A Geometric-Optics Proof of a Theorem on Boundary Control Given a Convex Function

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Abstract

In the area of boundary control of hyperbolic equations, the tools of geometric optics have sometimes proven to be very powerful. In geometric optics, authors including Littman [8] and Bardos, Lebeau and Rauch [1] have established under various circumstances that, if every bicharacteristic curve of the hyperbolic equation must cross a point on the boundary where the controls can be applied, then the equation can be controlled— and the time required is just the maximum time needed for a bicharacteristic to reach that part of the boundary.

Now that these results are in place, they allow for theorems on boundary control which do not require new integral inequalities for particular situations. Rather, assumptions are made on the geometry of the domain Ω of the equation. For instance, Gulliver and Littman [3] show that every bicharacteristic will cross the boundary, and hence control will be attained, so long as chords between points of the boundary are unique and the boundary is locally convex. They go on to give several examples of regions where this holds.

The present paper uses geometric optics to prove one of the main theorems in the important paper "Inverse/Observability Estimates for Second-Order Hyperbolic Equations with Variable Coefficients" by Lasiecka, Triggiani and Yao [5]. In that paper, the authors use Carleman estimates to show that the equation is controlled if there is a positive function v on Ω which is strictly convex with respect to the metric defined by the coefficients of the equation, if that convex function has non-positive outward normal derivative on the uncontrolled part of the boundary. The time needed for control is a function of the maximum value of v on Ω and a lower bound on its convexity. Here we will show that control in the same time is established by a simpler geometric optics argument—in fact it comes down to a short calculus computation on the value of v along a bicharacteristic of the equation.

1 Introduction

In this paper we use a geometric optics idea to prove a theorem that Lasiecka, Triggiani and Yao [5] proved using Carleman estimates. Among the advantages in doing so is that the computations become shorter, and the geometric methods could be applied without major revision to a broad range of situations. However, the geometric optics results cited are limited in that they require rather strong smoothness assumptions which are not needed in the approach by Carleman estimates. Given a Riemannian manifold M and a domain Ω with $\overline{\Omega}$ compact and $\overline{\Omega} \subset M$, whose boundary is written as a disjoint union $\partial\Omega = \Gamma = \Gamma_0 \bigcup \Gamma_1$ and $T > 0$, Γ_1 relatively open in Γ, we consider a hyperbolic partial differential equation of the form

$$u_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \textit{lower order terms} = 0 \text{ in } Q = \Omega \times (0,T) \quad (1)$$

$$u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1 \text{ in } \Omega \times \{0\}$$

$$u = g \text{ on } \Sigma_1 = \Gamma_1 \times (0,T),\ u = 0 \text{ on } \Sigma_0 = \Gamma_0 \times (0,T). \quad (2)$$

The problem that we deal with is: can we design the control function g so that the solution u satisfies

$$u(T,\cdot) = 0,\ u_t(T,\cdot) = 0 \text{ on } \Omega \times \{T\};$$

i.e. can we make the wave vanish by time T?

First I will outline some results of geometric optics. Then I will outline the theorem of Lasiecka, Triggiani and Yao and their proof. Finally I will show how the same theorem (modulo smoothness assumptions) follows more quickly by geometric optics.

2 Geometric optics

The first thing that we do is define a new geometry on $Q = \Omega \times (0,T)$. The coefficients $(a_{ij}(x))$ of equation (1) are positive definite for a hyperbolic equation, and so their inverse $(g_{ij}(x)) = (a_{ij}(x))^{-1}$ defines a Riemannian metric on Ω. It is well known that in solutions u of equation (1), singularities and other effects travel along bicharacteristics $c : \mathbb{R} \to T^*(\Omega)$, which are the graphs of unit-speed geodesics of $\gamma : \mathbb{R} \to \Omega$ under the metric g. These bicharacteristics are also called the *rays of geometric optics.* (By abuse of language the geodesic γ itself will sometimes be called a bicharacteristic or ray.)

If a ray γ reaches the boundary Γ, various things can happen. We say that a bicharacteristic c *crosses* the boundary Γ at x if γ would continue into $M \setminus \overline{\Omega}$ if the boundary Γ did not intervene. If c crosses the uncontrolled boundary Γ_0 at a point $x = \gamma(t_0)$, then usually c becomes a *broken bicharacteristic*, that is, γ

is reflected back into Ω by the familiar reflection law: the tangent vector to the reflected geodesic is in the same plane in $T_x(\overline{\Omega})$ as the tangent to the incident ray and the normal ν to the boundary, and the angle of reflection equals the angle of incidence. See Lemma 4 below. Here by "usually" I mean, when $\gamma'(t_0)$ is not tangent to Γ. Other possibilities are gliding rays and diffractive points.

A *gliding ray* is a bicharacteristic which follows the boundary Γ_0 for some interval of time. These can occur if a ray intersects Γ_0 tangentially, and Γ_0 has positive normal curvature in the direction of that ray. (Imagine a segment of the boundary is the graph of $f(x) = x^3$ on $(-\infty, \infty)$, Ω lies above the graph of f, and a ray approaches from the negative x-axis. Then the ray will follow the graph of f after the intersection.) When a gliding ray does occur, unless the bicharacteristic meets the boundary with infinite order of contact, it can occur only as the limit of broken geodesics as the number of reflection points becomes infinite. (see [4], p.441) We will assume a boundary such that these are the only case of gliding rays possible.

A *diffractive point* is a point where a ray meets Γ_0 tangentially, and Γ_0 has negative normal curvature in the direction of the ray. In this case there is no loss of smoothness in $\gamma(t)$ at t_0. That is, γ brushes Γ_0 and reenters Ω but does not change direction or velocity.

A bicharacteristic that has been reflected or which may have become a gliding ray or encountered diffractive points is called a *compressed generalized bicharacteristic.* All of these cases are treated in fuller detail in [4], section 24.3, and in [1], section 2.

If c crosses Γ_1 at time t, then in the language of [1], $c(t) \in T^*(\overline{\Omega})$ is a *nondiffractive point.* In that case we can set Dirichlet conditions g on Γ_1 that absorb the effects propagating along γ. Here is an outline of the argument for that, as presented in [1]:

Step 1. γ would continue unchanged, leaving Ω, if the boundary did not intervene. If Γ has such an effect on u, it follows that u must leave a considerable trace on Γ. This is expressed in terms of strong local/microlocal estimates.

Step 2. But, as effects propagate along the bicharacteristics, similar estimates can be applied backwards along the bicharacteristics that pass through a given controlled region of Γ. And so if all bicharacteristics reach a given region Γ_1 of Γ, we can control the values of u throughout Ω by setting conditions on Γ_1.

Step 3. Our estimates from propagation of singularities introduce some lower-order terms. We can use a compactness argument to show that the effect of these lower-order terms is small or non-existent.

This gives us our

Theorem 1 <u>General Principle of Geometric Optics</u>: *If the coefficients of* (1) *are* C^∞, $\partial\Omega$ *is* C^∞, *no bicharacteristic of* (1) *in* $\overline{\Omega}$ *meets* $\partial\Omega$ *with infinite order of contact, and if every compressed generalized bicharacteristic* c *crosses* $T^*(\Gamma_1 \times (0,T))$ *(in the language of [1], passes through a nondiffractive point of* $T^*(\Gamma_1 \times (0,T))$*), then equation* (1) *is controlled in time* $T + \epsilon$ *for any* $\epsilon > 0$.

See the paper of Bardos, Lebeau and Rauch [1], where the result is stated and proven for a domain $\Omega \subset \mathbb{R}^n$. We believe that the result also holds for manifolds. Indeed, in later papers Lebeau [6] and Lebeau and Robbiano [7] take the theorem to be proven when $\overline{\Omega} \subset M$ is a compact connected Riemannian manifold with boundary.

Littman [8] proved that this same conclusion holds for hyperbolic equations of order m with C^∞ coefficients in domains with C^m-smooth boundary, when the whole lateral boundary $\partial\Omega \times [0,T]$ can be controlled and every bicharacteristic crosses the lateral boundary. It is worth noting in Littman's proof that (1) the needed fact is that every bicharacteristic crosses the lateral boundary, and (2) Littman states that the theorem is true with less-than-C^∞ coefficients, but he wanted to keep the proof simple.

An inverse result is shown by Ralston [9]–with the standard wave equation in a bounded set $\Omega \subset \mathbb{R}^n$, if there is a reflected bicharacteristic which intersects $\partial\Omega \times [0,T]$ only in $\Gamma_0 \times [0,T]$, then control in time T is impossible.

3 Theorem of Lasiecka, Triggiani and Yao

Lasiecka, Triggiani and Yao deal with equation (1) in the form $u_{tt} + Au = F_1(u) + f$, when the coefficients of A satisfy $a_{ij}(x) \in C^1(\overline{\Omega})$, F_1 is linear first-order with $L_\infty(Q)$ coefficients, $f \in L_2(Q)$, Ω is a C^2 bounded open set in $\mathbb{R}^n$, and the controlled and uncontrolled parts of the boundary Γ_1 and Γ_0 are topologically separated, i.e. $\overline{\Gamma_0} \bigcap \overline{\Gamma_1} = \emptyset$.

The existence is assumed of a non-negative C^2 convex function v on $\overline{\Omega}$. v is strictly convex with respect to the metric already discussed on Ω in that the Hessian $D_g^2 v$ of v satisfies

$$D_g^2 v(X,X) \geq 2\rho |X|_g^2 \quad \forall X \in T_x(\Omega)$$

for some constant $\rho > 0$. It is also assumed that the gradient of v, $h(x) = \nabla_g v(x)$, satisfies $\langle h(x), \nu(x) \rangle \leq 0$ on Γ_0, where $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$.

Then control is attained for

$$T > T_0 = 2\left(\frac{max_{x \in \overline{\Omega}} v(x)}{\rho}\right)^{\frac{1}{2}} = 2\left(\frac{K}{\rho}\right)^{\frac{1}{2}}.$$

Example 2 *In $(\mathbb{R}^n, eucl)$, let Ω be the ball of radius R centered at the origin, B_R, $v(x) = |x|^2$, and consider the standard wave equation, i.e. $a_{ij}(x)$ is the identity matrix. Then $\rho = 1$ and $K = R^2$, so $T_0 = 2\left(\frac{K}{\rho}\right)^{\frac{1}{2}} = 2\left(\frac{R^2}{1}\right)^{\frac{1}{2}} = 2R$. And in this example at least, that bound is sharp: Γ_1 must be the entire boundary, and a ray can travel almost a diameter of the disk before crossing out of the boundary.*

Lasiecka, Triggiani and Yao's proof is by the Hilbert Uniqueness Method, but with the variation that they have geometrized everything, as for instance the use of the Hessian and gradient of v with respect to the Riemannian metric g mentioned above. This idea is of great benefit because it makes their calculations free of the particular coordinates on Ω. So in many steps of their HUM proof we find Riemannian inner products where Euclidean dot products would arise in other papers, for instance.

They start by multiplying each side of the equation $u_{tt} + Au = F_1(u) + f$ by $e^{\tau\phi(x,t)}[h(u) - \phi_t u_t]$ (where $\tau > 0$ is a constant and $\phi(x,t)$ is designed to satisfy certain useful bounds), and then after a long integration by parts and simplification this becomes

$$\left(\overline{BT}_u\right)|_{\Sigma} + ... \geq k_{\phi,\tau}[E(T) + E(0)]$$

Here $\overline{BT}_u$ denote boundary terms, and $E(t)$ is the energy of the solution at time t. But the assumptions that $u = 0$ and $\langle h(x), \nu(x)\rangle \leq 0$ on Γ_0 imply that the contribution of $\left(\overline{BT}_u\right)|_{\Sigma_0}$ is negative, so the inequality still holds with $\left(\overline{BT}_u\right)|_{\Sigma}$ replaced by $\left(\overline{BT}_u\right)|_{\Sigma_1}$. A few steps later we arrive at

$$\int_0^T \int_{\Gamma_1} \left(\frac{\partial\psi}{\partial\nu_A}\right) d\Sigma_1 \geq c_T \|\{\psi_0, \psi_1\}\|^2,$$

which is equivalent to controllability by the standard duality argument.

4 The present proof

Finally we come to this author's proof of a weakened version of the theorem discussed in Section 3 above. Note that we do not assume that $\overline{\Gamma_0} \bigcap \overline{\Gamma_1} = \emptyset$, because the results of geometric optics that we use only require that $\Gamma_1 \neq \emptyset$ be open in $\partial\Omega$.

Theorem 3 *Suppose that the region Ω and the coefficients of equation* (1) *are C^∞, $\partial\Omega \in C^\infty$ and that $\partial\Omega$ does not allow infinite order of contact with bicharacteristics, that $\Gamma_1 \neq \emptyset$ is open in $\partial\Omega$, and that a convex function v exists, $v : \overline{\Omega} \to [0, K]$, with*

$$D_g^2 v(X, X) \geq 2\rho |X|_g^2 \quad \forall X \in T_x(\Omega),$$

and for $h(x) = \nabla_g v(x)$, $\langle h(x), \nu(x)\rangle \leq 0$ on Γ_0. Then equation (1) *is controlled for any time $T > T_0 = 2\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$.*

Proof. I will show that every ray of geometric optics must cross the controlled boundary Γ_1 within time T_0.

We know (cf. [2], for example) that in the interior Ω, a bicharacteristic c of equation (1) is the graph of a unit-speed geodesic with respect to g, call it $\gamma : [0, T] \to \overline{\Omega}$. Define the tangent vector field $X(t) := \gamma'(t)$, and the function $f(t) := v(\gamma(t))$.

As v is a strictly convex function in $\overline{\Omega}$ which is assumed to be non-negative, we have $0 \le f(t) \le \max_{x\in\overline{\Omega}} v(x) = K$ for all time t such that $\gamma(t) \in \overline{\Omega}$. Also, since γ has unit speed, $D^2v(X(t), X(t)) \ge 2\rho|X(t)|_g^2 = 2\rho$. But then,

$$f'(t) = \frac{d}{dt}v(\gamma(t)) = \langle \nabla_g v(t), X(t)\rangle$$

and so

$$\begin{aligned} f''(t) = \frac{d^2}{dt^2}v(\gamma(t)) &= \langle \frac{d}{dt}\nabla_g v, X\rangle + \langle \nabla_g v, \nabla_X X\rangle \\ &= \langle D_X(\nabla_g v), X\rangle + 0 = D^2v(X,X) \ge 2\rho. \end{aligned}$$

for those intervals of time when γ is a smooth curve in Ω, since γ is a geodesic, so on such intervals $f(t)$ is a C^2 function of t whose second derivative is always at least 2ρ.

To complete the proof of Theorem 3, we will need a lemma concerning what happens when a geodesic crosses the uncontrolled boundary, Γ_0. For the time being we assume that gliding rays do not occur.

Lemma 4 *Suppose a bicharacteristic segment $c : (t_i - \epsilon, t_i) \to T^*(\Omega)$ of equation (1) would cross Γ_0 at time t_i with finite order of contact. Then c will continue after reflection at time t_i as a piecewise-smooth broken bicharacteristic in $T^*(\overline{\Omega})$. In particular, its associated geodesic curve $\gamma(t)$ and $f(t) = v(\gamma(t))$ are continuous functions. At the reflection point t_i, define $X^- = \lim_{t\to t_i^-}\gamma'(t)$ and $X^+ = \lim_{t\to t_i^+}\gamma'(t)$, and also $f'(t_i^+) = \lim_{t\to t_i^+} f'(t)$ and $f'(t_i^-) = \lim_{t\to t_i^-} f'(t)$. Then if $\nu(\gamma(t_i))$ is the outward unit normal vector to Γ_0 at $\gamma(t_i)$ and μ is tangent to Γ_0 at $\gamma(t_i)$, we have*

$$\langle X^+, \nu\rangle = -\langle X^-, \nu\rangle, \tag{3}$$

and

$$\langle X^+, \mu\rangle = \langle X^-, \mu\rangle. \tag{4}$$

If $\langle \nabla_g v, \nu\rangle \le 0$ at $f(t_i)$, then a nonnegative jump in $f'(t)$ occurs at t_i:

$$f'(t_i^+) \ge f'(t_i^-).$$

Proof. The geometric law of reflection is proved in, for instance, [4], section 24.2, and in [9], section 2.2. For the last inequality, concerning f', note that generally

$$f'(t) = \langle X, \nabla_g v(\gamma(t))\rangle = \langle X, h(\gamma(t))\rangle.$$

Hence at the point of reflection,

$$f'(t_i^+) - f'(t_i^-) = \langle X^+ - X^-, h(\gamma(t))\rangle.$$

But $X^+ - X^-$ is a negative multiple of the outward normal vector ν. For X^+, X^- and ν lie in the same plane. Let μ be a unit vector in that plane which is tangent to Γ_0. Then

$$X^+ - X^- = \langle X^+ - X^-, \nu\rangle\nu + \langle X^+ - X^-, \mu\rangle\mu.$$

But by (3) and (4), we have

$$X^+ - X^- = -2\langle X^-, \nu\rangle\nu = -k\nu,$$

and k is nonnegative because γ approaches the reflection point from within Ω. By assumption, on Γ_0 we have $\langle h(x), \nu(x)\rangle \leq 0$, so

$$f'(t_i^+) - f'(t_i^-) = -\langle k\nu(\gamma(t)), h(\gamma(t))\rangle \geq 0,$$

and the lemma is proved. ■

Now we return to the proof of Theorem 3.

Recall that if the bicharacteristic $c(t)$ encounters a diffractive point at some time t_d, then there is no loss in smoothness of its associated geodesic $\gamma(t)$ or of $f(t)$. Hence $f(t)$ is still a C^2 function with $f''(t) \geq 2\rho$ in a neighborhood of t_d.

Let $t_0 \in [0, T]$ denote the time at which $f(t)$ attains its minimum. If $t_0 = 0$, then $f'(t_0^+) \geq 0$; if $t_0 = T$, then $f'(t_0^-) \leq 0$. If $t_0 \neq 0, T$, and $\gamma(t_0)$ is an interior point in Ω, then $f'(t_0) = 0$. In the case where $\gamma(t_0)$ is a point of reflection on Γ_0, $f'(t_0^-) \leq 0$ and $f'(t_0^+) \geq 0$. Hence in any case where they are defined, the left- and right-hand limits satisfy $f'(t_0^-) \leq 0$ and $f'(t_0^+) \geq 0$.

Let t be a point in the domain of γ with $\gamma(t) \in \Omega \bigcup \Gamma_0$. Let $t > t_0$, and suppose that between t_0 and t there are reflection points $\gamma(t_i), i = 1, 2, ..., n$, where n could be zero or infinite. Then by the lemma, at each reflection point there is a non-negative jump in $f'(t)$, which we shall denote $J(t_i) = f'(t_i^+) - f'(t_i^-)$. Then

$$f'(t) - f'(t_0^+) = \int_{t_0}^{t} f''(\tau)d\tau + \sum_{i=1}^{n} J(t_i) \geq 2\rho(t - t_0),$$

and so $f'(t_0^+) \geq 0$ implies $f'(t) \geq 2\rho(t - t_0)$. But then

$$f(t) - f(t_0) = \int_{t_0}^{t} f'(\sigma)d\sigma \geq \int_{t_0}^{t} 2\rho(\sigma - t_0)d\sigma = \rho(t - t_0)^2$$

But again $f(t_0) \geq 0$, so $f(t) \geq \rho(t - t_0)^2$.

If on the other hand $t < t_0$, then $t_0 > 0$, and we have $f'(t_0^-) \leq 0$. Suppose that between t and t_0 there intervene reflection points $\gamma(t_i), i = 1, 2, ..., m$ as before. Then

$$f'(t_0^-) - f'(t) = \int_{t}^{t_0} f''(\tau)d\tau + \sum_{i=1}^{m} J(t_i) \geq 2\rho(t_0 - t).$$

Hence $f'(t) \leq 2\rho(t - t_0)$, and

$$f(t_0) - f(t) = \int_t^{t_0} f'(\sigma) d\sigma \leq \int_t^{t_0} 2\rho(\sigma - t_0) d\sigma = -\rho(t - t_0)^2.$$

But $f(t_0) \geq 0$, so $f(t) \geq \rho(t - t_0)^2$.

And so in all cases we have

$$f(t) \geq \rho(t - t_0)^2.$$

But $f(t) \leq max_{x \in \overline{\Omega}} v(x) =: K$, so $\rho(t - t_0)^2 \leq K$, and $|t - t_0| \leq \left(\frac{K}{\rho}\right)^{\frac{1}{2}}$. Thus $t = |t - 0| \leq |t - t_0| + |t_0 - 0| \leq 2\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$, for all time t with $\gamma(t) \in \Omega \bigcup \Gamma_0$.

Hence if $t > T_0 = 2\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$, then γ has left Ω through the controlled boundary Γ_1, and the equation is controlled.

Now suppose that a bicharacteristic γ_g includes a gliding ray as a segment. This ray is the limit of a sequence of broken geodesics γ_i, by our assumption on the smoothness of $\partial\Omega$. (Recall the discussion of gliding rays in Section 2 above.) For each of these broken geodesics, $\gamma_i(t) \in \Omega$ implies $t \leq T_0$, and so passing to the limit, $\gamma_g(t) \in \Omega$ implies $t \leq T_0$. So again $t > T_0$ implies that $\gamma_g(t)$ has crossed out of Ω through Γ_1, and the equation is controlled.

■

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