ON KAMPÉ DE FÉRIET AND LAURICELLA FUNCTIONS OF MATRIX ARGUMENTS – I

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ABSTRACT

In the present paper we propose to define the $\Psi_A^{(n)}, \Xi_1^{(n)}$ and $\Phi_D^{(n)}$ functions of matrix arguments. In continuation of our previous studies [12, 13], we intend to prove five results in this paper - one for the Kampé de Fériet function $F^{1:2;2}_{1:1;1}$ and one each for the four Lauricella functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ of matrix arguments, using some of the definitions of Mathai [4].

INTRODUCTION

The Kampé de Fériet functions and Lauricella functions of scalar arguments have been an active field of research in special functions for a very long time. The extension of the properties of these functions for the matrix arguments case has been done prominently by Mathai [2, 4 – 9], besides G. Pederzoli [10]. Mathai [2] has also applied these functions to statistical distributions. In this study we have used some of the definitions of Mathai and have given our definitions for the functions $\Psi_A^{(n)}, \Xi_1^{(n)}$ and $\Phi_D^{(n)}$ of matrix arguments. Using these definitions we have established our results for the functions cited in the abstract. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all other symbols used are the same as in the works of Mathai [3, 4].

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1. Preliminary Definitions

We quote the following definitions of Mathai [4] which will be required by us in proving our results.

DEFINITION 1.1: The Kampé de Fériet function

$$F^{r:q;k}_{s:m;n} = F^{r:q;k}_{s:m;n}\left[\begin{matrix}(a_r):(b_q);(c_k); \\ (\alpha_s):(\beta_m);(\gamma_n);\end{matrix} -X,-Y\right]$$

of matrix arguments is defined as that function for which the M-transform (matrix-transform) is the following:

$$M\left(F^{r:q;k}_{s:m;n}\right)=[\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$F^{r:q;k}_{s:m;n}\left[\begin{matrix}(a_r):(b_q);(c_k); \\ (\alpha_s):(\beta_m);(\gamma_n);\end{matrix} -X,-Y\right]dXdY]$$

$$=\frac{\{\prod_{j=1}^{s}\Gamma_p(\alpha_j)\}\{\prod_{j=1}^{m}\Gamma_p(\beta_j)\}\{\prod_{j=1}^{n}\Gamma_p(\gamma_j)\}}{\{\prod_{j=1}^{r}\Gamma_p(a_j)\}\{\prod_{j=1}^{q}\Gamma_p(b_j)\}\{\prod_{j=1}^{k}\Gamma_p(c_j)\}}\times$$

$$\frac{\{\prod_{j=1}^{r}\Gamma_p(a_j-\rho_1-\rho_2)\}\{\prod_{j=1}^{q}\Gamma_p(b_j-\rho_1)\}\{\prod_{j=1}^{k}\Gamma_p(c_j-\rho_2)\}}{\{\prod_{j=1}^{s}\Gamma_p(\alpha_j-\rho_1-\rho_2)\}\{\prod_{j=1}^{m}\Gamma_p(\beta_j-\rho_1)\}\{\prod_{j=1}^{n}\Gamma_p(\gamma_j-\rho_2)\}}\Gamma_p(\rho_1)\Gamma_p(\rho_2) \quad ......(1.1)$$

for $\mathrm{Re}(\rho_1,\rho_2,a_j-\rho_1-\rho_2, j=1,\cdots,r;\alpha_j-\rho_1-\rho_2, j=1,\cdots,s;b_j-\rho_1,$
$j=1,\cdots,q;\beta_j-\rho_1, j=1,\cdots,m;c_j-\rho_2, j=1,\cdots,k;\gamma_j-\rho_2, j=1,\cdots,n)>(p-1)/2,$

where Re (.) denotes the real part of (.).

DEFINITION 1.2: The Lauricella function

$$F^{(n)}_A = F^{(n)}_A(a,b_1,\cdots,b_n;c_1,\cdots,c_n;-X_1,\cdots,-X_n)$$

of matrix arguments is defined as a function for which the M-transform is the following:

$$M(F_A^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$

$$F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n) dX_1 \cdots dX_n]$$

$$= \frac{\{\prod_{j=1}^{n} \Gamma_p(c_j)\}\{\prod_{j=1}^{n} \Gamma_p(b_j - \rho_j)\}\Gamma_p(a - \rho_1 - \cdots - \rho_n)\{\prod_{j=1}^{n} \Gamma_p(\rho_j)\}}{\Gamma_p(a)\{\prod_{j=1}^{n} \Gamma_p(b_j)\}\{\prod_{j=1}^{n} \Gamma_p(c_j - \rho_j)\}} \quad ......(1.2)$$

for $\mathrm{Re}\,(b_j - \rho_j, c_j - \rho_j, \rho_j, a - \rho_1 - \cdots - \rho_n) > (p-1)/2\,;\ j = 1, \cdots, n.$

DEFINITION 1.3: The Lauricella function

$$F_B^{(n)} = F_B^{(n)}(a_1, \cdots, a_n, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$$

of matrix arguments is defined as that function which has the following M-transform:

$$M(F_B^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$

$$F_B^{(n)}(a_1, \cdots, a_n, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) dX_1 \cdots dX_n]$$

$$= \frac{\Gamma_p(c)\{\prod_{j=1}^{n} \Gamma_p(a_j - \rho_j)\Gamma_p(b_j - \rho_j)\Gamma_p(\rho_j)\}}{\{\prod_{j=1}^{n} \Gamma_p(a_j)\Gamma_p(b_j)\}\Gamma_p(c - \rho_1 - \cdots - \rho_n)} \quad ......(1.3)$$

for $\mathrm{Re}\,(a_j - \rho_j, b_j - \rho_j, \rho_j, c - \rho_1 - \cdots - \rho_n) > (p-1)/2\,;\ j = 1, \cdots, n.$

DEFINITION 1.4: The Lauricella function

$$F_D^{(n)} = F_D^{(n)}(a, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$$

of matrix arguments is defined as that function which has the following M-transform:

$$M(F_D^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$
$$F_D^{(n)}(a, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) dX_1 \cdots dX_n]$$
$$= \frac{\Gamma_p(c)\{\prod_{j=1}^{n} \Gamma_p(b_j - \rho_j)\}\Gamma_p(a - \rho_1 - \cdots - \rho_n)\{\prod_{j=1}^{n} \Gamma_p(\rho_j)\}}{\Gamma_p(a)\{\prod_{j=1}^{n} \Gamma_p(b_j)\}\Gamma_p(c - \rho_1 - \cdots - \rho_n)} \quad \ldots\ldots(1.4)$$

for $\mathrm{Re}\,(\rho_j, b_j - \rho_j, a - \rho_1 - \cdots - \rho_n, c - \rho_1 - \cdots - \rho_n) > (p-1)/2\,;\ j = 1, \cdots, n.$

Now we give our definitions for the functions $\Psi_A^{(n)}, \Xi_1^{(n)}$ and $\Phi_D^{(n)}$ of matrix arguments.

DEFINITION 1.5: The $\Psi_A^{(n)}$- function of matrix arguments

$$\Psi_A^{(n)} = \Psi_A^{(n)}(\alpha, \beta_1, \cdots, \beta_{n-1}; \gamma_1, \cdots, \gamma_n; -X_1, \cdots, -X_n)$$

is defined as that function for which the M-transform is the following:

$$M(\Psi_A^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$
$$\Psi_A^{(n)}(\alpha, \beta_1, \cdots, \beta_{n-1}; \gamma_1, \cdots, \gamma_n; -X_1, \cdots, -X_n) dX_1 \cdots dX_n]$$
$$= \frac{\Gamma_p(\alpha - \rho_1 - \cdots - \rho_n)}{\Gamma_p(\alpha)} \frac{\{\prod_{i=1}^{n-1} \Gamma_p(\beta_i - \rho_i)\}}{\{\prod_{i=1}^{n-1} \Gamma_p(\beta_i)\}} \frac{\{\prod_{j=1}^{n} \Gamma_p(\gamma_j)\Gamma_p(\rho_j)\}}{\{\prod_{j=1}^{n} \Gamma_p(\gamma_j - \rho_j)\}} \quad \cdots(1.5)$$

for $\mathrm{Re}\,(\alpha - \rho_1 - \cdots - \rho_n, \beta_i - \rho_i, \gamma_j - \rho_j, \rho_j) > (p-1)/2\,;\ i = 1, \cdots, n-1\,;\ j = 1, \cdots, n.$

DEFINITION 1.6: The $\Xi_1^{(n)}$- function of matrix arguments

$$\Xi_1^{(n)} = \Xi_1^{(n)}(a_1, \cdots, a_n, b_1, \cdots, b_{n-1}; c; -X_1, \cdots, -X_n)$$

is defined as that function for which the M-transform is the following:

$$M(\Xi_1^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$
$$\Xi_1^{(n)}(a_1,\cdots,a_n,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)dX_1\cdots dX_n]$$
$$= \frac{\{\prod_{i=1}^{n}\Gamma_p(a_i-\rho_i)\}}{\{\prod_{i=1}^{n}\Gamma_p(a_i)\}} \frac{\{\prod_{j=1}^{n-1}\Gamma_p(b_j-\rho_j)\}}{\{\prod_{j=1}^{n-1}\Gamma_p(b_j)\}} \frac{\Gamma_p(c)\{\prod_{i=1}^{n}\Gamma_p(\rho_i)\}}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \quad \cdots(1.6)$$

for Re $(a_i-\rho_i, b_j-\rho_j, c-\rho_1-\cdots-\rho_n, \rho_i) > (p-1)/2$; $i=1,\cdots,n$; $j=1,\cdots,n-1$.

DEFINITION 1.7: The $\Phi_D^{(n)}$- function of matrix arguments

$$\Phi_D^{(n)} = \Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)$$

is defined as that function which has the following M-transform :

$$M(\Phi_D^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$
$$\Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)dX_1\cdots dX_n]$$
$$= \frac{\Gamma_p(a-\rho_1-\cdots-\rho_n)\}}{\Gamma_p(a)} \frac{\{\prod_{i=1}^{n-1}\Gamma_p(b_i-\rho_i)\}}{\{\prod_{i=1}^{n-1}\Gamma_p(b_i)\}} \frac{\Gamma_p(c)\{\prod_{j=1}^{n}\Gamma_p(\rho_j)\}}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \quad \cdots(1.7)$$

for Re $(a-\rho_1-\cdots-\rho_n, b_i-\rho_i, c-\rho_1-\cdots-\rho_n, \rho_j) > (p-1)/2$; $i=1,\cdots,n-1$; $j=1,\cdots,n$.

2. The Kampé de Fériet Function of Matrix Arguments

THEOREM 2.1:

$$F_{1:1;1}^{1:2;2}\left[\begin{matrix}\alpha:\beta,\lambda;\beta',\lambda'; \\ \gamma:\ \mu;\ \mu';\end{matrix} -X,-Y\right] = \frac{\Gamma_p(\mu)\Gamma_p(\mu')}{\Gamma_p(\lambda)\Gamma_p(\mu-\lambda)\Gamma_p(\lambda')\Gamma_p(\mu'-\lambda')} \int_0^I \int_0^I |U|^{\lambda-(p+1)/2} \times$$
$$|V|^{\lambda'-(p+1)/2}|I-U|^{\mu-\lambda-(p+1)/2}|I-V|^{\mu'-\lambda'-(p+1)/2} \times$$

Continued to the next page…..

$$F_1(\alpha,\beta,\beta';\gamma;-U^{1/2}XU^{1/2},-V^{1/2}YV^{1/2})dUdV \quad \cdots(2.1)$$

for $\mathrm{Re}\,(\lambda,\lambda',\mu-\lambda,\mu'-\lambda') > (p-1)/2.$

PROOF: From definition (1.1) we deduce that,

$$M\left(F^{1:2;2}_{1:1;1}\right) = [\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2} \times$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\alpha:\beta,\lambda;\beta',\lambda'; \\ \gamma:\ \mu;\ \ \mu';\end{matrix} -X,-Y\right]dXdY\ ]$$

$$= [\frac{\Gamma_p(\gamma)\Gamma_p(\mu)\Gamma_p(\mu')\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)\Gamma_p(\lambda-\rho_1)\Gamma_p(\beta'-\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\lambda)\Gamma_p(\beta')\Gamma_p(\lambda')\Gamma_p(\gamma-\rho_1-\rho_2)\Gamma_p(\mu-\rho_1)\Gamma_p(\mu'-\rho_2)} \times$$

$$\Gamma_p(\lambda'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)] \quad \cdots\cdots(2.2)$$

for $\mathrm{Re}\,(\rho_1,\rho_2,\alpha-\rho_1-\rho_2,\beta-\rho_1,\lambda-\rho_1,\beta'-\rho_2,\lambda'-\rho_2,\gamma-\rho_1-\rho_2,$

$\mu-\rho_1,\mu'-\rho_2) > (p-1)/2.$

Now taking the M-transform of the right side of eq. (2.1) with respect to the variables X, Y and the parameters ρ_1,ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2} \times$$

$$F_1(\alpha,\beta,\beta';\gamma;-U^{1/2}XU^{1/2},-V^{1/2}YV^{1/2})dXdY \quad \cdots\cdots(2.3)$$

Applying the transformations,

$X_1 = U^{1/2}XU^{1/2}$, $Y_1 = V^{1/2}YV^{1/2}$ (implying thereby $dX_1 = |U|^{(p+1)/2}dX$,

$dY_1 = |V|^{(p+1)/2}dY$ and $|X_1| = |U||X|$, $|Y_1| = |V||Y|$)

in the expression (2.3) and then making use of Mathai's [4] definition of M-transform of an Appell's function F_1 we get,

$$|U|^{-\rho_1}|V|^{-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)} \quad \cdots(2.4)$$

Substituting this expression on the right side of eq. (2.1) and integrating out the variables U and V in the resulting expression by using a type-1 Beta integral we finally obtain $M\left(F^{1:2;2}_{1:1;1}\right)$ as given by eq. (2.2), which establishes the theorem.

3. The Lauricella Functions of Matrix Arguments

THEOREM 3.1:

$$F_A^{(n)}(\alpha,\beta_1,\cdots,\beta_n;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)$$

$$=\frac{1}{\Gamma_p(\beta_n)}\int_{T>0} e^{-\mathrm{tr}(T)}|T|^{\beta_n-(p+1)/2}\Psi_A^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1};\gamma_1,\cdots,\gamma_n;$$

$$-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dT \qquad \cdots\cdots(3.1)$$

for $\mathrm{Re}(\beta_n)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (3.1) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\Psi_A^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1};\gamma_1,\cdots,\gamma_n;$$

$$-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n \qquad \cdots\cdots(3.2)$$

which on applying the transformation

$$Y_n=T^{1/2}X_nT^{1/2} \text{ (with } dY_n=|T|^{(p+1)/2}dX_n \text{ and } |Y_n|=|T||X_n|\text{)}$$

and then using definition (1.5) yields,

$$|T|^{-\rho_n}\frac{\Gamma_p(\alpha-\rho_1-\cdots-\rho_n)}{\Gamma_p(\alpha)}\frac{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i-\rho_i)\}}{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i)\}}\frac{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j)\Gamma_p(\rho_j)\}}{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j-\rho_j)\}} \qquad \cdots\cdots(3.3)$$

Substituting this expression on the right side of eq. (3.1) and then integrating out T in the resulting expression by using a Gamma integral produces $M(F_A^{(n)})$ as given by eq. (1.2), thereby proving the theorem.

THEOREM 3.2:

$$F_B^{(n)}(\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n;\gamma;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(\beta_n)}\int_{T>0} e^{-tr(T)}|T|^{\beta_n-(p+1)/2}\,\Xi_1^{(n)}(\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_{n-1};\gamma;$$
$$-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dT \qquad \cdots\cdots(3.4)$$

for $\mathrm{Re}\,(\beta_n)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (3.4) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we obtain,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\,\Xi_1^{(n)}(\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_{n-1};\gamma;$$
$$-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n \qquad \cdots\cdots(3.5)$$

On applying the same transformation as in the previous theorem and using the definition (1.6) the above expression yields,

$$|T|^{-\rho_n}\frac{\{\prod_{i=1}^{n}\Gamma_p(\alpha_i-\rho_i)\}}{\{\prod_{i=1}^{n}\Gamma_p(\alpha_i)\}}\frac{\{\prod_{j=1}^{n-1}\Gamma_p(\beta_j-\rho_j)\}}{\{\prod_{j=1}^{n-1}\Gamma_p(\beta_j)\}}\frac{\Gamma_p(\gamma)\{\prod_{i=1}^{n}\Gamma_p(\rho_i)\}}{\Gamma_p(\gamma-\rho_1-\cdots-\rho_n)} \qquad \cdots\cdots(3.6)$$

Substituting this expression on the right side of eq. (3.4) and integrating out T in the resulting expression by using a Gamma integral generates $M(F_B^{(n)})$ as given by eq. (1.3) hence, completing the proof.

THEOREM 3.3:

$$F_C^{(n)}(\alpha,\beta;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0} e^{-tr(T)}|T|^{\alpha-(p+1)/2}\,\Psi_2^{(n)}(\beta;\gamma_1,\cdots,\gamma_n;-T^{1/2}X_1T^{1/2},\cdots,$$
$$-T^{1/2}X_nT^{1/2})dT \qquad \cdots\cdots(3.7)$$

for $\mathrm{Re}\,(\alpha)>(p-1)/2$.

PROOF: The Lauricella function $F_C^{(n)}$ of matrix arguments and the $\Psi_2^{(n)}$- function of matrix arguments have been defined through their M- transforms in eqs.(1.2) and (1.3) respectively of the authors' paper [13]. In the proof this theorem we shall make use of these definitions.

Taking the M-transform of the right side of eq. (3.7) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\Psi_2^{(n)}(\beta;\gamma_1,\cdots,\gamma_n;-T^{1/2}X_1T^{1/2},\cdots,-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n \qquad \cdots\cdots(3.8)$$

Making use of the transformations

$Y_j=T^{1/2}X_jT^{1/2}$, so that , $dY_j=|T|^{(p+1)/2}dX_j$ and $|Y_j|=|T||X_j|$ for $j=1,\cdots,n$

in the above expression and then applying definition (1.3) of the authors' paper [13], the outcome is

$$|T|^{-\rho_1-\cdots-\rho_n}\frac{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j)\Gamma_p(\rho_j)\}\Gamma_p(\beta-\rho_1-\cdots-\rho_n)}{\Gamma_p(\beta)\{\prod_{j=1}^{n}\Gamma_p(\gamma_j-\rho_j)\}} \qquad \cdots\cdots(3.9)$$

Substituting this expression on the right side of eq. (3.7) and then integrating out T in the resulting expression by using a Gamma integral, we obtain $M(F_C^{(n)})$ as given by eq. (1.2) of our paper [13], thus finishing the proof.

THEOREM 3.4:

$$F_D^{(n)}(\alpha,\beta_1,\cdots,\beta_n;\gamma;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(\beta_n)}\int_{T>0}e^{-tr(T)}|T|^{\beta_n-(p+1)/2}\Phi_D^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1};\gamma;$$
$$-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dT \qquad \cdots\cdots(3.10)$$

for $\mathrm{Re}\,(\beta_n)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (3.10) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\Phi_D^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1};\gamma;$$

$$-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n \qquad \cdots\cdots(3.11)$$

On applying the same transformation as in theorem (3.1) and then using the definition (1.7) the expression (3.11) leads us to

$$|T|^{-\rho_n}\frac{\Gamma_p(\alpha-\rho_1-\cdots-\rho_n)}{\Gamma_p(\alpha)}\frac{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i-\rho_i)\}}{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i)\}}\frac{\Gamma_p(\gamma)\{\prod_{j=1}^{n}\Gamma_p(\rho_j)\}}{\Gamma_p(\gamma-\rho_1-\cdots-\rho_n)} \qquad \cdots\cdots(3.12)$$

Substituting this expression on the right side of eq. (3.10) and integrating out T in the resulting expression by using a Gamma integral gives $M(F_D^{(n)})$ in conformity with eq. (1.4), which proves the result.

Further results for the Lauricella and related functions of matrix arguments will appear in one of our future communications.

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