Regularity of axially symmetric flows in a half-space in three dimension

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Abstract

We study axially symmetric solutions with no swirl of the three dimensional Navier-Stokes equations in a half-space. We prove that *suitable weak solutions* in this case are Hölder continuous up to the boundary at all points except for the origin. For interior points this implies smoothness in the spatial variables. Hölder continuity at the origin remains as an open problem.

Key words. axially symmetric flow, Navier-Stokes equations, no swirl

AMS subject classifications. 76D03, 76D05

1 Introduction

We consider a vector field in a half-space $\mathbb{R}^3_+$ which vanishes at $\partial\mathbb{R}^3_+$. If a vector field u is invariant under rotation around x_3-axis, we say that it is axially symmetric, in other words, $u(\mathbf{R}(x)) = \mathbf{R}(u(x))$ for every rotation $\mathbf{R}$ about the x_3-axis. If, moreover, an axially symmetric vector field v is invariant under reflection by every plane containing x_3-axis, we say it is axially symmetric with no swirl, that is to say, v is axially symmetric and $v(\mathbf{T}(x)) = \mathbf{T}(v(x))$ for every reflection $\mathbf{T}$ as above.

In this paper we study the regularity of axially symmetric solutions with no swirl of the Navier-Stokes equations in $\mathbb{R}^3_+$. When $\mathbb{R}^3_+$ is replaced by $\mathbb{R}^3$, it is known that such solutions are regular (see [3], [7] and [15]).

Our main result is that *suitable weak solutions* of the Navier-Stokes equations are locally Hölder continuous up to the boundary $(\bar{\mathbb{R}}^3_+ \setminus \{\mathbf{0}\}) \times (0,\infty)$. It follows that in $\mathbb{R}^3_+ \times (0,\infty)$ the solutions are smooth in spatial variables x.

The main tools are the partial regularity results of *suitable weak solutions* of the Navier-Stokes equations (see [1], [4], and [10] for interior case and see [12] for boundary case) and the maximum principle for the azimuthal component of vorticity, which was also used to prove full regularity in the case of $\mathbb{R}^3$.

Our result implies that the only possible singular point for axially symmetric solutions with no swirl in $\mathbb{R}^3_+$ would be the origin. It seems to be open whether or not singularity may occur at the origin and so we leave it as an open problem.

The plan of the paper is as follows:

In section 2, we introduce notation and definitions and review some well-known facts for our proof, and finally state our main theorem.

In section 3, we present the proof of the main theorem.

2 Preliminaries and main result

In this section, we introduce notation and definitions and also recall some well-known results used later, and finally state our main theorem. Let us begin with notation.

• We denote by $\mathbb{R}^3_+$ as a half-space of three dimension $\mathbb{R}^3$ and write the origin of $\mathbb{R}^3$ as $\mathbf{0}$.

• For a given point $(x,t) \in \mathbb{R}^3_+ \times I$, we denote by $B_{x,r} \subset \mathbb{R}^3_+$ the ball of radius r centered at x where $0 < r < \operatorname{dist}(x, \partial\mathbb{R}^3_+)$. We also denote a parabolic ball by $Q_{(x,t),r} = B_{x,r} \times (t-r^2, t)$ where $0 < r < \min\{\operatorname{dist}(x,\partial\mathbb{R}^3_+), \sqrt{t}\}$.

• If x is located on the boundary of $\mathbb{R}^3_+$, then we write a half ball of radius r as $B^+_{x,r} = B_{x,r} \cap \mathbb{R}^3_+$ where $B_{x,r} \subset \mathbb{R}^3$. Similarly if $(x,t) \in \partial\mathbb{R}^3_+ \times I$, a parabolic half ball at (x,t) is defined by $Q^+_{(x,t),r} = B^+_{x,r} \times (t-r^2, t)$ for $0 < r < \sqrt{t}$.

• Let $\Omega \subset \mathbb{R}^3$ be a domain. For $1 \le q \le \infty$, $W^{k,q}(\Omega)$ denote the usual Sobolev space, i.e. $W^{k,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), 0 \le |\alpha| \le k\}$. As usual, $W^{k,q}_0(\Omega)$ is defined the completion of $\mathcal{C}^\infty_0(\Omega)$ in $W^{k,q}(\Omega)$.

• Let $1 \le q, r \le \infty$ and $I = (0,\infty)$. $L^r(I; W^q(\Omega))$ is the Banach space consisting of all measurable functions with a finite norm

$$||u||_{L^r(I;W^q(\Omega))} = (\int_I ||u(\cdot,t)||^r_{W^q(\Omega)}\,\mathrm{d}t)^{\frac{1}{r}}.$$

The Navier-Stokes equations are expressed in the Cartesian coordinates x, y, and z in a half-space $\mathbb{R}^3_+$ as follows:

$$\left.\begin{array}{c} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \\ \nabla\cdot u = 0 \end{array}\right\} \text{ in } \mathbb{R}^3_+ \times (0,\infty) \tag{1}$$

with initial and boundary conditions

$$\left\{\begin{array}{ll} u(x,0) = u_0 & \text{when } t = 0 \\ u = 0 & \text{on } \partial\mathbb{R}^3_+ \times (0,\infty) \end{array}\right. \tag{2}$$

where $u : \mathbb{R}^3_+ \times (0,\infty) \to \mathbb{R}^3$ and $p : \mathbb{R}^3_+ \times (0,\infty) \to \mathbb{R}$ are unknown vector field and pressure respectively and ν is the kinematic viscosity, and f and u_0 are prescribed external force and initial condition, respectively. From now on, f and u_0 are, for simplicity, assumed to be smooth and compactly supported and we denote $I = (0,\infty)$ for simplicity. A solution u of (1) with initial and homogeneous boundary conditions (2) is called a *suitable weak solution* if the following conditions are satisfied:

1. u and p, which are in the class
$$u \in L^\infty(I; L^2(\mathbb{R}^3_+)^3) \cap L^2(I; W_0^{1,2}(\mathbb{R}^3_+)^3), \quad p \in L^{\frac{5}{3}}(I; L^{\frac{5}{3}}(\mathbb{R}^3_+)),$$
solve (1) in a weak sense:
$$\int_{\mathbb{R}^3_+ \times I} (u\xi_t - \nu \nabla u : \nabla \xi - (u\nabla)u\xi + f\xi)\,\mathrm{d}x\,\mathrm{d}t = 0$$
for all $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^3_+ \times I; \mathbb{R}^3)$ with $\nabla \cdot \xi = 0$ and
$$\int_{\mathbb{R}^3_+} u(\cdot, t)\nabla\phi\,\mathrm{d}x = 0$$
for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3_+)$ and a.e. $t \in I$.
2. u satisfies the following global energy inequality:
$$\begin{aligned} & \frac{1}{2}\int_{\mathbb{R}^3_+} |u(\cdot,t)|^2\,\mathrm{d}x + \nu \int_{\mathbb{R}^3_+ \times I} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le\quad & \frac{1}{2}\int_{\mathbb{R}^3_+} |u_0|^2\,\mathrm{d}x + \int_{\mathbb{R}^3_+ \times I} f \cdot u\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$
for almost all $t \in I$.
3. u and p satisfy the following local energy inequality:
$$\begin{aligned} & \int_{\mathbb{R}^3_+} |u(x,t)|^2\phi(x,t)\,\mathrm{d}x + 2\nu\int_{\mathbb{R}^3_+ \times I} |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t \\ \le\quad & \int_{\mathbb{R}^3_+ \times I} |u|^2(\partial_t\phi + \nu\Delta\phi) + (|u|^2 + 2p)u \cdot \nabla\phi + 2f \cdot u\phi\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3}$$
for a.e. $t \in I$ and for all non-negative function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3 \times I)$.

The existence of *suitable weak solutions* was proved in [1] and slight modified definition of it are observed in other contexts (see [4], [10] and [12]). In this paper, we follow the definition shown in [1].
We first define a regular or singular point of a *suitable weak solution* u.

Definition 2.1 *We say a point $(x,t) \in \bar{\mathbb{R}}^3_+ \times I$ is a regular point when a* suitable weak solution *u is bounded in a neighborhood $Q_{(x,t),r}$ (or $Q^+_{(x,t),r}$) for some $0 < r < \min\{\, dist(x, \partial\mathbb{R}^3_+),\ \sqrt{t}\,\}$ (or $0 < r < \sqrt{t}$) for $x \in \mathbb{R}^3_+$ (or $x \in \partial\mathbb{R}^3_+$). Otherwise it is called a singular point. In addition, we say u is regular at (x,t) if it is a regular point. Similarly we say u is singular at (x,t) if (x,t) is a singular point.* □

It is well-known that weak solutions are smooth in spatial variables and Hölder continuous in time in a neighborhood of an interior regular point (see [13]). At the boundary such solutions are Hölder continuous at each regular point, while the higher regularity seems to be open (see [12]).

On the other hand, it is also well-known that weak solutions are smooth and unique for a short time for given smooth data f and u_0 (see e.g. Theorem 3.2 in [14, page 22] or Theorem 9.3 in [2, page 80]). Here we recall a well known result regarding a Hausdorff measure of possible singular set of time (see e.g. [8], [11], and [14]).

Theorem 2.2 *Let u be a weak solution of the Navier-Stokes equations (1). Then there exists a closed set $\mathcal{S} \subset I$ whose $\frac{1}{2}$ dimensional Hausdorff measure vanishes, such that u is regular in $\bar{\mathbb{R}}^3_+ \times (I \setminus \mathcal{S})$.*

Remark 2.3 *It should be mentioned that if* suitable weak solutions *which are axially symmetric in $\mathbb{R}^3_+$ have a singular point, then singularity can occur only on the x_3-axis. This arguments are based on the results that 1-dimensional parabolic Hausdorff measure of singular set is zero for the interior case proved in [1] (see also [4], and [10]) and for the boundary case proved recently in [12]. Therefore, it suffices to investigate behavior of solutions near x_3-axis provided that it is axially symmetric.* $\square$

We conclude this section by stating main theorem and its proof will be given in next section.

Main Theorem Let u be a *suitable weak solution* of (1) which is axially symmetric with no swirl in a half space $\mathbb{R}^3_+$. Then u is regular for every $(x,t) \in \bar{\mathbb{R}}^3_+ \times I$ unless $x = \mathbf{0}$. Therefore, it is Hölder continuous in $(\bar{\mathbb{R}}^3_+ \setminus \{\mathbf{0}\}) \times I$.

3 The proof of main Theorem

We recall that the partial regularity results imply that all points $(\mathbf{x},t)$ with $x_1^2 + x_2^2 \neq 0$ are regular because, as mentioned earlier, singularity cannot happen away from the axis of symmetry for axially symmetric solutions. Therefore, we only have to prove that every point in $\{\,(\mathbf{x},t) \in \bar{\mathbb{R}}^3_+ \times I \;:\; x_3 > 0\,\}$ is regular. In the sequel, we only consider a fixed *suitable weak solution* u of (1) which is axially symmetric with no swirl. Here we assume that f and u_0 are smooth and compactly supported. For convenience, we denote $\mathcal{Z}^+ = \{\,\vec{z} = (0,0,z) \in \mathbb{R}^3_+ \;:\; z > 0\,\}$. Let us start with a simple observation.

Lemma 3.1 *There exist two sequences $(z_i')_{i=1}^{\infty}$ and $(z_i'')_{i=1}^{\infty}$ such that $z_i' \searrow 0$ and $z_i'' \nearrow \infty$, and every point in $\{\,(\mathbf{x},t) \in \mathcal{Z}^+ \times I \,:\, x_3 = z_i' \text{ or } x_3 = z_i''\,\}$ is regular.*

Proof. We only show the validity to the case of a decreasing sequence $(z_i')_{i=1}^{\infty}$. The other part can be proved by the similar argument. Suppose there is no such sequence. Then there is an interval $J_{\mathcal{Z}^+} \equiv (0,\delta)$ such that for every $\vec{z} = (0,0,z) \in \mathcal{Z}^+$ with $z \in J_{\mathcal{Z}^+}$, u is singular at $(\vec{z}, t_z)$ for some time $t_z \in I$. We collect all such points and denote it by

$$\mathcal{S}_\delta = \{(\vec{z}, t_z) \in \mathcal{Z}^+ \times I : u \text{ is singular at } (\vec{z}, t_z) \text{ where } \vec{z} = (0,0,z), z \in J_{\mathcal{Z}^+}\}.$$

Note that $\mathcal{S}_\delta$ is a subset of possible singular set and it can be easily checked that 1-dimensional parabolic Hausdorff measure of $\mathcal{S}_\delta$ is finite, not zero. In fact, $\mathcal{P}^1(\mathcal{S}_\delta) \geq \delta > 0$. However, in [1], it was proved that the 1-dimensional parabolic Hausdorff measure of a possible singular set is zero, which leads to a contradiction. Therefore, such sequence must exist. The existence of an increasing sequence $(z_i^{''})_{1=1}^\infty$ can be proved by the similar argument, and therefore we omit the details. This completes the proof. □

Let $\{\vec{z}_i^0 = (0,0,z_i')\}_{i=1}^\infty$ and $\{\vec{z}_i^\infty = (0,0,z_i^{''})\}_{i=1}^\infty$ be the sequences obtained in the previous Lemma. Without loss of generality, we assume that $z_1' < z_1^{''}$ because $z_i' \searrow 0$ and $z_i^{''} \nearrow \infty$ as $i \to \infty$. Next, we define a set $\mathcal{Z}_l^+ \subset \mathbb{R}_+^3$ as follows:

$$\mathcal{Z}_l^+ \equiv \{\vec{x} \in \mathcal{Z}^+ \,:\, z_l' < x_3 < z_l^{''}\} \quad \text{for each} \quad l \in \mathbb{N}.$$

We also consider a set $\mathcal{S}_l \subset I$, which is related with $\mathcal{Z}_l^+$ and defined as follows:

$$\mathcal{S}_l \equiv \{t \in I \,:\, u \text{ is singular for some } (\vec{x}, t) \in \mathcal{Z}_l^+ \times I\} \quad \text{for each} \quad l \in \mathbb{N}.$$

Our aim is to show that $\mathcal{S}_l = \emptyset$ for all $l \in \mathbb{N}$, which implies our main result. Suppose that this is not the case. Then there exists $m \in \mathbb{N}$ such that $\mathcal{S}_m \neq \emptyset$ and then we consider

$$t_m \equiv \inf_{t \in I} \mathcal{S}_m, \quad \mathcal{S}_m \equiv \{t \in I : u \text{ is singular for some } (\vec{z}, t) \in \mathcal{Z}_m^+ \times I\}. \tag{4}$$

Lemma 3.2 *Suppose $\mathcal{S}_m \neq \emptyset$ for some $m \in \mathbb{N}$. Let t_m be the number defined in (4). Then t_m is a strictly positive number in I and moreover there exists $\vec{z}_m \in \mathcal{Z}_m^+$ such that u is singular at $(\vec{z}_m, t_m)$.*

Proof. We note first that t_m is strictly bigger than 0 because u is smooth for a short time interval depending on given smooth data f and u_0 (see e.g. Theorem 3.2 in [14, page 22] or Theorem 9.3 in [2, page 80]). We claim that there exists $\vec{z}_m = (0,0,z_m) \in \mathcal{Z}_m^+$ such that u is singular at $(\vec{z}_m, t_m)$. Indeed if t_m is isolated in $\mathcal{S}_m$, then obviously there is a point $\vec{z}_m \in \mathcal{Z}_m^+$ such that u is singular at that point. On the other hand, if t_m is a limit point in $\mathcal{S}_m$, then there is a sequence of point $(\vec{z}_{m,j}, t_{m,j})_{j=1}^\infty \in \mathcal{Z}_m^+ \times I$ such that $t_{m,j} \searrow t_m$. On the other hand, since $[z_m', z_m^{''}]$ is compact, $\{\vec{z}_{m,j}\}$ must have a limit point, say $\vec{z}_m$, and therefore an appropriate subsequence of $(\vec{z}_{m,j}, t_{m,j})$, which we relabel as $(\vec{z}_{m,j}, t_{m,j})$, converges to $(\vec{z}_m, t_m)$. We note that z_m must be located in $[z_m', z_m^{''}]$, that is $z_m' \leq z_m \leq z_m^{''}$. However z_m cannot be z_m' nor $z_m^{''}$, because z_m' and $z_m^{''}$ were chosen at the beginning to satisfy that u is regular at (z_m', t) and $(z_m^{''}, t)$ for all $t \in I$. Our assertion is completed by noting that singular set is closed. This completes the proof. □

Remark 3.3 *It is worth noting that there exists positive numbers r such that u is bounded in $B_{\vec{z}_m^0,r} \times (0,t_m]$ and $B_{\vec{z}_m^\infty,r} \times (0,t_m]$. Indeed, according to Theorem 2.2, there exists $t_0 > 0$ such that $u(\cdot,t)$ is regular everywhere provided that $t < t_0$. Combining facts that u is regular for all time at $\vec{z}_m^0$ and $\vec{z}_m^\infty$ and $[t_0,t_m]$ is compact set, we can say that there exist positive numbers r_1, r_2 such that u is bounded in $B_{\vec{z}_m^0,r_1} \times (0,t_m]$ and $B_{\vec{z}_m^\infty,r_2} \times (0,t_m]$, respectively. By choosing $r = \min\{r_1, r_2\}$, we completes our claim.* $\square$

Next step is to investigate the vorticity equation. If flow is axially symmetry with no swirl, then with the aid of the cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

velocity vector u is converted as follows:

$$u^r e_r + u^z e_z = u^x e_x + u^y e_y + u^z e_z$$

where e_x, e_y and e_z are the basis vectors with unit length in the Cartesian coordinates and e_r and e_z are the basis vector with unit length in the cylindrical coordinates (note that azimuthal component u^θ vanishes because it has no swirl). In addition, each component satisfies following relation

$$u^r(r,z) = u^x\cos\theta + u^y\sin\theta, \quad u^z(r,z) = u^z(x,y,z).$$

We can also see that the system (1) can be written in the cylindrical coordinates (see e.g. [6, page 48-49]).

Now we consider the vorticity vector $w = \nabla \times u$. For simplicity, we assume that outer force $f = 0$. The good advantage of a flow with no swirl is that the vector w has only azimuthal component, i.e. $w^\theta = u^r_z - u^z_r$. More precisely, $w = \nabla \times u = (0, w^\theta, 0) = (0, u^r_z - u^z_r, 0)$ and it solves the following single equation in $\mathbb{R}^3_+ \times I$:

$$\frac{\partial w}{\partial t} + u^r\frac{\partial w}{\partial r} + u^z\frac{\partial w}{\partial z} - \frac{u^r}{r}w - \nu[\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}] = 0. \tag{5}$$

with smooth initial condition $w_0 = \nabla \times u_0$ and boundary $w = u^r_z - u^z_r$ on $\partial\mathbb{R}^3_+ \times I$. Now we define an axially scalar function $\xi \equiv \frac{1}{r}w$ and simple calculations show that it satisfies the following equation in $\mathbb{R}^3_+ \times I$:

$$\xi_t - \nu(\frac{\partial^2\xi}{\partial r^2} + \frac{\partial^2\xi}{\partial z^2}) + u^r\frac{\partial\xi}{\partial r} + u^z\frac{\partial\xi}{\partial z} - 3\nu\frac{1}{r}\frac{\partial\xi}{\partial r} = 0. \tag{6}$$

We can apply the maximum principle to ξ in (6) by using the fact that ξ is a scalar function. The first step is to consider ξ to be a function defined in five dimensional space $\mathbb{R}^3_+ \times \mathbb{R}^2 \times I$ by introducing another independent two variables z_1, z_2 although ξ is independent of the variables z_1 and z_2. For clarity, we denote the extended function by $\tilde{\xi}$, which is defined as follows:

$$\tilde{\xi} : \mathbb{R}^3_+ \times \mathbb{R}^2 \times I \to \mathbb{R} \text{ such that } \tilde{\xi}(x,y,z,z_1,z_2,t) = \xi(x,y,z,t). \tag{7}$$

In the same manner, we can also extend u^r, u^z, denoted by $\tilde{u}^r, \tilde{u}^z$, into $\mathbb{R}^3_+ \times \mathbb{R}^2 \times I$. Redefining $r^2 = x^2 + y^2 + z_1^2 + z_2^2$, we consider the following equations:

$$\tilde{\xi}_t - \nu(\frac{\partial^2 \tilde{\xi}}{\partial r^2} + \frac{\partial^2 \tilde{\xi}}{\partial z^2}) + \tilde{u}^r \frac{\partial \tilde{\xi}}{\partial r} + \tilde{u}^z \frac{\partial \tilde{\xi}}{\partial z} - 3\nu \frac{1}{r}\frac{\partial \tilde{\xi}}{\partial r} = 0 \tag{8}$$

Then (8) becomes

$$\tilde{\xi}_t - \nu \tilde{\Delta} \tilde{\xi} + \tilde{u}^r \partial_r \tilde{\xi} + \tilde{u}^z \partial_z \tilde{\xi} = 0 \tag{9}$$

where $\tilde{\Delta}$ indicates the Laplace operator in five dimension. To sum up, (6) is converted to 5-dimensional parabolic equation (9) in $\mathbb{R}^3_+ \times \mathbb{R}^2 \times I$. Note that the equation (8) (or (9)) is reduced to (6) when $\mathbb{R}^3_+ \times \mathbb{R}^2 \times I$ is restricted to $\mathbb{R}^3_+ \times I$.

We argue as follows. We first show that $\tilde{\xi}$ is regular at $(\vec{z}_m, 0, 0, t_m)$, which implies ξ is regular at $(\vec{z}_m, t_m)$, too. Therefore u is also regular at $(\vec{z}_m, t_m)$, which is contrary to the assumption that u is singular at $(\vec{z}_m, t_m)$. Therefore $\mathcal{S}_m$ must be empty, which makes our argument complete. Without any confusion, we denote $\vec{z}_m = (\vec{z}_m, 0, 0) \in \mathbb{R}^3_+ \times \mathbb{R}^2$ and $\mathbb{R}^3_+ \times \mathbb{R}^2 \times I = \mathbb{R}^4 \times \mathbb{R}_+ \times I$ by interchanging coordinates. Now we are ready to prove the main theorem.

The Proof of Main Theorem We first show that $\tilde{\xi}$ is regular at $(\vec{z}_m, t_m)$. As mentioned in Remark 3.3, there exists positive number r_1 such that u is bounded in $B_{\vec{z}^0_m, r} \times (0, t_m]$ and $B_{\vec{z}^\infty_m, r} \times (0, t_m]$ for all $0 < r \leq r_1$. On the other hand, there exists $r_2 > 0$ such that u is smooth at $t = t_m - r_2^2$, which is due to Theorem 2.2 because the set of possible singular time is of $\frac{1}{2}$ Hausdorff measure zero. Without loss of generality, we may take $r_2 < r_1$. We denote r_2 by r and define $\Omega = [0, r) \times (z'_m, z''_m) \subset \mathbb{R}^4 \times \mathbb{R}_+$ where $[0, r) = \{\, y \in \mathbb{R}^4 \, : \, |y| < r \,\}$, and consider parabolic domains

$$Q = \Omega \times (t_m - r^2, t_m - \epsilon), \quad Q^\epsilon = \Omega \times (t_m - r^2, t_m - \epsilon)$$

where ϵ is an arbitrary small positive number with $\epsilon < \frac{r^2}{4}$.

We note first that $\tilde{\xi}$ is regular on $Q_0 \equiv \Omega \times \{t_m - r^2\}$ because $\sup_{Q_0} |\tilde{\xi}| \leq C(\sup_{Q_0} |\xi| + \sup_{Q_0} |\nabla \xi|)$, which is bounded because of our choice of r. Hence $\tilde{\xi}$ is bounded on $\Omega \times \{t_m - r^2\}$. For convenience we denote $M_0 = \sup_{Q_0} |\tilde{\xi}|$. We also show that $\tilde{\xi}$ is bounded on other parabolic boundary of Q. Note that other parabolic boundary of Q is composed of three parts, which are denoted by ∂Q_i for $i = 1, 2, 3$,

$$\begin{aligned}
\partial Q_1 &\equiv [0, r) \times \{z'_m\} \times (t_m - r^2, t_m), \\
\partial Q_2 &\equiv [0, r) \times \{z''_m\} \times (t_m - r^2, t_m), \\
\partial Q_3 &\equiv \{r\} \times (z'_m, z''_m) \times (t_m - r^2, t_m).
\end{aligned}$$

It is obvious that $\tilde{\xi}$ is bounded on ∂Q_i for $i = 1, 2$ because z'_m and z''_m were chosen such a way that $\tilde{\xi}$ is regular on them. In addition, since ∂Q_3 is the part strictly away from z-axis and boundary, $\tilde{\xi}$ is also regular at every point on ∂Q_3, which implies that $\tilde{\xi}$ is bounded on ∂Q_3. Let $M_i = \sup_{Q_i} |\tilde{\xi}|$ for $i = 1, 2, 3$ and $M = \max\{M_i : i = 0, 1, 2, 3\}$.

Now we claim that $\tilde{\xi}$ is bounded by M in a parabolic domain Q^ϵ. Indeed, $\tilde{\xi}$ is bounded on each parabolic boundaries of Q^ϵ, denoted by $\partial_p Q^\epsilon$, because they are subset of parabolic boundaries $\partial_p Q$ of Q.

On the other hand, since u is smooth in spatial variable and each spatial derivative is regular in Q^ϵ, so are $\tilde{\xi}$ and each spatial derivative of $\tilde{\xi}$ where we used that $\tilde{\xi}$ is axially symmetric. Moreover, we can see that all spatial derivatives of $\tilde{\xi}$ are absolutely continuous with respect to time variable (see e.g. [13]), and therefore ξ_t is also absolutely continuous in Q^ϵ by the equation (9), which enable us to apply the maximum principle to $\tilde{\xi}$ in Q^ϵ (see e.g. Theorem 2.12 in [9, page 15] and Chap 3.7 in [5]). Therefore we obtain $\sup_{Q^\epsilon} |\tilde{\xi}| \leq \sup_{\partial_p Q^\epsilon} |\tilde{\xi}|$, which is bounded by M. Since the upper bound M is independent of ϵ, passing to the limit, we obtain $\operatorname{ess\,sup}_{\mathrm{Q}} |\tilde{\xi}| \leq \mathrm{M}$. Therefore, there exists $\rho > 0$ such that $\tilde{\xi}$ is bounded by M in $Q_{(\vec{z}_m, t_m), \rho} \subset Q$, which means $\tilde{\xi}$ is regular at $(\vec{z}_m, t_m) \in \mathbb{R}^4 \times \mathbb{R}_+ \times I$. Thus, automatically ξ is regular at $(\vec{z}_m, t_m) \in \mathbb{R}^3_+ \times I$, which immediately implies that u is also regular at $(\vec{z}_m, t_m) \in \mathbb{R}^3_+ \times I$. However, it is contrary to the statement given in Lemma 3.2 under the assumption that $\mathcal{S}_m \neq \emptyset$ for some $m \in \mathbb{N}$. Hence $\mathcal{S}_l$ must be an emptyset for all $l \in \mathbb{N}$. Hölder continuity follows that u is locally bounded. This completes the proof. $\square$

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