Dynamics of the Thermohaline Circulation under Wind Forcing

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Abstract

The ocean thermohaline circulation, also called meridional overturning circulation, is caused by water density contrasts. This circulation has large capacity of carrying heat around the globe and it thus affects the energy budget and further affects the climate. We consider a thermohaline circulation model in the meridional plane under external wind forcing. We show that, when there is no wind forcing, the stream function and the density fluctuation (under appropriate metrics) tend to zero exponentially fast as time goes to infinity. With rapidly oscillating wind forcing, we obtain an averaging principle for the thermohaline circulation model. This averaging principle provides convergence results and comparison estimates between the original thermohaline circulation and the averaged thermohaline circulation, where the wind forcing is replaced by its time average. This establishes the validity for using the averaged thermohaline circulation model for numerical simulations at long time scales.

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1 Introduction

In addition to the wind-driven surface circulation, the ocean also exhibits a large meridional overturning circulation called the thermohaline circulation. The ocean is heated (thus made less dense) where pure freshwater is evaporated (water thus made saltier and denser), and vice versa. The global thermohaline circulation involves water masses sinking at high latitudes and upwelling at lower latitudes. The process is maintained by water density contrasts in the ocean, which themselves are created by atmospheric forcing , namely, heat and water exchange via evaporation and condensation. During the thermohaline circulation, water masses carry heat (or cold) around the globe. Thus, it is believed that the global ocean thermohaline circulation plays an important role in the climate [28].

A two-dimensional thermohaline circulation model involves the Navier-Stokes equations for momentum (in the meridional plane) together with the convection-diffusion equations for temperature and for salinity. Due to the linear equation of state (relating fluid density with temperature and salinity), these latter two equations may be replaced by a single convection-diffusion equation for density or density fluctuation. We consider the thermohaline circulation on a fluid domain in the vertical meridional xz-plane, as in, e.g., [1]:

$$\begin{aligned} \Delta\psi_t + J(\Delta\psi, \psi) &= \rho_x + \nu\Delta^2\psi + f(x,z,t) \ , && (1.1) \\ \rho_t + J(\rho, \psi) &= -N^2\psi_x + \frac{\nu}{Pr}\Delta\rho \ , && (1.2) \end{aligned}$$

where $\psi(x, z, t)$ is the stream function which defines the velocity field $(u, w) = (\psi_z, -\psi_x)$; $\rho(x, z, t)$ is the density fluctuation from the mean density; $\nu > 0$ is the viscosity; $N^2 > 0$ is the mean buoyancy frequency and is taken as a constant; Pr is the Prandtl number. Finally, $J(a, b) = a_x b_y - a_y b_x$ is the Jacobian operator and $\Delta = \partial_{xx} + \partial_{zz}$ is the Laplace operator. Note that $\frac{\partial}{\partial t} + J(\cdot, \psi) = \frac{\partial}{\partial t} + u\partial_x + w\partial_z$ is the material derivative. The wind forcing term $f(x, z, t)$ is to be specified below.

In [1], the author made some numerical simulation for (1.1), (1.2) with periodic boundary conditions in both x and z for ψ and ρ and $f \equiv 0$. Therefore, we assume that ψ and ρ are periodic (with period 1) in x and z, and also assume that ψ and ρ have zero mean.

In some recent work on the thermohaline circulation, the wind forcing is ignored [27, 29, 20]. In this paper, we will consider the impact of wind forcing on the thermohaline circulation, while considering the evolution of the fluid density fluctuation (rather than the fluid density itself).

In the first part of this paper, we obtain the exponential decay estimates for the stream function ψ and density fluctuation ρ with $f \equiv 0$ (no external wind forcing). In the second part, we consider the effect of wind forcing on the stream function and density fluctuation. We obtain an averaging principle for rapidly oscillating wind forcing, which provides convergence results and comparison estimates between the original thermohaline circulation and the averaged thermohaline circulation. This establishes the validity for using the averaged thermohaline circulation model for numerical simulations at long time scales.

2 Exponential Decay: Without Wind Forcing

In this section, we consider the long time behavior of the stream function and the density fluctuation in the thermohaline circulation. We first briefly comment on the local existence for (1.1)-(1.2) with periodic boundary conditions (with period 1) in both x and z for ψ and ρ (with zero mean), we introduce some notations:

$$\int = \int_D dxdz,$$

where $D = \{(x,z) : 0 \leq x, z \leq 1\}$ is the periodic fluid domain.

$$H = L^2_{per0} = \{u : \ \ u \in L^2(D), u \text{ is perodic both in } x \text{ and } z, \int u = 0\}, \quad \text{with norm } \|\cdot\|;$$

$$V = H^1_{per0} = \{u : \ \ u \in H, \nabla u \in H\}, \ \ H^2_{per0} = \{u : \ \ u \in H, \nabla u \in H, \triangle u \in H\}, \ etc.$$

In fact, by a result in [4], we know $\|\triangle u\|$ is equivalent to $\|u\|^2_{H_{per0}}$.
Define the vorticity $\omega = \triangle\psi$. It is well known that $\triangle^{-1}$ exists for $\triangle$ with periodic boundary and zero mean, then (1.1) can be written as

$$\omega_t + J(\omega, \triangle^{-1}\omega) = \rho_x + \nu\triangle\omega + f(x,z,t). \tag{2.1}$$

Since the nonlinear Jacobian term is continuous from $V \times V \to H \times H$, by the theory of [11], we have the following local existence result for (2.1) and (1.2):

Lemma 2.1 (**Local Existence**) *Let $(\omega_0, \rho_0) \in V \times V$(initial values, that is $\psi_0 \in H^3_{per0}$) and $f \in L^\infty(0,T;H)$, then (2.1) and (1.2) with periodic boundary conditions in both x and z for ψ and ρ with zero mean has a unique local solution satisfying*

$$\omega \in L^\infty(0,T;V) \cap L^2(0,T;H^2_{per0}) \ \ \rho \in L^\infty(0,T;V) \cap L^\infty(0,T;H^2_{per0}),$$

that is, (1.1) and (1.2) has a unique local solution satisfying

$$\psi \in L^\infty(0,T;H^3_{per0}) \cap L^2(0,T;H^4_{per0}) \ \ \rho \in L^\infty(0,T;V) \cap L^\infty(0,T;H^2_{per0}),$$

where T depends on (ω_0, ρ_0).

We need the following properties and estimates (see [2]) of the Jacobian operator $J : H^1_0 \times H^1_0 \to L^1$ in the sequel:

$$\int_D J(f,g)hdxdy = -\int_D J(f,h)gdxdy, \int_D J(f,g)gdxdy = 0,$$

$$|\int_D J(f,g)dxdy| \leq \|\nabla f\|\|\nabla g\|,$$

for all $f, g, h \in H_0^1$.
Now, we derive some a priori estimates for the solution of (1.1) and (1.2) with $f \equiv 0$ to ensure the global existence. For $f \neq 0$, the estimates are almost the same.

Multiplying (1.1) by ψ, performing an integration by parts and using the periodic boundary conditions, we conclude

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|^2 + \int \partial_x \rho\psi + \nu\|\triangle\psi\|^2 = 0. \tag{2.2}$$

Similarly, we get

$$\frac{1}{2}\frac{d}{dt}\|\rho\|^2 - N^2\int \partial_x \rho\psi + \frac{\nu}{Pr}\|\nabla\rho\|^2 = 0. \tag{2.3}$$

Multiplying (2.2) by N^2 and adding to (2.3), we have

$$\frac{1}{2}\frac{d}{dt}(N^2\|\nabla\psi\|^2 + \|\rho\|^2) + \nu(N^2\|\triangle\psi\|^2 + \frac{1}{Pr}\|\nabla\rho\|^2) = 0. \tag{2.4}$$

By the Poincaré inequality and taking $\alpha = \frac{\nu}{\lambda_1}\min\{1, \frac{1}{Pr}\}$($\lambda_1$ is the smallest eigenvalue of $-\triangle$ with periodic boundary and zero mean), we obtain

$$N^2\|\nabla\psi\|^2 + \|\rho\|^2 + \int_0^t [\nu(N^2\|\triangle\psi\|^2 + \frac{1}{Pr}\|\nabla\rho\|^2)]dt$$

$$\leq e^{-\alpha t}(N^2\|\nabla\psi_0\|^2 + \|\rho_0\|^2). \tag{2.5}$$

If we only want to know whether the solution tends to zero as $t \to \infty$, we could use the following special Gronwall Lemma. We omit the details for this asymptotics here, and we will concentrate on the exponential decay of the solution in the sequel.

Lemma 2.2 *If a non-negative differential function f satisfies*

$$f'(t) + \alpha_1 f(t) \leq g(t),$$

where $\alpha_1 > 0$ and $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$.

PROOF. We first have

$$f(t) \leq e^{-\alpha_1 t}f(0) + e^{-\alpha_1 t}\int_0^t e^{\alpha_1\tau}g(\tau)d\tau. \tag{2.6}$$

Since $\lim_{t\to\infty} g(t) = 0$, for any given $\epsilon > 0$, there exists a T such that

$$g(t) \leq \epsilon, \quad \text{for } t \geq T.$$

So, (2.6) can be written as

$$f(t) \le e^{-\alpha_1 t} f(0) + \frac{\epsilon}{\alpha_1}(1 - e^{-\alpha_1 t}) + e^{-\alpha_1 t} \int_0^T e^{\alpha_1 \tau} g(\tau) d\tau. \tag{2.7}$$

Let $t \to \infty$ and note that ϵ is arbitrary, the result is thus obtained.

Now, we turn to the exponential decay estimates for the solutions.
Multiplying (1.1) and (1.2) by $-\triangle\rho$ and $\triangle\psi$ respectively, integrating by parts, using the periodic boundary conditions, we have

$$\frac{1}{2}\frac{d}{dt}\|\triangle\psi\|^2 = \int \partial_x \rho \triangle\psi - \nu\|\nabla\triangle\psi\|^2, \tag{2.8}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|^2 - \int J(\rho,\psi)\triangle\rho = -N^2 \int \partial_x\psi\triangle\rho - \frac{\nu}{Pr}\|\triangle\rho\|^2. \tag{2.9}$$

Note that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\triangle\psi\|^2 &= -\int \rho\triangle\psi_x - \nu\|\nabla\triangle\psi\|^2 \\
&\le \frac{1}{4\delta\nu}\|\rho\|^2 - (1-\delta)\nu\|\nabla\triangle\psi\|^2 (\ \forall\, 0 < \delta < 1) \\
&\le \frac{1}{4\delta\nu}\|\rho\|^2 - \frac{(1-\delta)\nu}{\lambda_1}\|\triangle\psi\|^2 \\
&\le \frac{1}{4\delta\nu} e^{-\alpha t}(N^2\|\nabla\psi_0\|^2 + \|\rho_0\|^2) - \frac{(1-\delta)\nu}{\lambda_1}\|\triangle\psi\|^2.
\end{aligned}$$

where the Poincaré inequality is used. By the Gronwall inequality and (2.5), we thus obtain

$$\|\triangle\psi\|^2 \le e^{-\frac{(1-\delta)\nu}{\lambda_1}t}\|\triangle\psi_0\|^2 +$$

$$e^{-\frac{(1-\delta)\nu}{\lambda_1}t} \frac{\lambda_1}{4\delta(1-\delta)\nu^2} \int_0^t e^{-\alpha\tau}(N^2\|\nabla\psi_0\|^2 + \|\rho_0\|^2) e^{\frac{(1-\delta)\nu}{\lambda_1}\tau} d\tau. \tag{2.10}$$

Noticing that $\alpha = \frac{\nu}{\lambda_1}\min\{1, \frac{1}{Pr}\}$, we know for every given Pr, there exists a $0 < \delta < 1$, such that

$$\delta_1 = \alpha - \frac{(1-\delta)\nu}{\lambda_1} > 0.$$

Hence (2.10) can be reduced to

$$\|\triangle\psi\|^2 \le e^{-\frac{(1-\delta)\nu}{\lambda_1}t}(\|\triangle\psi_0\|^2 + \frac{\lambda_1}{4\delta(1-\delta)\delta_1\nu^2}(N^2\|\nabla\psi_0\|^2 + \|\rho_0\|^2)). \tag{2.11}$$

Multiplying (2.8) by N^2, adding to (2.9), using integrating by parts and the fact that $\int[\int \partial_x \nabla\rho\nabla\psi + \int \partial_x\nabla\psi\nabla\rho] = 0$, we conclude that

$$\frac{1}{2}\frac{d}{dt}(N^2\|\triangle\psi\|^2+\|\nabla\rho\|^2)-\int J(\rho,\psi)\triangle\rho=-N^2\nu\|\nabla\triangle\psi\|^2-\frac{\nu}{Pr}\|\triangle\rho\|^2. \tag{2.12}$$

Now we only need to estimate $-\int J(\rho,\psi)\triangle\rho=\int(\rho_z\psi_x-\rho_x\psi_z)(\rho_{xx}+\rho_{zz})$. Note that

$$\begin{aligned}
\int \rho_z\psi_x\rho_{xx} &= \frac{1}{2}\int \rho_x^2\psi_{xz}-\int \rho_x\rho_z\psi_{xx},\\
\int \rho_z\psi_x\rho_{zz} &= -\frac{1}{2}\int \rho_z^2\psi_{xz},\\
-\int \rho_x\psi_z\rho_{xx} &= \frac{1}{2}\int \rho_x^2\psi_{xz},\\
-\int \rho_x\psi_z\rho_{zz} &= -\frac{1}{2}\int \rho_z^2\psi_{xz}+\int \rho_x\rho_z\psi_{zz}.
\end{aligned}$$

Now we need the following lemma about the equivalence between $\|u\|_{H^2}$ and $\|\triangle u\|$ for $u\in H^2_{per0}$.

Lemma 2.3 *For every $u\in H^2_{per0}$, we have*

$$\|u\|_{H^2}\le a_1\|\triangle u\|,$$

where $a_1=\sqrt{1+\lambda_1+\lambda_1^2}$.

PROOF. Since

$$\|\triangle u\|^2=\int((\frac{\partial^2 u}{\partial x^2})^2+(\frac{\partial^2 u}{\partial z^2})^2+(\frac{\partial^2 u}{\partial x\partial z})^2). \tag{2.13}$$

Adding $\|u\|^2_{H^1}$ to both side of (2.13), and using Poincaré inequality, the proof of this lemma is complete.

Using this lemma, we imply that

$$\begin{aligned}
|\int J(\rho,\psi)\triangle\rho| &= |\int(\rho_x^2-\rho_z^2)\psi_{xz}-\int \rho_x\rho_z(\psi_{xx}-\psi_{zz})|\\
&\le \int(|\rho_x|^2+|\rho_z|^2)|\psi_{xz}|+\int |\rho_x||\rho_z|(|\psi_{xx}|+|\psi_{zz}|)\\
&\le a_1(2+\sqrt{2})\|\triangle\psi\|[(\int |\rho_x|^4)^{\frac{1}{2}}+\int |\rho_z|^4)^{\frac{1}{2}}).
\end{aligned}$$

By the following inequality from [5]

$$\|u\|_{L^4}\le a_2\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}, u\in H^1_{per0}(D), a_2=(\frac{1}{4\pi^2}+\frac{\sqrt{2}}{\pi}+2)^{\frac{1}{4}}.$$

Therefore,

$$|\int J(\rho,\psi)\triangle\rho|\le a_1^2a_2^2(2+\sqrt{2})\|\triangle\psi\|\|\nabla\rho\|\|\triangle\rho\|$$

$$\leq \frac{\delta_2\nu}{Pr}\|\triangle\rho\|^2 + \frac{a_1^4a_2^4(6+2\sqrt{2})Pr}{\delta_2\nu}\|\triangle\psi\|^2\|\nabla\rho\|^2, \tag{2.14}$$

Combining (2.12) with (2.14) and using the Poincaré inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}(N^2\|\triangle\psi\|^2 + \|\nabla\rho\|^2)$$

$$\leq -N^2\nu\|\nabla\triangle\psi\|^2 + (\frac{a_1^4a_2^4(6+2\sqrt{2})Pr}{4\delta_2\nu}\|\triangle\psi\|^2 - (1-\delta_2)\frac{\nu}{\lambda_1 Pr})\|\nabla\rho\|^2. \tag{2.15}$$

By (2.11), there exists a t_1 such that

$$\frac{a_1^4a_2^4(6+2\sqrt{2})Pr}{4\delta_2\nu}\|\triangle\psi\|^2 - (1-\delta_2)\frac{\nu}{\lambda_1 Pr} = -\delta_3 < 0,\ 0 < \delta_3 < 1.$$

Let $\Phi_0 = \|\triangle\psi_0\|^2 + \frac{\lambda_1}{4\delta(1-\delta)\delta_1\nu^2}(N^2\|\nabla\psi_0\|^2 + \|\rho_0\|^2)$. It follows that

$$e^{-\frac{(1-\delta)\nu}{\lambda_1}t}\Phi_0 < \frac{4\delta_2(1-\delta_2)\nu^2}{a_1^4a_2^4(6+2\sqrt{2})\lambda_1 Pr^2}.$$

Since $4\delta_2(1-\delta_2) \leq 1$, so t_1 can be chosen as

$$t_1 > \frac{\lambda_1}{(1-\delta)\nu}\ln[\frac{aa_1^4a_2^4(6+2\sqrt{2})\lambda_1 Pr^2\Phi_0}{\nu^2}].$$

By (2.15), we know

$$\|\nabla\rho\|^2 \leq M_0,\quad 0 \leq t \leq t_1 \tag{2.16}$$

and

$$\|\nabla\rho\|^2 \leq e^{-\alpha_2 t}(N^2\|\triangle\psi_0\|^2 + \|\nabla\rho_0\|^2),\ t > t_1. \tag{2.17}$$

Here $\alpha_2 = \min\{\delta_3, \frac{N^2\nu}{\lambda_1}\}$ and M_0 is constant depending on t_1, $\|\triangle\psi_0\|^2, \|\nabla\rho\|^2, \nu, Pr$ and λ_1.

The estimates (2.16) and (2.17) tell us that the mean-square norm of the density (fluctuation) gradient, $\|\nabla\rho\|$, is uniformly bounded up to some time instant and then decay exponentially fast.

Remark 2.4 *If ν is large enough or Pr is small enough, we also can have*

$$\frac{a_1^4a_2^4(6+2\sqrt{2})Pr}{4\delta_2\nu}\|\triangle\psi\|^2 - (1-\delta_2)\frac{\nu}{\lambda_1 Pr} < 0,$$

that is

$$Pr < \frac{\nu}{a_1^2 a_2^2 (2+\sqrt{2})} \sqrt{\frac{1}{(\lambda_1 e^{-\frac{(1-\delta)\gamma}{\lambda_1}t}(\|\triangle\psi_0\|^2 + \frac{\lambda_1}{4\delta(1-\delta)\delta_1\nu^2}(N^2\|\nabla\psi_0\|^2 + \|\rho_0\|^2)))}}$$

or ν satisfying

$$e^{-\frac{(1-\delta)\nu}{\lambda_1}t}(\|\triangle\psi_0\|^2 + \frac{\lambda_1}{4\delta(1-\delta)\delta_1\nu^2}(N^2\|\nabla\psi_0\|^2 + \|\rho_0\|^2))) \leq 1$$

i.e.,

$$\nu \geq \sqrt{\frac{a_1^4 a_2^4 (6+2\sqrt{2}) Pr^2 \lambda_1}{4\delta_2(1-\delta_2)}}.$$

Now, we derive the estimates for $\|\nabla\triangle\psi\|$ when the initial value $\psi_0 \in H^3_{per0}$. By a similar process as above, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\triangle\psi\|^2 - \int J(\triangle\psi,\psi)\triangle^2\psi = -\int \partial_x\rho\triangle^2\psi - \nu\|\triangle^2\psi\|^2. \tag{2.18}$$

By the same discussion as in deriving (2.14), we have

$$|\int J(v,u)\triangle v| \leq a_1^2 a_2^2(2+\sqrt{2})\|\triangle u\|\|\nabla v\|\|\triangle v\|, \quad \text{for } u,v \in H^2_{per0}. \tag{2.19}$$

We use inequality (2.19) for $u = \psi$ and $v = \triangle\psi$, we have

$$|\int J(\triangle\psi,\psi)\triangle^2\psi| \leq a_1^2 a_2^2(2+\sqrt{2})\|\triangle\psi\|\|\nabla\triangle\psi\|\|\triangle^2\psi\|$$

$$\leq \frac{\nu}{4}\|\triangle^2\psi\|^2 + \frac{a_1^4 a_2^4(6+2\sqrt{2})}{\nu}\|\triangle\psi\|^2\|\nabla\triangle\psi\|^2 \tag{2.20}$$

and

$$\int |\partial_x\rho\triangle^2\psi| \leq \frac{\nu}{4}\|\triangle^2\psi\|^2 + \frac{1}{\nu}\|\nabla\rho\|^2. \tag{2.21}$$

By (2.18), (2.20) and (2.21), we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\triangle\psi\|^2 \leq \frac{1}{\nu}\|\nabla\rho\|^2 + \frac{a_1^4 a_2^4(6+2\sqrt{2})}{\nu}\|\triangle\psi\|^2\|\nabla\triangle\psi\|^2 - \frac{\nu}{2}\|\triangle^2\psi\|^2.$$

Thus we conclude by using the Poincaré inequality,

$$\frac{d}{dt}\|\nabla\triangle\psi\|^2 \leq \frac{2}{\nu}\|\nabla\rho\|^2 + (\frac{a_1^4 a_2^4(12+4\sqrt{2})}{\nu}\|\triangle\psi\|^2 - \frac{\lambda_1\nu}{2})\|\nabla\triangle\psi\|^2. \tag{2.22}$$

Using (2.11) and for λ_1 large enough, there exists $\alpha_3 > 0$ such that

$$\frac{a_1^4 a_2^4(12+4\sqrt{2})}{\nu}\|\triangle\psi\|^2 - \lambda_1\nu = -\alpha_3 < 0,\ (\alpha_2 > \alpha_3).$$

Hence

$$\frac{d}{dt}\|\nabla\triangle\psi\|^2 + \alpha_3\|\nabla\triangle\psi\|^2 \le \frac{2}{\nu}\|\nabla\rho\|^2,$$

by the Gronwall's inequality, we have

$$\|\nabla\triangle\psi\|^2 \le e^{-\alpha_3 t}\|\nabla\triangle\psi_0\|^2 + \frac{2}{\nu}\int_0^t e^{-\alpha_3(t-\tau)}\|\nabla\rho\|^2 d\tau.$$

For $t \le t_1$, using (2.16), we have

$$\|\nabla\triangle\psi\|^2 \le e^{-\alpha_3 t}\|\nabla\triangle\psi_0\|^2 + \frac{2}{\nu\alpha_3}M_0 \doteq M_1. \tag{2.23}$$

For $t > t_1$, using (2.17) and $\alpha_2 > \alpha_3$, we get

$$\begin{aligned}
\|\nabla\triangle\psi\|^2 &\le e^{-\alpha_3 t}\|\nabla\triangle\psi_0\|^2 + \frac{2}{\nu}(N^2\|\triangle\psi_0\|^2 + \|\nabla\rho_0\|^2)\int_0^t e^{-\alpha_3(t-\tau)-\alpha_2\tau}d\tau \\
&= e^{-\alpha_3 t}(\|\nabla\triangle\psi_0\|^2 + \frac{2}{\nu}(N^2\|\triangle\psi_0\|^2 + \|\nabla\rho_0\|^2))\int_0^t e^{-(\alpha_2-\alpha_3)\tau}d\tau. \\
&\le e^{-\alpha_3 t}(\|\nabla\triangle\psi_0\|^2 + \frac{2}{\nu}(N^2\|\triangle\psi_0\|^2 + \|\nabla\rho_0\|^2)).
\end{aligned} \tag{2.24}$$

If λ_1 is not so large, by (2.11), there exist $\alpha_4 > 0$ and $t_2 > 0$ large enough such that

$$\frac{a_1^4 a_2^4(12+4\sqrt{2})}{\nu}\|\triangle\psi\|^2 - \lambda_1\nu = -\alpha_4 < 0, \quad \text{for } t \ge t_2\ (\alpha_2 > \alpha_4).$$

Thus (2.22) can be written as

$$\frac{d}{dt}\|\nabla\triangle\psi\|^2 + \alpha_4\|\nabla\triangle\psi\|^2 \le \frac{2}{\nu}\|\nabla\rho\|^2.$$

By the the same argument as we obtain the estimates of (2.23) and (2.24), we could get similar estimates.

The estimates (2.23) and (2.24) tell us that the mean-square norm of the vorticity gradient, $\|\nabla\omega\|$, is uniformly bounded up to some time instant and then decay exponentially fast.

Thus, by the above estimates (2.5), (2.11), (2.16), (2.17), (2.23) and (2.24), which hold in the case of no wind forcing ($f = 0$), we obtain the main theorem in this section:

Theorem 2.5 *(***Exponential Decay in the Case of No Wind Forcing***) Let the initial conditions for the vorticity and density fluctuation (ω_0, ρ_0) be in $V \times V$ (i.e., $\psi_0 \in H^3_{per0}$). Then, when there is no external wind forcing, $\|\psi\|_{H^3_{per0}}$ and $\|\rho\|_{H^1_{per0}}$ tend to zero exponentially fast as $t \to \infty$.*

That is, under appropriate norms or metrics, the stream function and density fluctuation tend to zero exponentially fast as time goes to infinity.
Moreover, there exists some time instant $T > 0$ such that, the mean-square norms for the density (fluctuation) gradient, $\|\nabla\rho\|$, and for the vorticity gradient, $\|\nabla\omega\|$, are uniformly bounded when $t \le T$ and exponentially decay when $t > T$. The time instant $T = \max\{t_1, t_2\}$ depends on $Pr,\ N^2,\ \nu, \lambda_1$ and initial values.

3 Averaging Principle: Rapidly Oscillating Forcing

In this section, we consider the averaging principle for the system of (2.1) and (1.2) under the rapidly oscillating forcing $f(x, z, t)$. We rewrite (2.1) and (1.2) as

$$\omega_t + \mathcal{A}_1\omega = -J(\omega, \triangle^{-1}\omega) + \rho_x + f(x, z, t), \tag{3.1}$$

$$\rho_t + \mathcal{A}_2\rho = -J(\rho, \triangle^{-1}\omega) - N^2\triangle^{-1}\omega_x\ , \tag{3.2}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the operator $-\nu\triangle$ and $-\frac{\nu}{Pr}\Delta$ with the periodic boundary conditions and zero mean. For the rest of this section, we concentrate on the system (3.1)-(3.2).
We assume that the forcing term f in (3.1) is rapidly oscillating, i.e., it has the form $f(x, y, t) = f(x, y, \eta t) \doteq f(\eta t)$, with parameter $\eta \gg 1$. We also assume that f has a well-defined time average. With such a forcing, it is desirable to understand the fluid dynamics in some averaged sense, and compare the averaged flows with the original (non-averaged) flows.
The main result of this section is an averaging principle for (3.1)–(3.2) with rapidly oscillating forcing on finite but large time intervals. This includes comparison estimate and convergence result (as $\eta \to \infty$) between (3.1)–(3.2) and its averaged motions.
Starting from the fundamental work of Bogolyubov [6] the averaging theory for ODE has been developed and generalized in a large number of works (see [7]–[9] and the references therein). Bogolyubov's main theorems have been generalized in [10] to the case of differential equations with bounded operator-valued coefficients. Some problems of averaging of differential equations with unbounded operator-valued coefficients have been considered in [11]–[14] in the framework of abstract parabolic equations. In [15], Ilyin considered the averaging principle for an equation of the form

$$\partial_t u = N(u) + f(\eta t), \tag{3.3}$$

where f is a given forcing function and $\eta \gg 1$ is a large dimensionless parameter, and f has a time average defined as

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds = f_0. \tag{3.4}$$

Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are sectorial operators. For a sectorial operator, one can define the fractional power of $\mathcal{A}$ as follows [11]:

$$\mathcal{A}^\alpha = (\mathcal{A}^{-\alpha})^{-1}, \text{ where } \mathcal{A}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\mathcal{A}t} dt.$$

The corresponding domains $D(\mathcal{A}^\alpha)$ are Banach spaces with norm given by

$$\|x\|_\alpha := \|x\|_{D(\mathcal{A}^\alpha)} = \|\mathcal{A}^\alpha x\|.$$

We recall some definitions and results to be used in the rest of this section.

Lemma 3.1 *[11] The following estimates are valid:*

$$\|e^{-\mathcal{A}t}\|_{L^2 \to L^2} \le K e^{-at}, \qquad t \ge 0, \tag{3.5}$$

$$\|\mathcal{A}^\alpha e^{-\mathcal{A}t}\|_{L^2 \to L^2} \le \frac{K_\alpha}{t^\alpha} e^{-at}, \;\; t > 0, \tag{3.6}$$

where K, K_α are positive constants.

Remark 3.2 *Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are different operators but both satisfy the conditions of* **Lemma 3.1**. *For the simplicity, we take the same constants for $\mathcal{A}_1$ and $\mathcal{A}_2$ when we use* **Lemma 3.1**.

Lemma 3.3 *[11] Given two sectorial operators A and B in L^2, let $D(A) = D(B)$, $Re\sigma(A) > 0, Re\sigma(B) > 0$, and for some $\alpha \in [0, 1)$. Let the operator $(A - B)A^{-\alpha}$ be bounded in L^2. Then for every $\gamma \in [0, 1)$, $D(A^\gamma) = D(B^\gamma)$, the two norms being equivalent.*

Setting

$$\tau = \eta t, \;\; \epsilon = \eta^{-1},$$

we rewrite the equations (3.1)-(3.2) in the so-called standard form

$$\omega_\tau + \epsilon \mathcal{A}_1 \omega + \epsilon J(\triangle^{-1}\omega, \omega) = \epsilon \rho_x + \epsilon f(x, y, \tau), \tag{3.7}$$

$$\rho_\tau + \epsilon \mathcal{A}_2 \rho + \epsilon J(\rho, \triangle^{-1}\omega) = -\epsilon N^2 \triangle^{-1} \omega_x \;, \tag{3.8}$$

We assume that f has a time average , $f_0(x, z)$, in $D(\mathcal{A}^\gamma)$; the value of γ will be specified later on. More precisely, let $f(\tau), f_0 \in \mathcal{A}^\gamma$ and suppose that

$$\|\mathcal{A}^\gamma (\frac{1}{T} \int_t^{t+T} f(\tau) d\tau - f_0)\| \le \min(M_\gamma, \sigma_\gamma(T)), \tag{3.9}$$

where $M_\gamma > 0, \sigma_\gamma(T) \to 0$, as $T \to \infty$.

We consider the averaged equation

$$\bar{\omega}_\tau + \epsilon\mathcal{A}_1\bar{\omega} + \epsilon J(\triangle^{-1}\bar{\omega}, \bar{\omega}) = \epsilon\bar{\rho}_x + \epsilon f_0(x, y), \tag{3.10}$$

$$\bar{\rho}_\tau + \epsilon\mathcal{A}_2\bar{\rho} + \epsilon J(\bar{\rho}, \triangle^{-1}\bar{\omega}) = -\epsilon N^2\triangle^{-1}\bar{\omega}_x\ , \tag{3.11}$$

By the method of [16]–[19], we know the semigroup S_t corresponding to equation (3.10)–(3.11) possesses absorbing sets in the space $\mathbf{H} = L^2_{per0} \times L^2_{per0}, \mathbf{V} = D(\mathcal{A}_1^{\frac{1}{2}}) \times D(\mathcal{A}_2^{\frac{1}{2}}) = H^1_{per0} \times H^1_{per0}$ and $D(\mathcal{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_2)$(in fact, $D(\mathcal{A}_1) = D(\mathcal{A}_2)$).Using Lemma 3.3, we know $D(\mathcal{A}_1^\gamma) = D(\mathcal{A}_2^\gamma)$, for $\gamma \in [0, 1]$. $\|\cdot\|$ and $\|\cdot\|_{\frac{1}{2}}$ denote the norm in L^2_{per0} and H^1_{per0}. These sets are certain balls $B(R_0)$ in these spaces, where R_0 is large enough. This means that for every bounded set B

$$S_tB \subset B(R_0),\ \text{for}\ t > t_0(B, R_0).$$

In addition, the semigroup is uniformly bounded in these spaces, that is, given any ball, in particular, the ball $B(R_0)$, there exists a ball $B(R)$ such that

$$S_tB(R_0) \subset B(R),\ \text{for}\ t > 0.$$

By increasing R we may assume that

$$S_tB(R_0) \subset B(R - r),\ \text{for}\ t > 0, r > 0,$$

where r is a positive constant. We consider the averaging principle in the space $\mathbf{V}$. Given a point ω_0 in $B_{\mathbf{V}}(R_0)$, we compare the trajectories (solutions) $(\omega(\tau), \rho(\tau))$ and $(\bar{\omega}(\tau), \bar{\rho}(\tau))$ of system (3.1–(3.2) and (3.7)–(3.8) starting from same initial point. Consider their difference on the interval $\tau \in [0, \frac{T}{\epsilon}]$, T being arbitrary but fixed. We suppose for the moment that $(\omega(\tau), \rho(\tau)) \in B_{\mathbf{V}}(R)$. Then the difference $z(\tau) = \omega(\tau) - \bar{\omega}(\tau), \Theta = \rho - \bar{\rho}$ satisfies the equations

$$\partial_\tau z + \epsilon\mathcal{A}_1 z(\tau) + \epsilon[J(\triangle^{-1}\omega, \omega) - J(\triangle^{-1}\bar{\omega}, \bar{\omega})] = \epsilon((\rho_x - \bar{\rho}_x) + (f(\tau) - f_0)), \tag{3.12}$$

$$\partial_\tau\Theta + \epsilon\mathcal{A}_2(\tau) + \epsilon[J(\triangle^{-1}\omega, \rho) - J(\triangle^{-1}\bar{\omega}, \bar{\rho})] = -\epsilon N^2(\triangle^{-1}\omega_x - \triangle^{-1}\bar{\omega}_x). \tag{3.13}$$

We first give the following lemma, the proof can be obtained by direct estimate.

Lemma 3.4 *The nonlinear operator $J(u, v)$ is a bounded Lipschitz map in the following sense:*

$$\|J(u_1, v_1) - J(u_2, v_2)\| \le$$

$$C_{\frac{1}{2}}(\|u_1\|_{\frac{1}{2}} + \|u_2\|_{\frac{1}{2}} + \|v_1\|_{\frac{1}{2}} + \|v_2\|_{\frac{1}{2}})(\|u_1 - v_1\|_{\frac{1}{2}} + \|u_2 - v_2\|_{\frac{1}{2}}), \tag{3.14}$$

where $C_{\frac{1}{2}}$ is some positive constants.

Inverting the linear operators $\mathcal{A}_1$ and $\mathcal{A}_2$ we come to the equivalent integral equations of (3.12) and (3.13)

$$\begin{aligned} z(\tau) &= -\epsilon \int_0^\tau e^{-\epsilon \mathcal{A}_1(\tau-s)}[J(\triangle^{-1}\omega,\omega) - J(\triangle^{-1}\bar{\omega},\bar{\omega})]ds \\ &+ \epsilon \int_0^\tau e^{-\epsilon \mathcal{A}_1(\tau-s)}(\rho_x - \bar{\rho}_x)ds + \epsilon \int_0^\tau e^{-\epsilon \mathcal{A}_1(\tau-s)}(f(s) - f_0)ds, \qquad (3.15) \\ \Theta(\tau) &= -\epsilon \int_0^\tau e^{-\epsilon \mathcal{A}_2(\tau-s)}[J(\triangle^{-1}\omega,\rho) - J(\triangle^{-1}\bar{\omega},\bar{\rho})]ds \\ &- \epsilon N^2 \int_0^\tau e^{-\epsilon \mathcal{A}_2(\tau-s)}(\triangle^{-1}\omega_x - \triangle^{-1}\bar{\omega}_x). \qquad (3.16) \end{aligned}$$

Using (3.6) and (3.14), we see that the $\|\cdot\|_{\frac{1}{2}}$-norm of the first term in the right hand side of (3.15) satisfies the inequality

$$\begin{aligned} &\|\epsilon \int_0^\tau \mathcal{A}_1^{\frac{1}{2}} e^{-\epsilon \mathcal{A}_1(\tau-s)}[J(\triangle^{-1}\omega,\omega) - J(\triangle^{-1}\bar{\omega},\bar{\omega})]ds\| \\ &\le \epsilon \int_0^\tau K_{\frac{1}{2}} C_{\frac{1}{2}} \epsilon^{-\frac{1}{2}} (\tau-s)^{-\frac{1}{2}} e^{-\epsilon a(\tau-s)} 2R \|z(s)\|_{\frac{1}{2}} ds \\ &= 2RK_{\frac{1}{2}} C_{\frac{1}{2}} \epsilon^{\frac{1}{2}} \int_0^\tau (\tau-s)^{-\frac{1}{2}} e^{-\epsilon a(\tau-s)} \|z(s)\|_{\frac{1}{2}} ds. \qquad (3.17) \end{aligned}$$

Let us estimate the second term in the right hand side of (3.15).

$$\|\epsilon \int_0^\tau \mathcal{A}_1 e^{-\epsilon \mathcal{A}_1(\tau-s)}(\rho_x - \bar{\rho}_x)ds\| \le K_{\frac{1}{2}} \epsilon^{\frac{1}{2}} \int_0^\tau (\tau-s)^{-\frac{1}{2}} e^{-\epsilon a(\tau-s)} \|\Theta(s)\|_{\frac{1}{2}} ds. \qquad (3.18)$$

Now let us estimate the third term in the right hand side of (3.15). Integrating by parts we have

$$\begin{aligned} &\|\epsilon \int_0^\tau e^{-\epsilon \mathcal{A}_1(\tau-s)}(f(s) - f_0)ds\|_{\frac{1}{2}} \\ &= \| -\epsilon e^{-\epsilon \mathcal{A}_1(\tau-s)} \int_s^\tau (f(t) - f_0)dt|_0^\tau + \epsilon^2 \int_0^\tau \mathcal{A}_1 e^{-\epsilon \mathcal{A}_1(\tau-s)} \int_s^\tau (f(s) - f_0)ds\|_{\frac{1}{2}} \\ &\le \|\epsilon \mathcal{A}_1^{\frac{1}{2}-\gamma} e^{-\epsilon \mathcal{A}_1 \tau} \mathcal{A}_1^\gamma \int_0^\tau (f(t) - f_0)dt\| \\ &+ \|\epsilon^2 \int_0^\tau \mathcal{A}_1^{\frac{3}{2}-\gamma} e^{-\epsilon \mathcal{A}_1(\tau-s)} \mathcal{A}_1^\gamma \int_s^\tau (f(s) - f_0)ds\|. \qquad (3.19) \end{aligned}$$

Using (3.6) and (3.9), we further have

$$\begin{aligned} \|\epsilon \mathcal{A}_1^{\frac{1}{2}-\gamma} e^{-\epsilon \mathcal{A}_1 \tau} \mathcal{A}_1^\gamma \int_0^\tau (f(t) - f_0)dt\| &\le \epsilon K_{\frac{1}{2}-\gamma} e^{-\epsilon a \tau} (\epsilon\tau)^{\gamma-\frac{1}{2}} \|\frac{1}{\tau} \int_0^\tau \mathcal{A}_1^\gamma (f(t) - f_0)dt\| \\ &= (\epsilon\tau)^{\frac{1}{2}+\gamma} K_{\frac{1}{2}-\gamma} e^{-\epsilon a \tau} \|\frac{1}{\tau} \int_0^\tau \mathcal{A}_1^\gamma (f(t) - f_0)dt\| \end{aligned}$$

$$\leq (\epsilon\tau)^{\frac{1}{2}+\gamma} K_{\frac{1}{2}-\gamma} \min(M_\gamma, \sigma_\gamma(\tau)) e^{-\epsilon a\tau} =: L(\tau). \tag{3.20}$$

For any $\delta > 0$, let τ_δ be so large that for $\tau \geq \tau_\delta, we have \sigma_\gamma \leq \delta$. Let ϵ_0 be so small that for $\epsilon < \epsilon_0$ the inequality $\frac{T}{\epsilon} > \tau_\delta$ is valid. Then

$$L(\tau) \leq G_{\gamma 1}(T,\epsilon) = e^{-\epsilon a\tau} \begin{cases} T^{\frac{1}{2}+\gamma} K_{\frac{1}{2}-\gamma}\delta, & \text{if } \tau \geq \tau_\delta, \\ (\epsilon\tau)^{\frac{1}{2}+\gamma} K_{\frac{1}{2}-\gamma} M_\gamma, & \text{if } \tau < \tau_\delta. \end{cases}$$

Let $\gamma > -\frac{1}{2}$. Note that τ_δ does not depend on ϵ. Taking $\delta \to 0$ and then $\epsilon \to 0$, we obtain

$$\|\epsilon e^{-\epsilon \mathcal{A}_1 \tau} \int_0^\tau (f(t) - f_0) dt\|_{\frac{1}{2}} \leq G_{\gamma 1}(T,\epsilon) \to 0 \text{ when } \epsilon \to 0. \tag{3.21}$$

$$\|\epsilon^2 \int_0^\tau \mathcal{A}_1^{\frac{3}{2}-\gamma} e^{-\epsilon \mathcal{A}_1(\tau - s)} \mathcal{A}_1^\gamma \int_s^\tau (f(s) - f_0) ds\| \leq K_{\frac{3}{2}-\gamma} \epsilon^{\frac{1}{2}+\gamma} \int_0^\tau \min(M_\gamma, \sigma_\gamma(u)) u^{\gamma - \frac{1}{2}} du$$

$$\leq K_{\frac{3}{2}-\gamma} M_\gamma \epsilon^{\frac{1}{2}+\gamma} \int_0^{\tau_\mu} u^{\gamma-\frac{1}{2}} du + K_{\frac{3}{2}-\gamma} \epsilon^{\frac{1}{2}+\gamma} \mu \int_0^{\frac{T}{\epsilon}} u^{\gamma - \frac{1}{2}} du$$

$$= K_{\frac{3}{2}-\gamma} (\gamma + \frac{1}{2})^{-1} (M_\gamma (\epsilon\tau_\mu)^{\frac{1}{2}+\gamma} + \mu T^{\frac{1}{2}+\gamma}) =: G_{\gamma 2}(T,\epsilon), \tag{3.22}$$

where for any $\mu > 0$ we have chosen τ_μ so large that $\sigma_\gamma(\tau) < \mu$ when $\tau > \tau_\mu$. Letting $\mu \to 0$ and then $\epsilon \to 0$ we obtain

$$G_{\gamma 2}(T,\epsilon) \to 0, \quad \epsilon \to 0.$$

Thus, by (3.17)–(3.22) we obtain the following inequality:

$$\|z(\tau)\|_{\frac{1}{2}} \leq K \epsilon^{\frac{1}{2}} \int_0^\tau (\tau - s)^{-\frac{1}{2}} (\|z(s)\|_{\frac{1}{2}} + \|\Theta(s)\|_{\frac{1}{2}}) ds + G_\gamma(T,\epsilon), \tag{3.23}$$

where $K = \max\{2RK_{\frac{1}{2}} C_{\frac{1}{2}}, K_{\frac{1}{2}}\}$ and $G_\gamma = G_{\gamma 1} + G_{\gamma 2} \to 0, \epsilon \to 0$.
Similar to the argument above for $z(\tau)$, and using (3.6) and (3.14), we get

$$\|\Theta\|_{\frac{1}{2}} \leq K_1 \epsilon^{\frac{1}{2}} \int_0^\tau (\tau - s)^{-\frac{1}{2}} (\|z(s)\|_{\frac{1}{2}} + \|\Theta(s)\|_{\frac{1}{2}}) ds, \tag{3.24}$$

where K_1 depends on $R, K_{\frac{1}{2}}, C_{\frac{1}{2}}$ and λ_1.
Adding (3.23) and (3.24), we finally obtain

$$\|z(\tau)\|_{\frac{1}{2}} + \|\Theta(\tau)\|_{\frac{1}{2}} \leq (K + K_1) \epsilon^{\frac{1}{2}} \int_0^\tau (\tau - s)^{-\frac{1}{2}} (\|z(s)\|_{\frac{1}{2}} + \|\Theta(s)\|_{\frac{1}{2}}) ds + G_\gamma(T,\epsilon). \tag{3.25}$$

Here we need the following fact.

Lemma 3.5 *[11] Let $\gamma \in (0,1]$ and for $t \in [0,T]$*

$$u(t) \leq a + b\int_0^t (t-s)^{\gamma-1}u(s)ds.$$

Then

$$u(t) \leq aE_\gamma((b\Gamma(\gamma))^{\frac{1}{\gamma}}t),$$

where the function $E_\gamma(z)$ is monotone increasing and $E_\gamma(z) \sim \gamma^{-1}e^z$ as $z \to \infty$.

Applying this lemma to the inequality (3.25) on $\tau \in [0, \frac{T}{\epsilon}]$, we obtain

$$\begin{aligned} \|z(t)\|_{\frac{1}{2}} + \|\Theta(\tau)\|_{\frac{1}{2}} &\leq \\ G_\gamma(T,\epsilon)E_{\frac{1}{2}}(\epsilon\tau\pi(K+K_1)^2) &\leq G_\gamma(T,\epsilon)E_{\frac{1}{2}}(T\pi(K+K_1)^2) := \eta_T^1(\epsilon). \end{aligned} \tag{3.26}$$

We thus have proved the proximity of solutions of (3.1) and (3.2) in $\mathbf{V}$, for the trajectory $(\omega(t), \rho(t))$ with initial condition $(\omega(0), \rho(0)) \in B_{\mathbf{V}}(R_0)$ stays in the ball $B(R)$ on the time interval $[0, \frac{T}{\epsilon}]$.

Let ϵ be so small that the right-hand side of (3.26) are less than $\frac{r}{2}$, where r is defined earlier in this section when we discuss absorbing sets. Suppose that the trajectory $(\omega(t), \rho(t))$ leaves the ball $B(R)$ during the interval $[0, \frac{T}{\epsilon}]$ and let τ^* be the first moment where $\|\omega(\tau^*)\|_{\frac{1}{2}} + \|\rho(\tau^*)\|_{\frac{1}{2}} = R$. However, on the interval $\tau \in [0, \tau^*]$ both trajectories stay in the ball $B(R)$ and what we have proved so far shows that the inequality

$$\|\omega(\tau) - \bar{\omega}(\tau)\|_{\frac{1}{2}} + \|\rho(\tau) - \bar{\rho}(\tau)\|_{\frac{1}{2}} \leq \frac{r}{2}$$

is valid. In particular, it is valid for $\tau = \tau^*$. This together with the inequality $\|\bar{\omega}(\tau^*)\|_{\frac{1}{2}} + \|\rho(\tau^*)\|_{\frac{1}{2}} \leq R - r$, which holds by the hypothesis of the following theorem and the property of the semigroup $S(t)$, gives the contradiction

$$\|\omega(\tau^*)\|_{\frac{1}{2}} + \|\rho(\tau^*)\|_{\frac{1}{2}} \leq \|\omega(\tau^*) - \bar{\omega}(\tau^*)\|_{\frac{1}{2}} + \|\rho(\tau^*) - \bar{\rho}(\tau^*)\|_{\frac{1}{2}} + \|\bar{\omega}(\tau^*)\|_{\frac{1}{2}} + \|\bar{\rho}(\tau^*)\|_{\frac{1}{2}} \leq R - \frac{r}{2},$$

since $\|\omega(\tau^*)\|_{\frac{1}{2}} + \|\rho(\tau^*)\|_{\frac{1}{2}} = R$.

Thus we have the main theorem in this section:

Theorem 3.6 *(**Averaging Principle in the Case of Rapidly Oscillating Wind Forcing**) Assume that the wind forcing has a time average. Let $T > 0$ be arbitrary and fixed. If $\gamma > -\frac{1}{2}$ and the initial conditions for the vorticity and density fluctuation $(\omega(0), \rho(0)) = (\bar{\omega}(0), \bar{\rho}(0))$ are in the absorbing ball $B_{\mathbf{V}}(R_0)$ (depending on γ), then for $\tau \in [0, \frac{T}{\epsilon}]$, we have the following comparison and convergence estimate between the thermohaline circulation and the averaged thermohaline circulation*

$$\|\omega(\tau) - \bar{\omega}(\tau)\|_{\frac{1}{2}} + \|\rho(\tau) - \bar{\rho}(\tau)\|_{\frac{1}{2}} \leq \eta_T^1(\epsilon) \to 0, as\ \epsilon \to 0,$$

where $\eta_T^1(\epsilon)$ is a decaying function defined in (3.26).

4 Summary

The ocean thermohaline circulation has important impacts on the climate. We have considered a thermohaline circulation model in the meridional plane under external wind forcing. We have shown that, when there is no wind forcing, the stream function and the density fluctuation (under appropriate metrics) tend to zero exponentially fast as time goes to infinity (Theorem 2.5). With rapidly oscillating wind forcing, we have obtained an averaging principle for the thermohaline circulation model (Theorem 3.6). This averaging principle provides convergence results and comparison estimates between the original thermohaline circulation and the averaged thermohaline circulation, where the wind forcing is replaced by its time average.

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