Examinations on a Three-Dimensional Differentiable Vector Field That Equals its Own Curl

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Abstract Consider the differential equation curl$f = f$ for a 3-dimensional differentiable vector field f. We prove that f is analytic and then prove an existence and uniqueness theorem for the differential equation with a prescribed boundary data. We also outline with a few variations Professor J. Ericksen's work on a unit vector field that equals its own curl.

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1 Introduction

Consider the differential equation

$$\text{curl} f = f \tag{1.1}$$

for a 3-dimensional vector field f. In [E], Ericksen proved that a smooth unit vector field $n(x) = n(x_1, x_2, x_3)$ satisfying (1.1) must be equal to

$$< \cos x_3, \sin x_3, 0 > \tag{1.2}$$

in an appropriate coordinate system. Ericksen's work was motivated by the fact that (1.2) represents an equilibrium configuration in many liquid crystals. Indeed this configuration is widely used in the liquid crystal display devices. Here we set out to examine (1.1) from the perspective of a differential equation. We establish the following two theorems:

Theorem 1.1 *A differentiable vector field $f(x)$ satisfying (1.1) is analytic.*

Theorem 1.2 *Suppose $\phi(x_1, x_2), \psi(x_1, x_2)$ are two analytic functions on a domain D on the $x_1 - x_2$ plane. Then in a neighborhood of D in the three dimensional space, there exists a unique vector field*

$$f(x) =< f_1(x), f_2(x), f_3(x) >$$

satisfying (1.1) as well as

$$f_1(x_1, x_2, 0) = \phi(x_1, x_2), \quad f_2(x_1, x_2, 0) = \psi(x_1, x_2). \tag{1.3}$$

The essence of these two theorems is that a differentiable vector field satisfying (1.1) is determined by two 2-dimensional analytic functions. In contrast to Ericksen's uniqueness theorem for the unit vector fields satisfying (1.1), these theorems tell us that there are a lot of more vector fields that satisfy (1.1) but are not of unit length.

In §2 we prove Theorem 1.1, in §3 we prove Theorem 1.2, and in §4 we outline Ericksen's proof with a few variations.

2 The Analyticity

Since (1.1) only involves the first derivatives of f, it is natural to assume that f is differentiable. Theorem 1.1 states that the fact that f satisfies (1.1) and that f is differentiable imply that f is analytic.

Recall the identity in vector analysis:

$$\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f. \tag{2.1}$$

Here $\nabla =< \partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ is the gradient operator, $\nabla \cdot f$ and $\nabla \times f$ are the divergence and curl of f respectively, and $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ is the Laplacian operator. Bear in mind the well known facts that the divergence of the curl of a smooth vector field is zero and that the curl of the gradient of a smooth function is the zero vector.

If f is assumed to have continuous second derivatives, then by (2.1) the equation curl$f = f$ implies $\nabla \cdot f = 0$ and

$$-\nabla^2 f = f. \tag{2.2}$$

Thus each component of f, f_i for $i = 1, 2, 3$, satisfies $-\nabla^2 f_i = f_i$. It follows from a well-known theorem for the elliptic partial differential equations that f must be analytic (cf. [ADN],[E],[F], [GT]).

Without the assumption that f has continuous second derivatives, we show that $\nabla \cdot f = 0$ and $-\nabla^2 f = f$ in the distributional sense. Let $\xi(x)$ be a scalar function and let $g(x) =< g_1(x), g_2(x), g_3(x) >$ be a vector function with the assumption that ξ, g are infinitely smooth and vanish outside a bounded domain where f is defined. Then using (1.1) and integration by parts,

$$-\int f \cdot (\nabla \xi) dx = -\int \nabla \times f \cdot (\nabla \xi) dx = \int f \cdot \nabla \times (\nabla \xi) dx = 0,$$

where the integrals are over the whole R^3. This equation says that $\nabla \cdot f = 0$ in the distributional sense.

Similarly, we have

$$\begin{aligned}
\int f \cdot g dx &= \int \nabla \times f \cdot g \\
&= -\int f \cdot (\nabla \times g) dx = -\int (\nabla \times f) \cdot (\nabla \times g) dx \\
&= \int f \cdot (\nabla \times (\nabla \times g)) dx = \int f (\nabla (\nabla \cdot g) - \nabla^2 g) dx \\
&= -\int f \nabla^2 g dx.
\end{aligned}$$

Here again the integrals are on the whole R^3. Thus f satisfies (2.2) in the distributional sense and therefore is analytic by the same well-known regularity theorem for elliptic partial differential equations we cited earlier.

3 The Existence and Uniqueness

The essential tool we use to prove Theorem 1.2 is the classical Cauchy-Kowalevskaya Theorem in partial differential equations. Many textbooks on partial differential equations have a proof on the theorem, and we refer the reader to [Ev] and [Di] for two such textbooks published recently.

First we notice that if $f_1(x_1,x_2,0)=\phi(x_1,x_2)$, $f_2(x_1,x_2,0)=\psi(x_1,x_2)$ on a domain D in the x_1-x_2 plane, then, by $\nabla\cdot f=0$ and $\nabla\times f=f$,

$$\left\{\begin{array}{rcl} f_1(x_1,x_2,0) &=& \phi \\ f_2(x_1,x_2,0) &=& \psi \\ f_3(x_1,x_2,0) &=& \frac{\partial\psi}{\partial x_1}-\frac{\partial\phi}{\partial x_2} \\ \frac{\partial}{\partial x_3}f_1(x_1,x_2,0) &=& f_2+\frac{\partial}{\partial x_1}f_3=\psi+\frac{\partial^2}{\partial x_1^2}\psi-\frac{\partial^2}{\partial x_1\partial x_2}\phi \\ \frac{\partial}{\partial x_3}f_2(x_1,x_2,0) &=& -f_1+\frac{\partial}{\partial x_2}f_3=-\phi+\frac{\partial^2}{\partial x_1\partial x_2}\psi-\frac{\partial^2}{\partial x_2^2}\phi \\ \frac{\partial}{\partial x_3}f_3(x_1,x_2,0) &=& -\frac{\partial}{\partial x_1}f_1-\frac{\partial}{\partial x_2}f_2=-\frac{\partial}{\partial x_1}\phi-\frac{\partial}{\partial x_2}\psi \end{array}\right. \tag{3.1}$$

Also, notice that if $f(x_1,x_2,0)$ satisfies (3.1), then $\nabla\cdot f=0$ and $\nabla\times f=f$ on D.

By the Cauchy-Kowalevskaya theorem, the following problem

$$\left\{\begin{array}{l} -\nabla^2 f=f \quad \text{in a neighborhood of} \quad D, \quad \text{and} \\ f(x_1,x_2,0),\ \frac{\partial}{\partial x_3}f(x_1,x_2,0) \quad \text{satisfy (3.1) on} \quad D \end{array}\right.$$

has a unique analytic solution. Our task is to verify that the solution f satisfies $\nabla\times f=f$.

We start with proving that $\nabla\cdot f=0$. Let $u(x)=\nabla\cdot f$. By $-\nabla^2 f=f$, u satisfies the equation $-\nabla^2 u=u$. Also, by (3.1) $u(x_1,x_2,0)=0$ on D. Furthermore, on D

$$\begin{array}{l} \frac{\partial}{\partial x_3}u(x_1,x_2,0)=\frac{\partial^2}{\partial x_1\partial x_3}f_1(x_1,x_2,0)+\frac{\partial^2}{\partial x_2\partial x_3}f_2(x_1,x_2,0)+\frac{\partial^2}{\partial x_3^2}f_3(x_1,x_2,0) \\ =\frac{\partial}{\partial x_1}(\psi+\frac{\partial^2}{\partial x_1^2}\psi-\frac{\partial^2}{\partial x_1\partial x_2}\phi)+\frac{\partial}{\partial x_2}(-\phi+\frac{\partial^2}{\partial x_1\partial x_2}\psi-\frac{\partial^2}{\partial x_2^2}\phi) \\ \quad +(-f_3(x_1,x_2,0)-\frac{\partial^2}{\partial x_1^2}f_1(x_1,x_2,0)-\frac{\partial^2}{\partial x_2^2}f_2(x_1,x_2,0)) \\ =\frac{\partial}{\partial x_1}(\psi+\frac{\partial^2}{\partial x_1^2}\psi-\frac{\partial^2}{\partial x_1\partial x_2}\phi)+\frac{\partial}{\partial x_2}(-\phi+\frac{\partial^2}{\partial x_1\partial x_2}\psi-\frac{\partial^2}{\partial x_2^2}\phi) \\ \quad -(\frac{\partial}{\partial x_1}\psi-\frac{\partial}{\partial x_2}\phi)-\frac{\partial^2}{\partial x_1^2}(\frac{\partial}{\partial x_1}\psi-\frac{\partial}{\partial x_2}\phi)-\frac{\partial^2}{\partial x_2^2}(\frac{\partial}{\partial x_1}\psi-\frac{\partial}{\partial x_2}\phi) \\ =\frac{\partial}{\partial x_1}\psi-\frac{\partial}{\partial x_2}\phi-\frac{\partial}{\partial x_1}\psi+\frac{\partial}{\partial x_2}\phi+\psi_{x_1x_1x_1}-\phi_{x_1x_1x_2}+\psi_{x_1x_2x_2}-\phi_{x_2x_2x_2} \\ \quad -\psi_{x_1x_1x_1}+\phi_{x_1x_1x_2}-\psi_{x_1x_2x_2}+\phi_{x_2x_2x_2} \\ =0. \end{array}$$

That is, u satisfies

$$\left\{\begin{array}{l} -\nabla^2 u=u \quad \text{in a neighborhood of} \quad D, \quad \text{and} \\ u(x_1,x_2,0)=\frac{\partial}{\partial x_3}u(x_1,x_2,0)=0 \quad \text{on} \quad D. \end{array}\right.$$

By the uniqueness in the Cauchy-Kowalevskaya theorem, u must be a zero function.

Now we come to the proof of $\nabla \times f = f$. Let $h = \nabla \times f$. Then $-\nabla^2 f = f$ implies $-\nabla^2 h = h$. Also, by (3.1), we have $h(x_1, x_2, 0) = f(x_1, x_2, 0)$ on D. We only need to show that $\frac{\partial}{\partial x_3} h(x_1, x_2, 0) = \frac{\partial}{\partial x_3} f(x_1, x_2, 0)$ on D. To this end, we note that by $-\nabla^2 f = f$ and $\nabla \cdot f = 0$, which we just proved,

$$\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f = f.$$

This leads us to

$$\nabla \times h = f.$$

Particularly on D we obtain from the equation above that

$$\begin{aligned} \frac{\partial}{\partial x_3} h_1 &= \frac{\partial}{\partial x_1} h_3 + f_2 = \frac{\partial}{\partial x_1} f_3 + f_2 = \frac{\partial}{\partial x_3} f_1 \\ \frac{\partial}{\partial x_3} h_2 &= \frac{\partial}{\partial x_2} h_3 - f_1 = \frac{\partial}{\partial x_2} f_3 - f_1 = \frac{\partial}{\partial x_3} f_2 \end{aligned}$$

Note that above we have used $\nabla \times f = f$ on D, which follows from (3.1).

As for $\frac{\partial}{\partial x_3} h_3$, by $\nabla \cdot h = 0$,

$$\frac{\partial}{\partial x_3} h_3 = -\frac{\partial}{\partial x_1} h_1 - \frac{\partial}{\partial x_2} h_2 = -\frac{\partial}{\partial x_1} f_1 - \frac{\partial}{\partial x_2} f_2 = \frac{\partial}{\partial x_3} f_3.$$

In summary, we have proved that $h = \nabla \times f$ satisfies

$$\begin{cases} -\nabla^2 h = h \quad \text{in a neighborhood of} \quad D, \text{ and} \\ h(x_1, x_2, 0) = f(x_1, x_2, 0), \quad \frac{\partial}{\partial x_3} h(x_1, x_2, 0) = \frac{\partial}{\partial x_3} f(x_1, x_2, 0) \quad \text{on} \quad D. \end{cases}$$

Once more, by the uniqueness in the Cauchy-Kowalevskaya theorem, we know that $h = \nabla \times f = f$ throughout the domain where f is defined. Theorem 1.2 is now proved.

4 Notes on Ericksen's Uniqueness

We outline Ericksen's proof on the uniqueness for a differentiable unit vector field $n(x)$ that equals its own curl. We essentially follow the original proof with a few variations.

In the proof, we need higher derivatives of $n(x)$. By Theorem 1.1, the vector field $n(x)$ is analytic and indeed has all the higher derivatives. We also need the following facts in tensor analysis.

We recall that the matrix $\nabla n = (\partial n_i/\partial x_j)_{3\times 3}$ is a second order tensor, $\nabla \times n$ is a first order tensor, and

$$|\nabla n|^2,\ (\nabla \cdot n),\ (n \cdot \nabla \times n),\ |n \times (\nabla \times n)|,\ \mathrm{tr}(\nabla n)^2$$

are all scalars. Above

$$\begin{aligned} |\nabla n|^2 &= \sum_{i,j=1,2,3} (\frac{\partial}{\partial x_j} n_i)^2, \\ \mathrm{tr}(\nabla n)^2 &= \sum_{i,j=1,2,3} \frac{\partial}{\partial x_j} n_i \frac{\partial}{\partial x_i} n_j. \end{aligned}$$

We also recall tht because $n(x)$ is a unit vector field,

$$\begin{aligned} |\nabla n|^2 &= (\nabla \cdot n)^2 + (n \cdot \nabla \times n)^2 \\ &\quad + |n \times (\nabla \times n)|^2 + ((\mathrm{tr}(\nabla n)^2 - (\nabla \cdot n)^2. \end{aligned} \tag{4.1}$$

This identity is relevant in the Oseen-Frank theory of liquid crystals, cf. [Ch], [F]. It can be verified as follows. At any point x_0 assume

$$n(x_0) = <0, 0, 1> . \tag{4.2}$$

Then, because

$$\sum_{i=1,2,3} n_i \frac{\partial}{\partial x_j} n_i = 0 \ \text{ for each } \ j = 1, 2, 3,$$

we have

$$\nabla n = \begin{pmatrix} \frac{\partial}{\partial x_1} n_1 & \frac{\partial}{\partial x_2} n_1 & \frac{\partial}{\partial x_3} n_1 \\ \frac{\partial}{\partial x_1} n_2 & \frac{\partial}{\partial x_2} n_2 & \frac{\partial}{\partial x_3} n_2 \\ \frac{\partial}{\partial x_1} n_3 & \frac{\partial}{\partial x_2} n_3 & \frac{\partial}{\partial x_3} n_3 \end{pmatrix} = \begin{pmatrix} \# & \# & \# \\ \# & \# & \# \\ 0 & 0 & 0 \end{pmatrix}.$$

A direct computation then gives (4.1). Note that the assumption (4.2) is without loss of generality because all the terms in (4.1) are scalars and are therefore frame invariant.

Essential to Ericksen's proof is to recognize that the matrix ∇n is rank-1 everywhere. By $\nabla \times n = n$, we obtain $-\nabla^2 n = n$. Then we obtain

$$\begin{aligned} 1 = |n|^2 &= -\sum_{i,j=1,2,3} n_i \frac{\partial^2}{\partial x_j^2} n_i \\ &= \sum_{i,j=1,2,3} \left(-\frac{\partial}{\partial x_j}\left(n_i \frac{\partial}{\partial x_j} n_i\right) + \frac{\partial}{\partial x_j} n_i \frac{\partial}{\partial x_j} n_i\right) \\ &= |\nabla n|^2, \end{aligned}$$

using

$$\sum_{i=1,2,3} n_i \frac{\partial}{\partial x_j} n_i = \frac{1}{2} \frac{\partial}{\partial x_j} |n|^2 = 0.$$

Combined with

$$(\nabla \cdot n) = 0,\ (n \cdot \nabla \times n) = 1,\ |n \times (\nabla \times n)| = 0,$$

which all follow from $\nabla \times n = n$, we use the equation (4.1) to obtain

$$\mathrm{tr}(\nabla n)^2 - (\nabla \cdot n)^2 = 0.$$

It is direct to verify that this term equals the negative of the second invariant of the matrix ∇n. Further, at any point x_0, assume (4.2) without loss of generality. Then because of $\nabla \times n = n$,

$$\nabla n = \begin{pmatrix} \# & \# & 0 \\ \# & \# & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That the second invariant of such a matrix vanishes implies that the matrix is rank-1 at the point.

Now that ∇n is known to be rank-1 everywhere, there exist vector fields

$$m = < m_1, m_2, m_3 >,\ \ l = < l_1, l_2, l_3 >$$

such that

$$\frac{\partial}{\partial x_j} n_i = m_i l_j, \quad \text{for each } \ i, j = 1, 2, 3. \tag{4.3}$$

In fact, because $\nabla \times n = n$ and thus ∇n is not degenerate, the vectors m, l can be chosen to be analytic themselves at least locally. Then it follows from $\nabla \cdot n = 0$ and $\nabla \times n = n$ that

$$0 = m \cdot l \quad \text{and} \quad n = -m \times l.$$

Subsequently, m, l can be chosen to be unit vector fields in addition to be analytic. Next we apply to (4.3)

$$-\nabla^2 n = n \ \text{ and } \ \frac{\partial^2}{\partial x_j \partial x_k} n_i = \frac{\partial^2}{\partial x_k \partial x_j} n_i.$$

After making dot products with m we come to the identities

$$\nabla \cdot l = 0 \ \text{ and } \ \nabla \times l = 0.$$

Thus

$$0 = \nabla \times (\nabla \times l) = \nabla(\nabla \cdot l) - \nabla^2 l = -\nabla^2 l.$$

Then we obtain

$$\begin{aligned} 0 = -l \cdot \nabla^2 l &= -\sum_{i,j=1,2,3} l_i \frac{\partial^2}{\partial x_j^2} l_i \\ &= \sum_{i,j=1,2,3} \left(-\frac{\partial}{\partial x_j}\left(l_i \frac{\partial}{\partial x_j} l_i\right) + \frac{\partial}{\partial x_j} l_i \frac{\partial}{\partial x_j} l_i\right) \\ &= |\nabla l|^2, \end{aligned}$$

using

$$\sum_{i=1,2,3} l_i \frac{\partial}{\partial x_j} l_i = \frac{1}{2} \frac{\partial}{\partial x_j} |l|^2 = 0.$$

Thus l is a constant unit vector field. (People who are familiar with harmonic maps may recall that a harmonic map h from R^3 to S^2 satisfies $-\nabla^2 h = |\nabla h|^2 h$. In our case the unit vector fields n and l are both such harmonic maps because of $-\nabla^2 n = n$ and $-\nabla^2 l = 0$. One readily observes that $|\nabla n|^2 = 1$ and $|\nabla l|^2 = 0$.)

Now without loss of generality assume that $l =< 0, 0, 1 >$. From $n = -m \times l$ it follows that $n_3 = 0$. Furthermore, by $m = n \times l$,

$$m_1 = n_2, \quad m_2 = -n_1, \quad m_3 = 0.$$

The equations (4.3) give

$$\frac{\partial}{\partial x_1} n_i = \frac{\partial}{\partial x_2} n_i = 0 \;\text{ for each }\; i = 1, 2,$$

and

$$\frac{\partial}{\partial x_3} n_1 = n_2, \quad \frac{\partial}{\partial x_3} n_2 = -n_1.$$

It then follows that $n =< n_1, n_2, n_3 >=< \cos x_3, \sin x_3, 0 >$ if we choose the origin appropriately.

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