ON LAURICELLA AND RELATED FUNCTIONS OF MATRIX ARGUMENTS - II

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ABSTRACT

In this paper we have proved six results – one each for the three Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$ and $F_D^{(n)}$ of matrix arguments, two for the function $\Phi_2^{(n)}$ and one for the function $\Phi_D^{(n)}$ of matrix arguments.

INTRODUCTION

The Lauricella functions of matrix arguments have been studied extensively by Mathai [4,5,6] and to some extent by Mathai and Pederzoli[7,8].We have already given the Mathai's definitions for the four Lauricella functions of matrix arguments in our previous papers [11,12].Here we are proving further results for the Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$ and $F_D^{(n)}$ and also for the functions $\Phi_2^{(n)}$ and $\Phi_D^{(n)}$ of matrix arguments. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [3,4].

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1.The Lauricella Functions of Matrix Arguments

THEOREM 1.1:

$$F_A^{(n)}(a,b_1,\cdots,b_n;c_1,\ldots,c_n;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(a)\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\int_{S>0}\int_{T_1>0}\cdots\int_{T_n>0} e^{-tr(S+T_1+\cdots+T_n)}\times$$
$$|S|^{a-(p+1)/2}|T_1|^{b_1-(p+1)/2}\cdots|T_n|^{b_n-(p+1)/2}{}_0F_1(\;;c_1;-S^{1/2}T_1^{1/2}X_1T_1^{1/2}S^{1/2})\times$$
$$\cdots{}_0F_1(\;;c_n;-S^{1/2}T_n^{1/2}X_nT_n^{1/2}S^{1/2})dSdT_1\cdots dT_n \qquad \cdots\cdots(1.1)$$

for $\mathrm{Re}(a,b_1,\cdots,b_n)>(p-1)/2.$

PROOF: Taking the M-transform of the right side of eq. (1.1) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$${}_0F_1(\;;c_1;-S^{1/2}T_1^{1/2}X_1T_1^{1/2}S^{1/2})\cdots\times$$
$${}_0F_1(\;;c_n;-S^{1/2}T_n^{1/2}X_nT_n^{1/2}S^{1/2})dX_1\cdots dX_n \qquad \cdots\cdots(1.2)$$

Applying the transformations

$$Y_j=S^{1/2}T_j^{1/2}X_jT_j^{1/2}S^{1/2},\ \text{whence},\ dY_j=|S|^{(p+1)/2}|T_j|^{(p+1)/2}dX_j$$

and $|Y_j|=|S||T_j||X_j|$ for $j=1,\cdots,n;$

to the expression (1.2) and then using the M- transform of a ${}_0F_1$ function (eq.(2.3.5) page 38 of Mathai [4]) gives us,

$$|S|^{-\rho_1-\cdots-\rho_n}|T_1|^{-\rho_1}\cdots|T_n|^{-\rho_n}\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)}{\Gamma_p(c_1-\rho_1)}\cdots\frac{\Gamma_p(c_n)\Gamma_p(\rho_n)}{\Gamma_p(c_n-\rho_n)} \qquad \cdots\cdots(1.3)$$

Substituting this expression on the right side of eq. (1.1) and then integrating out the variables S, $T_1, \cdots, T_n$ in the resulting expression by using a Gamma integral produces $M(F_A^{(n)})$ as given eq. (1.2) of the authors' paper [12]. Thus the theorem is proved.

THEOREM 1.2:

$$F_B^{(n)}(a_1,\cdots,a_n,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$$

$$=\frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)\cdots\Gamma_p(a_n)\Gamma_p(b_n)}\int_{S_1>0}\cdots(2n)\cdots\int_{T_n>0}\times$$

$$e^{-tr(S_1+T_1+\cdots+S_n+T_n)}|S_1|^{a_1-(p+1)/2}|T_1|^{b_1-(p+1)/2}\cdots\times$$

$$|S_n|^{a_n-(p+1)/2}|T_n|^{b_n-(p+1)/2}\,{}_0F_1(\;;c;-T_1^{1/2}S_1^{1/2}X_1S_1^{1/2}T_1^{1/2}-\cdots$$

$$-T_n^{1/2}S_n^{1/2}X_nS_n^{1/2}T_n^{1/2})dS_1dT_1\cdots dS_ndT_n \qquad \cdots\cdots(1.4)$$

for $\text{Re}(a_i,b_i)>(p-1)/2,\ i=1,\cdots,n.$

PROOF: Taking the M-transform of the right side of eq.(1.4) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively,we get,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$${}_0F_1(\;;c;-T_1^{1/2}S_1^{1/2}X_1S_1^{1/2}T_1^{1/2}-\cdots-T_n^{1/2}S_n^{1/2}X_nS_n^{1/2}T_n^{1/2})dX_1\cdots dX_n \qquad \cdots\cdots(1.5)$$

Making use of the transformations,

$Y_i=T_i^{1/2}S_i^{1/2}X_iS_i^{1/2}T_i^{1/2}$, so that, $dY_i=|T_i|^{(p+1)/2}|S_i|^{(p+1)/2}dX_i$, and,

$|Y_i|=|T_i||S_i||X_i|$ for $i=1,\cdots,n;$

in the expression (1.5) and then using the theorem (3.3) page 55 of Mathai [4] yields,

$$|T_1|^{-\rho_1}|S_1|^{-\rho_1}\cdots|T_n|^{-\rho_n}|S_n|^{-\rho_n}\frac{\Gamma_p(c)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \qquad \cdots\cdots(1.6)$$

Substituting this expression on the right side of eq. (1.4) and then integrating out the variables $S_1, T_1, \cdots, S_n, T_n$ in the resulting expression by using a Gamma integral we obtain $M(F_B^{(n)})$ as given by eq. (1.3) of our paper [12], whence the proof.

THEOREM 1.3:

$$F_D^{(n)}(a, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$$

$$= \frac{1}{\Gamma_p(a)\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\int_{T>0}\int_{S_1>0}\cdots\int_{S_n>0} \times$$

$$e^{-tr(T+S_1+\cdots+S_n)}|T|^{a-(p+1)/2}|S_1|^{b_1-(p+1)/2}\cdots\times$$

$$|S_n|^{b_n-(p+1)/2}\,{}_0F_1(\;;c;-T^{1/2}S_1^{1/2}X_1S_1^{1/2}T^{1/2}-\cdots$$

$$-T^{1/2}S_n^{1/2}X_nS_n^{1/2}T^{1/2})dTdS_1\cdots dS_n \qquad \cdots\cdots(1.7)$$

for $\mathrm{Re}(a, b_i) > (p-1)/2, \; i = 1, \cdots, n.$

PROOF: This theorem can be proved in a similar fashion as theorem (1.2) above. The transformations to be used are $Y_i = T^{1/2}S_i^{1/2}X_iS_i^{1/2}T^{1/2}$, for $i = 1, \cdots, n$. Finally $M(F_D^{(n)})$ is obtained, as given eq. (1.4) of the authors' paper [12].

2. Related Functions of Matrix Arguments

THEOREM 2.1:

$$\Phi_D^{(n)}(a, b_1, \cdots, b_{n-1}; c; -X_1, \cdots, -X_n)$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I |U|^{a-(p+1)/2} |I-U|^{c-a-(p+1)/2} \times$$

$$\left|I + U^{1/2} X_1 U^{1/2}\right|^{-b_1} \cdots \left|I + U^{1/2} X_{n-1} U^{1/2}\right|^{-b_{n-1}} e^{-\mathrm{tr}(UX_n)} dU \qquad \cdots (2.1)$$

for $\mathrm{Re}\,(a, c-a) > (p-1)/2$ and $0 < U < I.$

PROOF: Taking the M-transform of the right side of eq. (2.1) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we obtain,

$$\int_{X_1 > 0} \cdots \int_{X_n > 0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_{n-1}|^{\rho_{n-1} - (p+1)/2} |X_n|^{\rho_n - (p+1)/2} \times$$

$$\left|I + U^{1/2} X_1 U^{1/2}\right|^{-b_1} \cdots \left|I + U^{1/2} X_{n-1} U^{1/2}\right|^{-b_{n-1}} e^{-\mathrm{tr}(UX_n)} dX_1 \cdots dX_n \cdots (2.2)$$

Applying the transformations

$Y_j = U^{1/2} X_j U^{1/2}$, with $dY_j = |U|^{(p+1)/2} dX_j$ and $|Y_j| = |U||X_j|$, for $j = 1, \cdots, n-1;$

to the expression (2.2) and then integrating out the variables $Y_1, \cdots, Y_{n-1}$ by using a type-2 Beta integral and the variables X_n by using a Gamma integral we are led to,

$$|U|^{-\rho_1 - \cdots - \rho_n} \frac{\Gamma_p(b_1 - \rho_1)\Gamma_p(\rho_1)}{\Gamma_p(b_1)} \cdots \frac{\Gamma_p(b_{n-1} - \rho_{n-1})\Gamma_p(\rho_{n-1})}{\Gamma_p(b_{n-1})} \Gamma_p(\rho_n) \cdots (2.3)$$

Substituting this expression on the right side of eq. (2.1) and integrating out the variable U in the resulting expression by using a type-1 Beta integral yields $M(\Phi_D^{(n)})$ as given by eq. (1.7) of the authors' paper [12].

THEOREM 2.2:

$$\Phi_2^{(n)}(b_1,...,b_n;c;-X_1,...,-X_n)$$
$$=\frac{2^{p(p-1)/2}\Gamma_p(c)}{(2\pi i)^{p(p+1)/2}}\int e^{tr(S)}|S|^{-c}\left|I+X_1S^{-1}\right|^{-b_1}\cdots\left|I+X_nS^{-1}\right|^{-b_n}dS\cdots\cdots(2.4)$$

where $S=S_1+iS_2$, S_1 and S_2 are real matrices with $S_1=S_1'>0$ and it being assumed that $S^{-1}=VV'$ with $V'X_jV>0,\ j=1,\cdots,n$ and $i=\sqrt{-1}$.

PROOF: Taking the M-transform of the right side of eq. (2.4) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\left|I+X_1S^{-1}\right|^{-b_1}\cdots\left|I+X_nS^{-1}\right|^{-b_n}dX_1\cdots dX_n\cdots(2.5)$$

Now observing that

$$\left|I+X_jS^{-1}\right|=\left|I+X_jVV'\right|=\left|I+V'X_jV\right|\text{ for }j=1,\cdots,n;$$

$|S|^{-1}=|V|^2$ and making use of the transformations,

$Y_j=V'X_jV$ (with $dY_j=|V|^{(p+1)}dX_j$ and $|Y_j|=|V|^2|X_j|$) for $j=1,\cdots,n$

and then integrating out the variables $Y_1,\cdots,Y_n$ by using a type-2 Beta integral, the last expression renders,

$$|S|^{\rho_1+\cdots+\rho_n}\frac{\Gamma_p(b_1-\rho_1)\Gamma_p(\rho_1)}{\Gamma_p(b_1)}\cdots\frac{\Gamma_p(b_n-\rho_n)\Gamma_p(\rho_n)}{\Gamma_p(b_n)}\cdots(2.6)$$

Substituting this expression on the right side of eq. (2.4) and using eq. (2.5.11) page 49 of Mathai [4] in the resulting expression produces $M(\Phi_2^{(n)})$, as given by eq. (1.4) of the authors' paper [11].

This theorem generalizes the theorem (4.5) page 63 of Mathai [4].

THEOREM 2.3:

$$|P|^{-\gamma}\left|I+P^{-1/2}X_1P^{-1/2}\right|^{-\beta_1}\cdots\left|I+P^{-1/2}X_nP^{-1/2}\right|^{-\beta_n}$$

$$=\frac{1}{\Gamma_p(\gamma)}\int_{T>0}e^{-\mathrm{tr}(PT)}|T|^{\gamma-(p+1)/2}\Phi_2^{(n)}(\beta_1,\cdots,\beta_n;\gamma;-T^{1/2}X_1T^{1/2},\cdots,$$

$$-T^{1/2}X_nT^{1/2})dT \qquad \cdots\cdots(2.7)$$

for $\mathrm{Re}\,(\gamma)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (2.7) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$$\Phi_2^{(n)}(\beta_1,\cdots,\beta_n;\gamma;-T^{1/2}X_1T^{1/2},\cdots,-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n\cdots(2.8)$$

Applying the transformations,

$Y_i=T^{1/2}X_iT^{1/2}$ with $dY_i=|T|^{(p+1)/2}dX_i$ and $|Y_i|=|T||X_i|$, for $i=1,\cdots,n$

in the above expression and then using eq. (1.4) of the authors' paper [11], we obtain,

$$|T|^{-\rho_1-\cdots-\rho_n}\frac{\Gamma_p(\gamma)\{\prod_{i=1}^{n}\Gamma_p(\beta_i-\rho_i)\Gamma_p(\rho_i)\}}{\{\prod_{i=1}^{n}\Gamma_p(\beta_i)\}\Gamma_p(\gamma-\rho_1-\cdots-\rho_n)} \qquad \cdots(2.9)$$

Substituting this expression on the right side of eq. (2.7) and integrating out T in the resulting expression by using a Gamma integral produces,

$$|P|^{-(\gamma-\rho_1-\cdots-\rho_n)} \frac{\{\prod_{i=1}^{n} \Gamma_p(\beta_i-\rho_i)\Gamma_p(\rho_i)\}}{\{\prod_{i=1}^{n} \Gamma_p(\beta_i)\}} \qquad \cdots(2.10)$$

Now taking the M-transform of the left side of eq. (2.7) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$

$$|P|^{-\gamma} \left|I+P^{-1/2}X_1P^{-1/2}\right|^{-\beta_1} \cdots \left|I+P^{-1/2}X_nP^{-1/2}\right|^{-\beta_n} dX_1\cdots dX_n \cdots(2.11)$$

On using the transformations,

$Z_i = P^{-1/2}X_iP^{-1/2}$ with $dZ_i = |P|^{-(p+1)/2}dX_i$ and $|Z_i| = |P|^{-1}|X_i|$ for $i = 1,\cdots,n$ in the above expression and integrating out the variables $Z_1,\cdots,Z_n$ by using a type-2 Beta integral the outcome is the same as in eq. (2.10) above, thus proving the result.

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