{3.10}$$

From (3.8) applied to $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_{r/2}$, and by Lemma 3.4,

$$[\boldsymbol{u}]^2_{\alpha,(\tau/2)r} = [\boldsymbol{v}]^2_{\alpha,(\tau/2)r} \le N r^{-5-2\alpha} \int_{Q_{r/2}} \left| \boldsymbol{u} - \boldsymbol{u}_{r/2} \right|^2 \le N r^{-3-2\alpha} \int_{Q_r} |\nabla \boldsymbol{u}|^2.$$

Putting the above inequalities together, we complete the proof. □

Theorem 3.2. *Let $\boldsymbol{u}$ be a solution of* (1.2) *with $A(x,t)$ satisfying:*

(**A**) $\lim_{\rho\to 0}\omega_A(\rho)=0$, *where ω_A is defined as in Definition 3.1.*

Then, $\boldsymbol{u}$ is locally Hölder continuous in Q. More precisely, let $\beta\in(0,\alpha)$, where α is the Hölder exponent in Theorem 3.1, *and let $Q'\Subset Q$. Then, there is Q'' such that $Q'\subset Q''\Subset Q$ and $[\boldsymbol{u}]_{C^{\beta,\beta/2}(Q')}\le N(\nu,\beta,\omega_A,Q',Q)\,\|\boldsymbol{u}\|_{L^2(Q'')}$.*

Proof. Let R_0 be a fixed number which will be specified later. Fix $z_0\in Q'$ and let $R>0$ be such that $2R\le R_0$ and $Q_{4R}=Q_{4R}(z_0)\subset Q$. We will show that $[\boldsymbol{u}]_{C^{\beta,\beta/2}(Q_R)}\le N(\nu,\beta,R)\,\|\boldsymbol{u}\|_{L^2(Q_{4R})}$. Then, the theorem will follow from a standard covering argument.

Fix $z=(x,t)\in Q_R$ and let $0<\rho<r\le R$. For any $z'=(y,t)\in Q_R$, denote $A_0(y)=A(y,t)$ and let $\boldsymbol{v}$ be a solution of the system

$$\boldsymbol{v}_t+\nabla\times[A_0\nabla\times\boldsymbol{v}]=0,\qquad \nabla\cdot\boldsymbol{v}=0\quad\text{in}\quad Q_{2r}(z)$$

with the boundary condition $\boldsymbol{v}=\boldsymbol{u}$ on $\partial_p Q_{2r}(z)$.

Let $\alpha=\alpha(\nu)$ be the Hölder exponent from Theorem 3.1. From (3.9) applied to $\boldsymbol{v}$, we have $\int_{Q_{2\rho}(z)}|\nabla\boldsymbol{v}|^2\le N(\rho/r)^{3+2\alpha}\int_{Q_{2r}(z)}|\nabla\boldsymbol{v}|^2$, and hence,

$$\int_{Q_{2\rho}(z)}|\nabla\boldsymbol{u}|^2\le N\left[\left(\frac{\rho}{r}\right)^{3+2\alpha}\int_{Q_{2r}(z)}|\nabla\boldsymbol{u}|^2+\int_{Q_{2r}(z)}|\nabla(\boldsymbol{u}-\boldsymbol{v})|^2\right].$$

Using the equations satisfied by $\boldsymbol{u}$ and $\boldsymbol{v}$, the function $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{v}$ satisfies

$$\boldsymbol{w}_t+\nabla\times[A_0\nabla\times\boldsymbol{w}]=\nabla\times[(A_0-A)\nabla\times\boldsymbol{u}],\qquad \nabla\cdot\boldsymbol{w}=0\quad\text{in}\quad Q_{2r}(z)$$

and $\boldsymbol{w}=0$ on $\partial_p Q_{2r}(z)$. By using $\boldsymbol{w}$ itself as a test function to the above equation we find $\int_{Q_{2r}(z)}|\nabla\boldsymbol{w}|^2\le N\omega_A(2r)\int_{Q_{2r}(z)}|\nabla\boldsymbol{u}|^2$ (recall $\nabla\cdot\boldsymbol{w}=0$) and hence

$$\int_{Q_{2\rho}(z)}|\nabla\boldsymbol{u}|^2\le N\left[\left(\frac{\rho}{r}\right)^{3+2\alpha}\int_{Q_{2r}(z)}|\nabla\boldsymbol{u}|^2+\omega_A(2r)\int_{Q_{2r}(z)}|\nabla\boldsymbol{u}|^2\right].$$

Now, choose R_0 such that $\omega_A(R_0)<\varepsilon_0$, where ε_0 is as in Lemma 2.2. From Lemma 3.4, Lemma 2.2, and Lemma 3.1, we get

$$\begin{aligned}\int_{Q_r(z)}|\boldsymbol{u}-\boldsymbol{u}_{z,r}|^2 &\le N(\nu)r^2\int_{Q_{2r}(z)}|\nabla\boldsymbol{u}|^2\\ &\le N(\nu,\beta,R)r^{5+2\beta}\int_{Q_{2R}}|\nabla\boldsymbol{u}|^2\le N(\nu,\beta,R)r^{5+2\beta}\int_{Q_{4R}}|\boldsymbol{u}|^2.\end{aligned}$$

Therefore, the theorem follows from Lemma 2.1. □

Let $A(x,t)\equiv a(t)B(x)$, where $a(t)$ is a positive, bounded scalar function and B is a symmetric matrix satisfying (1.1). The hypothesis (**A**) in Theorem 3.2 is not satisfied in this case unless $a(t)$ is assumed to be continuous. However, in such a case, it turns out that we don't need to impose any continuity assumptions on $a(t)$ to get the Hölder estimate for $\boldsymbol{u}$, as we will show next.

Theorem 3.3. *Let $\boldsymbol{u}$ be a solution of* (1.2) *with $A(x,t)=a(t)B(x)$, where $a(t)$ is a scalar function such that $\nu\le a(t)\le\nu^{-1}$ and $B(x)$ is a symmetric matrix satisfying* (1.1). *Then, for any $R>0$ such that $Q_{7R}=Q_{7R}(x_0)\subset Q$, $\boldsymbol{u}$ is Hölder continuous in Q_R and*

$$[\boldsymbol{u}]_{C^{\alpha,\alpha/2}(Q_R)}\le N(\nu)R^{-(5+2\alpha)/2}\,\|\boldsymbol{u}\|_{L^2(Q_{7R})},\tag{3.11}$$

where $\alpha = \alpha(\nu) \in (0, 1/2]$.

Proof. We may assume $z_0 = (0,0)$ and $R = 1$ as before. From the assumption that $A = a(t)B(x)$, we see that (1.2) becomes

$$\boldsymbol{u}_t + a(t)\nabla\times[B(x)\nabla\times\boldsymbol{u}] = 0, \qquad \nabla\cdot\boldsymbol{u} = 0. \tag{3.12}$$

Using $\eta^2 a^{-1} B\boldsymbol{u}_t$ as a test function in (3.12) and proceeding as in the proof of Lemma 3.2, we obtain the estimate

$$\int_{Q_6} |\boldsymbol{u}_t|^2 + \sup_{-6^2\le s\le 0}\int_{B_6} |\nabla\boldsymbol{u}(\cdot,s)|^2 \le N(\nu)\,\|\boldsymbol{u}\|^2_{L^2(Q_7)}. \tag{3.13}$$

In particular, we have $\boldsymbol{u}_t \in L^2(Q_6)$.

Next, by taking the curl in the equation (1.2) and denoting $\boldsymbol{w} = \nabla\times\boldsymbol{u}$, we get

$$\boldsymbol{w}_t + a\nabla\times[\nabla\times B\boldsymbol{w}] = 0. \tag{3.14}$$

Using $\eta^2 a^{-1} B\boldsymbol{w}_t$ as a test function in (3.14) and proceeding as above, we find

$$\int_{Q_5} |\boldsymbol{w}_t|^2 + \sup_{-5^2\le s\le 0}\int_{B_5} |\nabla\times B\boldsymbol{w}|^2 \le N(\nu)\int_{Q_6} |\nabla\times B\boldsymbol{w}|^2. \tag{3.15}$$

Since $\nabla\times B\boldsymbol{w} = -a^{-1}\boldsymbol{u}_t$ from (3.12), the right hand side of (3.15) is estimated as $\int_{Q_6} |\nabla\times B\boldsymbol{w}|^2 \le N(\nu)\,\|\boldsymbol{u}\|_{L^2(Q_7)}$ from (3.13). Hence, it follows that

$$\|\boldsymbol{u}_t\|_{L^{2,\infty}(Q_5)} \le N(\nu)\,\|\nabla\times B\boldsymbol{w}\|_{L^{2,\infty}(Q_5)} \le N(\nu)\,\|\boldsymbol{u}\|^2_{L^2(Q_7)}.$$

The rest of proof goes exactly as in that of Theorem 3.1, and therefore, the details are omitted. □

Corollary 3.1. *Let* $\beta \in (0,\alpha)$*, where* α *is the Hölder exponent in* Theorem 3.3*. Suppose* $\boldsymbol{u}$ *is a solution of* (1.2) *with* $A(x,t)$ *satisfying the following assumption: there exists* $A_0(x,t) = a(t)B(x)$ *such that*

($\mathbf{A}'$) $\quad |A(z) - A_0(z)| \le \varepsilon$ *for all* $z \in Q_{2R} \subset Q$*,*

where $a(t)$ *and* $B(x)$ *satisfy the assumptions in* Theorem 3.3*. Then, there exists* $\varepsilon_0 = \varepsilon_0(\nu,\beta)$ *such that if* $\varepsilon < \varepsilon_0$*,* $\boldsymbol{u}$ *is Hölder continuous in* Q_R *and*

$$[\boldsymbol{u}]_{C^{\beta,\beta/2}(Q_R)} \le N(\nu,\beta,R)\,\|\boldsymbol{u}\|_{L^2(Q_{2R})}.$$

Proof. From Theorem 3.3 and Lemma 3.5, we see that the estimate (3.9) holds. Hence, the proof of Theorem 3.2 works here. □

Remark 3.2. We may also consider the system (1.2) with an inhomogeneous term:

$$\left.\begin{aligned} \boldsymbol{u}_t + \nabla\times[A(x,t)\nabla\times\boldsymbol{u}] = \nabla\times\boldsymbol{f} \\ \nabla\cdot\boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in} \quad Q. \tag{3.16}$$

Let $\beta \in (0,\alpha)$, where α is the Hölder exponent in Theorem 3.1. Suppose $\boldsymbol{f} \in M^{2,3+2\beta}(Q)$ and $A(x,t)$ satisfies ($\mathbf{A}$) or ($\mathbf{A}'$). Let $\boldsymbol{u}$ be a solution of (3.16). It can be easily verified, by modifying the proof of Theorem 3.2, that $\boldsymbol{u}$ is Hölder continuous in $Q' \Subset Q$ and that it satisfies the following estimate:

$$[\boldsymbol{u}]_{C^{\beta,\beta/2}(Q')} \le N(\nu,\beta,Q',Q)\left(\|\boldsymbol{u}\|_{L^2(Q)} + \|\boldsymbol{f}\|_{M^{2,3+2\beta}(Q)}\right).$$

□

4. Applications

We first consider a certain system arising in a Maxwell's equations. Let a conductive material occupy a bounded domain $\Omega \subset \mathbb{R}^3$. Let $\boldsymbol{E}$ and $\boldsymbol{H}$ be the electric and the magnetic fields in Ω. Under certain assumptions, Yin derived the following mathematical model (see [9], [11]):

$$\left.\begin{aligned} (\epsilon \boldsymbol{E})_t + \sigma \boldsymbol{E} = \nabla\times\boldsymbol{H}, \\ (\mu \boldsymbol{H})_t + \nabla\times\boldsymbol{E} = 0, \\ \operatorname{div}(\mu\boldsymbol{H}) = 0, \end{aligned}\right\} \quad \text{in } Q, \tag{4.1}$$

where ϵ, μ, and σ are the electric permittivity, the magnetic permeability and the electric conductivity of the material, respectively. If the electrical displacement $\boldsymbol{D} = \epsilon\boldsymbol{E}$ is negligible and μ is a constant, the above system can be reduced to

$$\left.\begin{aligned} \boldsymbol{H}_t + \nabla\times[\varrho(x,t)\nabla\times\boldsymbol{H}] = 0, \\ \operatorname{div}\boldsymbol{H} = 0, \end{aligned}\right\} \quad \text{in } Q, \tag{4.2}$$

where ϱ is the resistivity of the material (see [9], [10]). The above system is the special case of the system (1.2), where $A = \varrho(x,t)I$ and I is the identity matrix. Therefore, our result implies that $\boldsymbol{H}$ is locally Hölder continuous in Q provided that $\varrho(x,t)I$ satisfies the assumption (A) or (A') of the previous section.

Theorem 4.1. *Let $\boldsymbol{H}$ be a weak solution of* (4.2). *Suppose that ϱI satisfies either (A) or (A'). Then $\boldsymbol{H}$ is locally Hölder continuous. In fact, the following estimate holds: for $R > 0$ such that $Q_{2R} = Q_{2R}(z_0) \subset Q$, there exists $\alpha \in (0, 1/2)$ such that*

$$[\boldsymbol{H}]_{C^{\alpha,\alpha/2}(Q_R)} \le N \|\boldsymbol{H}\|_{L^2(Q)}.$$

Next, we consider the case when temperature affects the electrical resistance. By taking the temperature effect into consideration, Yin derived the following mathematical model (see Yin [9], [10]):

$$\left.\begin{aligned} \boldsymbol{H}_t + \nabla\times[\varrho(u)\nabla\times\boldsymbol{H}] = 0 \\ \nabla\cdot\boldsymbol{H} = 0 \\ u_t - \Delta u = \varrho(u)\,|\nabla\times\boldsymbol{H}|^2 \end{aligned}\right\} \quad \text{in} \quad Q = \Omega\times(0,T), \tag{4.3}$$

where $\boldsymbol{H}$ and u are the magnetic field and the temperature of the material, respectively, and $\varrho(u)$ is the electrical resistivity of the material.

Let us assume that

$$\lim_{r\to 0} \omega_{\varrho(u)}(r) = 0, \tag{4.4}$$

where $\omega_{\varrho(u)}(r)$ is defined as in Definition 3.1. The assumption (4.4) above is an analogue of the assumption (A) in the previous section. Roughly speaking, this assumption is to say that the resistivity $\varrho(u)$ varies continuously in time, uniformly throughout the material Ω. For instance, the assumption (4.4) is satisfied if the scalar function ϱ is Lipschitz continuous and the temperature u is continuous in time variable so that $\lim_{r\to 0}\omega_u(r) = 0$. We will show that, under the assumption (4.4), a pair of weak solution $(\boldsymbol{H}, u)$ of the system (4.3) is locally Hölder continuous. We remark that in [10], the continuity of $\varrho(u)$ in both spatial and time directions is assumed to get the same result.

Theorem 4.2. *Let $(\boldsymbol{H}, u)$ be a pair of weak solutions of* (4.3). *Suppose that $\varrho(u)$ satisfies* (4.4). *Then $(\boldsymbol{H}, u)$ is locally Hölder continuous.*

The following lemmas will be used in the proof of Theorem 4.2 above. The proof of Lemma 4.1 below is standard (see, e.g. [8, Lemma 3]) and it will be omitted.

Lemma 4.1. *Let u be a solution of*

$$u_t - \Delta u = \nabla \cdot \boldsymbol{f} + g \quad in \quad Q. \tag{4.5}$$

Then the following estimate holds: for $R > 0$ such that $Q_{2R} \subset Q$

$$\int_{Q_R} |u - u_R|^2 \leq N \left(R^2 \int_{Q_{2R}} |\nabla u|^2 + R^2 \int_{Q_{2R}} |\boldsymbol{f}|^2 + R^4 \int_{Q_{2R}} |g|^2 \right).$$

Lemma 4.2. *Let u be a smooth solution of (4.5) in Q and let $R > 0$ such that $Q_{2R} \subset Q$. Assume that $\boldsymbol{f} \in M^{2,3+2\alpha}(Q_{2R})$ and $g \in M^{2,1+2\alpha}(Q_{2R})$. Then, $\nabla u \in M^{2,3+2\alpha}(Q_R)$ and*

$$\|\nabla u\|^2_{M^{2,3+2\alpha}(Q_R)} \leq N(R) \left(\|u\|^2_{L^2(Q_{2R})} + \|\boldsymbol{f}\|^2_{M^{2,3+2\alpha}(Q_{2R})} + \|g\|^2_{M^{2,1+2\alpha}(Q_{2R})} \right).$$

Sketch of proof. Fix $z \in Q_R$ and $r \in (0, R)$. Let v be the solution of

$$v_t - \Delta v = 0 \quad \text{in} \quad Q_r(z), \qquad v = u \quad \text{on} \quad \partial_p Q_r(z).$$

Then, $\int_{Q_\rho(z)} |\nabla v|^2 \leq N(\rho/r)^5 \int_{Q_r(z)} |\nabla v|^2$ for all $\rho \leq r$. Let $w := u - v$. Note that w satisfies $w_t - \Delta w = \nabla \cdot \boldsymbol{f} + g$ in $Q_r(z)$ and $w = 0$ on $\partial_p Q_r(z)$.

Denote $F = \|\boldsymbol{f}\|^2_{M^{2,3+2\alpha}(Q_{2R})}$ and $G = \|g\|^2_{M^{2,1+2\alpha}(Q_{2R})}$. One can easily check that $\int_{Q_r(z)} |\nabla w|^2 \leq N(F + G) r^{3+2\alpha}$. Then, using Lemma 2.2, one may conclude that $\int_{Q_r(z)} |\nabla u|^2 \leq N[(r/R)^{3+2\alpha} \int_{Q_R} |\nabla u|^2 + (F + G) r^{3+2\alpha}]$. On the other hand, from the energy inequality, $\int_{Q_R} |\nabla u|^2 \leq N(R^{-2} \int_Q |u|^2 + \int_{Q_{2R}} |\boldsymbol{f}|^2 + R^2 \int_{Q_{2R}} |g|^2)$. Therefore, we derived $r^{-3-2\alpha} \int_{Q_r(z)} |\nabla u|^2 \leq N(R)\{\|u\|^2_{2;Q_{2R}} + F + G\}$. $\square$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We may assume $\|\boldsymbol{H}\|_2 = \|\boldsymbol{H}\|_{2;Q} < \infty$. Fix $z_0 \in Q$ and let $R > 0$ be such that $Q_{8R} = Q_{8R}(z_0) \Subset Q$. It is enough to show that $(\boldsymbol{H}, u)$ is Hölder continuous in Q_R. From Theorem 3.2, there is $0 < \alpha < 1/2$ such that

$$[\boldsymbol{H}]_{C^{\alpha,\alpha/2}(Q_{8R})} \leq N \|\boldsymbol{H}\|_2,$$

where N is a constant depending on the given data. Thus, using the triangle inequality, we find that $\boldsymbol{H}$ is locally bounded and

$$\|\boldsymbol{H}\|_{L^\infty(Q_{4R})} \leq N \|\boldsymbol{H}\|_2. \tag{4.6}$$

It remains to show that u is Hölder continuous in Q_R. Using the following vector identity $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{G} - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})$ and (4.3), we observe that

$$\varrho(u) |\nabla \times \boldsymbol{H}|^2 = \nabla \cdot [\boldsymbol{H} \times (\varrho(u) \nabla \times \boldsymbol{H})] + \boldsymbol{H} \cdot \boldsymbol{H}_t,$$

and thus the last equation of (4.3) can be rewritten as

$$u_t - \Delta u = \nabla \cdot [\boldsymbol{H} \times (\varrho(u) \nabla \times \boldsymbol{H})] + \boldsymbol{H} \cdot \boldsymbol{H}_t =: \nabla \cdot \boldsymbol{f} + g. \tag{4.7}$$

Fix $z \in Q_R$ and $r \in (0, R)$. Using (4.6) and proceeding as in (3.10) in the proof of Lemma 3.5

$$\int_{Q_{2r}(z)} |\boldsymbol{f}|^2 \leq N \|\boldsymbol{H}\|_2^2 \int_{Q_{2r}(z)} |\nabla \boldsymbol{H}|^2 \leq N r^{3+2\alpha} \|\boldsymbol{H}\|_2^4. \tag{4.8}$$

Similarly, using (4.6) and applying Lemma 3.3 to $\boldsymbol{H}$ (recall $2\alpha \le 1$),

$$\int_{Q_{2r}(z)} |g|^2 \le N \|\boldsymbol{H}\|_2^2 \int_{t-4r^2}^{t} \int_{Q_{2r}(z)} |\boldsymbol{H}_t|^2 \le N r^{1+2\alpha} \|\boldsymbol{H}\|_2^4 . \tag{4.9}$$

Now, we apply Lemma 4.1 to (4.7) in $Q_{2r}(z)$ using (4.8) and (4.9)

$$\int_{Q_r(z)} |u - u_{z,r}|^2 \le N r^2 \int_{Q_{2r}(z)} |\nabla u|^2 + N r^{5+2\alpha} \|\boldsymbol{H}\|_2^4 .$$

Hence, taking the average

$$\fint_{Q_r(z)} |u - u_{z,r}|^2 \le N r^{-3} \int_{Q_{2r}(z)} |\nabla u|^2 + N r^{2\alpha} \|\boldsymbol{H}\|_2^4 . \tag{4.10}$$

Applying Lemma 4.2 to (4.7) and using (4.8) and (4.9), we conclude

$$r^{-3-2\alpha} \int_{Q_{2r}(z)} |\nabla u|^2 \le N \left(\|u\|_{L^2(Q_{4R})}^2 + \|\boldsymbol{H}\|_2^4 \right) . \tag{4.11}$$

Finally, by combining (4.10) and (4.11), we have

$$\fint_{Q_r(z)} |u - u_{z,r}|^2 \le N r^{2\alpha} \left(\|u\|_{L^2(Q_{4R})} + \|\boldsymbol{H}\|_{L^2(Q)}^2 \right)^2 .$$

Therefore, $[u]_{\alpha,\alpha/2;Q_R} < \infty$. This completes the proof. □

Next we consider the time-dependent Stokes system with measurable coefficients in diffusive term

$$\left. \begin{array}{c} \boldsymbol{u}_t - A(x,t)\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \operatorname{div} \boldsymbol{u} = 0 \end{array} \right\} \quad \text{in} \quad Q \equiv \mathbb{R}^3 \times (0,\infty). \tag{4.12}$$

Here $\boldsymbol{u}$ and p are the velocity field and the pressure associated with $\boldsymbol{u}$ and $\boldsymbol{f}$ is a given external force. $A(x,t)$ is a 3×3 symmetric matrix satisfying the ellipticity condition (1.1). The following theorem is the last application of our results from Section 3.

Theorem 4.3. *Let $\boldsymbol{f} \in M^{2,3+2\beta}(Q)$, where $0 < \beta < \alpha$ and α is the Hölder exponent in* Theorem 3.1. *Suppose $\boldsymbol{u}_t, \nabla p \in L^2(Q)$ and $u \in L^2\left((0,\infty); W^{2,2}(\mathbb{R}^3)\right)$ such that $\boldsymbol{u}$ solves the system* (4.12). *Suppose that $A(x,t)$ satisfies* (A) *or* (A$'$). *Then $\nabla \boldsymbol{u}$ is locally Hölder continuous and satisfies the following estimate: for $R > 0$ such that $Q_R = Q_R(z_0) \subset Q$, we have*

$$[\nabla \boldsymbol{u}]_{C^{\beta,\beta/2}(Q_R)} \le N \left(\|\nabla \boldsymbol{u}\|_{L^2(Q)} + \|\boldsymbol{f}\|_{M^{2,3+2\beta}(Q)} \right),$$

where $N = N(\nu, R)$.

Proof. By taking the curl in (4.12), we have the following equations:

$$\boldsymbol{w}_t + \nabla \times (A(x,t)\nabla\times\boldsymbol{w}) = \nabla \times \boldsymbol{f}, \qquad \boldsymbol{w} = \nabla\times\boldsymbol{u} \quad \text{in } Q,$$

where we used the identity $\Delta\boldsymbol{u} = -\nabla\times\nabla\times\boldsymbol{u} + \nabla(\nabla\cdot\boldsymbol{u})$. Then it is easy to see that $\boldsymbol{w}$ is locally Hölder continuous in Q and the following estimate holds:

$$[\boldsymbol{w}]_{C^{\beta,\beta/2}(Q_{2R})} \le N(\|\boldsymbol{w}\|_{L^2(Q_{4R})} + \|\boldsymbol{f}\|_{M^{2,3+2\beta}(Q_{4R})}),$$

where $N = N(\nu, R)$. Using the Biot-Savart law, $\boldsymbol{u}$ can be recovered in terms of $\boldsymbol{w}$ as follows:

$$\boldsymbol{u}(x,t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \boldsymbol{w}(y,t)}{|x-y|^3}\, dy.$$

It is easy to see that $\nabla \boldsymbol{u} \in C^{\beta,\beta/2}_{\mathrm{loc}}(Q)$ and the following estimate holds:

$$[\nabla \boldsymbol{u}]_{C^{\beta,\beta/2}(Q_R)} \le N[\boldsymbol{w}]_{C^{\beta,\beta/2}(Q_{2R})} \le N\left(\|\nabla \boldsymbol{u}\|_{L^2(Q)} + \|\boldsymbol{f}\|_{M^{2,3+2\beta}(Q)}\right).$$

This completes the proof. □

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