Interferometric GPS Ambiguity Resolution *

T. Craig Poling
Lockheed Martin Naval Electronics and Surveillance Systems, Eagan, MN U.S.A.

Aleksandar Zatezalo
Siemens AG, Munich, Germany

July 1, 2001

Abstract. The Maximum a Posteriori Ambiguity Search (MAPAS) method for GPS ambiguity resolution first introduced in [8], is generalized to accommodate: (1) satellite switches caused by satellites rising or falling below the horizon or obstructing terrain, and (2) cycle slips due to temporary loss of lock on satellite signals. It is shown that MAPAS and generalized MAPAS are equivalent to Bayesian estimation. The generalized MAPAS method is successfully applied to real GPS satellite data with cycle slips and satellite switches due to satellite obstruction.

Keywords: MAPAS method, estimation, modelling, interferometric GPS

1. Introduction

The Global Positioning System (GPS) is a system of satellites in space used to identify unknown positions on land, sea, air, or in space.

The position of the satellites is known at all times. Each satellite sends out signals continuously, and both the signals and their transmission times are precisely controlled by atomic clocks.

The current GPS systems comprises 25 satellites at heights of 20200 km above the earth. The number of satellites a receiver can see or detect is variable, but occasionally it may be up to 12. In order to identify its position by interferometric methods the receiver produces a reference carrier and beats this carrier against the received signals to extract carrier phase shifts. These phase shifts may be used to obtain millimeter level relative positioning and precise orientation in space.

GPS has been used by astronomers to achieve earth orientation accuracies of 5.6×10^{-8} degrees. These accuracies are within a factor of four of those achieved by the most precise very long baseline interferometry (VLBI) methods using quasars (e.g., 1.4×10^{-8} degrees which is smaller than the angle a postage stamp in San Francisco subtends as viewed from New York).

GPS receiver measurements have also been used to perform global ionospheric and tropospheric tomography of the atmospheric index of refraction. The mathematical model can be reduced to finding unknown

* We wish to thank Prof. Dr. Avner Friedman for his help and support.

integers called the integer ambiguities. Once these numbers are found, the receiver can determine precise relative positions and orientations in space as long as it maintains carrier lock to the satellites. There are many methods to resolve the integer ambiguities, but each has its limitations, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], and references therein. This is an area of current research.

In this article we investigate a method for resolving integer ambiguities to a given statistical confidence level in a minimum amount of time. We also address the satellite loss of lock, or cycle slip problem, on integer ambiguity resolution.

In Section 2 we develop a linearized mathematical model of Interferometric GPS. In Section 3 we state the problem of estimating the baseline vector and then reformulate it to a the problem of resolving integer ambiguities. In Section 4 the problem of resolving integer ambiguities is reduced into lower dimensional problem. Our generalized MAPAS method of resolving this smaller number of ambiguities is described. The results of applying this method to real data are discussed in Section 5. In Section 6 we state some open problems in the area of interferometric GPS estimation.

2. Mathematical Model for Interferometric GPS

As in [13] and [5] we introduce notation where subscripts refer to receivers and superscripts refer to satellites. A quantity $*$ from receiver r to satellite j is denoted by $*^j_r$.

A difference of a quantity $*$ between receivers r_1 and r_2 and the satellite j is denoted by

$$*^j_{r_1,r_2} := *^j_{r_2} - *^j_{r_1},$$

which is the so-called single difference.

The difference of the quantity $*$ between receivers r_1 and r_2 and satellites j and k is defined as the difference between the single differences $*^k_{r_1,r_2}$ and $*^j_{r_1,r_2}$ i.e.

$$*^{j,k}_{r_1,r_2} := *^k_{r_1,r_2} - *^j_{r_1,r_2} = *^k_{r_2} - *^k_{r_1} - *^j_{r_2} + *^j_{r_1}.$$

Let c be the speed of light in vacuum, z_{r_1} and z_{r_2} positions in the Earth Centered Earth Fixed (ECEF) coordinate frame of the two receiver antennas r_1 and r_2 respectively, s^l position of the GPS satellite in the ECEF coordinate frame indexed by $l \in \{1, 2, \ldots, 25\}$, K_{s^l}, $l \in \{1, 2, \ldots, 25\}$ the unit vector in the ECEF coordinate frame from z_{r_1} to s^l, d distance in meters from z_{r_1} to z_{r_2}, and τ^l_m travel time of the GPS

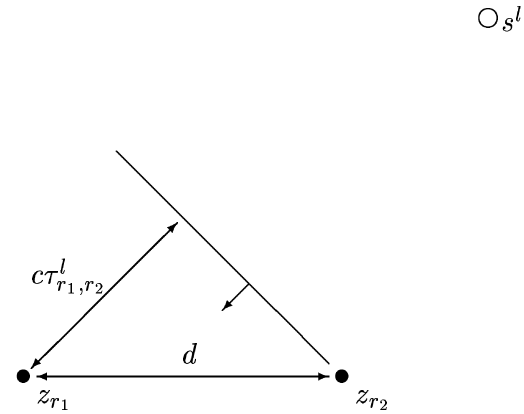


Figure 1. GPS signal wavefront from satellite s^l hitting receiver r_2 and approaching receiver r_1.

signal's wavefront from the satellite $l \in \{1, 2, \dots, 25\}$ to the antenna $m \in \{r_1, r_2\}$.

Then

$$\tau^l_{r_2,r_1} = \tau^l_{r_1} - \tau^l_{r_2}, \tag{1}$$

is the travel time delay between the plane wave arrival from the satellite l at antenna positions z_{r_2} and z_{r_1} respectively. This is shown in Figure 1.

If we denote by v^T the transpose of the vector v, then we have

$$c\tau^l_{r_2,r_1} = (z_{r_2} - z_{r_1})^T K_{s^l}, \tag{2}$$

for $l \in \{1, \dots, 25\}$, i.e. the distance from receiver r_1 to the wavefront of the signal of satellite l when that wavefront is already at the receiver r_2 is the scalar product of the vector from r_1 to r_2 and the unit vector from the receiver r_1 to the satellite s^l.

Here we assume that the unit vectors from the receiver r_1 to the satellite s^l and the receiver r_2 to the satellite s^l are parallel since the distance between the receivers under consideration is relatively small with respect to their distance to the satellites.

From [14] we have that the GPS signal from satellite l to antenna r_1 at the location z_{r_1}, for $t \geq T_0$ is

$$Re\left[A\tilde{x}_{s^l}\left(t + t_{s^l} - \tau^l_{r_1}\right) e^{2\pi i\left(f_c + f^l_d\right)\left(t + t_{s^l} - \tau^l_{r_1}\right)}\right],$$

where $\tilde{x}_{s^l}$ is the complex envelope of the GPS signal transmitted by the l'th GPS satellite, t_{s^l} is the satellite clock offset from GPS system time, A is the amplitude factor that accounts for transmission loss and satellite and receiver antenna patterns, f_c is the carrier frequency, and f_d^l is the Doppler shift, $f_{r_1}^l = f_c + f_d^l$ is the Doppler shifted frequency at z_{r_1}, and T_0 is the initial epoch when measurement begins. The receiver at z_{r_1} produces a reference carrier

$$Re\left[e^{2\pi i f_0(t+t_0)}\right],$$

where t_0 is the oscillator clock offset from GPS system time of the receiver r_1 and f_0 is the reference frequency produced by this receiver. The receiver beats this carrier against the received signal.

We assume that the reference carrier frequency produced at receiver r_1 is equal to the satellite carrier frequency $f_0 = f_c$, and we neglect the Doppler shift f_d^l relative to the size of the carrier $f_{r_1}^l = f_c + f_d^l \approx f_c$, see [14].

With these assumptions the beat phase produced at the receiver r_1 at time $t \geq T_0$ can be represented as

$$-2\pi f_c \tau_{r_1}^l(t) + 2\pi f_c(t_{s^l} - t_0).$$

When a receiver is switched on at the epoch T_0 it measures the instantaneous fractional beat phase where the integer part representing the number of cycles between the satellite s^l and the receiver r_1 is unknown.

Denoting that integer by $N_{r_1}^l$ and introducing measurement noise $\nu_{r_1}^l(t)$, $t \geq T_0$ which is normally distributed with mean zero and variance σ^2 and independent for different indexes r_1, l, and t, we have the measurement equation

$$\phi_{r_1}^l(t) := f_c \tau_{r_1}^l(t) + f_c(t_{r_1} - t_{s^l}) - N_{r_1}^l + \nu_{r_1}^l(t), \quad t \geq T_0. \tag{3}$$

The left hand side of (3) is the so called integral plus fractional part of the Integrated Carrier Phase (ICP).

From (1) and (3), for $t \geq T_0$, we have the so called first differences

$$\begin{aligned} \phi_{r_1,r_2}^l(t) &= \phi_{r_2}^l(t) - \phi_{r_1}^l(t) \\ &= -f_c \tau_{r_2,r_1}^l(t) + f_c(t_{r_2} - t_{r_1}) - N_{r_1,r_2}^l + \nu_{r_1,r_2}^l(t). \end{aligned} \tag{4}$$

By considering (4) for $l = l_1$ and $l = l_2$ where $l_1 \neq l_2$ i.e. the two equations for two different satellites l_1 and l_2, we have for $t \geq T_0$ the so called double differences

$$\begin{aligned} \phi_{r_1,r_2}^{l_1,l_2}(t) &= \phi_{r_1,r_2}^{l_2}(t) - \phi_{r_1,r_2}^{l_1}(t) \\ &= -f_c\left(\tau_{r_2,r_1}^{l_2} - \tau_{r_2,r_1}^{l_1}\right) - N_{r_1,r_2}^{l_1,l_2} + \nu_{r_1,r_2}^{l_1,l_2}(t). \end{aligned} \tag{5}$$

The purpose of taking first differences is to cancel satellite clock offsets $t_{s^{l_1}}$ and $t_{s^{l_2}}$ and the second differences to cancel the receiver clock offsets t_{r_1} and t_{r_2}.

Both satellite and receiver clock offset errors are with respect to true GPS system time.

A model where the Doppler shift is not assumed negligible relative to the carrier and possible consequences of such modelling is considered in [5].

The travel time from satellite l to antenna m can be written as

$$\tau_m^l = \frac{d(m,l)}{c} + T_m^l + I_m^l + \beta_m^l,$$

where $d(m,l)$ is the distance from antenna m to satellite l, T_m^l is the tropospheric delay, I_m^l is the ionospheric delay and β_m^l is multipath and other error source delays. For short baselines the signal from the l'th satellite to each receiver in the baseline travels approximately through the same region of the atmosphere. Therefore, the tropospheric and ionospheric delays along the paths are approximately equal i.e. $T_{r_1}^l \approx T_{r_2}^l$ and $I_{r_1}^l \approx I_{r_2}^l$ respectively, and approximately cancel when double differences are taken.

Multipath induced errors at the different receivers are not the same and they do not cancel in the double differences. The residual errors after double differencing due to multipath and atmospheric path differences are absorbed into the measurement noise term $\nu_{r_1,r_2}^{l_1,l_2}(t)$ in equation (6).

As the baseline lengths increase, the path differences through the atmosphere increases and $\nu_{r_1,r_2}^{l_1,l_2}(t)$ increases. The double differenced integer ambiguity $N_{r_1,r_2}^{l_1,l_2}$ in equation (6) assumes an integer value which must be estimated by statistical estimation. When the noise $\nu_{r_1,r_2}^{l_1,l_2}(t)$ in equation (6) reaches a level that no longer allows us to estimate the correct integer $N_{r_1,r_2}^{l_1,l_2}$ then we must model atmospheric effects and subtract them from $\nu_{r_1,r_2}^{l_1,l_2}(t)$.

From the relation $f_c\lambda_c = c$, where $\lambda_c = 19.029367$ cm is the GPS wavelength, and from equations (5) and (2), we have for $t \geq T_0$

$$\begin{aligned}\phi_{r_1,r_2}^{l_1,l_2}(t) = \lambda_c^{-1}\left(K_{s^{l_2}}(t) - K_{s^{l_1}}(t)\right)^T\left(z_{r_2}(t) - z_{r_1}(t)\right) \\ -N_{r_1,r_2}^{l_1,l_2} + \nu_{r_1,r_2}^{l_1,l_2}(t),\end{aligned} \tag{6}$$

where we note that the receivers r_1 and r_2 could be moving in time.

Equation (6) is valid for smaller baselines. In the case of long baseline, i.e. longer than 20 km it was noted in [8] that atmospheric effects should be considered. For baselines larger than a few meters and smaller than 20 km we can use pseudoranges to find an approximate estimate of the position of the receiver r_2, $\tilde{z}_{r_2}$ which can be displaced from z_{r_2} by up to a few meters. Then we have for $t \geq T_0$

$$\tau_{r_2,r_1}^l(t) = \tau_{\tilde{r}_2,r_1}^l(t) + \tau_{r_2,\tilde{r}_2}^l(t), \tag{7}$$

where

$$\tau^l_{\tilde{r}_2,r_1}(t) = \tau^l_{r_1}(t) - \tau^l_{\tilde{r}_2}(t),$$
$$\tau^l_{r_2,\tilde{r}_2}(t) = \tau^l_{\tilde{r}_2}(t) - \tau^l_{r_2}(t),$$

$\tau^l_{\tilde{r}_2}(t)$ is the travel time from the satellite l to the approximated position $\tilde{z}_{r_2}$ of the receiver r_2.

Suppose that we know the position of the receiver r_1 (or its approximation) which together with approximation $\tilde{z}_{r_2}$ allow us to calculate $\tau^l_{r_1}(t)$ and $\tau^l_{\tilde{r}_2,r_1}(t)$, $t \geq T_0$.

Substituting (7) into (5) we obtain

$$\tilde{\phi}^{l_1,l_2}_{r_1,r_2}(t) = \lambda_c^{-1}\left(\tilde{K}_{s^{l_2}}(t) - \tilde{K}_{s^{l_1}}(t)\right)^T (z_{r_2}(t) - z_{r_1}(t)) \tag{8}$$
$$-N^{l_1,l_2}_{r_1,r_2} + \nu^{l_1,l_2}_{r_1,r_2}(t),$$

where

$$\tilde{\phi}^{l_1,l_2}_{r_1,r_2}(t) := \phi^{l_1,l_2}_{r_1,r_2}(t) + f_c\left(\tau^{l_1}_{\tilde{r}_2,r_1}(t) - \tau^{l_2}_{\tilde{r}_2,r_1}(t)\right), \tag{9}$$

and $\tilde{K}_{s^l}(t)$ is the unit vector from the estimated position $\tilde{z}_{r_2}$ of receiver r_2 to the satellite l. If the second term on the right hand side of (9) is known from the noisy measurements then we have to alter the noise term in (8) (the third term on the right hand side in (8)). In other words we have to replace $\nu^{l_1,l_2}_{r_1,r_2}(t)$ with $\tilde{\nu}^{l_1,l_2}_{r_1,r_2}(t)$ which depends on $\nu^{l_1,l_2}_{r_1,r_2}(t)$ and the noise of the measurements from which we get $\tau^l_{\tilde{r}_2,r_1}(t)$, $t \geq T_0$. Similar considerations for long baselines are given in [8].

Atmospheric refraction effects have to be considered in interferometric GPS modelling for satellites which have very low elevation with respect to the receivers, i.e. usually less than seven degrees. The standard deviation of $\nu^{l_1,l_2}_{r_1,r_2}(t)$ increases significantly for low elevation satellites.

Observe that the unknown baseline $z_{r_2}(t) - z_{r_1}(t)$, $t \geq T_0$ enters linearly into observation equation (6) and implicitly into equation (8) as part of the scalar product which makes its estimation simpler and less time consuming. Although the model (6) is simple it has achieved millimeter level relative positioning accuracies and microradian attitude accuracies over short baselines.

3. Statement of the Interferometric GPS Integer Ambiguity Estimation Problem

We start our measurements of the ICP at epoch T_0. The measurements or observations are taken at equidistant time steps, i.e. we have discrete measurements of the ICP's at $t_0 = T_0, t_1, t_2, \ldots$, where $t_{i+1} - t_i = \Delta t$, for $i = 0, 1, 2, \ldots$.

One of the satellites is chosen as a reference satellite for the purpose of forming double differences (5) and this choice is assumed at least for some time during the integer ambiguity resolution procedure. Here we will consider equation (6). Let us consider that the baseline vector is just several meters long. The equivalent approach to the one considered in this paper to the problem of finding the baseline vector $z_{r_2}(t) - z_{r_1}(t)$, $t \geq T_0$ when it is longer than several meters and smaller than 20 km is to use equation (8) instead of equation (6).

Let $p_k \in \mathbb{N}$ be the number of satellite signals which we are receiving at the time instant t_k, $k = 0, 1, 2, \ldots$. Then, from (6) with l_1^k selected as a reference satellite, and $t = t_k$, $k = 0, 1, 2, \ldots$ we have the system of equations

$$\phi_{r_1,r_2}^{l_1^k,l_j^k}(t_k) = \lambda_c^{-1} \left(K_{s^{l_j^k}}(t_k) - K_{s^{l_1^k}}(t_k) \right)^T (z_{r_2}(t_k) - z_{r_1}(t_k)) \tag{10}$$
$$-N_{r_1,r_2}^{l_1^k,l_j^k} + \nu_{r_1,r_2}^{l_1^k,l_j^k}(t_k), \quad j = 2, \ldots, p_k, \quad k = 0, 1, \ldots.$$

Let for $j = 1, \ldots, p_k - 1$ and $k = 0, 1, \ldots$, $y_k^j := \phi_{r_1,r_2}^{l_1^k,l_{j+1}^k}(t_k)$, $a_k^j := \lambda_c^{-1} \left(K_{s^{l_{j+1}^k}}(t_k) - K_{s^{l_1^k}}(t_k) \right)$, $N_k^j := N_{r_1,r_2}^{l_1^k,l_{j+1}^k}$, $\nu_k^j := -\nu_{r_1,r_2}^{l_1^k,l_{j+1}^k}(t_k)$, and $x_k := z_{r_2}(t_k) - z_{r_1}(t_k)$. Then, from equation (10), and simple manipulation, for $j = 1, \ldots, p_k - 1$, $k = 0, 1, \ldots$, we have

$$\left(a_k^j\right)^T x_k = y_k^j + N_k^j + \nu_k^j. \tag{11}$$

As long as there is carrier lock on the satellite's signals, the integer ambiguities in equations (5), (6), (8), (10), and (11) are constant. When a temporary loss of signal lock is experienced, the phase measurements related to the particular satellite can not be performed. The integer ambiguities in equations (10) and (11) are indexed by time to account for cycle slips and changes in the order and choice of the satellite signals. Once the signal is re-acquired the initial integer ambiguity related to the measurement has a different value.

System (11) can be represented in the matrix form

$$A^k x_k = y_k + N_k + \nu_k, \quad k = 0, 1, 2, \ldots, \tag{12}$$

where

$$A^k := \begin{bmatrix} (a_k^1)^T \\ (a_k^2)^T \\ \vdots \\ \left(a_k^{p_k-1}\right)^T \end{bmatrix},$$

$$y_k := \left(y_k^1, y_k^2, \ldots, y_k^{p_k-1}\right)^T, \quad N_k := \left(N_k^1, \ldots, N_k^{p_k-1}\right)^T,$$

and $\nu_k := \left(\nu_k^1, \nu_k^2, \ldots, \nu_k^{p_k-1}\right)^T$.

For each $j = 1, \ldots, p_k - 1$, integer ambiguities N_k^j are constant between the cycle slips and between appearances and disappearances of the signals from the satellites l_1^k and l_j^k, $j = 2, \ldots, p_k$, i.e. during uninterrupted signal reception.

We state the problem as follows.

For every $n = 0, 1, 2, \ldots$ the problem is to estimate x_n from the noisy measurement sequence y_k and the almost exact sequence A^k, $k = 0, 1, \ldots, n$.

At time t_k we have p_k signals from satellites indexed by $l_1^k, \ldots, l_{p_k}^k \in \{1, 2, \ldots, 25\}$. Let l_1^k be the reference satellite, l_2^k, l_3^k, l_4^k set of primary satellites, and the rest of the satellites $l_5^k, \ldots, l_{p_k}^k$ are the so called secondary satellites. We divide equation (12) into two equations for $k = 0, 1, 2, \ldots$

$$A_p^k x_k = y_{k,p} + N_{k,p} + \nu_{k,p}, \tag{13}$$

and

$$A_s^k x_k = y_{k,s} + N_{k,s} + \nu_{k,s}, \tag{14}$$

where

$$A_p^k := \begin{bmatrix} (a_k^1)^T \\ (a_k^2)^T \\ (a_k^3)^T \end{bmatrix}, \qquad A_p^k := \begin{bmatrix} (a_k^4)^T \\ \vdots \\ \left(a_k^{p_k-1}\right)^T \end{bmatrix},$$

$$y_{k,p} := \left(y_k^1, y_k^2, y_k^3\right)^T, \quad N_{k,p} := \left(N_k^1, N_k^2, N_k^3\right)^T,$$

$$\nu_{k,p} := \left(\nu_k^1, \nu_k^2, \nu_k^3\right)^T, \quad y_{k,s} := \left(y_k^4, \ldots, y_k^{p_k-1}\right)^T,$$

$$N_{k,s} := \left(N_k^4, \ldots, N_k^{p_k-1}\right)^T, \text{and} \quad \nu_{k,s} := \left(\nu_k^4, \ldots \nu_k^{p_k-1}\right)^T.$$

We chose the reference satellite and the set of primary satellites such that the 3×3 matrix A_p^k is invertible for all $k = 0, 1, 2, \ldots$. Since no four satellites lie in a plane this is always possible. If the matrix A_p^k is close to singular then applying an inverse of A_p^k to both sides of equation (13) can cause large errors in estimating x_k. Here we assume that the right hand side of (13) is only known approximatively. The size of the relative error in x_k is determined by the condition number of A_p^k.

By choosing an appropriate reference satellite and set of primary satellites, and taking the inverse of A_p^k on both sides of equation (13) we get an expression for x_k which we substitute into equation (14) to obtain

$$A_s^k \left(A_p^k\right)^{-1} (y_{k,p} + N_{k,p} + \nu_{k,p}) = y_{k,s} + N_{k,s} + \nu_{k,s}. \tag{15}$$

Let $\alpha_k := A_s^k \left(A_p^k\right)^{-1} y_{k,p} - y_{k,s}$ and $\eta_k := \nu_{k,s} - A_s^k \left(A_p^k\right)^{-1} \nu_{k,p}$, $k = 0, 1, 2, \ldots$. Then from (15) and after simple manipulations, for $k = 0, 1, 2, \ldots$, we have

$$\alpha_k = N_{k,s} - A_s^k \left(A_p^k\right)^{-1} N_{k,p} + \eta_k. \tag{16}$$

The covariance matrix of the noise term in (16), η_k, is calculated in the Appendix.

Since α_k is only the transformation of the measurements $y_{k,p}$ and $y_{k,s}$ the problem of estimating x_k by using equation (12) was transformed into the problem of estimating integer ambiguities $N_{k,s}$ and $N_{k,p}$ using equation (16). The idea is that the noise ν_k is relatively small so that estimating x_k from (13) or (12) will yield a satisfactory answer.

4. The Generalized MAPAS Method

Since the number of integer ambiguities $N_k \in \mathbb{N}^{p_k - 1}$ could be quite large in applications and is definitely larger than the number of integer ambiguities $N_{k,p} \in \mathbb{N}^3$, the generalized MAPAS method is introduced which reduces the problem of finding N_k to the problem of finding $N_{k,p}$. This reduces the computational cost of estimating x_k, $k = 0, 1, \ldots$.

For each $k = 0, 1, 2, \ldots$, let γ_k^j be integer index such that $t_{\gamma_k^j}$ is epoch when the most recent cycle slip with the respect to epoch t_k of satellite l_j^k occurs for $j = 1, \ldots, p_k$. If there were no cycle slips before or at the time t_k then $t_{\gamma_k^j}$ is the time when the signal from the satellite l_j^k was first acquired which could have started at beginning of the observation process at epoch T_0 or later. We note that $\gamma_k^j \leq k$ for $j = 1, \ldots, p_k$, $k = 0, 1, \ldots$.

The motivation for the definitions below will become clear after inequality (19). For every $j = 1, \ldots, p_k - 4$ and $N \in \mathbb{N}^3$ let

$$\tilde{N}_{0,s}^j(N) := \alpha_0^j + \left(A_s^0 \left(A_p^0\right)^{-1}\right)_j,$$

$$\tilde{N}_{k,s}^j(N) := \left(\sum_{m=1}^{p_{k-1}-4} \left(k - \sigma_{k-1}^{m+4}\right) \tilde{N}_{k-1,s}^m(N) I_{l_{m+4}^{k-1} = l_{j+4}^k} + \alpha_k^j\right.$$

$$+\left(A_s^k\left(A_p^k\right)^{-1}\right)_j\Bigg)\Big/\left(k-\sigma_k^{j+4}+1\right),$$

$$\hat{N}_{k,s}^j := \left[\tilde{N}_{k,s}^j\right],$$

where $\sigma_k^j := \max\{\gamma_k^1, \gamma_k^2, \gamma_k^3, \gamma_k^4, \gamma_k^j\}$, α_k^j and $\left(A_s^k\left(A_p^k\right)^{-1}N\right)_j$ are jth coordinates of α_k and $A_s^k\left(A_p^k\right)^{-1}N$, respectively, $[*]$ is the closest integer to the quantity $*$, i.e. the so called round operator, and $I_{l_{m+4}^{k-1}=l_{j+4}^k}$ is equal to one if $l_{m+4}^{k-1} = l_{j+4}^k$ and zero otherwise.

If $l_{j+4}^k = l_{j+4}^i$, for $i = \sigma_k^j, \ldots, k-1$, $j = 1, \ldots, p_k - 4$, then we have

$$\hat{N}_{k,s}^j(N) = \left[\frac{1}{k-\sigma_k^{j+4}+1}\sum_{i=\sigma_k^{j+4}}^{k}\left(\alpha_i^j + \left(A_s^j\left(A_p^i\right)^{-1}\right)_j\right)\right]. \quad (17)$$

By setting $N = N_{k,p}$ into (17) and by (16), for $j = 1, \ldots, p_k - 4$, we get

$$\hat{N}_{k,s}^j\left(N_{k,p}\right) = \left[N_{k,s}^j + \frac{1}{k-\sigma_k^{j+4}+1}\sum_{i=\sigma_k^{j+4}}^{k}\eta_i^j\right]. \quad (18)$$

If for some $j \in \{1, \ldots, p_k - 4\}$ and $k \in \{0, 1, 2, \ldots\}$ we have

$$\left|\frac{1}{k-\sigma_k^{j+4}+1}\sum_{i=\sigma_k^{j+4}}^{k}\eta_i^j\right| < \frac{1}{2} \quad (19)$$

then by (18) we have $\hat{N}_{k,s}^j\left(N_{k,p}\right) = N_{k,s}^j$.

Since the expected values of η_k's are zero and the covariance matrices are uniformly bounded (through choice of satellites), as a consequence of Bernstein's theorem, see [16] p. 24, the inequality (19) will occur for large enough k. This is under the assumption that the cycle slips do not occur until inequality (19) is satisfied and the received signals arrive continuously from the same satellites. Since the noise covariances are relatively small (19) should happen relatively fast. This is the motivation for the definition of $\hat{N}_{k,s}^j(N)$, for $N \in \mathbb{N}^3$.

Let for $N \in \mathbb{N}^3$ and $k = 0, 1, 2, \ldots$,

$$\beta_k(N) := \hat{N}_{k,s}(N) - A_s^k\left(A_p^k\right)^{-1}N,$$

where $\hat{N}_{k,s}(N) = \left(\hat{N}_{k,s}^1, \ldots, \hat{N}_{k,s}^{p_k-4}\right)^T$.

The quantity $\beta_k(N)$, $k = 0, 1, 2, \dots$ are the so called predicted measurements for given $N \in \mathbb{N}^3$. The so called prediction error is then defined for $N \in \mathbb{N}^3$ and $k = 1, 2, \dots$, by

$$\delta_k(N) := \alpha_k - \beta_k(N).$$

In [8], under the assumption $N_{n,p} = N_p$, for $n = 0, 1, \dots$, the following conditional probability density is given

$$P\left(N = N_p | \delta_1(N), \dots, \delta_n(N)\right) = \frac{f\left(\delta_1(N), \dots, \delta_n(N) | N = N_{n,p}\right) P\left(N = N_p\right)}{\sum_{M \in \mathcal{N}} f\left(\delta_1(M), \dots, \delta_n(M) | M = N_{n,p}\right) P\left(M = N_p\right)}, \tag{20}$$

for $N \in \mathcal{N}$, where $\mathcal{N} \subset \mathbb{N}^3$ is the given set of integer ambiguities chosen such that $N_p \in \mathcal{N}$,

$$f\left(\delta_1(N), \dots, \delta_n(N) | N = N_{n,p}\right) = \prod_{i=1}^{n} f\left(\delta_i(N) | N = N_p\right),$$

$$f\left(\delta_k(N) | N = N_p\right) := \frac{1}{(2\pi)^{\frac{p_k - 4}{2}} \left(\det \Sigma_k\right)^{\frac{1}{2}}} e^{-\frac{\delta_k(N)^T \Sigma_k^{-1} \delta_k(N)}{2}},$$

and Σ_k is the covariance matrix of noise term η_k in (16).

The assumption that $N_{n,p} = N_p$ for all n, is justified in [8] by assuming that during integer ambiguity resolution the satellite signals in use are free of cycle slips and when the cycle slips occur the integer ambiguity resolution restarts. It also means that signals from the same satellites are used all the time.

Here we formulate a generalized approach where appearance of new satellite signals, disappearance of existing satellite signals, and cycle slips can be taken into consideration and appropriately modelled.

If for some $k \in \{1, 2, \dots\}$ (19) is true, then from (16) we have

$$\alpha_k = \hat{N}_{k,s}(N_{k,p}) - A_s^k \left(A_p^k\right)^{-1} N_{k,p} + \eta_k. \tag{21}$$

Then under the same assumption, from (21) we have that the conditional probability of α_k given $N_{k,p} = N$ is

$$p\left(\alpha | N_{k,p} = N\right) = \frac{1}{(2\pi)^{\frac{p_k - 4}{2}} \left(\det \Sigma_k\right)^{\frac{1}{2}}} e^{-\frac{(\alpha - \beta_k(N))^T \Sigma_k^{-1} (\alpha - \beta_k(N))}{2}}, \tag{22}$$

and if we further assume $N_{k,p} = N$ for $k = 0, 1, 2, \dots$, then we have

$$p\left(\alpha_k | N_{k,p} = N\right) = f\left(\delta_k(N) | N = N_p\right). \tag{23}$$

Let $\mathcal{Y}_n$ be a complete sigma algebra generated by the sequence of observations $\alpha_0, \alpha_1, \ldots, \alpha_n$. Then we recursively calculate the conditional probability density of $N = N_{n,p}$, for $N \in \mathcal{N}_n$, $n = 1, 2, \ldots$, by using Bayes rule. We obtain

$$P\left(N = N_{n,p} | \mathcal{Y}_n\right) = \frac{p\left(\alpha_n | N_{n,p} = N\right) P\left(N = N_{n,p} | \mathcal{Y}_{n-1}\right)}{\sum_{M \in \mathcal{N}_n} p\left(\alpha_n | N_{n,p} = M\right) P\left(M = N_{n,p} | \mathcal{Y}_{n-1}\right)}, \tag{24}$$

$$P\left(N = N_{0,p} | \mathcal{Y}_0\right) = \frac{1}{|\mathcal{N}_0|}, \quad N \in \mathcal{N}_0, \tag{25}$$

where in general $|\mathcal{N}|$ is the number of elements in a set $\mathcal{N}$ and $\mathcal{N}_k$ is the set of integer ambiguities chosen at time step $t = t_k$.

The set $\mathcal{N}_k$ has fewer elements with every k since we discard all $N \in \mathcal{N}_{n-1}$ when the conditional probabilities $P\left(N = N_{n-1,p} | \mathcal{Y}_{n-1}\right)$ are smaller than the prescribed threshold P_{min}.

Instead of choosing a uniform probability distribution function for the initialization (25) of the recursive procedure (24) we can use any other if possible, more suitable initial distribution. This of course could very much depend on the particular interferometric GPS application.

The question is, how to get $P\left(N = N_{n-1,p} | \mathcal{Y}_{n-1}\right)$, $N \in \mathcal{N}_{n-1}$ from $P\left(N = N_{n,p} | \mathcal{Y}_{n-1}\right)$, $N \in \mathcal{N}_n$ in recursive procedure (24). We proceed as follows.

If no cycle slips are detected at $t = t_n$ and the set of primary satellites and the reference satellite is the same as for $t = t_{n-1}$, and since $\mathcal{N}_n \subset \mathcal{N}_{n-1}$, we take for $N \in \mathcal{N}_n$

$$P\left(N = N_{n,p} | \mathcal{Y}_{n-1}\right) = P\left(N = N_{n-1,p} | \mathcal{Y}_{n-1}\right).$$

Otherwise if the cycle slip occurs on the signal from one of the primary satellites or the reference satellite at $t = t_n$, then the integer ambiguity estimation can be restarted by using pseudoranges utilizing knowledge about the dynamics of x_k, $k = 0, 1, 2, \ldots$ together with the probability distribution of x_{n-1} obtained from the previous measurements. The same can be done if we have to choose a different reference satellite signal and/or a different primary satellite signal.

In the stationary case, when $x = x_k$ for $k = 0, 1, 2, \ldots$ the situation is better. At $t = t_{n-1}$, for $N \in \mathcal{N}_{n-1}$, we have already assigned probability $P\left(N = N_{n-1,p} | \mathcal{Y}_{n-1}\right)$, $n = 1, 2, \ldots$, to an estimate of x defined for $n = 1, 2, \ldots$, by

$$\hat{x}(N) := \frac{1}{n - \tilde{\sigma}_{n-1}} \sum_{j=\tilde{\sigma}_{n-1}}^{n-1} \left(A_p^j\right)^{-1} \left(y_{j,p} + N\right),$$

where $\tilde{\sigma}_{n-1} = \max\{\gamma^1_{n-1}, \gamma^2_{n-1}, \gamma^3_{n-1}, \gamma^4_{n-1}\}$.

Therefore we can define

$$\mathcal{N}_n = \left\{ M \in \mathbb{N}^3 : M = \left[A^n_p \hat{x}(N) - y_{n,p} \right], N \in \mathcal{N}_{n-1} \right\},$$

and to each $M \in \mathcal{N}_n$ we assign the same probability as for $\hat{x}(N)$ if $M = \left[A^n_p \hat{x}(N) - y_{n,p} \right]$.

If we have the same initial condition and the assumptions under which (20) is valid then the left hand sides of (20) and (24) are equal, i.e. the procedure defined by (24) is a generalization of (20). To see this, we divide the numerator and denominator on the right hand side in (20) by $\sum_{\tilde{M} \in \mathcal{N}} f\left(\delta_1(\tilde{M}), \ldots, \delta_{n-1}(\tilde{M}) | \tilde{M} = N_p\right) P\left(\tilde{M} = N_p\right)$ and from equation (23) we get

$$\begin{aligned} &P\left(N = N_p | \delta_1(N), \ldots, \delta_n(N)\right) \\ &= \frac{p(\alpha_k | N_p = N) P\left(N = N_p | \delta_1(N), \ldots, \delta_{n-1}(N)\right)}{\sum_{M \in \mathcal{N}} p(\alpha_k | N_p = M) P\left(M = N_p | \delta_1(M), \ldots, \delta_{n-1}(M)\right)}, \end{aligned}$$

which by induction implies the equality.

The M-ary Sequential Probability Ratio Test (MSPRT) developed in [15] and [17], was applied to the integer ambiguity resolution procedure in [8] and [18]. It is used for choosing the stopping criteria given the error of choosing the false integer ambiguity. This property along with the reduction in computational costs makes the generalized MAPAS method very attractive for applications.

5. Real Data Example

The real data we use here for estimating integer ambiguities in interferometric GPS are obtained from the IMA Summer Program for Graduate Students of July 22-31, 1998, see [5]. The epochs (measurements) are separated approximately by one second. We used measurements from the beginning of the data set. The satellites are indexed by integers from 1 to 32. The algorithm is written in Matlab.

We start with satellites indexed by 6, 16, 17, 23, 26, and 28. At epoch 234 the signal from the new satellite indexed by 21 appeared and at epoch 264 the signal from the same satellite disappeared. We did not go beyond epoch 503 since all of our cases could be resolved by that epoch.

We chose satellite 17 as a reference satellite and satellites 6, 16, and 23 as the primary satellites.

We have cycle slips on the signals of the secondary satellites at epochs 8, 14, 131, and 477. One cycle slip appears during the integer

Table I. MAPAS for different $\tilde{d}$

$\tilde{d}$ in (m)	ASN	S_0	T_S
1	24	209	t_{372}
2	116	1701	t_{485}
3	315	5709	t_{503}
4	642	13539	t_{503}
5	1111	26455	t_{503}
6	1723	45650	t_{503}
7	2483	72464	t_{503}
8	3394	108205	t_{503}
9	4456	154101	t_{503}
10	5658	211374	t_{503}
11	6999	281313	t_{503}

ambiguity resolution on the reference satellite and that is at epoch 488. The cycle slips on the primary satellites appear at epochs 210, 223, 241, 266, 285, and 387.

For initialization we chose the uniform distribution in a 3D ball with radius $\tilde{d}$ and the center at the receiver r_1. This can be considered as a simulation of integer ambiguity resolution when the baseline is more than several meters long. In that case we would have an estimate of the position of the receiver r_2 which here we can consider to be the position of the receiver r_1 i.e. $z_{r_1} \approx \tilde{z}_{r_2}$. We choose radius $\tilde{d}$ such that it is larger than the length of the true baseline.

In [18] $P_0 := \frac{1}{1+D} \approx 1 - D$, for small D, where D is the decision parameter, and P_0 is the threshold for stopping the integer ambiguity procedure, i.e. when for some $k \in \mathbb{N}$ the probability of some $N \in \mathcal{N}_k$ is larger than P_0 the integer ambiguity procedure is stopped with decision $N_p^k = N$. For such $k \in \mathbb{N}$, $T_S := t_k$ is called the stopping time. The total error probability ε is given by

$$\varepsilon = 1 - \sum_{M \in \mathcal{N}} P\left(N_p = M\right) \tilde{P}\left(N_p = M | M\right),$$

where $\tilde{P}\left(N_p = N | M\right)$ is the probability that $N \in \mathcal{N}$ is accepted assuming M is the correct ambiguity. In [18] it is shown that

$$\varepsilon \leq \frac{D}{1+D} \approx D,$$

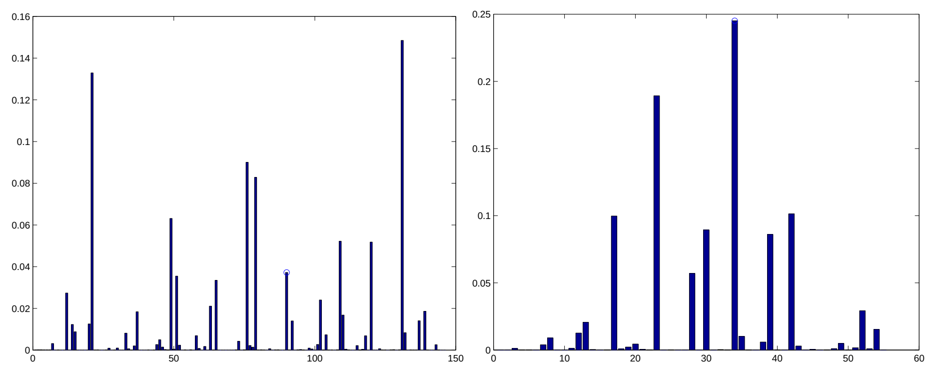
for small D. Therefore if the desired error probability is ε then for small $\varepsilon > 0$ we chose $P_0 = 1 - \varepsilon$ which is also the so called confidence level, i.e. the probability of resolving the true integer ambiguity.

In [1], equation (10), the covariance matrix of the carrier phase measurement, is altered by adding the term $\frac{1}{n^2}I$, where I is the unit matrix of an appropriate dimension. Changing the covariance matrix in [1] is justified by the Simulated Annealing technique. The decrease of temperature in Simulated Annealing is compared in [1] to the decrease of the additional uncertainty added to the data as time grows.

We use the same idea. We add to the covariance matrix Σ_k, $k = 1, 2, \ldots$ of the noise η_k in Equation (16) the quantity $\frac{\Delta}{k} I_{p_k - 4}$, $\Delta > 0$ i.e.

$$\hat{\Sigma}_n := \Sigma_n + \frac{\Delta}{n} I_{p_n - 4}, \quad n = 1, 2, \ldots. \tag{26}$$

By transformation (26) we decrease the average number of samples since thresholds P_0 and P_{min} can then be taken significantly smaller and larger, respectively. For example, to resolve the ambiguity correctly for $\Delta = 0$, P_{min} and P_0 have to be 10^{-25} and $1 - 10^{-15}$, respectively, and for $\Delta = 40$, they can be 10^{-6} and $1 - 10^{-6}$, respectively. For smaller ambiguity sets these values could be chosen even more suitably. This observation could mean that further studies of the noise with possible self tuning estimators of its variances and means could significantly improve the performance of the method.



a. Epoch t_{165} b. Epoch t_{225}

Figure 2. Conditional probability distributions of the integer ambiguities at two different epoch.

It is interesting to note that the number of epochs needed to resolve the integer ambiguities does not decrease for $\Delta > 0$. Indeed, in our example, fewer epochs are needed to come to the final decision even though the probability of the true integer ambiguity is larger for $\Delta >$

0 during the estimation process. Therefore the stopping time is not improved by the altering covariance matrix Σ_k, $k = 1, 2, \ldots$ by (26) but during the integer ambiguity procedure estimate of the integer ambiguity is better, i.e. the highest probability is more often assigned to the true integer ambiguity.

There are several important parameters which influence performance of the method in practice. These are thresholds P_0 and P_{min}, data uncertainty parameter Δ and the initial probability distribution.

The parameters which indicate performance of the method are the Average Sample Number (ASN) defined for n such that $t_n = T_S$ by ASN $:= \frac{\sum_{i=0}^{n} |\mathcal{N}_i|}{n+1}$, initial number of integer ambiguities S_0 which in our case directly depends on $\tilde{d}$, and the stopping time T_S.

The parameters ASN and S_0 are important indicators for overall computation time which together with the stopping time T_S give a general idea of the performance for the on-line applications i.e. estimation during data acquisition. In Table I we present performance parameters of the example. We set $P_{min} = 10^{-6}$ which is set the same for different $\tilde{d}$. When $\tilde{d} < 8$ we can set $P_{min} = 10^{-5}$ which would give $T_S = 395$ i.e. much smaller stopping time.

In Figures 2.a. and 2.b. we give plots of conditional probabilities of integer ambiguity sets $\mathcal{N}_k$ for $k = 165$ and $k = 225$ respectively, calculated during the same integer ambiguity resolution procedure. The parameters chosen during the integer ambiguity procedure are $\tilde{d} = 5$ meters, $\Delta = 40$, $P_{min} = 10^{-6}$, and $P_0 = 1 - 10^{-6}$ i.e. we have one of the cases from Table I. The tip of the probability of the true integer is circled. The sizes of the sets $\mathcal{N}_{165}$ and $\mathcal{N}_{225}$ are 146 and 55 respectively. Of the initial 26455 candidate integer ambiguities only a few remain which have a significant probability mass at epoch t_{225}. Observe that the correct integer ambiguity has been resolved by epoch $t = t_{503}$. The ambiguity resolution method developed in this report performed well on real selective availability (SA) corrupted GPS data, produced the correct integer ambiguities, and associated confidence levels, and is applicable in the presence of GPS cycle slips.

Since the intentional corruption of the GPS satellite signals by SA has been turned off by the U.S. Department of Defense (DoD) the GPS standard positioning service (SPS) which is available to nonmilitary users provides initial absolute positioning accuracies in the 10–20 meter range worldwide, see [10]. For distances up to 3000 km Wide Area Differential GPS (DGPS) provides relative positioning accuracies on the order of 5–10 meters and for distances up to 200 km local area Code Differential GPS provides relative positioning accuracies on the order of 0.5–1 meters. The method described here improves these initial accuracies to subcentimeter accuracies when the baselines are short. Our choices of $\tilde{d}$ in Table I are consistent with expected initial uncertainty in z_{r_2}. The real data used in this report was taken when the SA was still turned on and hence the satellite signals were being intentionally degraded by the U.S. DoD.

6. Open Problems

For the stationary case like the one we considered in this article, the generalized MAPAS method is very similar (if not equivalent) to the general Particle Nonlinear Filtering method described in [1], [2], [3], and [4]. In those articles the Particle Nonlinear Filtering method is further developed for Gaussian Shape Distributions. The similarity comes from the Bayes update step which is the same as Step 4 of Algorithm 1 in [1] and equation (24) in Section 4 of this article. In our case, the initial condition is the uniform distribution in a ball which is equivalent to associating to each integer ambiguity one particle with equal weight. Then during the integer ambiguity resolution we apply Bayes rule and then discard particles whose weight decreases below a specified threshold. In this case particles are not moving since we do not have any baseline dynamics. It would be very interesting to compare these two methods when we have a moving baseline with prescribed dynamics. The advantages and disadvantages of each method can be shown by comparing their performance during cycle slips.

In this article we have begun to address the problem of resolving GPS integer ambiguities to a given statistical confidence level in a minimum amount of time. Different optimization objective functions will lead to different ambiguity resolution methods. Some promising areas of future research include:

(1) Evaluating the time required to resolve the integer ambiguities to a specified confidence level as a function of satellite and receiver geometry and motion,

(2) Evaluating the accuracy of interferometric position and attitude estimates as a function of satellite and receiver geometry and motion,

(3) Recommending optimal satellite selection strategies and corresponding Interferometric GPS integer ambiguity resolution methods which achieve the highest position and attitude accuracies given a specified confidence level or time to resolve the integer ambiguities,

(4) Developing methods for determining when one or more GPS satellites or GPS receivers have broken and are giving bad data,

(5) Developing methods for dropping bad satellites and receivers and optimally reconfiguring the system to use the best available remaining good satellites and receivers,

(6) Statistically characterizing the measurement noise now that the intentional corruption of the GPS satellite signals by Selective Availability (SA) has been removed from the signals,

(7) Improving the atmospheric modeling to extend interferometric methods to longer baselines,

(8) Evaluating how to optimally use GPS pseudolites to improve GPS precision, reliability and robustness.

(9) Defining how future satellite carrier frequencies, waveforms and orbits should be selected to ensure rapid, reliable, and robust ambiguity resolution and corresponding high precision localization accuracy.

The GPS satellite system has reached a level of maturity and sophistication that future advances will require more rigorous mathematical analysis. GPS is a wonderful source of numerous unsolved mathematics problems.

Appendix

Here we are calculating the covariance matrix of the noise η_k, $k = 0, 1, \ldots$, which is the third term on the right hand side in equation (21). It is used in calculation of the right hand side of (22) which is then used in Bayes rule update (24).

In Section 3 just before equation (11) we defined ν_k by

$$\nu_k = -\left(\nu_{r_1,r_2}^{l_1^k,l_2^k}, \ldots, \nu_{r_1,r_2}^{l_1^k,l_{p_k}^k}\right)^T.$$

Therefore we have

$$E\left(\left(\nu_k^i\right)^2\right) = E\left(\left(\nu_{r_1,r_2}^{l_1^k,l_{i+1}^k}\right)^2\right) = E\left(\left(\nu_{r_1,r_2}^{l_{i+1}^k} - \nu_{r_1,r_2}^{l_1^k}\right)^2\right) \tag{27}$$

$$= E\left(\left(\nu_{r_2}^{l_{i+1}^k} - \nu_{r_1}^{l_{i+1}^k} - \nu_{r_2}^{l_1^k} + \nu_{r_1}^{l_1^k}\right)^2\right) = 4\sigma^2.$$

Also from definition, for $i \neq j$, we have

$$E\left(\nu_k^i \nu_k^j\right) = E\left(\left(\nu_{r_2}^{l_{i+1}^k} - \nu_{r_1}^{l_{i+1}^k} - \nu_{r_2}^{l_1^k} + \nu_{r_1}^{l_1^k}\right)\left(\nu_{r_2}^{l_{j+1}^k} - \nu_{r_1}^{l_{j+1}^k}\right.\right. \tag{28}$$

$$\left.\left. - \nu_{r_2}^{l_1^k} + \nu_{r_1}^{l_1^k}\right)\right) = E\left(\left(\nu_{r_1}^{l_1^k} - \nu_{r_2}^{l_1^k}\right)^2\right) = 2\sigma^2.$$

Since ν_k is a linear combination of normally distributed random variables with mean zero and covariance σ^2, it is also normally distributed with mean zero and $p_k - 1 \times p_k - 1$ covariance matrix C_k derived from (27) and (28) i.e. we have

$$C_k = \sigma^2 \begin{bmatrix} 4 & 2 & \dots & 2 \\ 2 & 4 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 4 \end{bmatrix},$$

which is $p_k - 1 \times p_k - 1$ matrix. We can matrix C_k split into block matrices

$$C_k = \begin{bmatrix} C_{k,p} & C_{k,ps} \\ C_{k,sp} & C_{k,s} \end{bmatrix}, \tag{29}$$

where $C_{k,p}$, $C_{k,ps}$, $C_{k,sp}$, and $C_{k,s}$ are 3×3, $3 \times p_k - 4$, $p_k - 4 \times 3$, and $p_k - 4 \times p_k - 4$ matrices, respectively. Also we have that $C_{k,p}$ and $C_{k,s}$ are covariance matrices of $\nu_{k,p}$ and $\nu_{k,s}$, respectively. Also we have $\mathrm{Cov}\,(\nu_{k,p}, \nu_{k,s}) = C_{k,ps}$ and $\mathrm{Cov}\,(\nu_{k,s}, \nu_{k,p}) = C_{k,sp}$.

By definition of η_k we have

$$\eta_k = \nu_{k,s} - A_s^k \left(A_p^k\right)^{-1} \nu_{k,p} = \left[-A_s^k \left(A_p^k\right)^{-1} I_{p_k-4}\right] \begin{bmatrix} \nu_{k,p} \\ \nu_{k,s} \end{bmatrix}, \tag{30}$$

where I_{p_k-4} is the unit $p_k - 4 \times p_k - 4$ matrix.

Therefore by linearity of probability expectation, (29), and (30), we have

$$\Sigma_k = \mathrm{Cov}\,(\eta_k) = \left[-A_s^k \left(A_p^k\right)^{-1} I_{p_k-4}\right] C_k \begin{bmatrix} -\left(A_s^k \left(A_p^k\right)^{-1}\right)^T \\ I_{p_k-4} \end{bmatrix}$$

$$= A_s^k \left(A_p^k\right)^{-1} C_{k,p} \left(A_s^k \left(A_p^k\right)^{-1}\right)^T - C_{k,sp} \left[A_s^k \left(A_p^k\right)^{-1}\right]^T$$

$$-A_s^k \left(A_p^k\right)^{-1} C_{k,ps} + C_{k,s}.$$

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