GEOMETRIC CHARACTERIZATION OF STRICTLY POSITIVE REAL REGIONS AND ITS APPLICATIONS

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Abstract: Strict positive realness (SPR) is an important concept in absolute stability theory, adaptive control, system identification, etc. This paper characterizes the strictly positive real (SPR) regions in coefficient space and presents a robust design method for SPR transfer functions. We first introduce the concepts of SPR regions and weak SPR regions and show that the SPR region associated with a fixed polynomial is unbounded, whereas the weak SPR region is bounded. We then prove that the intersection of several weak SPR regions associated with different polynomials can not be a single point. Furthermore, we show how to construct a point in the SPR region from a point in the weak SPR region. Based on these theoretical development, we propose an algorithm for robust design of SPR transfer functions. This algorithm works well for both low order and high order polynomial families. Illustrative examples are provided to show the effectiveness of this algorithm.

Keywords: Uncertain Systems, Robustness, Design, Strict Positive Realness, Transfer Functions, Polynomials.

1 Introduction

The notion of strict positive realness (SPR) of transfer functions plays an important role in absolute stability theory, adaptive control and system identification[1-23]. Motivated by Kharitonov's seminal theorem on the robust stability for a family of polynomials, a number of recent papers has concentrated on the strict positive realness for a family of transfer functions. In the spirit of Kharitonov, the robust SPR analysis and design problems were first formulated by Dasgupta and Bhagwat[8]. They showed that every transfer function in an interval transfer function family is strictly positive real if and only if sixteen prescibed vertex transfer functions in this family are strictly positive real. The sixteen critical vertex transfer functions can be constructed explicitly using Kharitonov's four vertex polynomials. This result was subsequently improved by Chapellat and Bhattacharyya, Wang and Huang, where only eight out of the sixteen critical vertex transfer functions need to be checked[11-12]. For a family of transfer functions with affine linearly correlated perturbations, or more generally, multilinearly correlated perturbations, Dasgupta, Anderson et al. showed that it suffices to check all vertices in order to ensure the strict positive realness of the entire family[14]. By resort to the concept of positive polynomial pairs and root interlacing properties, Hollot and Huang solved the robust SPR design problem for low order and structured families[9-10]. Anderson et al. considered the general robust SPR design problem, and by using the Hilbert transform, provided a constructive method[14]. Betser and Zeheb made some further improvements[15].

This paper characterizes SPR regions in coefficient space and presents a robust design method for SPR transfer functions. We first introduce the concepts of SPR regions and weak SPR regions and give a complete characterization of them. We show that the SPR region associated with a fixed polynomial is unbounded, whereas the weak SPR region is bounded. We then prove that the intersection of several weak SPR regions associated with different polynomials can not be a single point. Furthermore, we show how to construct a point in the SPR region from a point in the weak SPR region. Based on these theoretical development, we propose an algorithm for robust design of SPR transfer functions. This algorithm works well for both low order and high order polynomial families. Illustrative examples are provided to show the effectiveness of this algorithm.

2 Preliminaries

Denote P^n as the n-th order real polynomial family, R^n as the n dimensional real field, and $H^n \subset P^n$ as the set of all n-th order Hurwitz stable polynomials.

In the following definitions, $b(\cdot) \in P^m, a(\cdot) \in P^n$, and $p(s) = b(s)/a(s)$ is a rational function.

Definition 1 $p(s)$ is said to be strictly positive real(SPR),denoted as $p(s) \in$SPR,if $b(s) \in P^n, a(s) \in H^n$, and $\mathrm{Re}[p(j\omega)] > 0$, $\forall \omega \in R$.

Definition 2 $p(s)$ is said to be weak SPR (WSPR),denoted as $p(s) \in$WSPR,if $b(s) \in P^{n-1}, a(s) \in H^n$, and $\mathrm{Re}[p(j\omega)] > 0$, $\forall \omega \in R$.

Definition 3 Given $a(s) \in H^n$, the set of the coefficients (in R^{n+1}) of all the $b(s)$'s in P^n such that $p(s) := \dfrac{b(s)}{a(s)} \in$SPR is said to be the SPR region associated with $a(s)$,denoted as Ω_a.

Definition 4 Given $a(s) \in H^n$, the set of the coefficients (in R^n) of all the $b(s)$'s in P^{n-1} such that $p(s) := \dfrac{b(s)}{a(s)} \in$WSPR is said to be the WSPR region associated with $a(s)$,denoted as Ω_a^W.

For notational convenience, $\Omega_a(\Omega_a^W)$ sometimes also stands for the set of all the polynomials $b(s)$ in $P^n(P^{n-1})$ such that $p(s) := \dfrac{b(s)}{a(s)} \in$SPR(WSPR).

From the definitions above, it is easy to get the following properties:

Property 1[1,4,9,14,19] If $p(s) \in$SPR(WSPR), then $|arg(b(j\omega)) - arg(a(j\omega))| < \dfrac{\pi}{2}$, $\forall \omega \in R$, where $arg(\cdot)$ stands for the argument of the complex number,and the difference of two arguments can differ by an integer number of 2π.

Property 2[9,10] Given $a(s) \in H^n$, Ω_a is a non-empty, open, convex cone in R^{n+1}.

Property 3[10,11] Given $a(s) \in H^n$, we have $\Omega_a \subset H^n, \Omega_a^W \subset H^{n-1}$.

The problem we are interested in is: Given a family of Hurwitz stable polynomials, how can find a fixed polynomial such that their ratios will be SPR-invariant? In what follows, we will first give some characterization of WSPR regions, and then propose an efficient design procedure for this problem.

3 Geometric Characterization of SPR Regions

By definition, an SPR (WSPR) transfer function times a positive integer is still SPR (WSPR). Thus, without loss of generality, let

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n \tag{1}$$

Denote as Ω_{1a} the set of the coefficients of all the $b(s) = s^n + x_1 s^{n-1} + \cdots + x_n \in P^n$, i.e., $(x_1, x_2, \cdots, x_n)$ in R^n , such that $p(s) = \dfrac{b(s)}{a(s)} \in$SPR; and denote as Ω_{1a}^W the set of the coefficients of all the $b(s) = s^{n-1} + x_1 s^{n-2} + \cdots + x_{n-1} \in P^{n-1}$, i.e., $(x_1, x_2, \cdots, x_{n-1})$ in R^{n-1} , such that $p(s) = \dfrac{b(s)}{a(s)} \in$WSPR. Obviously, we have

$$\{1\} \times \Omega_{1a} = \left\{ (1, \frac{b_1}{b_0}, \frac{b_2}{b_0}, \cdots, \frac{b_n}{b_0}) \right| \left. \forall (b_0, b_1, b_2, \cdots, b_n) \in \Omega_a \right\} \tag{2}$$

$$\{1\} \times \Omega_{1a}^W = \left\{ (1, \frac{b_1}{b_0}, \frac{b_2}{b_0}, \cdots, \frac{b_{n-1}}{b_0}) \right| \left. \forall (b_0, b_1, b_2, \cdots, b_{n-1}) \in \Omega_a^W \right\} \tag{3}$$

For notational convenience, $\Omega_{1a}(\Omega_{1a}^W)$ sometimes also stands for the corresponding polynomial set.

As we know[9,10], Ω_a is a non-empty, open, convex cone in R^{n+1}. Thus, Ω_a is an unbounded set in R^{n+1}. In what follows, we will show that Ω_{1a} is also an unbounded set in R^n.

Theorem 1 Given $a(s) \in H^n$, Ω_{1a} is a non-empty, open, unbounded convex set in R^n.

Proof Obviously, we have $a(s) \in \Omega_a$. If the leading coefficient of $a(s)$ is a_0, then

$$(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \cdots, \frac{a_n}{a_0}) \in \Omega_{1a}.$$

Thus, Ω_{1a} is not empty.

Moreover, $(1, x_1, x_2, \cdots, x_n) \in \Omega_a, \forall (x_1, x_2, \cdots, x_n) \in \Omega_{1a}$. By Property 2, Ω_a is open. Thus, these exists $\delta > 0$, such that, when $\sqrt{(1-y_0)^2 + (x_1-y_1)^2 + \cdots + (x_n-y_n)^2} < \delta$, we have $(y_0, y_1, y_2, \cdots, y_n) \in \Omega_a$. For this δ, if $(z_1, z_2, \cdots, z_n) \in R^n$ satisfies

$$\sqrt{(x_1-z_1)^2 + \cdots + (x_n-z_n)^2} < \delta.$$

Then, we have

$$\sqrt{(1-1)^2 + (x_1-z_1)^2 + \cdots + (x_n-z_n)^2} < \delta.$$

Hence $(1, z_1, z_2, \cdots, z_n) \in \Omega_a$. Thus, we have $(z_1, z_2, \cdots, z_n) \in \Omega_{1a}$. Namely, Ω_{1a} is an open set.

By definition, Ω_{1a} is convex.

In what follows, we will prove that Ω_{1a} is unbounded. For this purpose, we first introduce some notations, which are needed in other proofs as well. Let

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n \tag{4}$$

$$b(s) = x_0 s^n + x_1 s^{n-1} + \cdots + x_n \in P^n \cup P^{n-1} \tag{5}$$

Then $\forall \omega \in R$, we have

$$\begin{aligned}\text{Re}[\frac{b(j\omega)}{a(j\omega)}] &= \frac{1}{|a(j\omega)|^2}\text{Re}[b(j\omega)a(-j\omega)] \\ &= \frac{1}{|a(j\omega)|^2}\sum_{l=0}^{n}(\sum_{k=0}^{n} a_k x_{2l-k}(-1)^{l+k})\omega^{2(n-l)} \\ &= \frac{1}{|a(j\omega)|^2}\sum_{l=0}^{n} c_l \omega^{2(n-l)}\end{aligned}$$

where $c_l := \sum_{k=0}^{n} a_k x_{2l-k}(-1)^{l+k}$, where $a_0 = 1$, and let $a_i = 0, x_i = 0$, when $i < 0$ or $i > n$, $l = 0, 1, \cdots, n$.

Introducing the matrices

$$H_a := \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & a_{2n-5} & \cdots & a_n \end{bmatrix},$$

$$E_n := \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{bmatrix},$$

$$A := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_2 & & & \\ a_4 & & E_n H_a E_n & \\ \vdots & & & \end{bmatrix},$$

$$b := \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix},$$

$$c := \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

where $a_i = 0$ when $i > n$. Then, it is easy to verify that

$$c = Ab \tag{6}$$

Since $a(s) \in H^n$, we know that $E_n H_a E_n$ is invertible. For any $d = [d_1, d_2, \cdots, d_n]^T \in R^n$ such that all elements of d are positive. Denote $\bar{a} = [-a_2, a_4, -a_6, a_8, \cdots, (-1)^n a_{2n}]^T$, where $a_i = 0$ when $i > n$. Let $\bar{b} = (E_n H_a E_n)^{-1}(d - \bar{a}) := [b_1, b_2, \cdots, b_n]^T$, Then obviously, we have $[1, d_1, d_2, \cdots, d_n]^T = A[1, b_1, b_2, \cdots, b_n]^T$, and $b_n = \dfrac{d_n}{a_n}$. By $c = Ab$, we know that $(b_1, b_2, \cdots, b_n) \in \Omega_{1a}$. On the other hand, due to the arbitrariness of d, d_n can be taken arbitrarily large. Therefore, b_n can also be arbitrarily large. Namely, Ω_{1a} is unbounded. This completes the proof.

Remark 1 Given two stable polynomials $a_1(s)$ and $a_2(s)$, existence of a polynomial $b(s)$ such that $\frac{b(s)}{a_1(s)}$ and $\frac{b(s)}{a_2(s)}$ are both SPR is tantamount to non-emptiness of the intersection of the two SPR regions associated with $a_1(s)$ and $a_2(s)$. Since Ω_a and Ω_{1a} are both unbounded sets. When dealing with robust SPR design problem, we must find the intersection of several unbounded sets (i.e., SPR regions), which is intractable. This is the reason that we introduce the concept of WSPR regions, which are bounded as shown below.

Theorem 2 Given $a(s) \in H^n$, Ω_{1a}^W is a non-empty, bounded convex set in R^{n-1}.

Proof

H_a, E_n and A were defined in the proof of Theorem 1.

Denote B as the $(n-1) \times (n-1)$ matrix formed by the first $n-1$ row and last $n-1$ column of the matrix $E_n H_a E_n$. Obviously, B is also invertible.

Denote $\bar{a} := [a_1, -a_3, a_5, -a_7, \cdots, (-1)^{n-1} a_{2(n-1)+1}]^T$ ($a_i = 0$ when $i > n$).

Let $\bar{b} := -B^{-1}\bar{a} = [b_1, b_2, \cdots, b_{n-1}]^T$. Since $a(s) \in H^n$, it is easy to verify that $b_{n-1} > 0$. Denote $b = [0, 1, b_1, b_2, \cdots, b_{n-1}]^T$. Let $c_0 = c_1 = \cdots = c_{n-1} = 0$, $c_n = a_n b_{n-1}$, in $c := [c_0, c_1, \cdots, c_n]$. Then, it is easy to verify that $c = Ab$ is true. Thus, we have $(b_1, b_2, \cdots, b_{n-1}) \in \Omega_{1a}^W$, Namely, Ω_{1a}^W is not empty.

By definition, Ω_{1a}^W is convex.

We now prove that Ω_{1a}^W is bounded.

For any $(x_1, x_2, \cdots, x_{n-1}) \in \Omega_{1a}^W$, we have

$$\frac{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}{s^n+a_1 s^{n-1}+\cdots+a_n} \in WSPR,$$

By Property 3, $s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1} \in H^{n-1}$. Moreover, $\forall \omega \in R$, we have

$$\mathrm{Re}\left(\frac{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}{s^n+a_1 s^{n-1}+\cdots+a_n}|_{s=j\omega}\right) > 0. \qquad (7)$$

Thus

$$\mathrm{Re}\left(\frac{s^n+a_1 s^{n-1}+\cdots+a_n}{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}|_{s=j\omega}\right) > 0, \quad \forall \omega \in R. \qquad (8)$$

Obviously

$$\frac{s^n+a_1 s^{n-1}+\cdots+a_n}{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}} = s + \frac{(a_1-x_1)s^{n-1}+(a_2-x_2)s^{n-2}+\cdots+(a_{n-1}-x_{n-1})s+a_n}{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}$$

Therefore

$$\begin{aligned} &Re\left(\frac{(a_1-x_1)s^{n-1}+(a_2-x_2)s^{n-2}+\cdots+(a_{n-1}-x_{n-1})s+a_n}{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}|_{s=j\omega}\right) \\ &= Re\left(\frac{s^n+a_1 s^{n-1}+\cdots+a_n}{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}|_{s=j\omega}\right) - Re(j\omega) \\ &= \mathrm{Re}\left(\frac{s^n+a_1 s^{n-1}+\cdots+a_n}{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}|_{s=j\omega}\right) > 0, \quad \forall \omega \in R. \end{aligned}$$

It is easy to see that

$$\frac{(a_1-x_1)s^{n-1}+(a_2-x_2)s^{n-2}+\cdots+(a_{n-1}-x_{n-1})s+a_n}{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}} \in \{\mathrm{SPR}\} \cup \{\mathrm{WSPR}\}.$$

Again, by Property 3, we have

$$(a_1-x_1)s^{n-1}+(a_2-x_2)s^{n-2}+\cdots +(a_{n-1}-x_{n-1})s+a_n \in H^{n-1} \cup H^{n-2} \qquad (9)$$

Hence

$$0 < x_1 \le a_1, 0 < x_2 < a_2, \cdots, 0 < x_{n-1} < a_{n-1} \qquad (10)$$

Namely

$$\Omega_{1a}^W \subset \{(x_1, x_2, \cdots, x_{n-1}) | \alpha(s) := \sum_{i=1}^{n} (a_i - x_i) s^{n-i} \in H^{n-1} \cup H^{n-2}, \text{where} \quad x_n = 0\}$$

$$\subset \{(x_1, x_2, \cdots, x_{n-1}) | 0 < x_1 \le a_1, 0 < x_2 < a_2, \cdots, 0 < x_{n-1} < a_{n-1}\}.$$

Thus, Ω_{1a}^W is bounded. This completes the proof.

Remark 2 It should be pointed out that Ω_{1a}^W is not an open set in R^{n-1}. In fact, from the proof of Theorem 2, we know that Ω_{1a}^W is tangent to the hyperplane $x_1 = a_1$ in R^{n-1}. And there exist some points of Ω_{1a}^W in this hyperplane. Thus, Ω_{1a}^W can not be an open set. Obviously,Ω_a^W is a non-empty, convex cone in R^{n-1}. Thus, Ω_a^W is also unbounded. One may be tempted to believe that Ω_a^W is not an open set either. Though Ω_{1a}^W is not an open set. The following theorem guarantees such a fact: when the intersection of two or more WSPR regions is not empty, then the intersection must be a region, not a single point. This means that Ackermann's counterexample (that the unstable region is an isolated point so that gridding the parameter space can lead to erroneous conclusions no matter how dense the gridding is[19]) does not happen in this case.

Theorem 3 Given $a(s) \in H^n$, if $(x_1, x_2, \cdots, x_{n-1}) \in \Omega_{1a}^W$, then for sufficiently small $\varepsilon > 0$, we have $(x_1 - \varepsilon, x_2 - \varepsilon, \cdots, x_{n-1} - \varepsilon) \in \Omega_{1a}^W$.

Proof $\forall (x_1, x_2, \cdots, x_{n-1}) \in \Omega_{1a}^W$, and $\forall \omega \in R$, we have

$$\mathrm{Re}\left(\frac{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}{s^n+a_1 s^{n-1}+\cdots+a_n}|_{s=j\omega}\right) > 0. \qquad (11)$$

$\forall \varepsilon > 0$, since

$$\begin{aligned} &Re\left(\frac{s^{n-1}+(x_1-\varepsilon)s^{n-2}+\cdots+(x_{n-1}-\varepsilon)}{s^n+a_1 s^{n-1}+\cdots+a_n}|_{s=j\omega}\right) \\ &= Re\left(\frac{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}{s^n+a_1 s^{n-1}+\cdots+a_n}|_{s=j\omega}\right) \\ &\quad + Re\left(\frac{(-\varepsilon)(s^{n-2}+s^{n-3}+\cdots+1)}{s^n+a_1 s^{n-1}+\cdots+a_n}|_{s=j\omega}\right) \\ &= Re\left(\frac{s^{n-1}+x_1 s^{n-2}+\cdots+x_{n-1}}{s^n+a_1 s^{n-1}+\cdots+a_n}|_{s=j\omega}\right) \\ &\quad + \frac{(-\varepsilon)}{|a(j\omega)|^2}(-\omega^{2(n-1)}+\tilde{c}(\omega)), \end{aligned}$$

where $\tilde{c}(\omega)$ is a real polynomial of order less or equal to $2(n-2)$. Thus, when $|\omega|$ is sufficiently large, the sign of $(-\varepsilon)(-\omega^{2(n-1)}+\tilde{c}(\omega))$ will be positive. Namely, there exists $\omega_1 > 0$ such that, for all $|\omega| \ge \omega_1$,

$$\mathrm{Re}(\frac{s^{n-1}+(x_1-\varepsilon)s^{n-2}+\cdots+(x_{n-1}-\varepsilon)}{s^n+a_1s^{n-1}+\cdots+a_n}|_{s=j\omega})>0. \quad (12)$$

Denote

$$M_1=\inf_{|\omega|\leq\omega_1}\mathrm{Re}(\frac{s^{n-1}+x_1s^{n-2}+\cdots+x_{n-1}}{s^n+a_1s^{n-1}+\cdots+a_n}|_{s=j\omega}), \quad (13)$$

$$N_1=\sup_{|\omega|\leq\omega_1}|\ \mathrm{Re}(\frac{1}{|a(j\omega)|^2}(\omega^{2(n-1)}-\tilde{c}(\omega))\ | \quad (14)$$

Then $M_1>0$ and $N_1>0$. Choosing $0<\varepsilon<\frac{M_1}{N_1}$, by simple computation, we have

$$\mathrm{Re}(\frac{s^{n-1}+(x_1-\varepsilon)s^{n-2}+\cdots+(x_{n-1}-\varepsilon)}{s^n+a_1s^{n-1}+\cdots+a_n}|_{s=j\omega})>0,\forall\omega\in R \quad (15)$$

Therefore,

$$\frac{s^{n-1}+(x_1-\varepsilon)s^{n-2}+\cdots+(x_{n-1}-\varepsilon)}{s^n+a_1s^{n-1}+\cdots+a_n}\in WSPR,$$

namely $(x_1-\varepsilon,x_2-\varepsilon,\cdots,x_{n-1}-\varepsilon)\in\Omega_{1a}^W$. This completes the proof.

The following theorem shows the relationship between Ω_{1a}^W and Ω_a, and plays an important role in robust SPR design.

Theorem 4 Given $a(s)\in H^n$, if $(x_1,x_2,\cdots,x_{n-1})\in\Omega_{1a}^W$, then $\forall(1,\alpha_1,\alpha_2,\cdots,\alpha_n)\in R^{n+1}$, we can take sufficiently small $\varepsilon>0$ such that $(0,1,x_1,x_2,\cdots,x_{n-1})+\varepsilon(1,\alpha_1,\alpha_2,\cdots,\alpha_n)\in\Omega_a$.

Proof Denote

$$b(s)=s^{n-1}+x_1s^{n-2}+\cdots+x_{n-1}, \quad (16)$$

$$\alpha(s)=s^n+\alpha_1s^{n-1}+\cdots+\alpha_n, \quad (17)$$

$$\tilde{b}(s)=b(s)+\varepsilon\alpha(s). \quad (18)$$

Since $(x_1,x_2,\cdots,x_{n-1})\in\Omega_{1a}^W$, we have

$$\mathrm{Re}(\frac{b(j\omega)}{a(j\omega)})>0,\forall\omega\in R. \quad (19)$$

We only need to show that, for sufficiently small $\varepsilon>0$,

$$\mathrm{Re}(\frac{\tilde{b}(j\omega)}{a(j\omega)})>0,\forall\omega\in R \quad (20)$$

Obviously, $\tilde{b}(s)$ and $a(s)$ have same order n. Thus, there exists $\omega_2>0$ such that, for all $|\ \omega\ |\geq\omega_2$, we have $\mathrm{Re}(\frac{\tilde{b}(j\omega)}{a(j\omega)})>0$.

Denote

$$M_2=\inf_{|\omega|\leq\omega_2}\mathrm{Re}(\frac{b(j\omega)}{a(j\omega)}),\quad N_2=\sup_{|\omega|\leq\omega_2}|\ \mathrm{Re}(\frac{\alpha(j\omega)}{a(j\omega)})\ | \quad (21)$$

Then $M_2>0$ and $N_2>0$. Choosing $0<\varepsilon<\frac{M_2}{N_2}$, by simple computation, we have

$$\mathrm{Re}(\frac{\tilde{b}(j\omega)}{a(j\omega)})>0,\forall\omega\in R \quad (22)$$

This completes the proof.

4 Applications in Robust Design of SPR Transfer Functions

Generally speaking, the design problem is more difficult than analysis problem, since it is usually constructive, i.e., it not only shows the existence of the solution, but also provides a constructive procedure to find it. In this section, we will propose an algorithm for robust design of SPR transfer functions. This algorithm works well for both low order and high order polynomial families. Illustrative examples are provided to show the effectiveness of this algorithm.

Suppose

$$F=\{a_i(s)=s^n+\sum_{l=1}^{n}a_l^{(i)}s^{n-l},i=1,2,\cdots,m.\} \quad (23)$$

How can we find a polynomial $b(s)$, such that $p_i(s):=\frac{b(s)}{a_i(s)}\in$SPR,$i=1,2,\cdots,m$?

As observed earlier, existence of such a polynomial $b(s)$ boils down to the condition that the intersection of the SPR regions associated with $a_i(s)$ is not empty. From the results in the previous section, we know that SPR regions are unbounded, whereas WSPR regions are bounded. Thus, by a computational consideration, we first consider the intersection of WSPR regions, and then construct a polynomial $b(s)$ by using the technique presented in the previous section.

Since SPR(WSPR) transfer functions with fixed numerator (or denominator) enjoy convexity property, namely, if there exists a polynomial $c(s)$, such that $\frac{c(s)}{a(s)}$ and $\frac{c(s)}{b(s)}$ are both SPR(WSPR),then, it is easy to verify that, for any $\alpha \geq 0, \beta \geq 0$ and $(\alpha, \beta) \neq (0,0)$, we have $\frac{c(s)}{\alpha a(s) + \beta b(s)}$ $\in$SPR(WSPR). Therefore, the assumptions made on F do not lose any generality. Actually, the method proposed in our paper also applies to convex combination of polynomials, interval polynomials, and more generally, polytopic polynomials and multilinearly perturbed polynomials[1,4,14].

By the results presented in the previous section, we propose the following design procedure:

Step 1. Test the robust stability of the convex hull of F, i.e., $\overline{F}$. If $\overline{F}$ is robustly stable, then go to Step 2; otherwise, print "there does not exist such a $b(s)$ "; (by Definitions 1 and 2)

Step 2. Let $\alpha_l = \min\{a_l^{(i)}, i = 1, 2, \cdots, m\}, l = 1, 2, \cdots, n-1$. Gridding the hyperrectangle

D:=

$$\{(x_1, x_2, \cdots, x_{n-1}) \mid 0 < x_l < \alpha_l, l = 1, 2, \cdots, n-1\} \tag{24}$$

according to the precision required; (by Theorem 2 and its proof)

Step 3. Take $b := (b_1, b_2, \cdots, b_{n-1})$ at each gridding point. Test if b belongs to $\cap_{i=1}^m \Omega_{1a_i}^W$ by the following steps:

1) Test if the $(n-1)$-th order polynomial with coefficients $(1, b_1, b_2, \cdots, b_{n-1})$ belongs to H^{n-1} (by Property 3),

2) For $i = 1, 2, \cdots, m$, test if the polynomial with coefficients $(a_1^{(i)} - b_1, a_2^{(i)} - b_2, \cdots, a_{n-1}^{(i)} - b_{n-1}, a_n^{(i)})$ belongs to $H^{n-1} \cup H^{n-2}$, respectively (by Theorem 2 and its proof),

3) Test if b belongs to $\Omega_{1a_i}^W, i = 1, 2, \cdots, m$.

If all 1), 2), 3) above are satisfied, go to Step 4; otherwise, move to the next gridding point and test 1), 2), 3) again; (If all gridding points have been tested, then print "there does not exist such a b in $\cap_{i=1}^m \Omega_{1a_i}^W$ with the given precision").

Step 4. Take a sufficiently small $\varepsilon > 0$ such that $(\varepsilon, 1, b_1, b_2, \cdots, b_{n-1}) \in \cap_{i=1}^m \Omega_{a_i}$. Hence, the n-th order polynomial with coefficients $(\varepsilon, 1, b_1, b_2, \cdots, b_{n-1})$ satisfies the design requirement. (by Theorem 4).

For the low order stable interval polynomial family or low order stable convex combination, existence of the solution to the design problem is always guaranteed[15,16]. Given adequate precision, our method will surely find a polynomial that satisfies the design requirement. As shown by numerous examples below, our method is also effective for higher order polynomial families.

Example 1

Let

$$\begin{aligned} F = \{ \quad & a_1(s) = s^4 + 11s^3 + 56s^2 + 88s + 1, \\ & a_2(s) = s^4 + 11s^3 + 56s^2 + 88s + 50, \\ & a_3(s) = s^4 + 89s^3 + 56s^2 + 88s + 1, \\ & a_4(s) = s^4 + 89s^3 + 56s^2 + 88s + 50\} \end{aligned} \tag{25}$$

the methods proposed in [8-10,14,15] do not work here. Using our method, it is easy to get $b(s) = s^3 + 3.3s^2 + 2.24s + 1.76 \in \cap_{i=1}^4 \Omega_{1a_i}^W$. Then let $c(s) := \varepsilon s^4 + b(s)$, where $\varepsilon > 0$ is sufficiently small, e.g., let $\varepsilon \leq 0.3$ (which is determined by Theorem 4),it is easy to check that the design requirement has been met.

Note that the example above is constructed by overbounding the line segment in [13] by an interval polynomial family. Thus, instead of dealing with two vertex polynomials as in [13], we must now deal with four Kharitonov's vertex polynomials.

In what follows, we will give two more examples of higher order polynomial families.

Example 2

Let

$$\begin{aligned} & F = \{ \\ & a_1(s) = s^7 + 9s^6 + 31s^5 + 71.5s^4 + 111.5s^3 + 109s^2 + 76s + 12.5, \\ & a_2(s) = s^7 + 9.5s^6 + 31s^5 + 71s^4 + 111.5s^3 + 109.5s^2 + 76s + 12, \\ & a_3(s) = s^7 + 9s^6 + 31.5s^5 + 71.5s^4 + 111s^3 + 109s^2 + 76.5s + 12.5, \\ & a_4(s) = s^7 + 9.5s^6 + 31.5s^5 + 71s^4 + 111s^3 + 109.5s^2 + 76.5s + 12\} \end{aligned} \tag{26}$$

It is easy to see that the convex hull $\overline{F}$ of F is robust stable. Using our method, it is easy to get $b(s) = s^6 + 7.2s^5 + 18.6s^4 + 42.6s^3 + 44.4s^2 + 43.6s + 15.2 \in \cap_{i=1}^4 \Omega_{1a_i}^W$.

Then let $c(s) := \varepsilon s^7 + b(s)$, where $\varepsilon > 0$ is sufficiently small, e.g., let $\varepsilon \leq 0.1$, it is easy to check that the design requirement has been met.

Example 3

Let

$$F=\{\quad a_1(s)=s^9+11s^8+52s^7+145s^6+266s^5+331s^4+280s^3+155s^2+49s+6,$$
$$a_2(s)=s^9+11s^8+52s^7+146s^6+265.5s^5+332s^4+278.5s^3+151s^2+48s+2\} \quad (27)$$

It can be verified that $a_2(s)-a_1(s)=s^6-0.5s^5+s^4-1.5s^3-4s^2-s-4$ satisfies the extended Alternating Hurwitz Minor Condition[7,21,22]namely, it is a convex direction for Hurwitz stability[7,21,22]. Moreover, it is easy to see that $a_1(s)$ and $a_2(s)$ are both Hurwitz stable polynomials. Thus, the convex hull $\overline{F}$ of F is robust stable[7,21,22]. Using our method, it is easy to get $b(s)=s^8+8.8s^7+41.6s^6+87s^5+159.3s^4+132.4s^3+111.4s^2+30.2s+9.6\in\Omega^W_{1a_1}\cap\Omega^W_{1a_2}$. Thus, let $c(s):=\varepsilon s^9+b(s),\varepsilon>0,\varepsilon$ sufficiently small, e.g., take $\varepsilon\le 0.07$, then the design requirement has been met.

Note that our design method is also effective when dealing with discrete time systems. Note also that, in the Examples 1-3, $b(s)$ is not unique. Using our method, we can get all such $b(s)$'s with given precision.

It should be pointed out that there is hardly any example with order higher than 4 in the literature. Recently, a sixth-order example of interval family was given in [23] as follows. Unfortunately, this example is incorrect.

Example 4
Suppose

$$\begin{aligned}F=\{\\ a_1(s)&=s^6+0.8s^5+58.06s^4+50.9s^3+1028.5s^2+163.82s+1042.5,\\ a_2(s)&=s^6+1.5s^5+58.06s^4+28.3s^3+1028.5s^2+376.36s+1042.5,\\ a_3(s)&=s^6+0.8s^5+68.62s^4+50.9s^3+755.47s^2+163.82s+3286.7,\\ a_4(s)&=s^6+1.5s^5+68.62s^4+28.3s^3+755.47s^2+376.36s+3286.7\},\end{aligned} \quad (28)$$

find a polynomial $b(s)$, such that

$$p_i(s):=\frac{b(s)}{a_i(s)}\in SPR, i=1,2,3,4.$$

By Definition 1 , Definition 2 and Property 3, a prerequisite of the robust SPR design problem is that the convex hull $\overline{F}$ of F is robustly stable. But it is easy to check that $\overline{F}$ is not robustly stable. In fact, $(1.0446\pm 5.8969i)$ are roots of $a_1(s)$ with positive real part; $(1.037\pm 4.9835i)$ are roots of $a_2(s)$ with positive real part; $(0.03291\pm 7.5026i)$ and $(0.68089\pm 2.4933i)$ are roots of $a_3(s)$ with positive real part; $(0.87123\pm 2.867i)$ are roots of $a_4(s)$ with positive real part. Thus, it does not make sense to consider the robust SPR design in this case.

It should also be pointed out that, for the vertex set of a general polytopic polynomial family $F=\{a_i(s)=s^n+\sum_{l=1}^n a_l^{(i)}s^{n-l}, i=1,2,\cdots,m.\}$, even if $\overline{F}$ is robustly stable,it is still possible that there does not exist a polynomial $c(s)$, such that, $c(s)/f(s)\in$WSPR, for all $f(s)\in\overline{F}$. Namely $\cap_{i=1}^m\Omega^W_{1a_i}=\phi$.

To see this, let us look at an example of a third order triangle polynomial family.

Proposition 1[15,16,18]

Let $a(s)=s^3+a_1s^2+a_2s+a_3\in H^3$, then

$$\begin{aligned}\Omega^W_{1a}=\;&\{(x,y)|a_2^2x^2+2(2a_3-a_1a_2)xy+a_1^2y^2\\&-2a_2a_3x-2a_1a_3y+a_3^2<0\}\\&\cup\{(x,y)|x\le a_1, a_2x-a_1y-a_3\ge 0, y>0\}.\end{aligned}$$

Example 5

Let

$$\begin{aligned}F=\{\quad &a_1(s)=s^3+2.6s^2+37s+64,\\ &a_2(s)=s^3+17s^2+83s+978,\\ &a_3(s)=s^3+15s^2+28s+415\}\end{aligned} \quad (29)$$

it is easy to verify that $a_i(s), i=1,2,3$, are Hurwitz stable. Moreover, all edges of $\overline{F}$, i.e., $\lambda a_i(s)+(1-\lambda)a_j(s),\lambda\in[0,1], i,j=1,2,3$, are also Hurwitz stable. Therefore, by Edge Theorem[1,6,7,20], $\overline{F}$ is robustly stable. On the other hand, by a direct computation using Proposition 1, we have $\Omega^W_{1a_1}\cap\Omega^W_{1a_2}\cap\Omega^W_{1a_3}=\phi$. Henceforth, there does not exist a polynomial $c(s)$ such that $c(s)/a_i(s)\in$ WSPR, $i=1,2,3$ (although $\Omega^W_{1a_1}\cap\Omega^W_{1a_2}\ne\phi,\Omega^W_{1a_1}\cap\Omega^W_{1a_3}\ne\phi$, and $\Omega^W_{1a_2}\cap\Omega^W_{1a_3}\ne\phi$).

Note that, in this example, though we have $\Omega^W_{1a_1}\cap\Omega^W_{1a_2}\cap\Omega^W_{1a_3}=\phi$, we do not know whether $\Omega_{a_1}\cap\Omega_{a_2}\cap\Omega_{a_3}$ is an empty set or not. This is a problem deserving further study.

For a fourth (or higher) order stable interval polynomial family (or stable convex combination of two polynomials), does there exist a polynomial such that their ratios are SPR-invariant? This is still an open problem[1,8,9,13,14]. From our numerous examples, it seems that such a polynomial can always be found.

Thus, we conjecture that this problem has a positive answer.

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