Computing the Smoothness Exponent of a Symmetric Multivariate Refinable Function

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Abstract:

Smoothness and symmetry are two important properties of a refinable function. It is known that the Sobolev smoothness exponent of a refinable function can be estimated by computing the spectral radius of certain finite matrix which is generated from a mask. However, the increase of dimension and the support of a mask tremendously increases the size of the matrix and therefore make the computation very expensive. In this paper, we shall present a simple algorithm to efficiently numerically compute the smoothness exponent of a symmetric refinable function with a general dilation matrix. By taking into account of symmetry of a refinable function, our algorithm greatly reduces the size of the matrix and enables us to numerically compute the Sobolev smoothness exponents of a large class of symmetric refinable functions. Step by step numerically stable algorithms and details of the numerical implementation are given. To illustrate our results by performing some numerical experiments, we construct a family of dyadic interpolatory masks in any dimension and we compute the smoothness exponents of their refinable functions in dimension three. Several examples will also be presented for computing smoothness exponents of symmetric refinable functions on the quincunx lattice and on the hexagonal lattice.

Key words: Eigenvalues of matrices, Smoothness exponent, regularity, multivariate refinable functions, symmetry, interpolating functions, quincunx dilation matrix

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1 Introduction

A $d \times d$ integer matrix M is called a **dilation matrix** if $\lim_{k\to\infty} M^{-k} = 0$. A dilation matrix M is **isotropic** if all the eigenvalues of M have the same modulus. We say that a is a **mask** on $\mathbb{Z}^d$ if a is a finitely supported sequence on $\mathbb{Z}^d$ such that $\sum_{\beta\in\mathbb{Z}^d} a(\beta) = 1$. Wavelets are derived from refinable functions via a standard multiresolution technique. A **refinable function** ϕ is a solution to the following refinement equation

$$\phi = |\det M| \sum_{\beta\in\mathbb{Z}^d} a(\beta)\phi(M\cdot -\beta), \tag{1.1}$$

where a is a mask and M is a dilation matrix. For a mask a on $\mathbb{Z}^d$ and a $d\times d$ dilation matrix M, it is known ([1]) that there exists a unique compactly supported distributional solution, denoted by ϕ_a^M throughout the paper, to the refinement equation (1.1) such that $\hat{\phi}_a^M(0) = 1$, where the Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx, \xi \in \mathbb{R}^d$ and can be naturally extended to tempered distributions. Symmetric multivariate wavelets and refinable functions have proved to be very useful in many applications. For example, 2D refinable functions and wavelets have been widely used in subdivision surfaces and image/mesh compression while 3D refinable functions have been used in subdivision volumes, animation and video processing, etc.

For a compactly supported function ϕ in $\mathbb{R}^d$, we say that the shifts of ϕ are **stable** if for every $\xi \in \mathbb{R}^d$, $\widehat{\phi}(\xi + 2\pi\beta) \neq 0$ for some $\beta \in \mathbb{Z}^d$. For a function $\phi \in L_2(\mathbb{R}^d)$, its **Sobolev smoothness exponent** is defined to be

$$\nu_2(\phi) := \sup\Big\{\nu \geqslant 0 \quad : \quad \int_{\mathbb{R}^d} |\widehat{\phi}(\xi)|^2(1+|\xi|^2)^\nu \, d\xi < \infty\Big\}. \tag{1.2}$$

Smoothness is one of the most important properties of a wavelet system. Therefore, it is of great importance to be able to numerically compute the smoothness exponent of a refinable function. Let a be a mask and M be a dilation matrix. We say that a satisfies the **sum rules** of order k with respect to the lattice $M\mathbb{Z}^d$ if

$$\sum_{\beta\in M\mathbb{Z}^d} a(\alpha+\beta)q(\alpha+\beta) = \sum_{\beta\in M\mathbb{Z}^d} a(\beta)q(\beta) \qquad \forall\ \alpha \in \mathbb{Z}^d, q \in \Pi_{k-1},$$

where Π_{k-1} denotes the set of all polynomials of total degree less than k. By convention, Π_{-1} is the empty set. Define a new sequence b from the mask a by

$$b(\alpha) := \sum_{\beta\in\mathbb{Z}^d} a(\alpha+\beta)\overline{a(\beta)}, \qquad \alpha \in \mathbb{Z}^d. \tag{1.3}$$

The **transition operator** $T_{b,M}$ associated with the sequence b and the dilation matrix M is defined by

$$[T_{b,M}u](\alpha) = |\det M| \sum_{\beta\in\mathbb{Z}^d} b(M\alpha-\beta)u(\beta), \qquad \alpha \in \mathbb{Z}^d,\ u \in \ell_0(\mathbb{Z}^d), \tag{1.4}$$

where $\ell_0(\mathbb{Z}^d)$ denotes the linear space of all finitely supported sequences on $\mathbb{Z}^d$. For a subset K of $\mathbb{Z}^d$, by $\ell(K)$ we denote the linear space of all finitely supported sequences on $\mathbb{Z}^d$ that vanish outside the set K.

When $\phi_a^M \in L_2(\mathbb{R}^d)$ and the shifts of ϕ_a^M are stable, if M is isotropic and a satisfies the sum rules of order k but not $k+1$, then it was demonstrated in [3, 4, 7, 15, 18, 20, 21, 26, 27, 29] in various forms under various conditions that

$$\nu_2(\phi_a^M) = -\frac{d}{2}\log_{|\det M|}\rho(T_{b,M}|_{V_{2k-1}}), \tag{1.5}$$

where $\rho(T_{b,M}|_{V_{2k-1}})$ is the spectral radius of the operator $T_{b,M}$ acting on the finite dimensional $T_{b,M}$-invariant subspace V_{2k-1} of $\ell(\Omega_{b,M})$ with the set $\Omega_{b,M}$ being defined to be

$$\Omega_{b,M} := \Big[\sum_{j=1}^{\infty} M^{-j}K\Big]\cap\mathbb{Z}^d \quad \text{and} \quad K := \{\alpha\in\mathbb{Z}^d \ : \ |\alpha|\leqslant k\}\cup\{\alpha\in\mathbb{Z}^d \ : \ b(\alpha)\neq 0\}, \tag{1.6}$$

and the slightly smaller subspace V_{2k-1} of $\ell(\Omega_{b,M})$ is defined to be

$$V_j := \{u\in\ell(\Omega_{b,M}) \ : \ \sum_{\beta\in\mathbb{Z}^d} u(\alpha)q(\beta) = 0 \qquad \forall\, q\in\Pi_j\}, \qquad j\in\mathbb{N}_0. \tag{1.7}$$

However, from the point of view of numerical computation, there are some difficulties in obtaining the Sobolev smoothness exponent of a refinable function via (1.5) by computing the quantity $\rho(T_{b,M}|_{V_{2k-1}})$ due to the following considerations:

D1. It is not easy to find a simple basis for the space V_{2k-1} by a numerically stable procedure to obtain a representation matrix of $T_{b,M}$ under such a basis. Theoretically speaking, if some elements in a numerically found basis of V_{2k-1} can not satisfy the equality in (1.7) exactly, then it will dramatically change the spectral radius since in general $T_{b,M}$ has significantly larger eigenvalues outside the subspace V_{2k-1}.

D2. When the dimension is greater than one and even when the mask has a relatively small support, in general, the dimensions of the spaces V_{2k-1} and $\ell(\Omega_{b,M})$ are very large. For example, for a 3D mask with support $[-7,7]^3$ and sum rules of order 4, we have $\dim(V_7) = 24269$ and $\dim(\ell(\Omega_{b,2I_3})) = 24389$. This makes the numerical computation using (1.5) very expensive or even impossible.

D3. In order to obtain the exact Sobolev smoothness exponent by (1.5), we have to check the assumption that the shifts of ϕ_a^M are stable which is a far from trivial condition to be verified.

Fortunately, the difficulty in **D1** was successfully overcome in Jia and Zhang [19], where they demonstrated that $\rho(T_{b,M}|_{V_{2k-1}})$ is the largest value in modulus in the set consisting of all the eigenvalues of $T_{b,M}|_{\ell(\Omega_{b,M})}$ excluding some known special eigenvalues. Note that $\ell(\Omega_{b,M})$ has a simple basis $\{\delta_\alpha \ : \ \alpha\in\Omega_{b,M}\}$, where $\delta_\alpha(\alpha)=1$ and $\delta_\alpha(\beta)=0$ for all $\beta\in\mathbb{Z}^d\backslash\{\alpha\}$.

On the other hand, both symmetry and smoothness of a wavelet basis are very important and much desired properties in many applications. It is one of the purposes in this paper to try to overcome the difficulty in **D2** for a symmetric refinable function. We shall demonstrate in Algorithm 2.1 in Section 2 that we can compute the Sobolev smoothness exponent of a symmetric refinable function by using a much smaller space than using the space $\ell(\Omega_{b,M})$. In Section 3, we shall see that for many refinable functions, it is not necessary to directly verify the stability assumption since they are already implicitly implied by the computation. Therefore, the difficulty in **D3** does not exist at all for many refinable functions (almost all interesting known examples fall into this class).

To give the reader some idea about how symmetry can be of help in computing the Sobolev smoothness exponents of symmetric refinable functions, we give the following comparison result in Table 1. See Section 2 for more detail and explanation of Table 1.

Mask	4D mask	3D mask	2D mask	2D mask	2D mask
Support	$[-5,5]^4$	$[-7,7]^3$	$[-27,27]^2$	$[-7,7]^2$	$[-12,12]^2$
Symmetry	full axes	full axes	hexagonal	full axes	hexagonal
Dilation matrix	$2I_4$	$2I_3$	$2I_2$	$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$
Method in [19]	194481	24389	8911	5601	$\geqslant 3241$
Algorithm 2.1	715	560	756	707	294

Table 1: The last two rows indicate the matrix sizes in computing the Sobolev smoothness exponents of symmetric refinable functions using both the method in [19] and the method in Algorithm 2.1 in Section 2 in this paper. This table demonstrates that Algorithm 2.1 can greatly reduce the size of matrix in computing the Sobolev smoothness exponent of a symmetric refinable function.

The structure of the paper is as follows. In Section 2, we shall present step by step numerically stable algorithms to numerically compute the smoothness exponent of a symmetric refinable function. In Section 3, we shall study the relation of the spectral radius of certain operator acting on different spaces. Such analysis enables us to overcome the difficulty in **D3** for a large class of masks. In Section 4, we shall apply the results in Sections 2 and 3 to several examples including refinable functions on quincunx lattice and hexagonal lattice. We shall also present a C^2 $\sqrt{3}$-interpolatory subdivision scheme in Section 4. In Section 5, we shall generalize the well known univariate interpolatory masks in Deslauriers and Dubuc [5] and the bivariate interpolatory masks in [11] to any dimension. Finally, we shall use the results in Sections 2 and 3 to compute Sobolev smoothness exponents of interpolating refinable functions associated with the interpolatory masks in Section 5 in dimension three.

Programs for computing the Sobolev and Hölder smoothness exponents of symmetric refinable functions based on the Algorithms 2.1 and 2.5 in Section 2, which come without warranty and are not yet optimized with respect to user interface, can be downloaded at `http://www.ualberta.ca/~bhan`.

2 Computing Smoothness Exponent Using Symmetry

In this section, taking into account of symmetry, we shall present an algorithm to efficiently numerically compute the Sobolev smoothness exponent of a symmetric multivariate refinable function with a general dilation matrix. As the main result in this section, Algorithms 2.1 and 2.5 are quite simple and can be easily implemented, though their proofs and some notation are relatively technical.

Before proceeding further, let us introduce some notation and necessary background. Let $\mathbb{N}_0$ denote all the nonnegative integers. For $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$, $|\mu| := \mu_1 + \cdots + \mu_d$, $\mu! := \mu_1! \cdots \mu_d!$ and $\xi^\mu := \xi_1^{\mu_1} \cdots \xi_d^{\mu_d}$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. For $\alpha \in \mathbb{Z}^d$ and $y \in \mathbb{R}^d$, we define

$$\nabla_\alpha u := u - u(\cdot - \alpha), \quad \nabla_y f := f - f(\cdot - y), \qquad u \in \ell_0(\mathbb{Z}^d), f \in L_p(\mathbb{R}^d).$$

For $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$, $\nabla^\mu := \nabla_{e_1}^{\mu_1} \cdots \nabla_{e_d}^{\mu_d}$, where e_j is the jth coordinate unit vector in $\mathbb{R}^d$. Let $\delta = \delta_0$ denote the sequence such that $\delta(0) = 1$ and $\delta(\beta) = 0$ for all $\beta \in \mathbb{Z}^d \backslash \{0\}$. Let M be a $d \times d$ dilation matrix and a be a mask on $\mathbb{Z}^d$. For $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}_0$, we define

$$\rho_k(a; M, p) := \max\{ \lim_{n \to \infty} \|\nabla^\mu S_{a,M}^n \delta\|_p^{1/n} \ : \ |\mu| = k, \ \mu \in \mathbb{N}_0^d\}, \tag{2.1}$$

where $\|u\|_p := (\sum_{\beta \in \mathbb{Z}^d} |u(\beta)|^p)^{1/p}$ for $u \in \ell_0(\mathbb{Z}^d)$ and $S_{a,M}$ is the **subdivision operator** given by

$$[S_{a,M} u](\alpha) := |\det M| \sum_{\beta \in \mathbb{Z}^d} a(\alpha - M\beta) u(\beta), \qquad \alpha \in \mathbb{Z}^d, u \in \ell_0(\mathbb{Z}^d).$$

Let M be a dilation matrix and λ_{max} be the spectral radius of M (When M is isotropic, then $\lambda_{max} = |\det M|^{1/d}$). When a mask a satisfies the sum rules of order k but not $k+1$, we define the following important quantity:

$$\nu_p(a; M) := -\log_{\lambda_{max}} \big[|\det M|^{-1/p} \rho_k(a; M, p) \big], \qquad 1 \leqslant p \leqslant \infty. \tag{2.2}$$

The above quantity $\nu_p(a; M)$ plays a very important role in characterizing the convergence of a subdivision scheme in a Sobolev space and in characterizing the L_p smoothness exponent of a refinable function. The partial derivative of a differentiable function f with respect to the jth coordinate is denoted by $D_j f$, $j = 1, \dots, d$, and for $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$, $D^\mu := D_1^{\mu_1} \cdots D_d^{\mu_d}$. By $W_p^k(\mathbb{R}^d)$ we denote the Sobolev space that consists of all functions f such that $D^\mu f \in L_p(\mathbb{R}^d)$ for all $\mu \in \mathbb{N}_0^d$ with $|\mu| \leqslant k$, equipped with the norm $\|f\|_{W_p^k(\mathbb{R}^d)} := \sum_{|\mu| \leqslant k} \|D^\mu f\|_p$. When $k = 0$, $W_p^0(\mathbb{R}^d)$ is the space $L_p(\mathbb{R}^d)$ and $\|\cdot\|_{W_p^0(\mathbb{R}^d)}$ is the L_p norm.

An iteration scheme can be employed to solve the refinement equation (1.1). Start with some initial function $\phi_0 \in W_p^k(\mathbb{R}^d)$ such that $\widehat{\phi}_0(0) = 1$ and $D^\mu \widehat{\phi}_0(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^d \backslash \{0\}$ and for all $\mu \in \mathbb{N}_0^d$ with $|\mu| \leqslant k$. We employ the iteration scheme $Q_{a,M}^n \phi_0$, $n \in \mathbb{N}_0$, where $Q_{a,M}$ is the linear operator on $L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$ given by

$$Q_{a,M} f := |\det M| \sum_{\beta \in \mathbb{Z}^d} a(\beta) f(M \cdot - \beta), \qquad f \in L_p(\mathbb{R}^d).$$

This iteration scheme is called a **subdivision scheme** ([1]). When the sequence $Q^n_{a,M}\phi_0$ converges in the Sobolev space $W^k_p(\mathbb{R}^d)$, then the limit function must be ϕ^M_a and we say that the subdivision scheme associated with mask a and dilation M converges in $W^k_p(\mathbb{R}^d)$. When M is an isotropic dilation matrix, the subdivision scheme associated with the mask a and dilation M converges in the Sobolev space $W^k_p(\mathbb{R}^d)$ if and only if $\rho_{k+1}(a;M,p) < |\det M|^{1/p-k/d}$ (we shall see in Section 3 that this is equivalent to $\nu_p(a;M) > k$). See [10] for the case $k=0$ and [17] for the case $p=2$ on the characterization of the convergence of a subdivision scheme.

The L_p smoothness of $f \in L_p(\mathbb{R}^d)$ is measured by its L_p **smoothness exponent**:

$$\nu_p(f) := \sup\{\nu \geqslant 0 \ : \ \|\nabla^n_y f\|_p \leqslant C\|y\|^\nu \ \forall\, y \in \mathbb{R}^d \quad \text{for some constant } C \text{ and for large enough positive integer } n\}. \tag{2.3}$$

When $p=2$, the above definition of $\nu_2(f)$ agrees with the definition in (1.5). By generalizing the results in [4, 3, 7, 9, 15, 18, 20, 21, 26, 27, 29] and references therein, we have

$$\nu_p(\phi^M_a) \geqslant \nu_p(a;M), \qquad 1 \leqslant p \leqslant \infty$$

and the equality holds when the shifts of ϕ^M_a are stable and M is an isotropic dilation matrix.

So, to compute the Sobolev smoothness exponent of a refinable function, we need to compute $\nu_2(a;M)$ and therefore, to compute $\rho_k(a;M,2)$. It is the purpose of this section to discuss how to efficiently compute $\rho_k(a;M,2)$ when a is a symmetric mask.

Let Θ be a finite subset of integer matrices whose determinants are ± 1. We say that Θ is a **symmetry group** with respect to a dilation matrix M if Θ forms a group under matrix multiplication and $M\theta M^{-1} \in \Theta$ for all $\theta \in \Theta$. Obviously, each element in a symmetry group induces a linear isomorphism on $\mathbb{Z}^d$.

Let Θ^A_d denote the set of all linear transforms on $\mathbb{Z}^d$ which are given by

$$\theta_{\pi,\varepsilon}(\alpha_1, \dots, \alpha_d) := (\varepsilon_1\alpha_{\pi(1)}, \dots, \varepsilon_d\alpha_{\pi(d)}), \qquad (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d, \tag{2.4}$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{-1,1\}^d$ and π is a permutation on $(1, \dots, d)$. Θ^A_d is called the **full axes symmetry group**. Obviously, Θ^A_d is a symmetry group with respect to the dilation matrix $2I_d$. It is also easy to check that Θ^A_2 is a symmetry group with respect to the quincunx dilation matrices $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Another symmetry group with respect to $2I_2$ is the following group which is called the **hexagonal symmetry group**:

$$\Theta^H = \left\{ \pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \pm\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}, \pm\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}, \pm\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \pm\begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}, \pm\begin{bmatrix}-1 & 0\\ -1 & 1\end{bmatrix} \right\}. \tag{2.5}$$

Such group Θ^H can be used to obtain wavelets on the hexagonal planar lattice (that is, the triangular mesh). For a symmetry group Θ and a sequence u on $\mathbb{Z}^d$, we define a new sequence $\Theta(u)$ as follows:

$$[\Theta(u)](\beta) := \frac{1}{\#\Theta} \sum_{\theta \in \Theta} u(\theta\beta), \qquad \beta \in \mathbb{Z}^d,\ u \in \ell_0(\mathbb{Z}^d), \tag{2.6}$$

where $\#\Theta$ denotes the cardinality of the set Θ. We say that a mask a is **invariant** under Θ if $\Theta(a) = a$. Obviously, for any sequence u, $\Theta(u)$ is invariant under Θ since $\Theta(\Theta(u)) = \Theta(u)$. When Θ is a symmetry group with respect to a dilation matrix M, then a is invariant under Θ implies that the refinable function ϕ_a^M is also invariant under Θ; that is, $\phi_a^M(\theta\cdot) = \phi_a^M$ for all $\theta \in \Theta$. We caution the reader that the condition $M\theta M^{-1} \in \Theta$ for all $\theta \in \Theta$ can not be removed in the definition of a symmetry group with respect to a dilation matrix M. For example, as a subgroup of Θ_2^A, $\Theta = \{\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\}$ is not a symmetry group with respect to the quincunx dilation matrices, though it is a symmetry group with respect to the dilation matrix $2I_2$. So even when a mask a is invariant under such a group Θ, the refinable function ϕ_a^M with the quincunx dilation matrix M may not be invariant under Θ.

Let $\mathbb{Z}_\Theta^d$ denote a subset of $\mathbb{Z}^d$ such that for every $\alpha \in \mathbb{Z}^d$, there exists a unique $\beta \in \mathbb{Z}_\Theta^d$ satisfying $\theta\beta = \alpha$ for some $\theta \in \Theta$. In other words, $\mathbb{Z}_\Theta^d$ is a set of complete representatives of the distinct cosets of $\mathbb{Z}^d$ under the equivalence relation induced by Θ on $\mathbb{Z}^d$.

Algorithm 2.1 *Let M be a $d \times d$ isotropic dilation matrix and let Θ be a symmetry group with respect to the dilation matrix M. Let a be a mask on $\mathbb{Z}^d$ such that $\sum_{\beta \in \mathbb{Z}^d} a(\beta) = 1$. Define the sequence b as in* (1.3). *Suppose that b is invariant under the symmetry group Θ and a satisfies the sum rules of order k but not $k+1$. The quantity $\nu_2(a; M)$ (or equivalently, $\rho_k(a; M, 2)$) is obtained via the following procedure:*

(a) *Find a finite subset K_Θ of $\mathbb{Z}_\Theta^d$ such that $\{M^{-1}(\theta\alpha + \beta) \ : \ \theta \in \Theta, \alpha \in K_\Theta, \beta \in \operatorname{supp} b\} \cap \mathbb{Z}^d \subseteq \{\theta\beta \ : \ \beta \in K_\Theta, \theta \in \Theta\}$ and $\dim(\Pi_{2k-1}|_{\{\theta\beta\,:\,\theta \in \Theta, \beta \in K_\Theta\}}) = \dim(\Pi_{2k-1})$;*

(b) *Obtain a $(\#K_\Theta) \times (\#K_\Theta)$ matrix T as follows:*

$$T[\alpha, \beta] := \frac{|\det M|}{\#\{\theta \in \Theta \ : \ \theta\beta = \beta\}} \sum_{\theta \in \Theta} b(M\beta - \theta\alpha), \qquad \alpha, \beta \in K_\Theta; \tag{2.7}$$

(c) *Let $\sigma(T)$ consist of the absolute values of all the eigenvalues of the square matrix T counting multiplicity. Then $\nu_2(a; M)$ is the smallest number in the following set*

$$\{-\frac{d}{2} \log_{|\det M|} \rho \ : \ \rho \in \sigma(T)\} \backslash \{j/2 \text{ with multiplicity } m_\Theta(j) \ : \ 0 \leqslant j < 2k\}, \tag{2.8}$$

where by default $\log_{|\det M|} 0 := -\infty$ and

$$m_\Theta(j) := \dim(\Theta(\Pi_j)) - \dim(\Theta(\Pi_{j-1})), \qquad j \in \mathbb{N}_0. \tag{2.9}$$

Before we give a proof to Algorithm 2.1, let us make some remarks and discuss how to compute the set K_Θ and the quantities $m_\Theta(j)$ in Algorithm 2.1. Since the matrix T in Algorithm 2.1 has a simple structure, it is not necessary to store the whole matrix T in order to compute its eigenvalues and many techniques from numerical analysis (such as the subspace iteration method and Arnoldi's method as discussed in [28]) can be exploited to further improve the efficiency in computing the eigenvalues of T. We shall not discuss such an issue here. The set K_Θ can be easily obtained as follows:

Proposition 2.2 *Let* $K_0 := \mathrm{supp}\, b \cup \{\theta\alpha \in \mathbb{Z}^d \; : \; |\alpha| \leqslant k, \theta \in \Theta\}$ *where* $\mathrm{supp}\, b := \{\beta \in \mathbb{Z}^d \; : \; b(\beta) \neq 0\}$*. Recursively compute*

$$K_j := K_{j-1} \cup [M^{-1}(K_{j-1} + \mathrm{supp}\, b) \cap \mathbb{Z}^d], \qquad j \in \mathbb{N}.$$

Then $K_j = K_{j-1}$ *for some* $j \in \mathbb{N}$*. Set* $K_\Theta := K_j \cap \mathbb{Z}^d_\Theta$*. Then* K_Θ *satisfies all the conditions in* (a) *of Algorithm 2.1.*

Proof: Note that $K_j \subseteq (\sum_{j=1}^{\infty} M^{-j} K_0) \cap \mathbb{Z}^d \subseteq \{\alpha \in \mathbb{Z}^d \; : \; |\alpha| < r\}$ for some finite integer r. Therefore, there must exist $j \in \mathbb{N}$ such that $K_j = K_{j-1}$ by $K_{i-1} \subseteq K_i$ for all $i \in \mathbb{N}$. Consequently, $M^{-1}(K_j + \mathrm{supp}\, b) \cap \mathbb{Z}^d = M^{-1}(K_{j-1} + \mathrm{supp}\, b) \cap \mathbb{Z}^d \subseteq K_j$. Since $K_0 \subseteq K_j$, we have $\dim(\Pi_{2k-1}) = \dim(\Pi_{2k-1}|_{K_0}) \leqslant \dim(\Pi_{2k-1}|_{\{\theta\beta \, : \, \theta\in\Theta, \beta\in K_\Theta\}}) \leqslant \dim(\Pi_{2k-1})$. ■

Let $O_j := \{\mu \in \mathbb{N}_0^d \; : \; |\mu| = j\}$. The set O_j can be ordered according to the lexicographic order. That is, $(\nu_1, \ldots, \nu_d)$ is less than $(\mu_1, \ldots, \mu_d)$ in lexicographic order if $|\nu| < |\mu|$ or $\nu_j = \mu_j$ for $j = 1, \ldots, i-1$ and $\nu_i < \mu_i$ for some i. For a $d \times d$ matrix A and any $j \in \mathbb{N}_0$, we define a $(\#O_j) \times (\#O_j)$ matrix $S(A, j)$ which is uniquely determined by

$$\frac{(Ax)^\mu}{\mu!} = \sum_{\nu \in O_j} [S(A,j)]_{\mu,\nu} \frac{x^\nu}{\nu!}, \qquad \mu \in O_j, j \in \mathbb{N}_0. \tag{2.10}$$

Note that $S(AB, j) = S(A, j)S(B, j)$ and $S(A^T, j) = S(A, j)^T$. Moreover, when $\lambda_1, \ldots, \lambda_d$ are all the eigenvalues of A, then $\lambda^\mu, \mu \in O_j$ are all the eigenvalues of $S(A, j)$, where $\lambda = (\lambda_1, \ldots, \lambda_d)$ since $S(A, j)$ is similar to $S(B, j)$ when A is similar to B.

The quantities $m_\Theta(j), j \in \mathbb{N}_0$ can be computed as follows.

Proposition 2.3 *Let* Θ *be a symmetry group. Then*

$$m_\Theta(j) = \mathrm{rank}\Big[\sum_{\theta \in \Theta} S(\theta, j)\Big], \qquad j \in \mathbb{N}_0. \tag{2.11}$$

In particular, when $-I_d \in \Theta$*, then* $m_\Theta(2j-1) = 0$ *for all* $j \in \mathbb{N}$*.*

Proof: For $\mu \in \mathbb{N}_0^d$, let q_μ be the sequence given by $q_\mu(\alpha) = \alpha^\mu/\mu!, \alpha \in \mathbb{Z}^d$. Note that

$$[\Theta(q_\mu)](\alpha) = \sum_{\theta\in\Theta} q_\mu(\theta\alpha) = \sum_{\theta\in\Theta} \frac{(\theta\alpha)^\mu}{\mu!} = \sum_{\theta\in\Theta} \sum_{\nu\in O_j} [S(\theta,j)]_{\mu,\nu} \frac{\alpha^\nu}{\nu!} = \sum_{\nu\in O_j} q_\nu(\alpha) \sum_{\theta\in\Theta} [S(\theta,j)]_{\mu,\nu}.$$

Since $q_\mu, \mu \in O_j$ are linearly independent, we have

$$m_\Theta(j) = \dim(\Theta(\Pi_j)) - \dim(\Theta(\Pi_{j-1})) = \dim(\text{span}\{\Theta(q_\mu) \ : \ \mu \in O_j\}) = \text{rank}\Big[\sum_{\theta\in\Theta} S(\theta, j)\Big].$$

When $-I_d \in \Theta$, we observe that $\sum_{\theta\in\Theta} S(\theta, j) = \sum_{\theta\in\Theta} S(-\theta, j) = (-1)^j \sum_{\theta\in\Theta} S(\theta, j)$. Therefore, $m_\Theta(2j-1) = 0$ for all $j \in \mathbb{N}$ since $\sum_{\theta\in\Theta} S(\theta, 2j-1) = 0$. ■

Note that $m_\Theta(j)$ depends only on the symmetry group Θ and is independent of the dilation matrix M. When Θ is a subgroup of the full axes symmetry group Θ_d^A, then $m_\Theta(j)$ can be easily determined since matrix $S(\theta, j)$ is very simple for every $\theta \in \Theta_d^A$. For example, $m_{\Theta_d^A}(2j) = \#\{(\mu_1, \ldots, \mu_d) \in \mathbb{N}_0^d \ : \ 0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_d, \ \mu_1 + \cdots + \mu_d = j\}$.

For the convenience of the reader, we list the quantities $m_\Theta(j)$ in Algorithm 2.1 for some well known symmetry groups in Table 2, where Θ_2^1 and Θ_2^2 are defined to be

$$\Theta_2^1 = \left\{\pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \pm\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\right\}, \qquad \Theta_2^2 = \left\{\pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \pm\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right\}.$$

	$m_\Theta(j), \quad j = 0, 2, 4, \ldots, 32$																
j	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
Θ_1^A	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Θ_2^A	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
Θ^H	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6
Θ_2^1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Θ_2^2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Θ_3^A	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30
Θ_4^A	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64

Table 2: The quantities $m_\Theta(j), j \in \mathbb{N}_0$ in Algorithm 2.1 for some known symmetry groups. Note that $m_\Theta(2j-1) = 0, j \in \mathbb{N}$ in this table.

For a sequence u on $\mathbb{Z}^d$, its **symbol** is given by

$$\widetilde{u}(\xi) = \sum_{\beta\in\mathbb{Z}^d} u(\beta) e^{-i\beta\cdot\xi}, \qquad \xi \in \mathbb{R}^d. \tag{2.12}$$

For $j = 1, \ldots, d$, let Δ_j denote the difference operator given by

$$\Delta_j u := -u(\cdot - e_j) + 2u - u(\cdot + e_j), \qquad u \in \ell_0(\mathbb{Z}^d),$$

and $\Delta^\mu := \Delta_1^{\mu_1} \cdots \Delta_d^{\mu_d}$ for $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}_0^d$. Define

$$\langle u, v\rangle := \sum_{\beta\in\mathbb{Z}^d} u(\beta)\overline{v(\beta)}, \qquad u, v \in \ell_0(\mathbb{Z}^d).$$

To prove Algorithm 2.1, we need the following result.

Theorem 2.4 *Let a be a finitely supported mask on $\mathbb{Z}^d$ and let b be the sequence defined in* (1.3). *Let Θ be a symmetry group with respect to a dilation matrix M. Suppose that b is invariant under Θ. Then $\Theta(T_{b,M}u) = T_{b,M}\Theta(u)$ for all $u \in \ell_0(\mathbb{Z}^d)$ and*

$$\rho_k(a; M, 2) = |\det M|^{1/2}\sqrt{\rho(T_{b,M}|_{W_k})}, \qquad k \in \mathbb{N}_0, \tag{2.13}$$

where $T_{b,M}$ is the transition operator defined in (1.4) *and W_k is the minimal $T_{b,M}$-invariant finite dimensional space which is generated by $\Theta(\Delta^\mu\delta)$, $\mu \in \mathbb{N}_0^d$ with $|\mu| = k$.*

Proof: Since Θ is a symmetry group with respect to the dilation matrix M and b is invariant under Θ, it is easy to directly check that $\Theta(T_{b,M}u) = T_{b,M}\Theta(u)$ for all $u \in \ell_0(\mathbb{Z}^d)$.

Note that $\widetilde{b}(\xi) = |\widetilde{a}(\xi)|^2 \geqslant 0$ for all $\xi \in \mathbb{R}^d$. Let $m := |\det M|$. By the Parseval identity, we have

$$\|\nabla^\mu S^n_{a,M}\delta\|_2^2 = \frac{1}{(2\pi)^d}\int_{[0,2\pi)^d} |\widetilde{\nabla^\mu S^n_{a,M}}\delta(\xi)|^2\, d\xi = \frac{m^n}{(2\pi)^d}\int_{[0,2\pi)^d} \widetilde{\Delta^\mu S^n_{b,M}}\delta(\xi)\, d\xi = m^n \Delta^\mu S^n_{b,M}\delta(0).$$

From the definition of the transition operator, it is easy to verify that

$$T^n_{b,M}\Delta^\mu\delta(0) = \langle T^n_{b,M}\Delta^\mu\delta, \delta\rangle = \langle \Delta^\mu\delta, S^n_{b,M}\delta\rangle = \langle \delta, \Delta^\mu S^n_{b,M}\delta\rangle = \Delta^\mu S^n_{b,M}\delta(0).$$

For a sequence u such that $\widetilde{u}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}^d$, we observe that $\|u\|_\infty = u(0)$ (see [8]). From the fact that $\widetilde{T^n_{b,M}\Theta(\Delta^\mu\delta)}(\xi) \geqslant 0$ and $\widetilde{\Delta^\mu S^n_{b,M}}\delta(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}^d$, it follows that

$$\begin{aligned}\|T^n_{b,M}\Theta(\Delta^\mu\delta)\|_\infty &= T^n_{b,M}\Theta(\Delta^\mu\delta)(0) = \Theta(T^n_{b,M}\Delta^\mu\delta)(0)\\ &= (\#\Theta)T^n_{b,M}\Delta^\mu\delta(0) = (\#\Theta)\Delta^\mu S^n_{b,M}(0) = (\#\Theta)m^{-n}\|\nabla^\mu S^n_{a,M}\delta\|_2^2.\end{aligned}$$

Since W_k is the minimal $T_{b,M}$-invariant subspace generated by $\{\Theta(\Delta^\mu\delta) \ :\ \mu \in \mathbb{N}_0^d, |\mu| = k\}$, we have

$$\begin{aligned}\rho(T_{b,M}|_{W_k}) &= \max\{\lim_{n\to\infty} \|T^n_{b,M}\Theta(\Delta^\mu\delta)\|_\infty^{1/n} \ :\ |\mu| = k, \mu \in \mathbb{N}_0^d\}\\ &= \max\{\lim_{n\to\infty} (\#\Theta)^{1/n} m^{-1}\|\nabla^\mu S^n_{a,M}\delta\|_2^{2/n} \ :\ |\mu| = k, \mu \in \mathbb{N}_0^d\} = m^{-1}\big(\rho_k(a; M, 2)\big)^2\end{aligned}$$

which completes the proof. ∎

Proof of Algorithm 2.1: Let $K := \{\theta\beta \ :\ \theta \in \Theta, \beta \in K_\Theta\}$. Then it is easy to check that both $\ell(K)$ and $\Theta(\ell(K))$ are invariant under $T_{b,M}$ (see [10, Lemma 2.3]). Since a satisfies the sum rules of order k, then the sequence b, which is defined in (1.3), satisfies the sum rules of order $2k$ and V_{2k-1} is invariant under $T_{b,M}$ (see [14, Theorem 5.2]), where $V_j := \{u \in \ell_0(\mathbb{Z}^d) \ :\ \sum_{\beta\in\mathbb{Z}^d} u(\beta)q(\beta) = 0 \quad \forall\, q \in \Pi_j\}$. Define $U_j := \Theta(\ell(K) \cap V_j)$, $j \in \mathbb{N}$. Let W_k denote the linear space in Theorem 2.4. Observe that $W_k \subseteq U_{2k-1} \subseteq V_{2k-1}$. By Theorem 2.4 and (1.5), we have $\rho_k(a; M, 2) = \sqrt{|\det M|\rho(T_{b,M}|_{U_{2k-1}})}$.

Since b satisfies the sum rules of order $2k$ and b is invariant under Θ, we have $T_{b,M}U_j \subseteq U_j$ for all $j = -1, 0, \ldots, 2k-1$. Therefore, we have $\mathrm{spec}(T_{b,M}|_{\Theta(\ell(K))}) = \mathrm{spec}(T_{b,M}|_{U_{2k-1}}) \cup \mathrm{spec}(T_{b.M}|_{\Theta(\ell(K))/U_{2k-1}})$, where $\mathrm{spec}(T)$ denotes the set of all the eigenvalues of T counting multiplicity and the liner space $\Theta(\ell(K))/U_{2k-1}$ is a quotient group under addition. Note that $U_{-1} = \Theta(\ell(K))$. Since $T_{b,M}U_j \subseteq U_j$ for all $j = -1, 0, \ldots, 2k-1$, the quotient group $\Theta(\ell(K))/U_{2k-1}$ is isomorphic to $U_{-1}/U_0 \oplus U_0/U_1 \oplus \cdots \oplus U_{2k-2}/U_{2k-1}$. Hence, $\mathrm{spec}(T_{b,M}|_{\Theta(\ell(K))/U_{2k-1}}) = \cup_{j=0}^{2k-1}\mathrm{spec}(T_{b,M}|_{U_{j-1}/U_j})$. By [19, Theorem 3.2] or by the proof of Theorem 3.1 in Section 3, we know that for any $j = 0, \ldots, 2k-1$, all the eigenvalues of $T_{b,M}|_{V_{j-1}/V_j}$ have modulus $|\det M|^{-j/d}$. Since U_{j-1}/U_j is a subgroup of V_{j-1}/V_j, we deduce that all the eigenvalues of $T_{b,M}|_{U_{j-1}/U_j}$ have modulus $|\det M|^{-j/d}$. (In fact, by duality, we can prove that for any $j = 0, \ldots, 2k-1$, $\mathrm{spec}(T_{b,M}|_{U_{j-1}/U_j}) = \mathrm{spec}(\tau|_{\Theta(\Pi_j \setminus \Pi_{j-1})})$, where $[\tau(q)](x) := q(M^{-1}x), q \in \Pi_{2k-1}$.) By duality, $\dim(U_{j-1}/U_j) = \dim(U_{j-1}) - \dim(U_j) = \dim(\Theta(\Pi_j)) - \dim(\Theta(\Pi_{j-1})) = m_\Theta(j)$. Note that $\{\Theta(\delta_\alpha) \ : \ \alpha \in K_\Theta\}$ is a basis of $\Theta(\ell(K))$ and the matrix T is the representation matrix of the linear operator $T_{b,M}$ acting on $\Theta(\ell(K))$ under the basis $\{\Theta(\delta_\alpha) \ : \ \alpha \in K_\Theta\}$. This completes the proof. ■

From the above proof, without the assumption that M is isotropic, we observe that $\rho_k(a; M, 2)$ is the largest number in the set $\sigma(T)\backslash\{|\lambda| \ : \ \lambda \in \mathrm{spec}(\tau|_{\Theta(\Pi_{2k-1})})\}$, where $\sigma(T)$ is defined in Algorithm 2.1 and $[\tau(q)](x) := q(M^{-1}x), x \in \mathbb{R}^d, q \in \Pi_{2k-1}$. Since Θ is a symmetry group with respect to the dilation matrix M, it is easy to see that $\tau\Theta(\Pi_j) \subseteq \Theta(\Pi_j)$ for all $j \in \mathbb{N}$, where $\Theta(\Pi_j) := \{\frac{1}{\#\Theta}\sum_{\theta\in\Theta} q(\theta x) \ : \ q \in \Pi_j\}$. In passing, we mention that the calculation of the Sobolev smoothness for a bivariate mask which is invariant under Θ_2^A with the dilation matrix $2I_2$ was also discussed by Zhang in [30]. When a mask has a nonnegative symbol, then we can also compute $\rho_k(a; M, \infty)$ in a similar way (see [10, Theorem 4.1]). For completeness, we present the following algorithm whose proof is almost identical to that of Algorithm 2.1.

Algorithm 2.5 *Let M be a $d \times d$ isotropic dilation matrix and let Θ be a symmetry group with respect to the dilation matrix M. Let a be a mask on $\mathbb{Z}^d$ such that $\sum_{\beta\in\mathbb{Z}^d} a(\beta) = 1$. Suppose that a is invariant under the symmetry group Θ, the symbol of a is nonnegative (i.e., $\widetilde{a}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}^d$), and a satisfies the sum rules of order k but not $k+1$. The quantity $\nu_\infty(a; M)$ (or equivalently, $\rho_k(a; M, \infty)$) is obtained via the following procedure:*

(a) *Find a finite subset K_Θ of $\mathbb{Z}_\Theta^d$ such that $\{M^{-1}(\theta\alpha + \beta) \ : \ \theta \in \Theta, \alpha \in K_\Theta, \beta \in \mathrm{supp}\, a\} \cap \mathbb{Z}^d \subseteq \{\theta\beta \ : \ \beta \in K_\Theta, \theta \in \Theta\}$ and $\dim(\Pi_{k-1}|_{\{\theta\beta \,:\, \theta\in\Theta, \beta\in K_\Theta\}}) = \dim(\Pi_{k-1})$;*

(b) *Obtain a $(\#K_\Theta) \times (\#K_\Theta)$ matrix T as follows:*

$$T[\alpha, \beta] := \frac{|\det M|}{\#\{\theta \in \Theta \ : \ \theta\beta = \beta\}} \sum_{\theta\in\Theta} a(M\beta - \theta\alpha), \qquad \alpha, \beta \in K_\Theta.$$

(c) *Let $\sigma(T)$ consist of the absolute values of all the eigenvalues of the square matrix T counting multiplicity. Then $\nu_\infty(a; M)$ is the smallest number in the following set*

$$\{-d \log_{|\det M|} \rho \ : \ \rho \in \sigma(T)\}\backslash\{j \text{ with multiplicity } m_\Theta(j) \ : \ j = 0, \ldots, k-1\}.$$

3 Relations Among $\rho_k(a; M, p), k \in \mathbb{N}_0$

In this section, we shall study the relations among $\rho_k(a; M, p), k \in \mathbb{N}_0$. Using such relations we shall be able to overcome the difficulty in **D3** in Section 1 in order to check the stability condition for certain refinable functions.

The main results in this section are as follows.

Theorem 3.1 *Let M be a dilation matrix. Let a be a finitely supported mask on $\mathbb{Z}^d$ such that $\sum_{\beta \in \mathbb{Z}^d} a(\beta) = 1$ and a satisfies the sum rules of order k with respect to the lattice $M\mathbb{Z}^d$. Let $\lambda_{min} := \min_{1 \leqslant j \leqslant d} |\lambda_j|$ and $\lambda_{max} := \max_{1 \leqslant j \leqslant d} |\lambda_j|$, where $\lambda_1, \ldots, \lambda_d$ are all the eigenvalues of M. Then*

$$\rho_j(a; M, p) = \max\{|\det M|^{1/p} \lambda_{min}^{-j},\ \rho_k(a; M, p)\} \qquad \forall\, 1 \leqslant p \leqslant \infty, 0 \leqslant j < k \tag{3.1}$$

and $|\det M|^{1/q - 1/p} \rho_j(a; M, p) \leqslant \rho_j(a; M, q) \leqslant \rho_j(a; M, p)$ for all $j \in \mathbb{N}_0$ and $1 \leqslant p \leqslant q \leqslant \infty$. Consequently, $\nu_p(a; M) \geqslant \nu_q(a; M) \geqslant \nu_p(a; M) + (1/q - 1/p) \log_{\lambda_{max}} |\det M|$.

We say that a mask a is an **interpolatory mask** with respect to the lattice $M\mathbb{Z}^d$ if $a(\beta) = 0$ for all $\beta \in M\mathbb{Z}^d \backslash \{0\}$. Let a and b be two finitely supported masks on $\mathbb{Z}^d$. Define a sequence c by $\widetilde{c}(\xi) = \widetilde{a}(\xi)\overline{\widetilde{b}(\xi)}, \xi \in \mathbb{R}^d$. If c is an interpolatory mask with respect to the lattice $M\mathbb{Z}^d$, then b is called a **dual mask** of a with respect to $M\mathbb{Z}^d$ and vice versa.

Let ϕ be a continuous function on $\mathbb{R}^d$. We say that ϕ is an **interpolating function** if $\phi(0) = 1$ and $\phi(\beta) = 0$ for all $\beta \in \mathbb{Z}^d \backslash \{0\}$. For discussion on interpolating refinable functions and interpolatory masks, the reader is referred to [5, 6, 11, 12, 24, 25] and references therein. For a compactly supported function ϕ on $\mathbb{R}^d$, we say that the shifts of ϕ are **linearly independent** if for every $\xi \in \mathbb{C}^d$, $\widehat{\phi}(\xi + 2\pi\beta) \neq 0$ for some $\beta \in \mathbb{Z}^d$. Clearly, if the shifts of ϕ are linearly independent, then the shifts of ϕ are stable. When ϕ is an interpolating function, then the shifts of ϕ are linearly independent. Given a finitely supported mask a on $\mathbb{Z}^d$, though a method was proposed in Hogan and Jia [13] to check whether the shifts of $\phi_a^{2I_d}$ are linearly independent or not. However, there are similar difficulties as mentioned in **D1** and **D2** in Section 1 when applying such a method in [13]. In fact, the procedure in [13] is not numerically stable and exact arithmetic is needed. Also see [23] on stability.

Given a mask a and a dilation matrix M, it is known that ϕ_a^M is an interpolating refinable function if and only if the mask a is an interpolatory mask and the subdivision scheme associated with mask a and dilation M converges in the L_∞ norm (equivalently, $\rho_1(a; M, \infty) < 1$, see [10]). However, in general, it is difficult to directly check the condition $\rho_1(a; M, \infty) < 1$. The following result is useful to indirectly check such a condition.

Corollary 3.2 *Let a be a finitely supported mask on $\mathbb{Z}^d$ and M be a dilation matrix. Suppose that b is a dual mask of a with respect to the lattice $M\mathbb{Z}^d$ and*

$$\nu_p(a; M) + \nu_q(b; M) > 0 \quad \textit{for some} \quad 1 \leqslant p, q \leqslant \infty \quad \textit{with} \quad 1/p + 1/q = 1. \tag{3.2}$$

Then the shifts of ϕ_a^M are linearly independent. Therefore, the shifts of ϕ_a^M are stable and when M is isotropic and $\nu_p(a;M) > 0$, $\nu_p(\phi_a^M) = \nu_p(a;M)$. In particular, when a is an interpolatory mask with respect to the lattice $M\mathbb{Z}^d$, if $\nu_2(a;M) > d/2$ (or more generally $\nu_p(a;M) > d/p$ for some $1 \leqslant p \leqslant \infty$), then the subdivision scheme associated with mask a and dilation M converges in the L_∞ norm and ϕ_a^M is a continuous interpolating refinable function.

Proof: Let $m := |\det M|$. Define a sequence c by $\widetilde{c}(\xi) = \widetilde{a}(\xi)\overline{\widetilde{b}(\xi)}$. By definition, c is an interpolatory mask with respect to the lattice $M\mathbb{Z}^d$. By [9, Theorem 5.2] and using Young's inequality, we have

$$\rho_{j+k}(c;M,\infty) \leqslant m^{-1}\rho_j(a;M,p)\rho_k(b;M,q) \qquad \forall\, j,k \in \mathbb{N}_0.$$

Note that $\nu_p(a;M) = -\log_{\lambda_{max}}[m^{-1/p}\rho_j(a;M,p)]$ and $\nu_q(b;M) = -\log_{\lambda_{max}}[m^{-1/q}\rho_k(b;M,q)]$ for some proper integers j and k. Therefore, $\rho_{j+k}(c;M,\infty) \leqslant \lambda_{max}^{-\nu_p(a;M)-\nu_q(b;M)} < 1$. It follows from Theorem 3.1 that $\rho_1(c;M,\infty) < 1$ and therefore, the subdivision scheme associated with mask c and dilation M converges in the L_∞ norm. Consequently, ϕ_c^M is an interpolating refinable function and so its shifts are linearly independent. Note that $\widehat{\phi_c^M}(\xi) = \widehat{\phi_a^M}(\xi)\overline{\widehat{\phi_b^M}(\xi)}$. Therefore, the shifts of ϕ_a^M must be linearly independent.

Note that δ is a dual mask of an interpolatory mask and for any $1 \leqslant q \leqslant \infty$, $\nu_q(\delta;M) = (1/q-1)\log_{\lambda_{max}} m \geqslant d/q - d$. The second part of Corollary 3.2 follows directly from the first part. The second part can also be proved directly. Since $\nu_p(a;M) > d/p$, by Theorem 3.1, we have $\rho_k(a;M,\infty) \leqslant \rho_k(a;M,p) = m^{1/p}\lambda_{max}^{-\nu_p(a;M)} < [m\lambda_{max}^{-d}]^{1/p} \leqslant 1$ for some proper integer k. By Theorem 3.1, we have $\rho_1(a;M,\infty) < 1$. So the subdivision scheme associated with the mask a and the dilation matrix M converges in the L_∞ norm and therefore, ϕ_a^M is a continuous interpolating refinable function. ∎

In particular, if b is a dual mask of a and $\nu_2(a;M) + \nu_2(b;M) > 0$, then the shifts of ϕ_a^M are linearly independent (In fact, when the shifts of ϕ_a^M are linearly independent, there exists a dual mask b of a such that $\nu_2(b;M)$ can be made arbitrarily large and in particular $\nu_2(a;M) + \nu_2(b;M) > 0$).

In order to prove Theorem 3.1, we need to introduce the concept of ℓ_p-norm joint spectral radius. Let $\mathcal{A}$ be a finite collection of linear operators on a finite-dimensional normed vector space V. For a positive integer n, $\mathcal{A}^n$ denotes the Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \{(A_1,\dots,A_n) \ : \ A_1,\dots,A_n \in \mathcal{A}\},$$

and for $1 \leqslant p \leqslant \infty$, we define

$$\|\mathcal{A}^n\|_p := \Big(\sum_{(A_1,\dots,A_n)\in\mathcal{A}^n} \|A_1\cdots A_n\|^p\Big)^{1/p}, \qquad \text{when} \quad 1 \leqslant p < \infty,$$

$$\|\mathcal{A}^n\|_\infty := \max\{\|A_1\cdots A_n\| \quad : \quad (A_1,\dots,A_n) \in \mathcal{A}^n\}, \qquad \text{when} \quad p = \infty,$$

where $\|\cdot\|$ denotes the operator norm given by $\|A\| := \sup\{\|Av\| \ : \ \|v\| = 1, v \in V\}$.

For any $1 \leqslant p \leqslant \infty$, the **$\ell_p$-norm joint spectral radius** (see [4, 11, 18] and references therein on ℓ_p-norm joint spectral radius) of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\geqslant 1} \|\mathcal{A}^n\|_p^{1/n}.$$

Let $\mathcal{E}$ be the set of complete representatives of the distinct cosets of $\mathbb{Z}^d/M\mathbb{Z}^d$. To relate the quantities $\rho_k(a; M, p)$ to the ℓ_p-norm joint spectral radius, we introduce the linear operator $T_\varepsilon (\varepsilon \in \mathcal{E})$ on $\ell_0(\mathbb{Z}^d)$ as follows:

$$T_\varepsilon u(\alpha) := |\det M| \sum_{\beta\in\mathbb{Z}^d} a(M\alpha - \beta + \varepsilon) u(\beta), \qquad \alpha \in \mathbb{Z}^d, \ u \in \ell_0(\mathbb{Z}^d). \tag{3.3}$$

For $\nu = (\nu_1, \dots, \nu_d)$ and $\mu = (\mu_1, \dots, \mu_d)$, We say that $\nu \leqslant \mu$ if $\nu_j \leqslant \mu_j$ for all $j = 1, \dots, d$.

Proof of Theorem 3.1: Let $m := |\det M|$. Let $K_0 = \operatorname{supp} a \cup \{\beta \in \mathbb{Z}^d \ : \ |\beta| \leqslant k\}$ and $K = \mathbb{Z}^d \cap \sum_{j=1}^\infty M^{-j} K_0$. Define

$$V_j = \{u \in \ell(K) \ : \ \langle u, q\rangle = \sum_{\beta\in\mathbb{Z}^d} u(\beta)\overline{q(\beta)} = 0 \qquad \forall\, q \in \Pi_j\}, \qquad j \in \mathbb{N}_0.$$

Since a satisfies the sum rules of order k, by [14, Theorem 5.2], $T_\varepsilon V_j \subseteq V_j$ for all $0 \leqslant j < k$ and for all $\varepsilon \in \mathcal{E}$. By [10, Theorem 2.5], we have

$$\rho_j(a; M, p) = \rho_p(\{T_\varepsilon|_{V_{j-1}} \ : \ \varepsilon \in \mathcal{E}\}), \qquad j = 0, \dots, k.$$

Note that $V_{j-1} = V_j \oplus W_j$, where $W_j := \operatorname{span}\{\nabla^\mu \delta \ : \ |\mu| = j, \mu \in \mathbb{N}_0^d\}$.

For any $\mu, \nu \in \mathbb{N}_0^d$ such that $|\nu| \leqslant |\mu| < k$, we have

$$\begin{aligned}\langle T_\varepsilon \nabla^\mu \delta, (M\cdot)^\nu/\nu!\rangle &= \sum_{\alpha\in\mathbb{Z}^d} [T_\varepsilon \nabla^\mu \delta](\alpha)(M\alpha)^\nu/\nu! \\ &= m \sum_{\beta\in\mathbb{Z}^d}\sum_{\alpha\in\mathbb{Z}^d} a(M\alpha - \beta + \varepsilon)[\nabla^\mu\delta](\beta)(M\alpha)^\nu/\nu!.\end{aligned}$$

Note that

$$\frac{(M\alpha)^\nu}{\nu!} = \frac{(M\alpha - \beta + \varepsilon + \beta - \varepsilon)^\nu}{\nu!} = \sum_{0\leqslant\eta\leqslant\nu} \frac{(M\alpha - \beta + \varepsilon)^{\nu-\eta}}{(\nu-\eta)!}\frac{(\beta-\varepsilon)^\eta}{\eta!}.$$

Since a satisfies the sum rules of order k, we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^d} a(M\alpha - \beta + \varepsilon)\frac{(M\alpha)^\nu}{\nu!} &= \sum_{0\leqslant\eta\leqslant\nu} \frac{(\beta-\varepsilon)^\eta}{\eta!} \sum_{\alpha\in\mathbb{Z}^d} a(M\alpha - \beta + \varepsilon)\frac{(M\alpha - \beta + \varepsilon)^{\nu-\eta}}{(\nu-\eta)!} \\ &= \sum_{0\leqslant\eta\leqslant\nu} \frac{(\beta-\varepsilon)^\eta}{\eta!} \sum_{\alpha\in\mathbb{Z}^d} a(M\alpha)\frac{(M\alpha)^{\nu-\eta}}{(\nu-\eta)!}.\end{aligned}$$

Thus, for $\nu, \mu \in \mathbb{N}_0^d$ such that $|\nu| \leqslant |\mu| < k$, we have

$$\langle T_\varepsilon \nabla^\mu \delta, \frac{(M\cdot)^\nu}{\nu!}\rangle = m \sum_{0\leqslant\eta\leqslant\nu} \sum_{\alpha\in\mathbb{Z}^d} a(M\alpha)\frac{(M\alpha)^{\nu-\eta}}{(\nu-\eta)!} \sum_{\beta\in\mathbb{Z}^d} [\nabla^\mu\delta](\beta)\frac{(\beta-\varepsilon)^\eta}{\eta!}.$$

It is evident that

$$\sum_{\beta\in\mathbb{Z}^d} [\nabla^\mu\delta](\beta)\frac{(\beta-\varepsilon)^\eta}{\eta!} = \langle \nabla^\mu\delta, \frac{(\cdot-\varepsilon)^\eta}{\eta!}\rangle = \delta(\mu-\eta) \qquad \forall\, |\eta| \leqslant |\mu|.$$

Therefore,

$$\langle T_\varepsilon \nabla^\mu\delta, (M\cdot)^\nu/\nu!\rangle = m\delta(\mu-\nu)\sum_{\alpha\in\mathbb{Z}^d} a(M\alpha) = \delta(\mu-\nu) \qquad \forall\, \varepsilon\in\mathcal{E},\ |\nu| \leqslant |\mu| < k. \tag{3.4}$$

On the other hand, for any $|\nu| \leqslant |\mu|$,

$$\begin{aligned}
\langle \sum_{|\eta|=|\mu|} S(M^{-1},|\mu|)_{\eta,\mu}\nabla^\eta\delta, (M\cdot)^\nu/\nu!\rangle &= \sum_{\alpha\in\mathbb{Z}^d}\sum_{|\eta|=|\mu|} S(M^{-1},|\mu|)_{\eta,\mu}[\nabla^\eta\delta](\alpha)\frac{(M\alpha)^\nu}{\nu!} \\
&= \sum_{\alpha\in\mathbb{Z}^d}\sum_{|\eta|=|\mu|} S(M^{-1},|\mu|)_{\eta,\mu}[\nabla^\eta\delta](\alpha)\sum_{|\lambda|=|\nu|} S(M,|\nu|)_{\nu,\lambda}\frac{\alpha^\lambda}{\lambda!} \\
&= \sum_{|\eta|=|\mu|}\sum_{|\lambda|=|\nu|} S(M^{-1},|\mu|)_{\eta,\mu}S(M,|\nu|)_{\nu,\lambda}\sum_{\alpha\in\mathbb{Z}^d}[\nabla^\eta\delta](\alpha)\frac{\alpha^\lambda}{\lambda!} \\
&= \sum_{|\eta|=|\mu|}\sum_{|\lambda|=|\nu|} S(M^{-1},|\mu|)_{\eta,\mu}S(M,|\nu|)_{\nu,\lambda}\delta(\eta-\lambda) \\
&= \delta(|\mu|-|\nu|)\sum_{|\eta|=|\mu|} S(M,|\mu|)_{\nu,\eta}S(M^{-1},|\mu|)_{\eta,\mu} \\
&= \delta(|\mu|-|\nu|)S(I_d,|\mu|)_{\nu,\mu} = \delta(\mu-\nu),
\end{aligned}$$

where $S(M^{-1},|\mu|)$ is defined in (2.10). Therefore, we have

$$T_\varepsilon\nabla^\mu\delta - \sum_{|\eta|=|\mu|} S(M^{-1},j)^T_{\mu,\eta}\nabla^\eta\delta \in V_j \qquad \forall\, |\mu| = j < k, \varepsilon\in\mathcal{E}.$$

Since $V_{j-1} = V_j \oplus W_j$ and $\{\nabla^\mu\delta\ :\ |\mu| = j, \mu\in\mathbb{N}_0^d\}$ is a basis for W_j, we have

$$T_\varepsilon|_{V_{j-1}} = \begin{bmatrix} S(M^{-1},j)^T & * \\ 0 & T_\varepsilon|_{V_j}\end{bmatrix}, \qquad \varepsilon\in\mathcal{E},\ 0\leqslant j<k.$$

Note that the spectral radius of $S(M^{-1},j)^T$ is λ_{min}^{-j} for all $j\in\mathbb{N}$. Therefore, we deduce that $\rho_p(\{T_\varepsilon|_{V_{j-1}}\ :\ \varepsilon\in\mathcal{E}\}) = \max\{|\det M|^{1/p}\lambda_{min}^{-j}, \rho_p(\{T_\varepsilon|_{V_j}\ :\ \varepsilon\in\mathcal{E}\})$. So (3.1) holds.

By the definition of the ℓ_p-norm joint spectral radius, using the Hölder inequality, we have $|\det M|^{1/q-1/p}\rho_p(\{T_\varepsilon|_{V_j}\ :\ \varepsilon\in\mathcal{E}\}) \leqslant \rho_q(\{T_\varepsilon|_{V_j}\ :\ \varepsilon\in\mathcal{E}\}) \leqslant \rho_p(\{T_\varepsilon|_{V_j}\ :\ \varepsilon\in\mathcal{E}\})$ (also see [10]). This completes the proof. ■

4 Some Examples of Symmetric Refinable Functions

In this section, we shall give several examples to demonstrate the advantages of the algorithms and results in Sections 2 and 3 on computing smoothness exponents of symmetric refinable functions.

Example 4.1 Let $M = 2I_2$. The interpolatory mask a for the butterfly scheme in [6] is supported on $[-3, 3]^2$ and is given by

$$\frac{1}{64}\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & 8 & 2 & -1 \\ 0 & 0 & 8 & 16 & 8 & 0 & 0 \\ -1 & 2 & 8 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a satisfies the sum rules of order 4 and a is invariant under the hexagonal symmetry group Θ^H. By Proposition 2.2, we have $\#K_{\Theta^H} = 11$ and by computing the eigenvalues of the 11×11 matrix T in Algorithm 2.1, we have

$$\{-\log_4 \rho \;:\; \rho \in \sigma(T)\} = \{0,\, 1,\, 2,\, 2.44077,\, 2.56925,\, 3,\, 3,\, 3.05923,\, 3.28397,\, 3.72404,\, 4\}.$$

So by Algorithm 2.1, $\nu_2(a; 2I_2) \approx 2.44077 > 1$. Therefore, by Corollary 3.2, $\phi_a^{2I_2}$ is an interpolating refinable function and $\nu_2(\phi_a^{2I_2}) = \nu_2(a; 2I_2) \approx 2.44077$. Note that the matrix size using the method in [19] is $\#\Omega_{b,2I_2} = 109$ which is much larger than the matrix size $\#K_{\Theta^H} = 11$ used in Algorithm 2.1.

Example 4.2 Let $M = 2I_2$. A family of bivariate interpolatory masks $RS_r (r \in \mathbb{N})$ was given in Riemenschneider and Shen [25] (also see Jia [16]) such that RS_r is supported on $[1-2r, 2r-1]^2$, RS_r satisfies the sum rules of order $2r$ with respect to the lattice $2\mathbb{Z}^2$ and RS_r is invariant under the hexagonal symmetry group Θ^H. Using the fact that the symbol of RS_r has the factor $[(1 + e^{-i\xi_1})(1 + e^{-i\xi_2})(1 + e^{-i(\xi_1+\xi_2)})]^r$, by taking out some of such factors, Jia and Zhang [19, Theorem 4.1] was able to compute the Sobolev smoothness exponents of $\phi_{RS_r}^{2I_2}$ for $r = 2, \ldots, 16$. In fact, when $r = 16$, to compute $\nu_2(\phi_{RS_{16}}^{2I_2})$, the method in [19, Theorem 4.1] has to compute the eigenvalues of two matrices of size 4743 (without factorization, the matrix size used in [19] is 11719). Without using any factorization, when $r = 16$, for any mask a which is supported on $[-31, 31]^2$ and is invariant under Θ^H, by Algorithm 2.1, we have $\#K_{\Theta^H} = 909$. So, to compute $\nu_2(\phi_{RS_{16}}^{2I_2})$, we only need to compute the eigenvalues of a matrix of size 909.

Example 4.3 Let $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ be the quincunx dilation matrix. The interpolatory mask a is supported on $[-3,3]^2$ and is given by

$$\frac{1}{64}\begin{bmatrix} 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 10 & 0 & -1 \\ 0 & 10 & 32 & 10 & 0 \\ -1 & 0 & 10 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

Note that a satisfies the sum rules of order 4 with respect to the quincunx lattice $M\mathbb{Z}^2$ and a is invariant under the full axes symmetry group Θ_2^A with respect to the dilation matrix M. This example was discussed in [19] and belongs to a family of quincunx interpolatory masks in [12]. By Algorithm 2.1, we have $\#K_{\Theta_2^A} = 46$ and $\nu_2(a;M) \approx 2.44792 > 1$. Therefore, $\nu_2(\phi_a^M) = \nu_2(a;M) \approx 2.44792$. Note that the matrix to compute $\nu_2(\phi_a^M)$ using method in [19] has size 481 (see [19]) which is much larger than the size 46 when using Algorithm 2.1. Note that the symbol of a is nonnegative. By Algorithm 2.5, we have $\#K_{\Theta_2^A} = 13$ and $\nu_\infty(a;M) \approx 1.45934 > 0$. Therefore, by Corollary 3.2, $\nu_\infty(\phi_a^M) = \nu_\infty(a;M) \approx 1.45934$. However, using method in [19], the matrix size is 129 (see [19]) which is much larger than the size 13 in Algorithm 2.5.

Example 4.4 Let $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. A family of quincunx interpolatory masks $g_r (r \in \mathbb{N})$ was proposed in [12] such that g_r is supported on $[-r,r]^2$, satisfies the sum rules of order $2r$ with respect to $M\mathbb{Z}^2$, is an interpolatory mask with respect to $M\mathbb{Z}^2$ and is invariant under the full axes symmetry group Θ_2^A. Note that the mask in Example 4.3 corresponds to the mask g_2 in this family. Since the symbols of g_r are nonnegative, the L_∞ smoothness exponents $\nu_\infty(\phi_{g_r}^M)$ were computed in [12] for $r = 1, \ldots, 8$. Using Algorithm 2.5, we are able to compute $\nu_\infty(\phi_{g_r}^M)$ for $r = 9, \ldots, 16$ in Table 3.

$\nu_\infty(\phi_{g_9}^M)$	$\nu_\infty(\phi_{g_{10}}^M)$	$\nu_\infty(\phi_{g_{11}}^M)$	$\nu_\infty(\phi_{g_{12}}^M)$	$\nu_\infty(\phi_{g_{13}}^M)$	$\nu_\infty(\phi_{g_{14}}^M)$	$\nu_\infty(\phi_{g_{15}}^M)$	$\nu_\infty(\phi_{g_{16}}^M)$
5.71514	6.21534	6.70431	7.18321	7.65242	8.11171	8.56039	8.99752

Table 3: The L_∞ (Hölder) smoothness exponents of the interpolating refinable functions $\phi_{g_r}^M$.

A coset by coset (CBC) algorithm was proposed in [9, 12] to construct quincunx biorthogonal wavelets. Some examples of dual masks of g_r, denoted by $(g_r)_k^s$, were constructed via [12, Theorem 5.2] and some of their Sobolev smoothness exponents were given in Table 4 in [12]. Note that the dual mask $(g_r)_k^s$ is supported on $[-k-r, r+k]^2$, satisfies the sum rules of order $2k$, has nonnegative symbol and is invariant under the full axes symmetry group Θ_2^A. However, in the paper [12] we are unable to complete the computation in Table 4 in [12] due to the difficulty mentioned in **D2** in Section 1. In fact, to compute $\nu_2(a;M)$ for a mask supported on $[-k,k]^2$, the set $\Omega_{b,M}$ defined in (1.6) is given by $\{(i,j)\in\mathbb{Z}^2 : |i|\leqslant 6k, |j|\leqslant 6k, |i-j|\leqslant 8k, |i+j|\leqslant 8k\}$. For example, in order to compute $\nu_2((g_4)_8^s;M)$, the set $\Omega_{b,M}$ consists of 16321 points which is beyond our ability to compute the eigenvalues of a 16321×16321 matrix. We now can complete the computation using Algorithm 2.1 in Section 2. Note that the quincunx dilation M here is denoted by Q in Table 4 in [12]. By computation, $\nu_2(\phi^M_{(g_4)_6^s})\approx 2.47477$ and rest of the computation is given in the following Table 4.

Table 4: Computing $\nu_2(\phi^M_{(g_r)_k^s})$ by Algorithm 2.1. The result here completes Table 4 in [12].

$\nu_2(\phi^M_{(g_1)_7^s})$	$\nu_2(\phi^M_{(g_2)_7^s})$	$\nu_2(\phi^M_{(g_3)_7^s})$	$\nu_2(\phi^M_{(g_4)_7^s})$	$\nu_2(\phi^M_{(g_1)_8^s})$	$\nu_2(\phi^M_{(g_2)_8^s})$	$\nu_2(\phi^M_{(g_3)_8^s})$	$\nu_2(\phi^M_{(g_4)_8^s})$
3.01166	2.92850	2.90251	2.91546	3.49499	3.38671	3.34268	3.32116

Example 4.5 Let $M=\begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix}$ be the dilation matrix in a $\sqrt{3}$-subdivision scheme ([22]). The interpolatory mask a is supported on $[-4,4]^2$ and is given by

$$\frac{1}{2187}\begin{bmatrix}0&0&0&0&7&4&0&4&7\\0&0&0&4&0&-32&-32&0&4\\0&0&0&-32&-20&0&-20&-32&0\\0&4&-32&0&312&312&0&-32&4\\7&0&-20&312&729&312&-20&0&7\\4&-32&0&312&312&0&-32&4&0\\0&-32&-20&0&-20&-32&0&0&0\\4&0&-32&-32&0&4&0&0&0\\7&4&0&4&7&0&0&0&0\end{bmatrix}.$$

Note that a satisfies the sum rules of order 6 with respect to the lattice $M\mathbb{Z}^2$ and a is invariant under the hexagonal symmetry group Θ^H with respect to the dilation matrix M. By Algorithm 2.1, we have $\#K_{\Theta^H}=38$ and $\nu_2(a;M)\approx 3.28036>1$. Therefore, by Corollary 3.2, ϕ_a^M is a C^2 interpolating refinable function and $\nu_2(\phi_a^M)=\nu_2(a;M)\approx 3.28036$. By estimate, the matrix size $\#\Omega_{b,M}$ using the method in [19] is greater than 361 which is much larger than the size 38 when using Algorithm 2.1. Note that the symbol of a is nonnegative. By Algorithm 2.5, we have $\#K_{\Theta^H}=11$ and $\nu_\infty(a;M)\approx 2.34654>0$. Therefore, by Corollary 3.2, $\nu_\infty(\phi_a^M)=\nu_\infty(a;M)\approx 2.34654$. Using method in [19], the matrix size $\#\Omega_{a,M}$ is greater than 85 which is much larger than the size 11 in Algorithm 2.5. Since $\phi_a^M\in C^2$, this example gives us a C^2 $\sqrt{3}$ interpolatory subdivision scheme.

5 Dyadic Interpolatory Masks in $\mathbb{R}^d$ and Some Examples in $\mathbb{R}^3$

In this section, based on the work [11], we shall present a general construction of dyadic interpolatory masks in any dimension with optimal sum rules and minimal support. This general construction includes the family of interpolatory masks for dimension one in [5] and for dimension two in [11] as special cases. To facilitate our discussion, we need the following result.

Lemma 5.1 *For a positive integer r, define*

$$\Lambda_r^d := \{\alpha \in \mathbb{N}_0^d \ : \ |\alpha| < r\} \tag{5.1}$$

and order the elements in Λ_r^d in the lexicographic order. Then for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0,1\}^d$, the square matrix $\big((2\beta+\varepsilon)^{2\mu}\big)_{\beta\in\Lambda_r^d,\mu\in\Lambda_r^d}$ is nonsingular and

$$\left|\det\left[\big((\varepsilon+2\beta)^{2\mu}\big)_{\beta\in\Lambda_r^d,\mu\in\Lambda_r^d}\right]\right| = \prod_{(\mu_1,\ldots,\mu_d)\in\Lambda_r^d}\prod_{i=1}^{d}\prod_{j=0}^{\mu_i-1}[(2\mu_i+\varepsilon_i)^2-(2j+\varepsilon_i)^2] \neq 0,$$

where by convention $\Pi_{j=0}^{-1} := 1$. Moreover, for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0,1\}^d$, the unique solution $\{c_\beta \ : \ \beta \in \Lambda_r^d\}$ to the following system of linear equations

$$\sum_{\beta\in\Lambda_r^d} c_\beta(2\beta+\varepsilon)^{2\mu} = f_\mu \qquad \forall\ \mu \in \Lambda_r^d \tag{5.2}$$

is given by the following recursive formula (first compute c_β for $|\beta| = r-1$, then compute c_β for $|\beta| = r-2$ and so on, finally compute c_0):

$$c_\beta = \sum_{\mu\in\Lambda_{|\beta|+1}^d} d_{\beta,\mu}^{\varepsilon}\left[f_\mu - \sum_{\alpha\in\Lambda_r^d,|\alpha|>|\beta|} c_\alpha(2\alpha+\varepsilon)^{2\mu}\right], \qquad \beta \in \Lambda_r^d, \tag{5.3}$$

where for $\beta = (\beta_1, \ldots, \beta_d), \mu = (\mu_1, \ldots, \mu_d) \in \Lambda_r^d$, $d_{\beta,\mu}^{\varepsilon} := \Pi_{i=1}^d d_{\beta_i,\mu_i}^{\varepsilon_i}$ and $d_{\beta_i,\mu_i}^{\varepsilon_i}$ are uniquely determined by

$$\sum_{\mu_i=0}^{r-1} d_{\beta_i,\mu_i}^{\varepsilon_i} x^{\mu_i} = \frac{\prod_{j=0}^{\beta_i-1}[x-(2j+\varepsilon_i)^2]}{\prod_{j=0}^{\beta_i-1}[(2\beta_i+\varepsilon_i)^2-(2j+\varepsilon_i)^2]}, \qquad x \in \mathbb{R}.$$

Proof: Obviously, the claim holds true for $r = 1$ since $\Lambda_1^d = \{0\}$. For a fixed $\mu = (\mu_1, \ldots, \mu_d) \in \Lambda_r^d$ with $|\mu| = r-1$, we can replace the column $\big((2\beta+\varepsilon)^{2\mu}\big)_{\beta\in\Lambda_r^d}$ of the matrix $F := \big((2\beta+\varepsilon)^{2\mu}\big)_{\beta\in\Lambda_r^d,\mu\in\Lambda_r^d}$ by a new column

$$\left(\prod_{i=1}^{d}\prod_{j=0}^{\mu_i-1}[(2\beta_i+\varepsilon_i)^2-(2j+\varepsilon_i)^2]\right)_{(\beta_1,\ldots,\beta_d)\in\Lambda_r^d}.$$

It is easily seen that this transform does not change the determinant of F. Let us simplify the new column. We observe that $\prod_{i=1}^d \prod_{j=0}^{\mu_i-1}[(2\beta_i+\varepsilon)^2-(2j+\varepsilon_i)^2] = 0$ for $(\beta_1, \dots, \beta_d) \in \Lambda_r^d \backslash \{\mu\}$ since for any $\beta \in \Lambda_r^d \backslash \{\mu\}$, we have $\beta_i < \mu_i$ for some $1 \leqslant i \leqslant d$. Now by induction on r, we have

$$|\det F| = \prod_{(\mu_1,\dots,\mu_s)\in\Lambda_r^d} \prod_{i=1}^{d} \prod_{j=0}^{\mu_i-1} [(2\mu_i+\varepsilon_i)^2 - (2j+\varepsilon_i)^2] \neq 0.$$

So F is a nonsingular square matrix.

Since F is nonsingular, there is a unique solution to (5.2). To derive the formula (5.3), we prove it by induction on r. When $r = 1$, it is evident that $\Lambda_1^d = \{0\}$ and the formula (5.3) gives us $c_0 = f_0$ which is the solution to (5.2) with $r = 1$. Suppose (5.3) is true for $r-1$. In the following, we demonstrate that (5.3) holds for r.

For any $\beta = (\beta_1, \dots, \beta_d) \in \Lambda_r^d$ with $|\beta| = r-1$, define a polynomial q_β by

$$q_\beta(x_1, \dots, x_d) := \frac{\Pi_{i=1}^d \Pi_{j=0}^{\beta_i-1} \left[x_i - (2j+\varepsilon_i)^2\right]}{\Pi_{i=1}^d \Pi_{j=0}^{\beta_i-1} \left[(2\beta_i+\varepsilon_i)^2 - (2j+\varepsilon_i)^2\right]}, \qquad (x_1, \dots, x_d) \in \mathbb{R}^d.$$

It is easy to see that

$$q_\beta\big((2\alpha_1+\varepsilon_1)^2, \dots, (2\alpha_d+\varepsilon_d)^2\big) = \delta(\alpha-\beta) \qquad \forall\, \alpha = (\alpha_1, \dots, \alpha_d) \in \Lambda_r^d \tag{5.4}$$

since for any $\alpha \in \Lambda_r^d \backslash \{\beta\}$, we have $\alpha_i < \beta_i$ for some $1 \leqslant i \leqslant d$.

Note that for any $\mu \in \Lambda_r^d$, the coefficient of x^μ in $q_\beta(x)$ is exactly $d_{\beta,\mu}^\varepsilon$. Therefore, $q_\beta(x) = \sum_{\mu\in\Lambda_r^d} d_{\beta,\mu}^\varepsilon x^\mu$ and (5.4) can be rewritten as

$$\sum_{\mu\in\Lambda_r^d} d_{\beta,\mu}^\varepsilon (2\alpha+\varepsilon)^{2\mu} = \delta(\alpha-\beta) \qquad \forall\ \alpha \in \Lambda_r^d,\ \beta \in \Lambda_r^d \quad \text{with} \quad |\beta| = r-1.$$

The above equality and (5.2) imply that

$$\begin{aligned} \sum_{\mu\in\Lambda_r^d} f_\mu d_{\beta,\mu}^\varepsilon &= \sum_{\mu\in\Lambda_r^d} \sum_{\alpha\in\Lambda_r^d} c_\alpha (2\alpha+\varepsilon)^{2\mu} d_{\beta,\mu}^\varepsilon = \sum_{\alpha\in\Lambda_r^d} c_\alpha \sum_{\mu\in\Lambda_r^d} d_{\beta,\mu}^\varepsilon (2\alpha+\varepsilon)^{2\mu} \\ &= \sum_{\alpha\in\Lambda_r^d} c_\alpha \delta(\alpha-\beta) = c_\beta. \end{aligned}$$

Hence, $c_\beta = \sum_{\mu\in\Lambda_r^d} f_\mu d_{\beta,\mu}^\varepsilon$ for all $\beta \in \Lambda_r^d$ with $|\beta| = r-1$. Therefore, from (5.2), we have

$$\sum_{\alpha\in\Lambda_{r-1}^d} c_\alpha (2\alpha+\varepsilon)^{2\mu} = f_\mu - \sum_{\beta\in\Lambda_r^d, |\beta|=r-1} c_\beta (2\beta+\varepsilon)^{2\mu}, \qquad \mu \in \Lambda_{r-1}^d.$$

By induction hypothesis, we obtain the formula (5.3). ∎

We are now in a position to describe the general construction of dyadic interpolatory masks with minimal support and optimal order of sum rules in $\mathbb{R}^d$.

Theorem 5.2 *Let $M = 2I_d$ be the dilation matrix. For each positive integer r, there exists a unique dyadic interpolatory mask g_r^d in $\mathbb{R}^d$ with the following properties:*

(a) *g_r^d is supported on the set $\{2\alpha + \varepsilon \; : \; \varepsilon \in \{-1,0,1\}^d, \; \alpha \in \mathbb{Z}^d, \; |\alpha| < r \}$;*

(b) *g_r^d is symmetric about all the coordinate axes;*

(c) *g_r^d satisfies the sum rules of order $2r$ with respect to the lattice $2\mathbb{Z}^d$.*

Proof: Using Lemma 5.1, the proof is similar to that of [11, Theorem 4.3]. Let a be a dyadic interpolatory mask satisfying all the conditions (a), (b) and (c). Then all the conditions in Theorem 5.2 can be translated into

$$\sum_{\beta \in \Lambda_r^d} 2^{Z(2\beta+\varepsilon)} a(2\beta+\varepsilon)(2\beta+\varepsilon)^{2\mu} = 2^{-d}\delta(\mu) \qquad \forall\, \mu \in \Lambda_r^d, \; \varepsilon \in \{0,1\}^d \backslash \{0\}, \tag{5.5}$$

where $Z(\alpha_1, \ldots, \alpha_d)$ denotes the cardinality of the set $\{i \in \{1, \ldots, d\} \; : \; \alpha_i \neq 0\}$. By Lemma 5.1, there is a unique solution $\{2^{Z(2\beta+\varepsilon)} a(2\beta+\varepsilon) \; : \; \beta \in \Lambda_r^d\}$ (and therefore, a unique solution $\{a(2\beta+\varepsilon) \; : \; \beta \in \Lambda_r^d\}$) to the system of linear equations in (5.5). So there is a unique interpolatory mask a satisfying all the conditions in Theorem 5.2. The reader is referred to [11, Theorem 4.3] for a more detailed proof for the case $d = 2$. ■

By the uniqueness, we see that each g_r^d in Theorem 5.2 is invariant under the full axes symmetry group Θ_d^A. From the construction, it is not difficult to verify that the number of nonzero coefficients in g_r^d is $2^d r^d/d! + O(r^{d-1})$. For example, g_r^3 has $\frac{4}{3}r^3 + 10r^2 + \frac{44}{3}r + 1$ nonzero coefficients. By the uniqueness of g_r^d in Theorem 5.2 again, we see that $g_r^1 (r \in \mathbb{N})$ were exactly the univariate Deslauriers–Dubuc interpolatory masks in [5] and $g_r^2 (r \in \mathbb{N})$ were the bivariate interpolatory masks proposed in [11]. In passing, we mention that Lemma 5.1 can be used in the CBC algorithm in [9] to construct biorthogonal wavelets without solving any equations.

In the following, let us give some examples of the above interpolatory masks in dimension three. Let $\mathbb{Z}^3_{\Theta_3^A} := \{(\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3 \; : \; \beta_1 \leqslant \beta_2 \leqslant \beta_3\}$. Clearly, if a is a mask invariant under the group Θ_3^A, then it is totally determined by all the coefficients $a(\beta), \beta \in \mathbb{Z}^3_{\Theta_3^A}$.

Example 5.3 The coefficients of the interpolatory mask g_2^3 on the set $\mathbb{Z}^3_{\Theta_3^A}$ are given by

$g_2^3(0,0,0) = 1/8,$	$g_2^3(0,0,1) = 9/128,$	$g_2^3(0,1,1) = 5/128,$	$g_2^3(1,1,1) = 11/512,$
$g_2^3(0,0,3) = -1/128,$	$g_2^3(0,1,3) = -1/256,$	$g_2^3(1,1,3) = -1/512,$	
$g_2^3(\alpha) = 0,$	for any other $\alpha \in \mathbb{Z}^3_{\Theta_3^A}$.		

Then g_2^3 satisfies the sum rules of order 4 and there are only 81 nonzero coefficients in the mask g_2^3. By Algorithm 2.1, we have $\#K_{\Theta_3^A} = 36$ and $\nu_2(g_2^3; 2I_3) \approx 2.44077 > 1.5$. Therefore, by Corollary 3.2, $\phi_{g_2^3}^{2I_3}$ is an interpolating refinable function and $\nu_2(\phi_{g_2^3}^{2I_3}) \approx 2.44077$. Note that $\#\Omega_{b,2I_3} = 965$ and $\#K_{\Theta_3^A} = 36$. Hence, Algorithm 2.1 can greatly reduce the dimension of the matrix to compute $\nu_2(g_2^3; 2I_3)$.

Example 5.4 The coefficients of the interpolatory mask g_3^3 on the set $\mathbb{Z}^3_{\Theta_3^A}$ are given by

$$
\begin{array}{lll}
g_3^3(0,0,0) = 1/8, & g_3^3(0,0,1) = 75/1024, & g_3^3(0,1,1) = 87/2048, \\
g_3^3(1,1,1) = 25/1024, & g_3^3(0,0,3) = -25/2048, & g_3^3(0,1,3) = -29/8192, \\
g_3^3(1,1,3) = -29/8192, & g_3^3(0,3,3) = 1/2048, & g_3^3(1,3,3) = 1/4096, \\
g_3^3(0,0,5) = 3/2048, & g_3^3(0,1,5) = 3/4096, & g_3^3(1,1,5) = 3/8192, \\
g_2^3(\alpha) = 0 \quad \text{for other } \alpha \in \mathbb{Z}^3_{\Theta_3^A}. & &
\end{array}
$$

Then g_3^3 satisfies the sum rules of order 6 and it has 171 nonzero coefficients. By Algorithm 2.1, we have $\#K_{\Theta_3^A} = 101$ and $\nu_2(g_3^3; 2I_3) \approx 3.17513 > 1.5$. Therefore, by Corollary 3.2, $\phi_{g_3^3}^{2I_3}$ is an interpolating refinable function and $\nu_2(\phi_{g_3^3}^{2I_3}) \approx 3.17513$. Note that $\#\Omega_{b,2I_3} = 3021$ and $\#K_{\Theta_3^A} = 101$. Hence, Algorithm 2.1 can greatly reduce the dimension of the matrix to compute $\nu_2(g_3^3; 2I_3)$.

The Sobolev smoothness exponents of $\phi_{g_r^3}^{2I_3}(r = 2, \ldots, 10)$ are presented in Table 5. By [11, Theorem 3.3] and [9, Theorem 5.1], we see that $g_r^3(r = 1, \ldots, 10)$ achieve the optimal Sobolev smoothness and optimal order of sum rules with respect to the support of their masks. In general, the Algorithms 2.1 and 2.5 roughly reduce the size of matrix to be $1/\#\Theta$ of the number of points in $\Omega_{b,M}$ in (1.6). Note that $\#\Theta_d^A = 2^d d!$ and $\#\Theta_3^A = 48$. So Algorithms 2.1 and 2.5 are very useful in computing the smoothness exponents of symmetric multivariate refinable functions.

$\nu_2(\phi_{g_2^3}^{2I_3})$	$\nu_2(\phi_{g_3^3}^{2I_3})$	$\nu_2(\phi_{g_4^3}^{2I_3})$	$\nu_2(\phi_{g_5^3}^{2I_3})$	$\nu_2(\phi_{g_6^3}^{2I_3})$	$\nu_2(\phi_{g_7^3}^{2I_3})$	$\nu_2(\phi_{g_8^3}^{2I_3})$	$\nu_2(\phi_{g_9^3}^{2I_3})$	$\nu_2(\phi_{g_{10}^3}^{2I_3})$
2.44077	3.17513	3.79313	4.34408	4.86202	5.36283	5.85293	6.33522	6.81143

Table 5: The Sobolev smoothness exponents of $\phi_{g_r^3}^{2I_3}$ for $r = 2, \ldots, 10$.

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