The infimum is attained at $\tilde{z}_5 = z_5/\rho$ if $|z_5| \leq \rho^4$ and at $\tilde{z}_5 = 0$ if $|z_5| \geq \rho^4$. This, together with the estimate (14) of $\tilde{d}$, gives the estimate of $d(x_1, x)$, with $c = \tilde{c}$ and $C = \tilde{C}$. The expression of ρ is easily derived from (10).

Having proven the result on a compact neighborhood N_{x_0} of each $x_0 \in \mathcal{V}_{46}$, let now S be a compact subset of $\mathcal{V}_{46}$. The union of the interiors V_{x_0} of N_{x_0}, $x_0 \in S$, is a covering of S by open sets, from which we can extract a finite covering $\cup V_i$; equation (12) holds on each V_i with constants c_i, C_i and ϵ_i. Setting $\epsilon = \min_i \epsilon_i$, $c = \min_i c_i$, and $C = \max_i C_i$, the thesis follows. ■

Note the following points.

- The estimate does not depend on the choice of the lifting.
- When x_1 is a singular point, the continuous function ρ equals zero and Theorem 2 is simply the Ball-Box Theorem at a singular point. On the other hand, when x_1 is regular and far enough from the singular locus, it may be certainly assumed that $\rho > \epsilon$ (reducing ϵ if needed). In this case, condition $d(x_1, x) < \epsilon$ implies $|z_5| \leq \rho^4$, and Theorem 2 turns out to be the Ball-Box Theorem at a regular point.
- A uniform estimate of the form (12)–(13) holds for compact subsets of the generic $\mathcal{V}_{ij}$, with the privileged coordinates defined by AP_{ij}^{nh} and $\rho_{ij} = \det \Gamma_r / \det \Gamma_{ij}$ in place of ρ. The same is true on compact subsets of $\mathcal{V}_r$, with the privileged coordinates defined by AP_r and $\rho = \rho_r = 1$ in place of ρ; in this case, the estimate (12)–(13) coincides with that of the classical Ball-Box theorem.

If $\mathcal{V}_{46}$ covers the whole state space, Theorem 2 directly provides a uniform estimation of d on $\mathbb{R}^5$. Even in the general case, however, it is possible to obtain the same result; in fact, given any compact subset $K \subset \mathbb{R}^5$, we can write $K = \left(\bigcup_{i,j} K_{ij}\right) \cup K_r$, having set $K_{ij} = K \cap \mathcal{D}_{ij}$ and $K_r = K \cap \mathcal{D}_r$. Estimate (12)–(13) holds on K_r as well as each K_{ij}; a uniform distance estimation over K is then obtained by computing the appropriate extremal values of c, C and ϵ over the subset.

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Control With Disturbance Preview and Online Optimization

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***Abstract*—We present a intuitive and self-contained formulation of a stability preserving receding horizon control strategy for a system where limited preview information is available for the disturbances. The simplicity of the derivation is due to (and its benefits somewhat offset by) a set of stringent and highly structured assumptions. The formulation uses a suboptimal value function for terminal cost, and relies on optimization strategies that only require a trivial improvement property, allowing implementation as an "anytime" algorithm. The nature of this strategy's performance is clarified with linear examples.**

***Index Terms*—Anytime, disturbance preview, model predictive control, receding horizon control.**

I. Introduction

Performance advances in microprocessors have spurred the interest in receding horizon, also termed model predictive, control strategies. An excellent review of the growth of the field is given in [1]. Of particular interest to this note are [2], [3], especially [4], [5], and the suboptimality results of [6].

We extend the methods of receding horizon control to the case where a discrete nonlinear dynamic system is driven by disturbances, and where consistent finite length previews of these disturbances are available. We consider the problem as a dynamic game between control and disturbance. From this perspective, it is generally the case that advanced knowledge of the disturbance is both desirable *and* expensive. Hence, in some cases a limited preview will be available through additional sensors, intelligence, or short term predictive models (e.g., the

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weather). While it may be conservative to assume that the disturbance works in the worst case manner beyond the prediction horizon, it may prove prudent in certain situations. From an aerospace perspective, limited finite previews of disturbances may correspond to situations where an outer guidance loop (based on current measurements) can only calculate a desired trajectory a limited interval into the future. References [7] and [8] consider a similar tracking problem where the signal comes from a stable exosystem.

Our approach, which expands the results in [9], uses a suboptimal value function as a terminal cost and requires improvement, not optimality, in the optimization step. The ability to terminate the optimization early (local minima or reduced computational resources) makes our approach implementable as an "anytime" algorithm.

In Section II, we introduce the dynamic system and our assumptions. Section III then gives the control objective and algorithm. In Section IV, an important lemma unlocks the performance and stability results that are shown in Section V. Section VI uses two linear examples to clarify the exact behavior that the theorems in Section V guarantee.

II. Definitions, Problem Setup, and Assumptions

First, we need a few mathematical preliminaries and notational conventions.

- *$\mathcal{K}$-functions:* $\alpha : [0, a) \to [0, \infty)$ is a $\mathcal{K}$-function if it is continuous and strictly increasing on $[0, a)$ and $\alpha(0) = 0$.
- $\mathbb{R}^+$, $\mathbb{Z}^+$: $\mathbb{R}^+ := \{v \in \mathbb{R} : v \geq 0\}$ and $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$.
- *Balls:* For $r \geq 0$ and $n \in \mathbb{Z}^+$, $B_r^n := \{\chi \in \mathbb{R}^n : \|\chi\| \leq r\}$. When n is clear from context, write B_r.
- $\xi_{[k,k+N-1]}$: Shorthand for the sequence $\{\xi_j\}_{j=k}^{k+N-1}$.
- *l_{2+} Spaces:* $\sum_{j=k_0}^{\infty} \|\xi_j\|^2 < \infty \Leftrightarrow \xi \in l_{2+}$.

The functions that define the dynamics and their assumptions follow.

- $f : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^n : f(0,0,0) = 0$ and f continuous on $\mathbb{R}^n \times 0 \times \mathbb{R}^m$.
- $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$: Continuous with $h(0,0) = 0$. Also, $\exists$ $\mathcal{K}$-function $\tau(\cdot)$, $h(\chi, v) > h(\chi, 0) \geq \tau(\|\chi\|)$, $\forall \chi \in \mathbb{R}^n$, $\forall v \neq 0 \in \mathbb{R}^m$. Additionally, $\sum_{j=k_0}^{\infty} h(\xi_j, \eta_j) < \infty \Leftrightarrow \xi \in l_{2+}$, $\eta \in l_{2+}$.
- $g : \mathbb{R}^l \to \mathbb{R}^+$: $g(0) = 0$, and $\exists \alpha_1 > 0 \in \mathbb{R}$ such that $g(\omega) \geq \alpha_1 \|\omega\|^2$, $\forall \omega \in \mathbb{R}^l$, and, $\forall \zeta \in l_{2+}$, $\sum_{j=k_0}^{\infty} g(\zeta_j) < \infty$.

We define our dynamic system as $x_{k+1} = f(x_k, w_k, u_k)$, where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^l$ is the disturbance, and $u_k \in \mathbb{R}^m$ is the control input. Also, let the error signal at each time step be defined as $h(x_k, u_k)$ and let $\sum_{j=k_0}^{\infty} [h(x_j, u_j) - g(w_j)]$ be the cost of a two player, (w, u), dynamic game. The developments of this note also follow when g is a function of the disturbance, w, and the state, x, as long as the requirements on h for (x, u) hold on g for (x, w).

With these system definitions we introduce the following system properties and their explicit assumptions. Define $\mathbf{X}$, $\mathbf{W}$ to be neighborhoods that contain the origins of $\mathbb{R}^n$, $\mathbb{R}^l$ as interior points, respectively.

- $\phi_N : \mathbb{R}^n \times \mathbb{R}^{l \times N} \times \mathbb{R}^{m \times N} \to \mathbb{R}^n$: The system flow function, which takes the system's state forward N steps in time, $x_{k+N} = \phi_N(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]})$. For completeness define $\phi_0(x_k, \cdot, \cdot) = x_k$.
- $\mu : \mathbb{R}^n \to \mathbb{R}^m$ The baseline controller, referred to as the legacy controller. Let Φ_j be the j-step flow of the system $x_{k+1} = f(x_k, 0, \mu(x_k))$. μ satisfies the following two assumptions.
  1) It meets the norm bound; $\forall N > 0, \exists r_N > 0, \exists \sigma_N \in \mathcal{K}$: $\forall x_k \in B_{r_N}^n$, $\|\{\mu(\Phi_{l-k}(x_k))\}_{l=k}^{k+N-1}\| \leq \sigma_N(\|x_k\|)$. Implying that, if x_k is small, $\mu(x_k)$ is as well.
  2) It provides the following invariance property $\forall x_k, w_k \in \mathbf{X} \times \mathbf{W}$, $f(x_k, w_k, \mu(x_k)) \in \mathbf{X}$.
- $V : \mathbf{X} \to \mathbb{R}^+$: V is a continuous positive–definite function s.t. $\forall x_k, w_k \in \mathbf{X} \times \mathbf{W}$

$$V\left(f(x_k, w_k, \mu(x_k))\right) + h(x_k, \mu(x_k)) - g(w_k) \leq V(x_k). \quad (1)$$

Using this inequality, V can be used as a Lyapunov function for the system under μ with $w := 0$ to prove local asymptotic stability with $\mathbf{X}$ as its region of attraction. Recursively on the flow of the system, the above inequality also gives $\forall x_{k_0} \in \mathbf{X}$, $V(x_{k_0}) \geq \sum_{j=k_0}^{\infty} [h(x_j, \mu(x_j)) - g(w_j)]$ if $w_k \in \mathbf{W}, \forall k$. Defining the worst case cost incurred starting from x_{k_0} under the system dynamics with $w_k \in \mathbf{W}, \forall k$, as $J_\mu(x_{k_0})$ with

$$J_\mu(x_{k_0}) := \sup_{w \in \mathbf{W}} \sum_{j=k_0}^{\infty} \left(h(x_j, \mu(x_j)) - g(w_j)\right) \quad (2)$$

we can use the recursive results to give the bound $V(x_{k_0}) \geq J_\mu(x_{k_0})$.
- $J_N : \mathbb{R}^n \times \mathbb{R}^{l \times N} \times \mathbb{R}^{m \times N} \to \mathbb{R}$: The cost-like function

$$\begin{aligned} &J_N(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]}) \\ &:= \sum_{j=k}^{k+N-1} \left[h\left(\underbrace{\phi_{j-k}(x_k, w_{[k,j-1]}, u_{[k,j-1]})}_{x_j}, u_j \right) - g(w_j) \right] \\ &\quad + V\left(\underbrace{\phi_N(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]})}_{x_{k+N}} \right). \end{aligned}$$

Clearly, J_N is only well defined when $x_N \in \mathbf{X}$.
- $\mathcal{A}_{C,K} : \mathbb{R}^{m \times N} \to 2^{\mathbb{R}^{m \times N}}$: The constrained optimization engine, with set valued range and constraint set K, gives trivial improvement, whereby if $u \in K \subseteq \mathbb{R}^{m \times N}$ and $v \in \mathcal{A}_{C,K}(u)$ then $C(v) \leq C(u)$ and $v \in K$.

  We will use $C(\cdot) = J_N(x_k, w_{[k,k+N-1]}, \cdot)$ and $K(\cdot) = K_N(x_k, w_{[k,k+N-1]}, \cdot)$ such that $\phi_j(x_k, w_{[k,j-1]}, \cdot) \in \mathbf{X}$, $\forall j \in \{1, \ldots, N\}$. If u is feasible, the optimization is always feasible with $v = u$.
- $\mathcal{S}_\mu : \mathbb{R}^n \times \mathbb{R}^{l \times N} \times \mathbb{R}^{m \times N} \to \mathbb{R}^{m \times N}$: A control sequence time shift that appends a control action from the baseline controller

$$\begin{aligned} &\mathcal{S}_\mu(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]}) \\ &\quad := \left\{ u_{[k+1,k+N-1]}, \mu\left(\phi_N\left(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]} \right) \right) \right\}. \end{aligned}$$

By the properties of μ, if $u \in K_N(x_k, w_{[k,k+N-1]}, \cdot)$, then $\mathcal{S}_\mu(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]}) \in K_N(x_{k+1}, w_{[k+1,k+N]}, \cdot)$.

III. Control Strategy

Using the legacy controller, the worst-case cost incurred starting from some state x_k is given by $J_\mu(x_k)$, and our goal is to use N steps of preview information about the disturbance, $w_{[k,k+N-1]}$, to choose a value of u_k that results in a lower cost and retains a guarantee of stability. We use $\mathcal{A}_{C,K}$, the constrained optimization engine, to minimize J_N, the incurred cost over an N step horizon including an appropriate terminal state cost as the base of the algorithm.

Receding Horizon Control (RHC) Algorithm:
To generate the control signal, u_k, with $x_k \in \mathbf{X}$ and $w_{[k,k+N-1]} \in \mathbf{W}^N$ known.
1. Compute a suboptimal control sequence $\hat{u}$, $k = k_0$ (initialization)

$$\hat{u} = \{\mu(\phi_j(x_k, w_{[k,k+N-1]}, \{\mu(x_l)\}_{l=k}^{k+j-1}))\}_{j=0}^{N-1}$$

$k > k_0$

$$\hat{u} = \mathcal{S}_\mu(x_{k-1}, w_{[k-1,k+N-2]}, u^{\bar{\star},k-1})$$

`then` $u^{\bar{\star},k} \in \mathcal{A}_{J_N(x_k, w_{[k,k+N-1]},\cdot), K_N(x_k,w_{[k,k+N-1]},\cdot)}(\hat{u})$.

`2. Set` $u_k = u_k^{\bar{\star},k}$`, increment` k `and repeat.`

Note that, in this algorithm, u_k is functionally dependent on $w_{[k,k+N-1]}$ and also that, due to the assumptions, the optimization is always feasible.

IV. Keystone Lemma

The following lemma shows that the cost-to-go of implementing one step of the control strategy and reoptimizing over the next N steps is no greater than the N step preview cost starting from x_k. The proof follows along the lines of the proof for [10, Th. 2].

Lemma 1: With the control strategy shown previously, and a given $x_k \in \mathbf{X}$, $w_{[k,k+N]} \in \mathbf{W}^{N+1}$

$$J_N(x_{k+1}, w_{[k+1,k+N]}, u^{\bar{\star},k+1}) + \left[h(x_k, u_k^{\bar{\star},k}) - g(w_k)\right] \leq J_N(x_k, w_{[k,k+N-1]}, u^{\bar{\star},k}).$$

Proof: Let $\bar{u}^{k+1}_{[k+1,k+N]} = \mathcal{S}_\mu(x_k, w_{[k,k+N-1]}, u^{\bar{\star},k}_{[k,k+N-1]})$, and note that $\bar{u}_j^{k+1} := u_j^{\bar{\star},k}, \forall j \in \{k+1,\ldots,k+N-1\}$. Following the definition and manipulating we get the equation chain that follows, where for size we let $k_1 := k+1$. Starting with the suboptimality of $\bar{u}$

$$\begin{aligned}
&J_N(x_{k+1}, w_{[k+1,k+N]}, u^{\bar{\star},k+1}) \\
&\leq J_N(x_{k+1}, w_{[k+1,k+N]}, \bar{u}^{k+1}) \\
&:= \sum_{j=k_1}^{k+N}\left[h\Big(\underbrace{\phi_{j-k_1}(x_{k_1}, w_{[k_1,j-1]}, \bar{u}^{k_1}_{[k_1,j-1]})}_{x_j,\text{ if } \bar{u}^{k_1} \text{ and } w \text{ are used}}, \bar{u}_j^{k_1}\Big) - g(w_j)\right] \\
&\quad + V\Big(\underbrace{\phi_N(x_{k_1}, w_{[k_1,k+N]}, \bar{u}^{k_1}_{[k_1,k+N]})}_{x_{k+N+1} \text{ if } \bar{u}^{k_1} \text{ and } w \text{ are used}}\Big) \\
&= -\Big(h(x_k, u_k^{\bar{\star},k}) - g(w_k)\Big) + \Big(h(x_k, u_k^{\bar{\star},k}) - g(w_k)\Big) \\
&\quad + \sum_{j=k_1}^{k+N-1}\left[h\Big(\underbrace{\phi_{j-k_1}(x_{k_1}, w_{[k_1,j-1]}, u^{\bar{\star},k}_{[k_1,j-1]})}_{x_j \text{ if } u^{\bar{\star},k} \text{ and } w \text{ are used}}, u_j^{\bar{\star},k}\Big) - g(w_j)\right] \\
&\quad + h\Big(\underbrace{\phi_{N-1}(x_{k_1}, w_{[k_1,k+N-1]}, \bar{u}^{k_1}_{[k_1,k+N-1]})}_{x_{k+N} \text{ if } \bar{u}^{k_1} \text{ and } w \text{ are used}}, \bar{u}^{k_1}_{k+N}\Big) \\
&\quad - g(w_{k+N}) + V\Big(\underbrace{\phi_N(x_{k_1}, w_{[k_1,k+N]}, \bar{u}^{k_1}_{[k_1,k+N]})}_{x_{k+N+1} \text{ if } \bar{u}^{k_1} \text{ and } w \text{ are used}}\Big) \\
&= -\Big(h(x_k, u_k^{\bar{\star},k}) - g(w_k)\Big) \\
&\quad + \sum_{j=k}^{k+N-1}\left[h\Big(\underbrace{\phi_{j-k}(x_k, w_{[k,j-1]}, u^{\bar{\star},k}_{[k,j-1]})}_{x_j \text{ if } u^{\bar{\star},k} \text{ and } w \text{ are used}}, u_j^{\bar{\star},k}\Big) - g(w_j)\right] \\
&\quad + h\Big(\underbrace{\phi_N(x_k, w_{[k,k+N-1]}, u^{\bar{\star},k}_{[k,k+N-1]})}_{x_{k+N} \text{ if } u^{\bar{\star},k} \text{ and } w \text{ are used}}, \bar{u}^{k_1}_{k+N}\Big) - g(w_{k+N}) \\
&\quad + V\Big(\underbrace{\phi_N(x_{k_1}, w_{[k_1,k+N]}, \bar{u}^{k_1}_{[k_1,k+N]})}_{x_{k+N+1} \text{ if } \bar{u}^{k_1} \text{ and } w \text{ are used}}\Big) \\
&\overset{(a)}{\leq} -\Big(h(x_k, u_k^{\bar{\star},k}) - g(w_k)\Big) \\
&\quad + \sum_{j=k}^{k+N-1}\left[h\Big(\underbrace{\phi_{j-k}(x_k, w_{[k,j-1]}, u^{\bar{\star},k}_{[k,j-1]})}_{x_j \text{ if } u^{\bar{\star},k} \text{ and } w \text{ are used}}, u_j^{\bar{\star},k}\Big) - g(w_j)\right] \\
&\quad + V\Big(\underbrace{\phi_N(x_k, w_{[k,k+N-1]}, u^{\bar{\star},k}_{[k,k+N-1]})}_{x_{k+N} \text{ if } u^{\bar{\star},k} \text{ and } w \text{ are used}}\Big) \\
&= -\Big(h(x_k, u_k^{\bar{\star},k}) - g(w_k)\Big) + J_N(x_k, w_{[k,k+N-1]}, u^{\bar{\star},k})
\end{aligned}$$

(a) is from the definition of V, which is valid since $x_{k+N} \in \mathbf{X}$, and the control at $k+N$ is from μ. ■

V. Performance and Stability

By bounding the cost of the RHC algorithm, we can show that this strategy achieves a cost that is no greater than our upper bound for the worst-case cost under the legacy controller.

Theorem 1: $\forall w \in l_{2+}$, with $w_k \in \mathbf{W}$, $\forall k$ and $\forall x_{k_0} \in \mathbf{X}$, the cost resulting from the application of the RHC algorithm is bounded as follows:

$$\sum_{j=k_0}^{\infty}\Big(h(x_j, u_j^{\bar{\star},j}) - g(w_j)\Big) \leq V(x_{k_0}).$$

Proof: Let $\epsilon > 0$. From Lemma 1

$$\begin{aligned}
&J_N(x_{k+1}, w_{[k+1,k+N]}, u^{\bar{\star},k+1}) \\
&\qquad \leq -\left[h(x_k, u_k^{\bar{\star},k}) - g(w_k)\right] + J_N(x_k, w_{[k,k+N-1]}, u^{\bar{\star},k}).
\end{aligned}$$

Since the state trajectory remains in $\mathbf{X}$ for all time, taking this relation for $x_{k_0}, \ldots, x_{k_0+L-2}$, $L > 0$, summing and canceling we have, with $k_L := k_0 + L - 1$

$$\begin{aligned}
&J_N(x_{k_L}, w_{[k_L,k_L+N-1]}, u^{\bar{\star},k_L}) \\
&\leq -\sum_{j=k_0}^{k_0+L-2}\left[h(x_j, u_j^{\bar{\star},j}) - g(w_j)\right] + J_N(x_{k_0}, w_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}).
\end{aligned}$$

If we rearrange the previous equation and use the definition of $J_N(x_{k_L}, w_{[k_L,k_L+N-1]}, u^{\bar{\star},k_L})$, we have

$$\begin{aligned}
&J_N(x_{k_0}, w_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) \\
&\geq \sum_{j=k_0}^{k_0+L-2}\Big(h(x_j, u_j^{\bar{\star},j}) - g(w_j)\Big) \\
&\quad + \sum_{j=k_L}^{k_L+N-1}\left[h(\underbrace{\phi_{j-k_L}(x_{k_L}, w_{[k_L,j-1]}, u^{\bar{\star},k_L}_{[k_L,j-1]})}_{x_j \text{ if } u^{\bar{\star},k_L} \text{ and } w \text{ are used}}, u_j^{\bar{\star},k_L}) - g(w_j)\right] \\
&\quad + V\Big(\underbrace{\phi_N(x_{k_L}, w_{[k_L,k_L+N-1]}, u^{\bar{\star},k_L}_{[k_L,k_L+N-1]})}_{x_{k_0+L-1+N} \text{ if } u^{\bar{\star},k_L} \text{ and } w \text{ are used}}\Big).
\end{aligned}$$

Since $V \geq 0$ and $h \geq 0$, we have the following:

$$\begin{aligned}
&J_N(x_{k_0}, w_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) \\
&\qquad \geq \sum_{j=k_0}^{k_0+L-2}\Big(h(x_j, u_j^{\bar{\star},j}) - g(w_j)\Big) - \sum_{j=k_0+L-1}^{k_0+L-1+N-1} g(w_j).
\end{aligned}$$

By our assumptions on g, we know that $\sum g$ converges and $g(w) \geq 0$, $\forall w$, thus, for L large enough

$$J_N(x_{k_0}, w_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) + \epsilon \geq \sum_{j=k_0}^{k_0+L-1}\left[h(x_j, u_j^{\bar{\star},j}) - g(w_j)\right].$$

Since the left-hand side of the previous equation does not depend on L and is a fixed number, it is a valid upper bound of the right-hand side for all L large enough. Taking $L \to \infty$ and rearranging yields

$$\begin{aligned}&\sum_{j=k_0}^{\infty}\left(h(x_j, u_j^{\bar{\star},j}) - g(w_j)\right)\\ &\leq J_N(x_{k_0}, w_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) + \epsilon\\ &\overset{(a)}{\leq} J_N(x_{k_0}, w_{[k_0,k_0+N-1]}, [\mu(x_{k_0}), \ldots, \mu(x_{k_0+N-1})]) + \epsilon\\ &\overset{(b)}{\leq} V(x_{k_0}) + \epsilon.\end{aligned}$$

Inequality (a) comes from the use of $\mathcal{A}_{C,K}$ to get $u^{\bar{\star},k_0}$ from the control sequence under μ. (b) results from applying (1) N times. ■

Lemma 2: Using the RHC algorithm with any known disturbance trajectory $w \in l_{2+}$ and $w_k \in \mathbf{W}$, $\forall k$, and any initial condition $x_{k_0} \in \mathbf{X}$ results in a state trajectory in l_{2+}.

Proof: Theorem 1 gives us

$$V(x_{k_0}) \geq \sum_{j=k_0}^{\infty}\left(h(x_j, u_j^{\bar{\star},j}) - g(w_j)\right).$$

We know that for $x \in \mathbf{X}$, $V(x) < \infty$, and by assumption $\sum g$ converges for any $w \in l_{2+}$, yielding

$$\sum_{j=k_0}^{\infty} h(x_j, u_j^{\bar{\star},j}) < \infty$$

which by the assumptions on h gives $x \in l_{2+}$. ■

This gives the result that if $w \in l_{2+}$ then $x \in l_{2+}$, implying $x_k \to 0$ as $k \to \infty$, and we can use result in the following theorem to have local asymptotic stability.

Theorem 2: Using the RHC algorithm with $w := 0$, the closed-loop system $x_{k+1} = f(x_k, 0, u_k^{\bar{\star},k})$ with

$$u^{\bar{\star},k} \in \mathcal{A}_{J_N(x_k,0,\cdot),K_N(x_k,0,\cdot)}\left(\mathcal{S}_\mu(x_{k-1}, 0, u^{\bar{\star},k-1})\right)$$

is locally asymptotically stable about the origin with region of attraction $\mathbf{X}$.

Proof: This proof is based on the time-varying Lyapunov stability proof in [6]. Using $V(0) = 0$ and the continuity of J_N (from continuity of f, h, g, and V) allows us to upper bound J_N around the origin, so $\exists\mathcal{K}$-function β and $\exists r_1 > 0$ such that $\forall \chi \in B_{r_1}^n \subseteq \mathbf{X}$, $\forall \nu \in B_{r_1}^{m\times N} \subseteq K_N(\chi, 0, \cdot)$, $J_N(\chi, 0, \nu) \leq \beta(\|\chi\| + \|\nu\|)$. Using the positivity of V and h along with the other assumptions on h we get $\forall \chi \in \mathbf{X}$, $\forall \nu \in K_N(\chi, 0, \cdot)$ it follows that $J_N(\chi, 0, \nu) \geq h(\chi, 0) \geq \tau(\|\chi\|)$. From the assumptions on μ, $\forall x_k \in B_{r_N}^n$, $\|\{\mu(\Phi_{l-k}(x_k))\}_{l=k}^{k+N-1}\| \leq \sigma_N(\|x_k\|)$. From the keystone lemma, $\forall k \geq k_0$, $J_N(x_k, 0, u^{\bar{\star},k}) \leq J_N(x_{k_0}, 0, u^{\bar{\star},k_0})$. Now, for any $\epsilon > 0$ pick $\delta_\epsilon > 0$ so that $\delta_\epsilon < \min(r_N, r_1)$, $\sigma_N(\delta_\epsilon) < r_1$, and $\beta(\delta_\epsilon + \sigma_N(\delta_\epsilon)) < \tau(\epsilon)$. Then for any $x_{k_0} \in B_{\delta_\epsilon}^n$ we have $\|\{\mu(\Phi_{l-k_0}(x_{k_0}))\}_{l=k_0}^{k_0+N-1}\| \leq \sigma_N(\|x_{k_0}\|) \leq \sigma_N(\delta_\epsilon) < r_1$, which implies that $\{\mu(\Phi_{l-k_0}(x_{k_0}))\}_{l=k_0}^{k_0+N-1} \in B_{r_1}^{m\times N}$.

We can now build the following chain for all $k \geq k_0$:

$$\begin{aligned}\tau(\|x_k\|) &\leq J_N(x_k, 0, u^{\bar{\star},k}) \leq J_N(x_{k_0}, 0, u^{\bar{\star},k_0})\\ &\leq J_N(x_{k_0}, 0, \{\mu(\Phi_{l-k_0}(x_{k_0}))\}_{l=k_0}^{k_0+N-1}).\end{aligned}$$

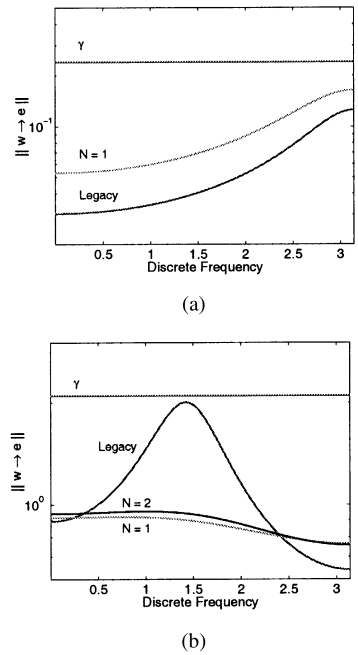

Fig. 1. (a) (Example 1): preview information causing worse disturbance rejection at all frequencies. (b) (Example 2): increased preview length leading to the worsening of disturbance rejection.

Using the upper bound on J_N, we get

$$\begin{aligned}\tau(\|x_k\|) &\leq \beta(\|x_{k_0}\| + \|\{\mu(\Phi_{l-k_0}(x_{k_0}))\}_{l=k_0}^{k_0+N-1}\|)\\ &\leq \beta(\delta_\epsilon + \sigma_N(\delta_\epsilon)) \leq \tau(\epsilon)\end{aligned}$$

showing that if $x_{k_0} \in B_{\delta_\epsilon}^n$ then $x_k \in B_\epsilon^n$ $\forall k \geq k_0$, yielding stability. Since w meets the conditions of Lemma 2 we have $x_k \to 0$, implying that, if $x_{k_0} \in \mathbf{X}$ then eventually $x_k \in B_{\delta_\epsilon}$, resulting in local asymptotic stability. ■

VI. What the Theorems Don't Say …

It is tempting to sloppily summarize the result in Theorem 1 as "the RHC controller always does as well as the legacy controller." This is false. In order to show that certain results are *not* true in general, we consider the simplest framework which fits within the theorem.

Given matrices A, B_1 and B_2 of appropriate dimension, with $\rho(A) < 1$, $0 \prec H = H^T \in \mathbb{R}^{(n+m)\times(n+m)}$, partition $H^{(1/2)} =: [C_1\ D_{12}]$, define $e := C_1 x + D_{12} u$, and let $\gamma > \left\|C_1(zI - A)^{-1}B_1\right\|_\infty$. Then there exists $X = X^T \succ 0$ such that

$$L_{\gamma,X} := \begin{bmatrix}X & 0\\ 0 & \gamma^2 I\end{bmatrix} - \begin{bmatrix}A^T & C_1^T\\ B_1^T & 0\end{bmatrix}\begin{bmatrix}X & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A & B_1\\ C_1 & 0\end{bmatrix} \succ 0$$

(if (A, B_1) and (A, C_1) are, respectively, controllable and observable and $\gamma = \left\|C_1(zI - A)^{-1}B_1\right\|_\infty$, then there exists $X = X^T \succ 0$ such that $L_{\gamma,X} \succeq 0$). Consider specific forms for f, g, h, μ and V, namely $\mu(x) \equiv 0$ for all x, $f(x, w, u) = Ax + B_1 w + B_2 u$, $g = \gamma^2 w^T w$

$$h(x, u) = \begin{bmatrix}x\\ u\end{bmatrix}^T H \begin{bmatrix}x\\ u\end{bmatrix}, \text{ and } V(x) = x^T X x.$$

This set of functions satisfies all the hypotheses of Theorem 1, with $\mathbf{X} = \mathbb{R}^n$ and $\mathbf{W} = \mathbb{R}^l$. Denote $\hat{G}_{\text{leg}}(z) := C_1(zI - A)^{-1}B_1$, which is both the closed and open-loop w to e transfer function. Note that $\|\hat{G}_{\text{leg}}\|_\infty < \gamma$.

TABLE I
DATA FOR THE EXAMPLES

Format	Example 1	Example 2
$\left[\begin{array}{c\|c\|c} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ \hline X & 0 & 0 \end{array}\right]_{\gamma}$	$\left[\begin{array}{c\|c\|c} -0.609 & -0.753 & 0.926 \\ \hline 0.052 & 0 & 0.039 \\ 0.039 & 0 & 0.287 \\ \hline 0.048 & 0 & 0 \end{array}\right]_{0.240}$	$\left[\begin{array}{cc\|c\|c} 0.088 & -0.664 & 1.592 & -1.148 \\ 0.664 & 0.088 & -0.973 & -1.486 \\ \hline 0.508 & 0.118 & 0 & 0.020 \\ 0.118 & 0.508 & 0 & -0.135 \\ 0.020 & -0.135 & 0 & 0.261 \\ \hline 0.759 & 0.162 & 0 & 0 \\ 0.162 & 0.896 & 0 & 0 \end{array}\right]_{2.097}$

$J_N(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]})$ is a quadratic function of x_k, $w_{[k,k+N-1]}$ and $u_{[k,k+N-1]}$. Since $H_{22} \succ 0$ and $X \succ 0$, J_N is positive–definite in $u_{[k,k+N-1]}$, and the minimizing control input $u^{\bar{\star},k}_{[k,k+N-1]}$ (as well as it's first entry, $u^{\bar{\star},k}_k$) is a linear function of x_k and $w_{[k,k+N-1]}$. With this feedback/feedforward law, Theorems 1 and 2 imply that the closed-loop system is LTI and stable, and for any initial condition $x_{k_0} = \chi$, and any $w \in l_2$

$$\sum_{j=k_0}^{\infty} \left(h(x_j, u_j^{\bar{\star},j}) - g(w_j) \right) \leq \chi^T X \chi.$$

So, starting from initial condition $x_{k_0} = 0$, and for any $w \in l_2$ we have $\|e\|_2 \leq \gamma \|w\|_2$.

Let T denote the (noncausal, linear, time-invariant, finite-dimensional) closed-loop map from w to e, and let $\hat{T}$ denote the transfer function. Let T_N denote the causal, closed loop map from an N-step advance of w to e. It is straightforward to write a realization for the causal T_N, and we use this in the upcoming computations. Both T and T_N have the same induced l_2 norm, which in the frequency domain is $\|\hat{T}\|_\infty = \|\hat{T}_N\|_\infty$. Therefore, we *must* have $\|\hat{T}_N\|_\infty \leq \gamma$, and in the special case where γ is equal to $\|\hat{G}_{\text{leg}}\|_\infty$ it follows that $\|\hat{T}_N\|_\infty \leq \|\hat{G}_{\text{leg}}\|_\infty$ (i.e., guaranteed performance retention/improvement in the $\|\cdot\|_\infty$ sense).

It is *definitely* not claimed that $\|\hat{T}_N(e^{j\theta})\| \leq \|\hat{G}_{\text{leg}}(e^{j\theta})\|$ for all θ. In fact, if $\gamma > \|\hat{G}_{\text{leg}}\|_\infty$, it is not even claimed that $\|\hat{T}_N\|_\infty \leq \|\hat{G}_{\text{leg}}\|_\infty$. Indeed, it is possible [see Example 1: data in Table 1 and results in Fig.1(a)] that

$$\bar{\sigma}\left[\hat{G}_{\text{leg}}(e^{j\theta})\right] < \bar{\sigma}\left[\hat{T}_N(e^{j\theta})\right] < \gamma \qquad \forall\theta \in [0, \pi].$$

For this example, starting from $x_{k_0} = 0$, for any $w \in l_{2+}$ using the RHC strategy, $u^{\bar{\star}}$, and separately the legacy control, $\mu(x)$, results in costs $\sum_{j=k_0}^{\infty} h(x_j, u_j^{\bar{\star},j}) > \sum_{j=k_0}^{\infty} h(x_j, \mu(x_j))$. Additionally, increasing the horizon length does not necessarily improve performance in the $\|\cdot\|_\infty$ sense, as is shown in Example 2 (data in Table 1 and results in Fig.1(b)], where the magnitude of $\hat{G}_{\text{leg}}$ and $\hat{T}_N$, with $N = \{1, 2\}$ are shown. In both examples the plotted value of γ (our choice) is an upper bound to $\|\hat{T}_N\|_\infty$, as expected. Based on this discussion, for any specific system, the following inequalities need not hold.

1) $\left\|\hat{T}_N(e^{j\theta})\right\| \leq \left\|\hat{G}_{\text{leg}}(e^{j\theta})\right\|$ for all θ.
2) $\left\|\hat{T}_N\right\|_\infty \leq \left\|\hat{G}_{\text{leg}}\right\|_\infty$.
3) For any initial condition $\xi = x_{k_0}$, and any $w \in l_2$, $\sum_{j=k_0}^{\infty} h(x_j^{\text{RHC}}, u_j^{\text{RHC}}) \leq \sum_{j=k_0}^{\infty} h(x_j^{\text{leg}}, u_j^{\text{leg}})$.
4) If $N > M > 0$, then $\left\|\hat{T}_N\right\|_\infty \leq \left\|\hat{T}_M\right\|_\infty$.

VII. CONCLUSION

We have shown that, in a discrete-time context, a receding horizon control algorithm with a suboptimal minimization step, can be used to take advantage of previews of exogenous signals, disturbances or tracking commands, to possibly improve worst-case performance over some nominal controller, while still guaranteeing stability of the closed-loop system. However, these results are based on the availability of consistent disturbance previews, which does not allow for the consideration of unknown disturbances or noise in this framework.

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