# Uniqueness for the determination of sound-soft defects in an inhomogeneous planar medium by acoustic boundary measurements

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**Abstract**

We consider the inverse problem of determining shape and location of sound-soft defects inside a known planar inhomogeneous and anisotropic medium through acoustic imaging at low frequency. We consider the case of acoustic boundary measurements, with different types of boundary conditions to be prescribed, and we prove that at most two, suitably chosen, measurements allow us to uniquely determine multiple defects under minimal regularity assumptions on the defects and the medium containing them. Finally we treat applications of these results to the case of inverse scattering.

## 1 Introduction

In this paper we prove some uniqueness results for the following kind of inverse boundary value problem.

Let us assume we are given a bounded planar domain $\Omega$, a $2\times 2$ tensor $A$ in $\Omega$ satisfying a uniform ellipticity condition and a measurable positive function $b$ in $\Omega$. Let $\Sigma$ be a closed set contained in $\Omega$ such that $\Omega\backslash\Sigma$ is connected.

Then the (direct) boundary value problem consists in finding a function $u$ solving

$$\begin{cases} \mathrm{div}(A\nabla u)+kbu=0 & \text{in } \Omega\backslash\Sigma \\ u=0 & \text{on } \Sigma \end{cases} \tag{1.1}$$

with $k$ a positive constant, and, satisfying, in addition, a certain boundary condition on the exterior boundary $\partial\Omega$, namely either a Dirichlet condition

$$u=\psi \quad \text{on } \partial\Omega \tag{1.2}$$

or a Neumann one

$$A\nabla u\cdot\nu=\eta \quad \text{on } \partial\Omega. \tag{1.3}$$

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The corresponding inverse boundary problem consists in the following. We assume that the set $\Sigma$ is unknown and we want to determine it through additional information, in particular through measurements on the boundary, on the solutions $u$ to boundary value problems of the kind (1.1), (1.2), or (1.1), (1.3) respectively, corresponding to one or more choices of the boundary datum $\psi$, or $\eta$ respectively. More precisely, we prescribe one or more, suitably chosen, boundary data $\psi$ (or respectively $\eta$) and we measure on $\Gamma$, a subarc of $\partial\Omega$, the value of $A\nabla u \cdot \nu$, (respectively of $u$), $u$ being the solution to (1.1), (1.2) (respectively (1.1), (1.3)).

We ask whether these additional measurements determine uniquely the set $\Sigma$, that is if there is a unique set $\Sigma$ which is compatible with the measurements performed.

The Helmholtz type equation $\mathrm{div}(A\nabla u) + kbu = 0$ in $\Omega\backslash\Sigma$ can model the wave pattern of a low amplitude sound wave with a fixed temporal frequency, given by $\sqrt{k}$, in a, possibly inhomogeneous and anisotropic, given medium. The boundary condition $u = 0$ on $\Sigma$ can model the presence of a *sound-soft* multiple defect buried into the medium. Therefore our inverse problem can be described as an acoustic imaging technique for the detection in a given medium of sound-soft defects or flaws such as, for example, obstacles or cracks.

In fact, there are two cases of particular interest. In the first one $\Sigma$ is constituted by the finite union of pairwise disjoint closed sets, where each of them can be represented as the graph of a simple open curve. Here $\Sigma$ models the presence of many cracks, that is fractures, in the medium. In the second case, $\Sigma$ is formed by a finite collection of pairwise disjoint closed sets, each of them being the region delimited by a simple closed curve, and models, therefore, the presence of finitely many obstacles in the medium.

There is a well-developed theory, as well as numerical methods, on the determination of defects, such as obstacles or cracks, through inverse scattering techniques associated to acoustic waves, see for instance the book by D. Colton and R. Kress, [8], and the recent survey paper [7] for the obstacle case and [16] for the crack case. Our approach is different since we do not perform measurements in the far-field, as in the scattering case, but on the (exterior) boundary of the medium obtaining additional information on the behaviour of the waves on the boundary. However, from the scattering data it is possible to obtain information on the near field and hence on the overdetermined boundary data. Therefore we can apply our results also to the scattering case and we generalize, at least in the planar case, some uniqueness results for the determination of obstacles in a known inhomogeneity contained in [15], by lowering the assumptions on the defects and the inhomogeneity containing them. See Section 4 for details.

Let us describe the main results of the paper, as well as the main assumptions we make on the data of the problem, which we shall keep as minimal as possible.

Concerning the unknown multiple defect $\Sigma$, we allow the greatest generality in its definition, in order to cover all the most interesting cases in applications. Namely we assume only that $\Sigma$ is composed by a finite collection of pairwise disjoint defects, where a defect $\sigma$ in $\Omega$ is a closed continuum contained in $\Omega$

such that $\Omega\backslash\sigma$ is connected. We recall that a *continuum* is a connected set with at least two points. The previously described cases of obstacles and cracks are clearly included in this broader definition. Indeed we allow obstacles and cracks to be present at the same time and no regularity assumption is made on the components of $\Sigma$, which may not even be curves. Our assumptions, however, guarantee that $\Sigma$ is not reduced to a single point and that $\Sigma$ does not break $\Omega$ into two or more pieces.

About the medium in which the defects are buried, we do not assume it either homogeneous or isotropic, that is the coefficients of the Helmholtz type equation involved, $A$ and $b$, verify only the following. We suppose that $A$ is a measurable, symmetric, bounded and uniformly positive definite $2 \times 2$ matrix in $\Omega$ and $b$ is a measurable, bounded and positive function in $\Omega$. We remark that no continuity or smoothness assumption is imposed on the coefficients.

We shall consider the case of waves at low frequency, that is we assume that $k$ is positive and less than a constant which depends on known quantities involving the geometry of $\Omega$, the region occupied by the medium, and the physical properties of the medium itself, that is the coefficients of the Helmholtz type equation, and, sometimes, on some *a priori* information concerning, roughly speaking, the size of the unknown multiple defect.

Then the following kind of unique determination will be proved, either by prescribing Dirichlet data on $\partial\Omega$ and measuring the corresponding Neumann ones on an open subarc $\Gamma$ of $\partial\Omega$, or by assigning Neumann conditions on $\partial\Omega$ and measuring the Dirichlet data on $\Gamma$.

First of all we show that one measurement is enough to uniquely determine $\Sigma$ in any of the following two cases. Either we assume that $\Sigma$ is composed by a collection of obstacles, and in this case any nontrivial prescribed boundary condition may be used, see Theorem 3.2. Or we let $\Sigma$ be any kind of multiple defect and prescribe a nonnegative boundary datum, see Theorem 3.3.

In the third uniqueness result, Theorem 3.4, we show that two suitably chosen measurements, corresponding to two prescribed boundary data still satisfying assumptions on their sign changing on $\partial\Omega$, although not as restrictive as the one in Theorem 3.3, allow us to uniquely determine the unknown multiple defect $\Sigma$.

The mathematical formulation of our inverse problem is similar to the one of determining a collection of defects, in particular cracks or cavities, inside a conductor body by performing electrostatic current and voltage measurements on the exterior boundary of the conductor itself. This problem, which has been referred to as the inverse crack problem, was introduced by A. Friedman and M. Vogelius who proved the first uniqueness result, [9]. Many other authors contributed to establish further uniqueness and stability results, see [20] for references on the subject.

In fact, in order to prove our results, we shall extend to the Helmholtz type equation the technique used in [2] to obtain a very general uniqueness result for the inverse crack problem, coupling it sometimes with reasonings close in spirit to some developed in [3] where the inverse crack problem in three dimensions is tackled.

The plan of the paper is as follows. In Section 2 we study qualitative properties of solutions to (1.1). In particular we introduce a generalized notion of nodal critical point, which follows from the one in [4] for elliptic equations without any lower order term, and we investigate the relationship between the presence of nodal critical points for $u$, solution to (1.1), and the behaviour of $u$ on $\partial\Omega$, see Proposition 2.2. In Section 3 we state and prove the uniqueness results for our inverse problem briefly outlined above. The applications to the scattering case are developed in Section 4.

### Acnowledgement

The author wish to thank Professor Giovanni Alessandrini for useful discussions on the subject of this work. The present work was initiated while the author was at Institut für Industriemathematik/SFB 013 of Johannes Kepler Universität Linz, Austria, and was supported by the Austrian Science Foundation FWF under project SFB 013 F1310.

## 2 Helmholtz type equation with low frequency in a medium containing sound-soft defects

We study existence and uniqueness of the solution to the Helmholtz type equation with low frequency in a planar medium which contains a finite number of sound-soft defects under different type of boundary conditions on the (exterior) boundary of the medium itself.

Then we shall describe some qualitative properties of such solutions, in particular we shall investigate the behaviour of their *nodal lines*, that is their level lines at level 0, see Proposition 2.2.

For any $z \in \mathbb{R}^2$ and any positive number $r$ we denote with $B_r(z)$ the open ball with centre $z$ and radius $r$. We also use the following notation for complex derivatives

$$f_{\bar{z}} = \frac{1}{2}(f_x + \mathrm{i} f_y), \qquad f_z = \frac{1}{2}(f_x - \mathrm{i} f_y),$$

where $f$ is usually a complex-valued function.

For any planar domain $\Omega$ and any real number $p$, $p \geq 1$, we denote, as usual, with $W^{1,p}(\Omega)$ the following Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \ \nabla u \in L^p(\Omega)\}$$

where $\nabla u$ denotes the gradient of $u$ in the sense of distributions. With $W_0^{1,p}(\Omega)$ we denote the closure in $W^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$, the space of $C^\infty(\Omega)$ functions whose supports are compactly contained in $\Omega$. When $p = 2$ we usually denote $W^{1,2}(\Omega)$ with $H^1(\Omega)$ and $W_0^{1,2}(\Omega)$ with $H_0^1(\Omega)$. We recall that a domain is said to be *Lipschitz* if its boundary $\partial\Omega$ can be locally represented as the graph of a Lipschitz function. If $\Omega$ is bounded and Lipschitz, then $H^{1/2}(\partial\Omega)$ denotes the trace space of $H^1(\Omega)$ on $\partial\Omega$. By $H^{-1/2}(\Omega)$ we shall denote the dual space to $H^{1/2}(\Omega)$. We recall that $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$.

## 2.1 Capacities

We shall make use of different notions of capacity, limiting ourselves to the planar case. The first kind of capacity we introduce is the so-called *Sobolev 2-capacity*. For any $E \subset \mathbb{R}^2$, we set

$$C_{1,2}(E) = \inf_{u \in S(E)} \int_{\mathbb{R}^2} |u|^2 + |\nabla u|^2$$

where

$$S(E) = \{u \in H^1(\mathbb{R}^n) : \ u = 1 \text{ a.e. in an open set containing } E\}.$$

If $S(E)$ is empty, then $C_{1,2}(E) = \infty$. The number $C_{1,2}(E)$ is called the Sobolev 2-capacity of $E$. For its basic properties we refer, for instance, to [12, Chapter 2]. We recall that if $K$ is a compact set contained in $\mathbb{R}^2$ then

$$C_{1,2}(K) = \inf_{u \in S(K) \cap C_0^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |u|^2 + |\nabla u|^2.$$

Our interest in capacity follows from the fact that capacity plays an important rôle in the following Poincaré type inequalities. We recall that for any bounded domain $\Omega \subset \mathbb{R}^2$ there exists a constant $C(\Omega)$ such that for any $u \in H_0^1(\Omega)$ we have

$$\int_\Omega u^2 \leq C(\Omega) \int_\Omega |\nabla u|^2. \tag{2.1}$$

This inequality is called *Poincaré inequality* and we have that

$$C(\Omega) \leq (|\Omega|/(2\pi))^{1/2},$$

see, for instance, [10, page 164]. Here $|\Omega|$ denotes the Lebesque measure of $\Omega$. An inequality of this kind holds for any $H^1$ function whose set of points where it vanishes (in a weak sense) has a strictly positive Sobolev 2-capacity.

Let us state this kind of generalization of Poincaré inequality, see [23, Section 4.5] for details. Let $\Omega$ be a bounded Lipschitz domain and let $\Sigma$ be a closed set contained in $\Omega$ with positive Sobolev 2-capacity. Then there exists a constant $C(\Sigma, \Omega)$ such that

$$\int_\Omega u^2 \leq C(\Sigma, \Omega) \int_\Omega |\nabla u|^2 \tag{2.2}$$

for any $u \in H^1(\Omega)$ verifying $u = 0$ in a weak sense on $\Sigma$. Furthermore we have that

$$C(\Sigma, \Omega) \leq C(C_{1,2}(\Sigma))^{-1/2} \tag{2.3}$$

where the constant $C$ depends on $\Omega$ only.

We wish to remark that this result, and in particular the estimate (2.3), is stated in [23] with $C_{1,2}(\Sigma)$ replaced by $B_{1,2}(\Sigma)$, where for any $E \subset \mathbb{R}^2$, $B_{1,2}(E)$ denotes the $(1,2)$-*Bessel capacity* of $E$ (see [23, Section 2.6] for its definition).

Since these two notions of capacity are equivalent, that is there exist two positive constants $A_1$ and $A_2$ such that

$$A_1 B_{1,2}(E) \leq C_{1,2}(E) \leq A_2 B_{1,2}(E)$$

for any set $E \subset \mathbb{R}^2$, and the definition of Bessel capacity is somewhat involved we have preferred to use the Sobolev 2-capacity. The equivalency of the two capacities may be found in [1], by coupling Proposition 2.3.13 with the remark after Definition 2.2.6 therein contained.

We shall need an estimate from above of $C(\Sigma, \Omega)$ in (2.2) in terms of the metric properties of $\Sigma$. Hence, by (2.3), it is enough to estimate from below $C_{1,2}(\Sigma)$ in terms of the same quantities. For planar sets we shall be able to obtain these estimates using known results concerning another notion of capacity, the *logarithmic capacity.* For any $E$ bounded and closed subset of $\mathbb{R}^2$, let $\gamma(E)$ be its logarithmic capacity. For a definition of logarithmic capacity and its basic properties we refer to [22, Chapter III]. Of particular interest to us are a theorem, [22, Theorem III.5], which states that if $\sigma$ is a closed and bounded continuum then $\gamma(\sigma) > 0$ and a stronger result, see [22, page 85], where, under the same hypothesis on $\sigma$, we have that

$$\gamma(\sigma) \geq \frac{\operatorname{diam}(\sigma)}{4}, \tag{2.4}$$

where $\operatorname{diam}(\sigma)$ denotes the diameter of $\sigma$.

We shall use (2.4) to deduce an estimate, from below, of $C_{1,2}(\sigma)$ where $\sigma$ is a bounded and closed continuum. In order to find a suitable relation between the Sobolev 2-capacity and the logarithmic capacity we need to consider two other notions of capacity.

The first is defined as follows. For any bounded domain $\Omega \subset \mathbb{R}^2$ and any $E \subset \Omega$ we define the 2-*capacity* of the *condenser* $(E, \Omega)$, $\operatorname{cap}_2(E, \Omega)$, as

$$\operatorname{cap}_2(E, \Omega) = \inf_{u \in S(E,\Omega)} \int_{\Omega} |\nabla u|^2$$

where

$$S(E, \Omega) = \{u \in H^1_0(\Omega) : \ u = 1 \text{ a.e. in an open set containing } E\}.$$

If $S(E, \Omega)$ is empty we set $\operatorname{cap}_2(E, \Omega) = \infty$. Also for this capacity, if $K$ is a compact set contained in $\Omega$, we have

$$\operatorname{cap}_2(K, \Omega) = \inf_{u \in S(E,\Omega) \cap C^\infty_0(\Omega)} \int_{\Omega} |\nabla u|^2.$$

For basic properties of the 2-capacity of condensers we refer again to [12, Chapter 2]. Here we simply recall that, for any $x_0 \in \mathbb{R}^2$ and any $0 < r < R$, the following formula holds

$$\operatorname{cap}_2(\overline{B_r(x_0)}, B_R(x_0)) = 2\pi(\log(R/r))^{-1}, \tag{2.5}$$

and that the following relation holds between the Sobolev 2-capacity and the 2-capacity of a condenser. There exists a positive constant $A_3$ such that for any $E \subset B_r(x_0)$ we have

$$(1 + A_3 r^2)^{-1} C_{1,2}(E) \leq \mathrm{cap}_2(E, B_{2r}(x_0)) \leq 16(1 + r^{-2}) C_{1,2}(E), \tag{2.6}$$

see [12, Theorem 2.38].

The other capacity we need to introduce is the so-called *hyperbolic capacity.* For any planar closed set $E$ contained in $B_1(0)$ we denote with $\gamma_-(E)$ the hyperbolic capacity of $E$. For the definition of this capacity we refer to [22, Section III.12]. By the definitions of logarithmic and hyperbolic capacities it is immediate to remark that for every $r$, $0 < r < 1$, if $E$ is a closed set contained in $B_r(0)$ we have

$$\frac{\gamma(E)}{1 + r^2} \leq \gamma_-(E) \leq \frac{\gamma(E)}{1 - r^2}. \tag{2.7}$$

The last link is provided by the *modulus of a ring domain*, see [22, Section III.13]. Let $\sigma$ be a closed continuum contained in $B_1(0)$. Then we consider the ring domain $B_1(0) \backslash \sigma$ and denote with $M(B_1(0) \backslash \sigma)$ its modulus. On one side, it is easy to prove, through the invariance of capacity of condensers and modulus of ring domains by conformal mappings and (2.5), that

$$\mathrm{cap}_2(\sigma, B_1(0)) = 2\pi M(B_1(0) \backslash \sigma)^{-1}. \tag{2.8}$$

On the other hand, Theorem III.54 in [22] states that

$$M(B_1(0) \backslash \sigma) = \log 1/\gamma_-(\sigma). \tag{2.9}$$

We collect all these results as follows. Let $\sigma$ be a closed continuum and let $\mathrm{diam}(\sigma)$ be its diameter. We take $x_0 \in \sigma$. Without loss of generality, we can assume $x_0 = 0$ and we denote $\tilde{\sigma} = \sigma \cap \overline{B_{1/3}(0)}$. It follows that $\tilde{\sigma}$ is a closed continuum contained in $B_{1/2}(0)$ and, by (2.6) and the monotonicity property of capacity, we infer immediately that

$$C_{1,2}(\sigma) \geq C_{1,2}(\tilde{\sigma}) \geq \mathrm{cap}_2(\tilde{\sigma}, B_1(0))/80.$$

On the other hand we have that $\mathrm{diam}(\tilde{\sigma}) \geq \min\{1/3, \mathrm{diam}(\sigma)\}$ and hence, by (2.4) and (2.7), we have

$$\gamma_-(\tilde{\sigma}) \geq \frac{4}{5}\gamma(\tilde{\sigma}) \geq \min\left\{\frac{1}{15}, \frac{\mathrm{diam}(\sigma)}{5}\right\}.$$

By (2.8) and (2.9) we infer that

$$\mathrm{cap}_2(\tilde{\sigma}, B_1(0)) = \frac{2\pi}{\log 1/\gamma_-(\tilde{\sigma})}.$$

By collecting the last three equations we have that for any closed continuum $\sigma$ the following estimates holds

$$C_{1,2}(\sigma) \geq \frac{\pi}{-40 \log\left(\min\left\{\frac{1}{15}, \frac{\mathrm{diam}(\sigma)}{5}\right\}\right)}.$$

Hence we deduce that

$$C_{1,2}(\sigma) \geq \frac{\pi}{40 \log\left(\max\left\{15, 5/\operatorname{diam}(\sigma)\right\}\right)}. \tag{2.10}$$

As an immediate corollary of this last result it follows, in particular, that a closed continuum $\sigma$ has a strictly positive Sobolev 2-capacity.

## 2.2 First eigenvalues

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let $A = A(z)$, $z \in \Omega$, be a $2 \times 2$ symmetric matrix whose entries are measurable and which satisfies the following uniform ellipticity condition for some positive constant $\lambda$

$$\lambda Id \leq A(z) \leq \lambda^{-1} Id \quad \text{for a.e. } z \in \Omega, \tag{2.11}$$

where $Id$ denotes the identity matrix.

Let $b = b(z)$, $z \in \Omega$, be a measurable function satisfying the following condition

$$\lambda \leq b(z) \leq \lambda^{-1} \quad \text{for a.e. } z \in \Omega. \tag{2.12}$$

We define the number $k_0(\Omega, A, b)$ in the following way

$$k_0(\Omega, A, b) = \min_{u \in H_0^1(\Omega),\ u \neq 0} \frac{\int_\Omega A\nabla u \cdot \nabla u}{\int_\Omega b u^2}. \tag{2.13}$$

We have that $k_0$ is a positive number and it is the first Dirichlet eigenvalue associated to the following boundary value problem

$$\begin{cases} -\operatorname{div}(A\nabla u) = kbu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{2.14}$$

We remark that the following monotonicity property holds

$$k_0(\Omega, A, b) \leq k_0(\Omega_1, A, b) \quad \text{for any domain } \Omega_1 \subset \Omega. \tag{2.15}$$

We also notice that if $A \equiv Id$ and $b \equiv 1$ we have that $k_0(\Omega, Id, 1)$ is the first eigenvalue of the Laplace operator and the following inequality holds

$$\lambda^2 k_0(\Omega, Id, 1) \leq k_0(\Omega, A, b) \leq \lambda^{-2} k_0(\Omega, Id, 1). \tag{2.16}$$

Finally, it is evident that $k_0(\Omega, Id, 1)$ is given by $1/C(\Omega)$ where $C(\Omega)$ is the best possible constant in (2.1). Therefore $k_0(\Omega, A, b)$ satisfies the following estimate

$$k_0(\Omega, A, b) \geq \frac{\lambda^2}{C(\Omega)} \geq \lambda^2 \left(\frac{2\pi}{|\Omega|}\right)^{1/2}. \tag{2.17}$$

For a review of theoretical results, as well as numerical methods, concerning the Dirichlet eigenvalues of the Laplace operator for 2-dimensional domains we refer to [17].

For any closed set $\Sigma$ contained in $\Omega$, we define the number $k_0(\Sigma,\Omega,A,b)$ as

$$k_0(\Sigma,\Omega,A,b)=\min_{u\in H^1(\Sigma,\Omega),\ u\neq 0}\frac{\int_\Omega A\nabla u\cdot\nabla u}{\int_\Omega bu^2} \tag{2.18}$$

where $H^1(\Sigma,\Omega)$ denotes the set of $H^1(\Omega)$ functions vanishing (in a weak sense) on $\Sigma$. We observe that if the Sobolev 2-capacity of $\Sigma$ is zero, then $H^1(\Sigma,\Omega)$ coincides with $H^1(\Omega)$ and therefore $k_0(\Sigma,\Omega,A,b)=0$ in this case. On the other hand, if $C_{1,2}(\Sigma)>0$ we have that $k_0(\Sigma,\Omega,A,b)$ is strictly positive and is linked to $k_0(\Sigma,\Omega,Id,1)$ by a formula completely analogous to (2.16). Furthermore, it is possible to evaluate $k_0(\Sigma,\Omega,Id,1)$ in terms of the best constant $C(\Sigma,\Omega)$ appearing in the Poincaré type inequality (2.2). As before $k_0(\Sigma,\Omega,Id,1)=1/C(\Sigma,\Omega)$. Therefore, using (2.3), $k_0(\Sigma,\Omega,A,b)$ can be bounded from below by a positive constant which depends on $\lambda$, $\Omega$ and the Sobolev 2-capacity of $\Sigma$ only, as follows

$$k_0(\Sigma,\Omega,A,b)\geq\frac{\lambda^2}{C(\Sigma,\Omega)}\geq\lambda^2C(C_{1,2}(\Sigma))^{1/2} \tag{2.19}$$

where $C$ is a positive constant depending on $\Omega$ only.

## 2.3 Nodal critical points

Given a bounded domain $\Omega$, we say that $\sigma\subset\Omega$ is a *defect* in $\Omega$ if $\sigma$ is a closed continuum such that $\Omega\backslash\sigma$ is connected.

A *multiple defect* in $\Omega$ will be a subset $\Sigma$ of $\Omega$ which is constituted by the union of a finite number of pairwise disjoint defects, $\sigma_1,\ldots,\sigma_N$, where $N$ is a positive integer. We have that $\Omega\backslash\Sigma$ is a connected open set and we have already noticed that $\Sigma$ has a strictly positive Sobolev 2-capacity. We introduce the following notation. Given a multiple defect $\Sigma=\bigcup_{i=1}^N\sigma_i$ we set

$$\delta(\Sigma)=\max_{i=1,\ldots,N}\{\mathrm{diam}(\sigma_i)\}.$$

In order to be able to compare the case when no defect is present in the medium, with a slight abuse of notation, we consider also the empty set as a multiple defect. We assume that

$$\delta(\Sigma)=0 \text{ if } \Sigma=\emptyset.$$

Therefore, in the sequel of the paper, we shall always assume that $\Sigma$ is either the empty set or a finite union of (nonempty) pairwise disjoint defects contained in $\Omega$.

We are interested to study functions $u$ satisfying in a weak sense, for a given positive constant $k$,

$$\begin{cases}\mathrm{div}(A\nabla u)+kbu=0 & \text{in } \Omega\backslash\Sigma\\ u=0 & \text{on } \Sigma\end{cases} \tag{2.20}$$

that is functions $u$ belonging to $H^1(\Sigma,\Omega)$ such that

$$\int_{\Omega\backslash\Sigma}A\nabla u\cdot\nabla\phi-kbu\phi=0\quad\text{for any } \phi\in H_0^1(\Omega\backslash\Sigma). \tag{2.21}$$

First of all we notice that functions $u$ satisfying (2.20) are continuous functions in $\Omega$. If $\Sigma$ is empty, this may be deduced by standard regularity estimates in the interior for elliptic equations. If $\Sigma$ is not empty, being every component of $\Sigma$ a continuum, we deduce that every point of $\partial\Sigma$ is regular for the Dirichlet problem for the Laplace equation with respect to the domain $\Omega\backslash\Sigma$, see, for instance, [22]. Hence, by standard regularity estimates in the interior and [18], we obtain that any solution $u$ to (2.20) belongs to $C(\Omega)$ and hence satisfies $u=0$ on $\Sigma$ also in a classical sense.

Weak solutions to an equation of the kind $\mathrm{div}(A\nabla u)+kbu=0$, with $k$ a positive constant, can be related, at least locally, with weak solutions to an elliptic equation in divergence form with no lower order terms, that is to functions $\tilde{u}$ satisfying $\mathrm{div}(\tilde{A}\nabla\tilde{u})=0$, with $\tilde{A}$ a symmetric matrix satisfying the uniform ellipticity condition (2.11), as follows.

Given a simply connected bounded Lipschitz domain $D$, a symmetric and measurable matrix $A$ satisfying (2.11) in $D$, a measurable function $b$ satisfying (2.12) in $D$, we assume that $u\in H^1(D)\cap C(\overline{D})$ solves in a weak sense

$$\mathrm{div}(A\nabla u)+kbu=0 \text{ in } D, \tag{2.22}$$

where $k$ is a positive constant. We assume that $k<k_0(D,A,b)$. We remark that for this condition to be satisfied we need only the measure of $D$ to be small enough. Hence all the following considerations might be applied locally inside $\Omega\backslash\Sigma$ for solutions to (2.20). For such a $k$ we have that there exists a unique weak solution $w$ to the boundary value problem

$$\begin{cases} \mathrm{div}(A\nabla w)+kbw=0 & \text{in } D \\ w=1 & \text{on } \partial D \end{cases}. \tag{2.23}$$

We have that $w$ here is continuous up to the boundary of $D$ and, by the strong maximum principle, is strictly positive in $D$. We consider the function $\tilde{u}=u/w$. We have that $\tilde{u}\in H^1(D)\cap C(\overline{D})$ and solves the following equation

$$\mathrm{div}(\tilde{A}\nabla\tilde{u})=0 \text{ in } D, \tag{2.24}$$

where $\tilde{A}=w^2A$. We notice that $\tilde{A}$ is measurable, symmetric and satisfies a uniform ellipticity condition in $D$, that is satisfies, for some positive $\lambda$, (2.11).

We also introduce the function $\tilde{v}$, the so-called *stream function* associated to $\tilde{u}$, namely a function satisfying almost everywhere in $D$

$$\nabla\tilde{v}=\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]\tilde{A}\nabla\tilde{u}.$$

The existence of a single-valued function satisfying this condition is guaranteed by the fact that $D$ is simply connected. The stream function is unique up to an additive constant and, furthermore, we have that $\tilde{v}$ is still a solution to an elliptic equation, namely $\tilde{v}$ solves

$$\mathrm{div}(\tilde{B}\nabla\tilde{v})=0 \text{ in } D,$$

where $\tilde{B} = (\det \tilde{A})^{-1}\tilde{A}$. Clearly, $\tilde{B}$ still verifies the uniform ellipticity condition (2.11). We remark that if $\tilde{u}$ is harmonic, then the notion of stream function coincides with the one of harmonic conjugate and, in this case, the complex valued function $\tilde{f} = \tilde{u} + \mathrm{i}\tilde{v}$ is known to be holomorphic, that is to satisfy $\tilde{f}_{\overline{z}} = 0$. In the more general case we have that $\tilde{f}$ satisfies a first order Beltrami type equation of the following kind

$$\tilde{f}_{\overline{z}} = \mu \tilde{f}_z + \nu \overline{\tilde{f}_z} \tag{2.25}$$

where $\mu$ and $\nu$ are measurable complex-valued coefficients in $D$ satisfying almost everywhere in $D$

$$|\mu| + |\nu| \leq \kappa < 1 \tag{2.26}$$

where $\kappa$ is a constant strictly less than one depending on the ellipticity and boundedness constants of $\tilde{A}$ only.

Any $H^1(D, \mathbb{C})$ solution $\tilde{f}$ of an equation of the type (2.25), (2.26), $\kappa$ being a constant such that $0 \leq \kappa < 1$, is called a *$\kappa$-quasiconformal function.* If, in addition, $\tilde{f}$ is also univalent, then $\tilde{f}$ is called a *$\kappa$-quasiconformal mapping.* We call quasiconformal function (respectively mapping) any $\kappa$-quasiconformal function (respectively mapping), for some $\kappa$, $0 \leq \kappa < 1$.

For solutions to an elliptic equation in two dimensions like (2.24), we can define, following [4], a generalized notion of critical points. Let $\tilde{u}$ solve (2.24) and let $\tilde{v}$ be its associated stream function. A representation theorem due to L. Bers and L. Nirenberg, see [6] and also [5], states that the quasiconformal function $\tilde{f} = \tilde{u} + \mathrm{i}\tilde{v}$ can be represented as $\tilde{f} = \tilde{F} \circ \chi$ where $\chi$ is a quasiconformal mapping, such that $\chi$ and its inverse $\chi^{-1}$ are Hölder continuous, and $\tilde{F}$ is a holomorphic function. If $\tilde{f} = \tilde{u} + \mathrm{i}\tilde{v}$ is not constant, then $z_0 \in D$ is called a *geometrical critical point* for $\tilde{u}$, and at the same time for $\tilde{v}$, if $\chi(z_0)$ is a critical point in the classical sense for $\tilde{U} = \Re \tilde{F}$. We remark that the definition is independent on the choice of the representation and coincides with the classical one if $\tilde{u}$ is smooth.

To any geometrical critical point we can also associate a generalized notion of index. Let $G$ be a smooth planar domain and let $E$ be a smooth vector field such that $E \neq 0$ on $\partial G$. Then we define the *index* of $E$ in $G$, $I(G, E)$, as $-$(*winding number*) of $E$ along $\partial G$, that is

$$I(G, E) = -\frac{1}{2\pi} \int_{\partial G} \mathrm{d} \arg(E).$$

If $z_0$ is an isolated zero of $E$ the index of $E$ at $z_0$ is given by

$$I(z_0, E) = \lim_{r \to 0} I(B_r(z_0), E).$$

For the present purposes, a complex valued function $g = g_1 + \mathrm{i}g_2$ will be identified with the vector field $E = \binom{g_1}{g_2}$.

The *geometric index* of $\nabla \tilde{u}$ at $z_0 \in D$, still denoted by $I(z_0, \nabla \tilde{u})$, will be defined as the index of $\nabla \tilde{U}$ at $\chi(z_0)$. We remark that, by this definition, for solutions of elliptic equations like (2.24) the index is positive if and only if $z_0$ is a geometrical critical point. Moreover we have that if $z_0$ is such that $\tilde{f}(z_0) = 0$ then

$$I(z_0, \tilde{f}) = I(\chi(z_0), \tilde{F}) = I(\chi(z_0), \nabla\tilde{U}) + 1 = I(z_0, \nabla\tilde{u}) + 1.$$

Let us also recall that the number of critical point inside a given domain, when they are counted with their multiplicity, is continuous with respect to $H^1$ convergence, see [4] for details.

We may give a geometric characterization of the geometric index in the following way. The geometric index of $\nabla\tilde{u}$ at $z_0$ is $n$, $n \geq 0$, if and only if, locally in a neighbourhood of $z_0$, the level set $\{\tilde{u} = \tilde{u}(z_0)\}$ is constituted by $n+1$ simple curves intersecting at $z_0$ only. Then, since $u$ and $\tilde{u}$ shares the same behaviour of the nodal lines, we can define *nodal geometrical critical points* for solutions to (2.22), along with their index, using their geometrical characterization, that is $z_0 \in D$ is a nodal geometrical critical point of index $n$, $n \geq 1$, for $u$ solution to (2.22) if the set $\{u = u(z_0) = 0\}$ is constituted, in a suitable neighbourhood of $z_0$, by $n + 1$ simple curves intersecting at $z_0$ only. Moreover the following proposition describing the local behaviour of the nodal lines can be proved.

**Proposition 2.1** *Let $u$ be a solution of* (2.22) *where $D$ is a bounded domain, $k$ is a positive number and $A$ and $b$ satisfy the previously stated assumptions, in particular they verify* (2.11) *and* (2.12) *respectively. Let $z_0$ be a point of $D$ such that $u(z_0) = 0$.*

*Then there exists a nonnegative integer, $n$, and an open neighbourhood of $z_0$, $U$, such that the following holds.*

*The set $\{z \in U : \ u(z) = 0\}$ is composed by $n+1$ simple curves intersecting at $z_0$ only, and the set $\{z \in U : \ u(z) \neq 0\}$ has exactly $2(n+1)$ connected components $D_1, \ldots, D_{2(n+1)}$. Moreover, we can find a Jordan curve $\gamma$ surrounding $z_0$ and contained in $U$ such that $\gamma \cap D_j$ is connected for every $j$. By following the parametrization on $\gamma$, we can assume that the open sets $D_j$, $j = 1, \ldots, 2(n+1)$, are ordered in a clockwise sense. We have that for any $j$, $j = 1, \ldots, 2(n+1)$, the sign of $u$ on $D_j$ is different from the sign of $u$ on $D_{j+1}$. Obviously, we identify $D_{2(n+1)+1}$ with $D_1$.*

PROOF. We can fix $r > 0$ small enough such that $\overline{B_r(z_0)} \subset D$ and $k < k_0(B_r(z_0), A, b)$. Let $w$ be the solution to (2.23) when $D$ is replaced by $B_r(z_0)$. The function $\tilde{u} = u/w$ satisfies in $B_r(z_0)$ an elliptic equation as in (2.24). Let $\tilde{v}$ be the stream function associated to $\tilde{u}$ in $B_r(z_0)$. We have that $\tilde{u}(z_0) = 0$ and, without loss of generality, we can assume $\tilde{v}(z_0) = 0$. We know that $\tilde{f} = \tilde{u} + \mathrm{i}\tilde{v}$, up to a quasiconformal change of coordinates, is holomorphic. Then, setting $n$ as the index of $\nabla\tilde{u}$ at $z_0$, the proposition follows from well-known properties of harmonic and holomorphic functions. □

Let again $D$ be a Lipschitz bounded domain and let $u$ and $u'$ belong to $H^1(D)$. Let $\Gamma$ be an open subarc of $\partial D$. Then we say that $u = u'$ in a weak sense on $\Gamma$ if for any $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\phi) \cap \partial D \subset \Gamma$ we have that $(u - u')\phi \in H_0^1(D)$. So $u = 0$ on $\Gamma$ if $u\phi \in H_0^1(D)$ for any $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\phi) \cap \partial D \subset \Gamma$.

For any $u$ solution to (2.22), where in this case we allow $k$ to be either zero or positive, we can define the conormal derivative of $u$ on the boundary, $A\nabla u\cdot\nu$, $\nu$ denoting the outer normal to $\partial D$, as follows. It is an element of $H^{-1/2}(\partial D)$ such that $(A\nabla u\cdot\nu)[\phi|_{\partial D}]=\int_D A\nabla u\cdot\nabla\phi-kbu\phi$ for any $\phi\in H^1(D)$.

If $u$ and $u'$ satisfy (2.22) with the same $k$, $k\geq 0$, we say that $A\nabla u\cdot\nu=A\nabla u'\cdot\nu$ in a weak sense on an open subarc $\Gamma$ of $\partial D$ if $(A\nabla u\cdot\nu-A\nabla u'\cdot\nu)[\phi|_{\partial D}]=0$ for any $\phi\in C_0^\infty(\mathbb{R}^2)$ such that $\mathrm{supp}(\phi)\cap\partial D\subset\Gamma$. Hence we say that $A\nabla u\cdot\nu=0$ in a weak sense on $\Gamma$ if $(A\nabla u\cdot\nu)[\phi|_{\partial D}]=0$ for any $\phi\in C_0^\infty(\mathbb{R}^2)$ such that $\mathrm{supp}(\phi)\cap\partial D\subset\Gamma$. We say also that $A\nabla u\cdot\nu\geq 0$ ($\leq 0$ respectively) in a weak sense on $\Gamma$ if $(A\nabla u\cdot\nu)[\phi|_{\partial D}]\geq 0$ ($\leq 0$ respectively) for any $\phi\in C_0^\infty(\mathbb{R}^2)$ such that $\mathrm{supp}(\phi)\cap\partial D\subset\Gamma$ and $\phi\geq 0$.

It has been proven in [4] that if two solutions of (2.22) with $k=0$ are such that $u=u'$ and $A\nabla u\cdot\nu=A\nabla u'\cdot\nu$ on an open subarc $\Gamma$ in a weak sense, then $u=u'$ in $D$. This result can be extended to the case when $k>0$ simply by the following considerations.

We can extend $A$ and $b$ on the whole plane by taking $A\equiv Id$ outside $D$ and $b\equiv 1$ outside $D$. We consider a point $z$ in $\Gamma$ and we consider the ball $B_r(z)$. We take $r$ small enough in such a way $k<k_0(B_r(z),A,b)$. Hence the problem (2.23), with $D$ replaced by $B_r(z)$, admits a unique positive solution $w$. It is well known, since [19], that this solution is continuous inside $B_r(z)$ and belongs to $W^{1,p}_{loc}(B_r(z))$ for some $p>2$. Then, if we take $r_1<r$, we have that $\tilde{u}=u/w$ and $\tilde{u}'=u'/w$ belong to $H^1(D\cap B_{r_1}(z))$. It is also clear that if $u=u'$ on $\Gamma$, then $\tilde{u}=\tilde{u}'$ on $\Gamma\cap B_{r_1}(z)$, in a weak sense.

Furthermore it is straighforward to notice that if $u=u'$ and $A\nabla u\cdot\nu=A\nabla u'\cdot\nu$ on $\Gamma$, then not only $\tilde{u}=\tilde{u}'$ on $\Gamma\cap B_{r_1}(z)$, but also $\tilde{A}\nabla\tilde{u}\cdot\nu=\tilde{A}\nabla\tilde{u}'\cdot\nu$ on $\Gamma\cap B_{r_1}(z)$. Therefore $\tilde{u}=\tilde{u}'$ in $B_{r_1}(z)\cap D$ and so $u=u'$ on this set. So far, we have obtained a local unique continuation property near the boundary. However, we can immediately deduce a global unique continuation property, that is we can infer that $u=u'$ on the whole domain $D$. In fact, it is well-known that any solution to (2.22) vanishing in an open subset of $D$ must vanish everywhere, hence since $u=u'$ in an open subset of $D$, they must coincide everywhere.

We have obtained some information concerning the local behaviour of the nodal lines of solutions to (2.20). For our purposes we need to study also a global behaviour of the nodal lines of these functions. In particular, we want to extract information about the behaviour of these solutions on the exterior boundary $\partial\Omega$ from information about their nodal critical points inside $\Omega\backslash\Sigma$.

We shall investigate two kinds of boundary conditions, namely Dirichlet and Neumann ones. We assume that $\Omega$ is a bounded, simply connected domain and we assume $\partial\Omega$ to be Lipschitz. Let $\Sigma$ be a multiple defect in $\Omega$ (possibly empty).

Let $\psi\in H^1(\Omega)$ and let $k$ be such that $0<k<k_0(\Omega\backslash\Sigma,A,b)$. Then the following Dirichlet type boundary value problem admits a unique weak solution

$$\begin{cases}\mathrm{div}(A\nabla u)+kbu=0 & \text{in } \Omega\backslash\Sigma\\ u=0 & \text{on } \Sigma\\ u=\psi & \text{on } \partial\Omega\end{cases}. \tag{2.27}$$

The weak formulation of (2.27) is the following. We look for a function $u\in$

$H^1(\Sigma,\Omega)$ such that its trace on $\partial\Omega$ is equal to the trace of $\psi$ on $\partial\Omega$ and such that

$$\int_{\Omega\backslash\Sigma} A\nabla u\cdot\nabla\phi - kbu\phi = 0 \quad \text{for any } \phi\in H^1_0(\Omega\backslash\Sigma).$$

If in addition we have that $\psi\in H^1(\Omega)\cap C(\overline{\Omega})$, then we infer that $u\in C(\overline{\Omega})$, see for instance [10, Theorem 8.31], and $u=\psi$ on $\partial\Omega$ is satisfied in a classical sense.

We consider also the case when a Neumann type condition is imposed on $\partial\Omega$. We assume that $\Sigma$ is not empty and we take $k$ such that $0<k<k_0(\Sigma,\Omega,A,b)$. For any $\eta\in H^{-1/2}(\partial\Omega)$, we have that the following mixed type boundary value problem

$$\begin{cases} \operatorname{div}(A\nabla v)+kbv=0 & \text{in } \Omega\backslash\Sigma\\ v=0 & \text{on } \Sigma\\ A\nabla v\cdot\nu=\eta & \text{on } \partial\Omega \end{cases} \tag{2.28}$$

admits a unique (weak) solution. The weak formulation of (2.28) is to find a function $v\in H^1(\Sigma,\Omega)$ such that

$$\int_{\Omega\backslash\Sigma} A\nabla v\cdot\nabla\phi - kbv\phi = \eta[\phi|_{\partial\Omega}] \quad \text{for any } \phi\in H^1(\Sigma,\Omega).$$

We recall that if $\eta\in L^2(\partial\Omega)$, then $\eta\in H^{-1/2}(\partial\Omega)$ and we have $\eta[\phi]=\int_{\partial\Omega}\eta\phi$ for any $\phi\in H^{1/2}(\partial\Omega)$. Also, if $\eta\in L^2(\partial\Omega)$, then $v$ is continuous up to the boundary of $\Omega$, that is $v\in C(\overline{\Omega})$.

The proof of this continuity result can be sketched as follows. We already know that $v$ is continuous in all $\Omega$, we need only to consider its continuity near $\partial\Omega$. Let us take a smooth simply connected domain $\Omega_1$ which is compactly contained in $\Omega$ and such that $\Sigma\subset\Omega_1$. Then the following Neumann problem

$$\begin{cases} \operatorname{div}(A\nabla\hat{v})=0 & \text{in } \Omega\backslash\overline{\Omega_1}\\ A\nabla\hat{v}\cdot\nu=\eta & \text{on } \partial\Omega\\ A\nabla\hat{v}\cdot\nu=-\frac{1}{|\partial\Omega_1|}\int_{\partial\Omega}\eta & \text{on } \partial\Omega_1 \end{cases} \tag{2.29}$$

admits a solution which is unique up to additive constants. Here $|\partial\Omega_1|$ denotes the length of $\partial\Omega_1$ and the integral $\int_{\partial\Omega}\eta$ is taken with respect to arclength. We have that $\hat{v}$ is continuous on $\overline{\Omega}\backslash\Omega_1$. This may be proved as follows. We study the behaviour of $\hat{v}$ locally near a point of the boundary of $\Omega\backslash\overline{\Omega_1}=\partial\Omega\cup\partial\Omega_1$. Let us fix a point a point $z_0$ in $\partial\Omega$ and let us investigate the continuity of $\hat{v}$ in a neighbourhood of $z_0$. The same kind of reasoning can be applied to points belonging to $\partial\Omega_1$. There is no guarantee that a global single-valued stream function associated to $\hat{v}$ exists in $\Omega\backslash\Omega_1$, since this domain is doubly connected. However, fixed $z_0\in\partial\Omega$, we can find a neighbourhood $U$ of $z_0$ such that $U\cap\Omega$ is a simply connected Lipschitz domain. Therefore, $\hat{w}$, the stream function associated to $\hat{v}$ on $U\cap\Omega$, is well-defined and single-valued and its trace on $\partial\Omega\cap U$ is equal to $\psi$, the antiderivative of $\eta$ along $\partial\Omega$, that is $\psi=\int\eta$ where the indefinite integral is taken with respect to arclength in the counterclockwise direction. We remark that $\psi$ is not necessarily single-valued on the whole $\partial\Omega$ (unless, actually, $\eta$ has zero mean) but it is well-defined on every proper open subset of it, and, furthermore, on any such set is Hölder continuous. Then by standard regularity estimates up to the boundary, we infer that $\hat{w}$ is Hölder continuous on $\overline{U_1 \cap \Omega}$, where we assume $U_1$ to be an open neighbourhood of $z_0$ compactly contained in $U$ and such that $U_1 \cap \Omega$ is a simply connected Lipschitz domain. Through a bi-Lipschitz transformation between $U_1 \cap \Omega$ and $B_1(0)$ and the representation theorem already cited, [6], we have that $\hat{v} + \mathrm{i}\hat{w} = \hat{F} \circ \chi$ where $\hat{F}$ is holomorphic in $B_1(0)$ and $\chi$ is a quasiconformal mapping between $B_1(0)$ and $U_1 \cap \Omega$ such that $\chi$ and its inverse are Hölder continuous up to the boundary. We deduce that $\hat{W} = \Im\hat{F}$ is Hölder continuous up to $\partial B_1(0)$, since $\hat{w}$ is Hölder continuous up to $\partial(U_1 \cap \Omega)$. Therefore, by Privaloff's Theorem, see [5, Part II, Chapter 6, Theorem 5, p. 279], also $\hat{v} = \Re\hat{F}$ is Hölder continuous up to the boundary of $B_1(0)$, and this, in turn, implies that $\hat{v}$ is Hölder continuous up to the boundary of $U_1 \cap \Omega$.

If we take $\Omega_2$ a smooth simply connected domain containing the closure of $\Omega_1$ and contained (compactly) in $\Omega$, then by the already mentioned theorem by N. P. Meyers, [19], and by standard regularity estimates in the interior, we can find a function $h$ such that $h$ is continuous in $\mathbb{R}^2$ with compact support contained in $\Omega\backslash\overline{\Omega_1}$, $h \in W^{1,q}(\Omega)$ for some $q > 2$ and $h = v - \hat{v}$ on $\partial\Omega_2$.

Then we have that $h_1 = v - \hat{v} - h$ satisfies the following boundary value problem

$$\begin{cases} \operatorname{div}(A\nabla h_1) = g + \operatorname{div}(G) & \text{in } \Omega\backslash\overline{\Omega_2} \\ A\nabla h_1 \cdot \nu = 0 & \text{on } \partial\Omega \\ h_1 = 0 & \text{on } \partial\Omega_2 \end{cases} \tag{2.30}$$

where $g \in L^p(\Omega\backslash\overline{\Omega_2})$ for any $p < \infty$ and $G$ belongs to $L^q(\Omega\backslash\overline{\Omega_2})$ for some $q > 2$. A bi-Lipschitz change of variables transforming $\Omega\backslash\overline{\Omega_2}$ onto an annulus does not modify the nature of the elliptic equation, of its right hand side and of its boundary conditions. So, we can assume without loss of generality that the boundary of $\Omega\backslash\overline{\Omega_2}$ is at least $C^1$ and we can apply a theorem by G. Stampacchia, [21], to ensure that $h_1 \in C(\overline{\Omega}\backslash\Omega_2)$. Since $v = h_1 + \hat{v}$ near $\partial\Omega$ and both $h_1$ and $\hat{v}$ are continuous up to $\partial\Omega$, the result is proven.

For solutions to (2.20), under some assumptions on $k$, the geometric index of a zero of $u$ provides information on the behaviour either of the boundary values of $u$ or of its conormal derivative. Roughly speaking, the number of sign changes either of $u|_{\partial\Omega}$ or of the conormal derivative of $u$ on $\partial\Omega$ can be estimated. We summarize this result in the following proposition.

**Proposition 2.2** *Let $u$ be a nonconstant $C(\overline{\Omega}) \cap H^1(\Omega)$ solution to* (2.20)*. Let $z_0 \in \Omega\backslash\Sigma$ be such that $u(z_0) = 0$ and let $n$ be the geometric index of $\nabla u$ at $z_0$.*

*Then the following holds*

*i) if $k < k_0(\Omega\backslash\Sigma, A, b)$, there exist $2(n+1)$ points on $\partial\Omega$, $y_j$, $j = 1, \dots, 2(n+1)$, ordered in clockwise sense along $\partial\Omega$, such that $(-1)^j u(y_j) > 0$ for any $j$, $j = 1, \dots, 2(n+1)$;*

*ii) if $k < k_0(\Sigma, \Omega, A, b)$, then there exist $2(n+1)$ open, connected and pairwise disjoint subarcs of $\partial\Omega$, $\Gamma_j$, $j = 1, \ldots, 2(n+1)$, ordered in clockwise sense, such that $A\nabla u \cdot \nu$ is not allowed to be nonnegative on $\Gamma_j$ for any $j$ even, $j = 1, \ldots, 2(n+1)$, and is not allowed to be nonpositive on $\Gamma_j$ for any $j$ odd, $j = 1, \ldots, 2(n+1)$.*

PROOF. We recall that $n$ is a nonnegative integer and it is positive if and only if $z_0$ is a geometrical critical point for $u$. We apply Proposition 2.1 to $z_0$. For any $j$, $j = 1 \ldots, 2(n+1)$, let $\tilde{D}_j$ be the connected component of $\{z \in \Omega : \ u(z) \neq 0\}$ containing $D_j$. In either cases, since clearly $k_0(\Sigma, \Omega, A, b) \leq k_0(\Omega \backslash \Sigma, A, b)$, we have that $k < k_0(\Omega \backslash \Sigma, A, b) \leq k_0(\tilde{D}_j, A, b)$ and so we can infer that $u|_{\partial \tilde{D}_j}$ is not identically equal to zero, for any $j = 1, \ldots, 2(n+1)$. Obviously $u = 0$ for any point of $\partial\tilde{D}_j \cap \Omega$, therefore there exists a point $y_j \in \partial\tilde{D}_j \cap \partial\Omega$ such that $u(y_j) \neq 0$. The fact that we can connect each $y_j$ with a point belonging to $D_j$ by a continuous curve which, but for its endpoint $y_j$, is contained in $\tilde{D}_j$ and the fact that the sign of $u$ on $D_j$ (and hence also on $\tilde{D}_j$) changes from one $j$ to the next one, allow us to prove that the order of the points $y_j$ on $\partial\Omega$ preserves the one defined by the sets $D_j$ and the sign of $u$ on $y_j$ is also changing from one $j$ to the next one. Here, and in the sequel, we clearly identify $j = 2(n+1)+1$ with $j = 1$. So, we have established part *i*) of the proposition.

For what concerns part *ii*), we consider the following reasoning. For any $j$, $j = 1, \ldots, 2(n+1)$, we define the function $u_j$ in the following way

$$u_j = \begin{cases} u & \text{in } \tilde{D}_j \\ 0 & \text{in } \Omega \backslash \tilde{D}_j \end{cases} .$$

We have that $u_j \in H^1(\Sigma, \Omega)$, therefore by the assumption on $k$ we obtain that $\int_\Omega A\nabla u_j \cdot \nabla u_j - kbu_j^2 > 0$. It is also easy to notice that

$$\int_\Omega A\nabla u_j \cdot \nabla u_j - kbu_j^2 = \int_\Omega A\nabla u \cdot \nabla u_j - kbuu_j > 0.$$

Therefore we deduce that

$$(A\nabla u \cdot \nu)[u_j|_{\partial\Omega}] > 0 \quad \text{for any } j, \ j = 1 \ldots, 2(n+1). \tag{2.31}$$

We have that $u_j$ is a continuous function on $\overline{\Omega}$ and is either nonpositive or nonnegative. By the previous construction, we are able to find $\Gamma_j$, $j = 1, \ldots, 2(n+1)$, open, connected and pairwise disjoint subarcs of $\partial\Omega$ such that $\{z \in \partial\Omega : \ u_j(z) \neq 0\} \subset \Gamma_j$. As before, the arcs $\Gamma_j$ are ordered in a clockwise sense and preserves the order of the sets $D_j$ and $u$ changes sign from one $\Gamma_j$, $j = 1 \ldots, 2(n+1)$, to the adjacent ones. That is, we can assume without loss of generality, that $u_j$ is nonnegative for $j$ even and nonpositive for $j$ odd, $j = 1 \ldots, 2(n+1)$.

By (2.31), we deduce that $A\nabla u \cdot \nu$ can not be nonnegative on $\Gamma_j$ if $u_j$ is nonnegative and $A\nabla u \cdot \nu$ can not be nonpositive on $\Gamma_j$ if $u_j$ is nonpositive. Hence the proof follows immediately. □

## 3 The main uniqueness results

In this section we prove uniqueness results for the determination of multiple sound-soft defects buried into a domain $\Omega$ by a finite number of boundary measurements. We are interested into two cases. In the first one, Dirichlet conditions are assigned on the exterior boundary $\partial\Omega$ and the corresponding Neumann data are measured on an open subarc $\Gamma$ of $\partial\Omega$. In the second case, we prescribe Neumann conditions and we measure the corresponding Dirichlet data on $\Gamma$. A suitable choice of the data to be prescribed on $\partial\Omega$ allows us to uniquely determine the multiple defect.

Let us consider the following framework. Let $\Omega$ be a Lipschitz, bounded and simply connected domain. Let $A$ be a $2\times 2$ measurable symmetric matrix defined in $\Omega$ and satisfying (2.11) and let $b$ be a measurable function in $\Omega$ satisfying (2.12), for some positive constant $\lambda$.

Let $\Sigma$ and $\Sigma'$ be two multiple defects in $\Omega$. We fix also $\Gamma$, an open subarc of $\partial\Omega$.

We prove the following auxiliary proposition.

**Proposition 3.1** *Let $k$ satisfy $0 < k < \min\{k_0(\Omega\backslash\Sigma, A, b), k_0(\Omega\backslash\Sigma', A, b)\}$.*

*Fix $\psi \in C(\overline{\Omega}) \cap H^1(\Omega)$ and let $u$ be the solution to (2.27) and let $u'$ be the solution to the same problem when $\Sigma$ is replaced by $\Sigma'$. Then if $A\nabla u\cdot\nu = A\nabla u'\cdot\nu$ on $\Gamma$ in a weak sense, we have $u = u'$ on $\Omega$.*

*Fixed $\eta \in L^2(\partial\Omega)$, if $v$ solves (2.28) and $v'$ solves the same problem with $\Sigma$ replaced by $\Sigma'$, and $v = v'$ on $\Gamma$, then $v = v'$ on $\Omega$.*

Proof. The proof follows the lines of that of Proposition 3.1 in [2]. We define $G$ as the connected component of $\Omega\backslash(\Sigma \cup \Sigma')$ such that $\partial\Omega$ is contained in $\partial G$. By the unique continuation property we have described in the previous section we immediately obtain that $u = u'$ in $G$ and indeed on $\overline{G}$, given the continuity of the functions involved. The same reasoning allows us to say that $v = v'$ on $\overline{G}$. We need to check that $u = u'$ and $v = v'$ also outside $G$. We shall treat only the first case since the second is completely analogous.

On $\partial G$ we have that $u = u' = 0$ since $\partial G$ is clearly contained in $\Sigma \cup \Sigma'$. Let $D$ be one of the connected components of $(\Omega\backslash\overline{G})\backslash\Sigma$. The boundary of $D$ is composed by points either belonging to $\Sigma$ or to $\partial G$. In either cases, we infer that $u \equiv 0$ on $\partial D$ and this implies, given the assumption on $k$, that $u = 0$ on $D$. Therefore we have that $u = 0$ on $\Omega\backslash\overline{G}$. By the same reasoning we obtain that $u'$ is equal to zero on $\Omega\backslash\overline{G}$ and therefore the two functions coincide everywhere.□

Already this auxiliary global unique continuation property provides us a uniqueness result with a single measurement in some particular cases. In the first one we consider a restriction on the kind of defects considered, namely we assume that the unknown multiple defect coincides with the closure of its interior. With this assumption, one measurement will be enough to determine the multiple defect and no assumption, but nontriviality, is imposed on the prescribed datum. In the second case, we make no restriction on the kind of defects considered, but we restrict the prescribed boundary datum to be, a part from nontrivial, either nonnegative or nonpositive on the whole $\partial\Omega$.

We describe these results in the following theorems.

**Theorem 3.2** *Let* $\Omega$*,* $A$*,* $b$ *and* $\Gamma$ *satisfy the previously described assumptions.*

*Let* $\Sigma$ *and* $\Sigma'$ *be two multiple defects in* $\Omega$ *such that* $\Sigma = \overline{\mathring{\Sigma}}$*, where* $\mathring{\Sigma}$ *denotes the interior part of* $\Sigma$*, and* $\Sigma' = \overline{\mathring{\Sigma}'}$*.*

*Let* $k$ *be such that* $0 < k < \min\{k_0(\Omega\backslash\Sigma, A, b), k_0(\Omega\backslash\Sigma', A, b)\}$*. Let us fix* $\psi \in C(\overline{\Omega}) \cap H^1(\Omega)$ *in such a way that* $\psi|_{\partial\Omega} \not\equiv 0$*. Then if* $u$*, solution to* (2.27)*, and* $u'$*, solution to* (2.27) *with* $\Sigma$ *replaced by* $\Sigma'$*, verify*

$$A\nabla u \cdot \nu = A\nabla u' \cdot \nu \quad \text{on } \Gamma$$

*we have that* $\Sigma = \Sigma'$*.*

*Let* $k$ *satisfy* $0 < k < \min\{k_0(\Sigma, \Omega, A, b), k_0(\Sigma', \Omega, A, b)\}$*. Fixed* $\eta \in L^2(\partial\Omega)$*,* $\eta \neq 0$*, if* $v$*, solution to* (2.28)*, and* $v'$*, solution to* (2.28) *where* $\Sigma$ *is replaced by* $\Sigma'$*, verify*

$$v = v' \quad \text{on } \Gamma,$$

*then* $\Sigma = \Sigma'$*.*

**Theorem 3.3** *Let* $\Omega$*,* $A$*,* $b$ *and* $\Gamma$ *be as before. Let* $\Sigma$ *and* $\Sigma'$ *be two multiple defects in* $\Omega$*.*

*We assume that* $k$ *verify* $0 < k < \min\{k_0(\Omega\backslash\Sigma, A, b), k_0(\Omega\backslash\Sigma', A, b)\}$*. If* $\psi \in C(\overline{\Omega}) \cap H^1(\Omega)$ *is such that* $\psi|_{\partial\Omega} \not\equiv 0$ *and* $\psi|_{\partial\Omega} \geq 0$*, then, letting* $u$ *and* $u'$ *be defined as before, if*

$$A\nabla u \cdot \nu = A\nabla u' \cdot \nu \quad \text{on } \Gamma$$

*we have that* $\Sigma = \Sigma'$*.*

*If* $k$ *is such that* $0 < k < \min\{k_0(\Sigma, \Omega, A, b), k_0(\Sigma', \Omega, A, b)\}$ *and* $\eta \in L^2(\partial\Omega)$*,* $\eta \neq 0$*, is such that* $\eta \geq 0$ *on* $\partial\Omega$*, then, being* $v$ *and* $v'$ *as in the previous theorem,*

$$v = v' \quad \text{on } \Gamma$$

*implies* $\Sigma = \Sigma'$*.*

PROOF OF THEOREM 3.2 AND THEOREM 3.3. By Proposition 3.1 we infer, in both cases, that $u = u'$ in $\Omega$ and $v = v'$ in $\Omega$.

In order to prove Theorem 3.2, we argue by contradiction. Assume that $\Sigma \neq \Sigma'$, then, without loss of generality, we can assume that $\Sigma'\backslash\Sigma \neq \emptyset$ and, by our assumption on $\Sigma$ and $\Sigma'$, $\Sigma'\backslash\Sigma$ actually contains an open ball. On this ball $u = u' = 0$, therefore, by unique continuation, $u = 0$ in $\Omega$ and this would contradict the nontriviality of $\psi$. The same reasoning, applied to $v$ and $v'$, provides a proof for the second part of Theorem 3.2.

Theorem 3.3 can be proved as follows. By Proposition 2.2 and the hypothesis on $\psi$ and $\eta$ respectively, we infer that $u$ and $v$ respectively are always different from zero inside $\Omega\backslash\Sigma$. Therefore, since $u = u'$ in $\Omega$, or in the second case since $v = v'$ in $\Omega$, and $u' = 0$ on $\Sigma'$, or $v' = 0$ on $\Sigma'$ respectively, we deduce that $\Sigma'\backslash\Sigma$ must be empty. Changing the role of $\Sigma$ and $\Sigma'$, we have that also $\Sigma\backslash\Sigma' = \emptyset$ and hence $\Sigma = \Sigma'$. □

Proposition 2.2, which describes the sign changes of the boundary data in terms of the index of nodal geometrical critical points inside the medium, is crucial also for the proof of the other uniqueness result we state in this paper. In this case we make no assumptions on the multiple defect to be determined, but those implicit on the hypothesis on $k$. A single measurement, unless of the particular kind described in Theorem 3.3, would therefore not be enough to uniquely determine the multiple defects, since we are not able to decide whether a part of a nodal line of $u$ is contained in the multiple defect or not. Thus the need to take at least a second measurement. In the following theorem we show that if we prescribe two suitably chosen boundary data, then the corresponding measurements uniquely identify the multiple defect.

**Theorem 3.4** *Let $\Omega$, $A$, $b$ and $\Gamma$ verify the previously described assumptions. Let $\Sigma$ and $\Sigma'$ be two multiple defects in $\Omega$. Let $\gamma_0$, $\gamma_1$ and $\gamma_2$ be three open subarcs of $\partial\Omega$ which are pairwise disjoint.*

*Let $k$ be such that $0 < k < \min\{k_0(\Omega\backslash\Sigma, A, b), k_0(\Omega\backslash\Sigma', A, b)\}$. Let $\psi_1$ and $\psi_2$ be two functions belonging to $C(\overline{\Omega}) \cap H^1(\Omega)$ and such that $\psi_1|_{\partial\Omega}$ and $\psi_2|_{\partial\Omega}$ are nontrivial, they coincide on $\gamma_0$, where both are nonincreasing, they are nondecreasing on $\gamma_1$ and $\gamma_2$ respectively, and are constant elsewhere.*

*Let $u_1$ and $u_2$ be the solutions to (2.27) with $\psi$ replaced by $\psi_1$ and $\psi_2$ respectively and let $u_1'$, $u_2'$ be the solutions to the same boundary value problems where $\Sigma$ is replaced by $\Sigma'$. Then, if*

$$A\nabla u_i \cdot \nu = A\nabla u_i' \cdot \nu \quad \text{on } \Gamma$$

*for any $i = 1, 2$, we have $\Sigma = \Sigma'$.*

*Suppose that $k$ verify $0 < k < \min\{k_0(\Sigma, \Omega, A, b), k_0(\Sigma', \Omega, A, b)\}$. Let $\eta_1$ and $\eta_2$ be two nontrivial $L^2(\partial\Omega)$ functions which are coinciding on $\gamma_0$, where both are nonnegative, are nonpositive on $\gamma_1$ and $\gamma_2$ respectively and are equal to zero elsewhere.*

*We set $v_1$, $v_2$ as the solutions to (2.28) with $\eta$ replaced by $\eta_1$ and $\eta_2$ respectively and $v_1'$, $v_2'$ as the solutions to the same boundary value problems if $\Sigma$ is replaced by $\Sigma'$. Then, if*

$$v_i = v_i' \quad \text{on } \Gamma$$

*for any $i = 1, 2$, we have $\Sigma = \Sigma'$.*

Proof. Let $\alpha$, $\beta$ be any two real numbers such that $\alpha^2 + \beta^1 = 1$. Let $\psi = \alpha\psi_1 + \beta\psi_2$. Then $u = \alpha u_1 + \beta u_2$ solves (2.27) and we have that $u$ has no nodal geometrical critical point in $\Omega\backslash\Sigma$. In fact, we can find two open subarcs of $\partial\Omega$ such that $\psi$ is nonincreasing on the first one, nondecreasing on the second one and constant elsewhere. Therefore, for any value $c$ attained by $\psi$ on $\partial\Omega$, the set $\{z \in \partial\Omega :\ \psi(z) = c\}$ has at most two connected components. Proposition 2.2

implies that if $u$ admits a nodal geometrical critical point, then the set $\{z \in \partial\Omega : u(z) = 0\}$ has at least four connected components and this would contradict the configuration of $\psi$. By an analogous reasoning, we have that also $u' = \alpha u'_1 + \beta u'_2$ has no nodal geometrical critical point in $\Omega\backslash\Sigma'$.

The second part of Proposition 2.2 provides, in an analogous way, that any linear combination of $v_1$ and $v_2$ (and of $v'_1$ and $v'_2$ respectively) with coefficients not both zero, has no nodal geometrical critical point in $\Omega\backslash\Sigma$ ($\Omega\backslash\Sigma'$ respectively).

Then the conclusion follows immediately from this claim, which we prove along the lines of the proof of Theorem 1.1 in [2].

Let $\omega_1$ and $\omega_2$ be two linearly independent functions verifying (2.20) and, at the same time, the same equation (2.20) with $\Sigma$ replaced by $\Sigma'$. Then, if $\Sigma'\backslash\Sigma$ is not empty, we can find two real numbers $\alpha$, $\beta$, with $\alpha^2 + \beta^2 = 1$, so that $\omega = \alpha\omega_1 + \beta\omega_2$ has a nodal geometrical critical point in $\Omega\backslash\Sigma$.

Let $z_0$ belong to $\Sigma'\backslash\Sigma$, then $\omega_i(z_0) = 0$ for any $i = 1, 2$. Let $r > 0$ be such that $B_r(z_0) \subset \Omega\backslash\Sigma$ and let $w$ be the solution to (2.23) in $B_r(z_0)$. We consider the functions $\tilde{\omega}_i = \omega_i/w$, $i = 1, 2$, and their stream functions $\tilde{\theta}_i$, $i = 1, 2$. Without loss of generality we can choose $\tilde{\theta}_i$, $i = 1, 2$, so that $\tilde{\theta}_1(z_0) = \tilde{\theta}_2(z_0) = 0$. We take a sequence of points $\{z_n\}$ in $\Sigma' \cap B_r(z_0)$, such that $z_n$ is different from $z_0$ for every $n$ and $z_n$ converges to $z_0$ as $n$ goes to infinity. We have that $\tilde{\omega}_i(z_n) = 0$ for any $i = 1, 2$ and any $n$. For every $n$ we can choose $\alpha_n$, $\beta_n$ such that $\alpha_n^2 + \beta_n^2 = 1$ and $(\alpha_n\tilde{\theta}_1 + \beta_n\tilde{\theta}_2)(z_n) = 0$. We can suppose, passing to a subsequence, that $\alpha_n$ and $\beta_n$ converges to $\alpha_0$, $\beta_0$ respectively as $n$ goes to infinity. We have that the function $\tilde{f}_n = \alpha_n(\tilde{\omega}_1 + \mathrm{i}\tilde{\theta}_1) + \beta_n(\tilde{\omega}_2 + \mathrm{i}\tilde{\theta}_2)$ has a zero in $z_n$ and in $z_0$. By the continuity property of the index we obtain that $\tilde{f}_0 = \alpha_0(\tilde{\omega}_1 + \mathrm{i}\tilde{\theta}_1) + \beta_0(\tilde{\omega}_2 + \mathrm{i}\tilde{\theta}_2)$ is such that $I(z_0, \tilde{f}_0) \geq 2$ and this implies that $z_0$ is a nodal geometrical critical point for $\omega = \alpha_0\omega_1 + \beta_0\omega_2$. □

We wish to make the following remarks about the assumptions on $k$ we have used in Theorem 3.2, Theorem 3.3 and Theorem 3.4.

The first assumption, that is when we prescribe Dirichlet data, is that $k$ must satisfy $0 < k < \min\{k_0(\Omega\backslash\Sigma, A, b), k_0(\Omega\backslash\Sigma', A, b)\}$. This hypothesis is satisfied for any multiple defect $\Sigma$ (including the empty set) if we assume $k < k_0(\Omega, A, b)$. Therefore, by (2.17), it will be enough to assume $0 < k < \lambda^2 \left(2\pi/|\Omega|\right)^{1/2}$.

The second assumption on $k$, when instead we prescribe Neumann data, consists in assuming $0 < k < \min\{k_0(\Sigma, \Omega, A, b), k_0(\Sigma', \Omega, A, b)\}$. First of all we notice that this condition is stronger than the one considered above and also implies that $\Sigma$ and $\Sigma'$ are not empty. Second, since the value of $k_0(\Sigma, \Omega, A, b)$, as we have seen in Section 2, can be bounded from below by a constant depending on $\lambda$, $\Omega$ and the Sobolev 2-capacity of $\Sigma$ only, see (2.19), any *a priori* information on the unknown multiple defect which allows us to estimate its capacity would provide also an estimate on the value of $k$ which can be used to apply the previously recalled theorems. In particular, let us assume that the unknown multiple defect $\Sigma$ is such that $\delta(\Sigma)$ is greater than or equal to a fixed positive constant $\delta$. Then we have that $\Sigma$ is not empty and verifies, using (2.10),

$$C_{1,2}(\Sigma) \geq \frac{\pi}{40 \log\left(\max\left\{15, 5/\delta(\Sigma)\right\}\right)} \geq \frac{\pi}{40 \log\left(\max\left\{15, 5/\delta\right\}\right)}.$$

Hence there exists a positive constant $\tilde{k}$, depending on $\lambda$, $\Omega$ and $\delta$ only, such that $\tilde{k} \leq k_0(\Sigma, \Omega, A, b)$ for every $\Sigma$ such that $\delta(\Sigma) \geq \delta$. Therefore, if we further assume that $\Sigma$ and $\Sigma'$ satisfy $\delta(\Sigma) \geq \delta$ and $\delta(\Sigma') \geq \delta$ respectively, $\delta$ being a fixed positive constant, then it will be enough to choose $k$ to be positive and less than the corresponding $\tilde{k}$.

We summarize these remarks in the following corollary.

**Corollary 3.5** *The conclusions of Theorem* 3.2*, Theorem* 3.3 *and Theorem* 3.4 *remain valid if we replace, in the Dirichlet case, the assumption that $k$ satisfies* $0 < k < \min\{k_0(\Omega \backslash \Sigma, A, b), k_0(\Omega \backslash \Sigma', A, b)\}$ *with*

$$0 < k < \lambda^2 \left(\frac{2\pi}{|\Omega|}\right)^{1/2}.$$

*In the Neumann case, if we further assume that $\delta(\Sigma) \geq \delta$ and $\delta(\Sigma') \geq \delta$, for a fixed positive constant $\delta$, then there exists a positive constant $C$ depending on $\Omega$ only such that if we replace the assumption that $k$ verifies* $0 < k < \min\{k_0(\Sigma, \Omega, A, b), k_0(\Sigma', \Omega, A, b)\}$ *with*

$$0 < k < \lambda^2 C \left[\log\left(\max\left\{15, \frac{5}{\delta}\right\}\right)\right]^{-1/2}$$

*we have that the conclusions of Theorem* 3.2*, Theorem* 3.3 *and Theorem* 3.4 *still hold.*

# 4 Applications to the inverse scattering case

In this section we describe how to obtain uniqueness results for the determination of sound-soft defects buried in an inhomogeneous and anisotropic medium by inverse scattering techniques, that is by measuring far-field data. We shall show that the far-field data provide us with information on the near field and, in turn, on boundary measurements. Applying the uniqueness results of the previous section, we are therefore able to determine uniquely a multiple sound-soft defect, of the most general type, in an inhomogeneous and anisotropic medium. Thus our results will be similar to the determination of obstacles inside an inhomogeneity described in [15], the main differences being in the fact that we deal with the planar case and that we consider a much wider class of the (unknown) admissible multiple defects and of the (known) inhomogeneities surrounding them.

We begin by investigating the existence and uniqueness of the solution to our forward scattering problem, considering the following framework. We shall essentially use the notations of [8], to which we refer also for a more comprehensive treatment of inverse scattering theory.

Let $A = A(z)$, $z \in \mathbb{R}^2$, be a $2 \times 2$ symmetric matrix whose entries are measurable and satisfying, for some fixed positive constant $\lambda$ and for almost every $z \in \mathbb{R}^2$, the uniform ellipticity condition (2.11). Let $b = b(z)$, $z \in \mathbb{R}^2$, be a measurable function verifying, with the same constant $\lambda$, (2.12) for almost every $z \in \mathbb{R}^2$.

Roughly speaking, we assume that outside a bounded domain the matrix $A$ coincides with the identity matrix and $b$ is identically equal to 1. More precisely we assume that there exists a positive constant $R$ such that

$$\mathrm{supp}(A - Id) \subset B_R(0), \quad \mathrm{supp}(b - 1) \subset B_R(0). \tag{4.1}$$

Let $k$ be a positive constant such that

$$0 < k < \frac{\sqrt{2}\lambda^2}{R}.$$

Finally, let $\Sigma$ be a multiple defect in $B_R(0)$, as defined in Section 2, page 9.

Let the incident field be given by the time-harmonic acoustic plane wave

$$u^i(z, t; d) = \mathrm{e}^{\mathrm{i}\sqrt{k}z \cdot d - \omega t}$$

where $\sqrt{k}$ is the wave number, $d$ is the direction of propagation and $\omega$ is the frequency.

Then the direct scattering problem will be to find the total field $u(z; d)$, $z \in \mathbb{R}^2$, given by the sum of the incident field $u^i(z; d) = \mathrm{e}^{\mathrm{i}\sqrt{k}z \cdot d}$ and of the scattered field $u^s(z; d)$, which is due to the presence of the inhomogeneity and of the sound-soft multiple defect. The total field $u$ satisfies, in a weak sense, the following problem

$$\begin{cases} \mathrm{div}(A\nabla u) + kbu = 0 & \text{in } \mathbb{R}^2 \backslash \Sigma \\ u = 0 & \text{on } \Sigma \end{cases}, \tag{4.2}$$

and verifies the so-called *Sommerfeld radiation condition*

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - \mathrm{i}\sqrt{k} u^s \right) = 0. \tag{4.3}$$

By solving (4.2) in a weak sense we mean that $u$ belongs to $C(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$, verifies

$$\int_{\mathbb{R}^2 \backslash \Sigma} A\nabla u \cdot \nabla \phi - kbu\phi = 0 \quad \text{for any } \phi \in C_0^\infty(\mathbb{R}^2 \backslash \Sigma)$$

and it is identically equal to zero on $\Sigma$. Concerning the Sommerfeld radiation condition, here $r = |z|$ and the limit has to be intended uniformly for all directions $z/|z|$.

From a physical point of view, the condition $u = 0$ on $\Sigma$ models the presence of many sound-soft defects, whereas the Sommerfeld radiation condition characterizes outgoing waves and therefore characterizes the scattered wave.

We notice that outside $B_R(0)$, the Helmholtz type equation $\mathrm{div}(A\nabla u) + kbu$ coincides with the Helmholtz equation $\Delta u + ku = 0$ and we have that this Helmholtz equation is satisfied outside $B_R(0)$ by the incident field $u^i$, the scattered field $u^s$ and the total field $u = u^i + u^s$.

Since in addition the scattered field satisfies the Sommerfeld condition it is said to be radiating and its asymptotic behaviour is the one of an outgoing spherical wave. More precisely

$$u^s(z;d) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{k}|z|}}{\sqrt{|z|}}\left\{u^s_\infty(\hat{z};d) + O\left(\frac{1}{|z|}\right)\right\} \tag{4.4}$$

as $|z|$ goes to $\infty$ uniformly in all directions $\hat{z} = z/|z|$.

The function $u^s_\infty(\cdot;d)$, which is defined on the unit circle, is called the *far-field pattern* of $u^s$. The inverse scattering problem will consist in identifying the multiple defect $\Sigma$ by measuring the far-field pattern of the scattered wave for one or more directions of propagation $d$ of the incident wave.

Existence and uniqueness of the solution to the direct problem (4.2)-(4.3) are an immediate consequence of the following proposition.

**Proposition 4.1** *Let $A$, $b$, $k$ and $\Sigma$ satisfy the previously stated assumptions for some constants $\lambda$, $R$. Let $g$, $f = (f_1, f_2)$ belong to $L^p(B_R(0))$ and let $\psi$ belong to $C(B_R(0)) \cap W^{1,p}(B_R(0)\backslash\Sigma)$ for some $p > 2$.*

*Then there exists a unique (weak) solution $u$ to the problem*

$$\begin{cases} \mathrm{div}(A\nabla u) + kbu = \mathrm{div}(f) + g & \text{in } \mathbb{R}^2\backslash\Sigma \\ u = \psi & \text{on } \partial\Sigma \\ \lim_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u}{\partial r} - \mathrm{i}\sqrt{k}u\right) = 0 & \end{cases} . \tag{4.5}$$

Proof. By a weak solution to (4.5) we mean the following. We look for a function $u$ in $C(\mathbb{R}^2)$ such that $u \in H^1(B_r(0)\backslash\Sigma)$ for every $r > 0$, it verifies

$$\int_{\mathbb{R}^2\backslash\Sigma} A\nabla u \cdot \nabla\phi - kbu\phi = \int_{B_R(0)} f \cdot \nabla\phi - g\phi \quad \text{for any } \phi \in C^\infty_0(\mathbb{R}^2\backslash\Sigma),$$

we have that $u = \psi$, both in the classical and weak sense, on $\partial\Sigma$ and the Sommerfeld condition holds, where the limit has to be intended uniformly for all directions $z/|z|$ and $r = |z|$.

We can always assume, without loss of generality, that the support of $\psi$ is compactly contained in $B_R(0)$ and, possibly by choosing a slightly greater constant $R$ without breaking the assumption on $k$, that also $g$ and $f$ are compactly supported in $B_R(0)$. That is we can suppose that there exists a constant $r$, $0 < r < R$, such that the following conditions, which we shall assume to be satisfied in the sequel of the proof, hold. The multiple defect $\Sigma$, and the supports of $A - Id$, $b - 1$, $\psi$, $g$ and $f$ are all (compactly) contained in $B_r(0)$. Let us denote by $h$ a cut-off function between $B_r(0)$ and $B_R(0)$, that is a function belonging to $C^\infty_0(B_R(0))$ which is identically equal to 1 in a neighbourhood of $\overline{B_r(0)}$.

The uniqueness of the solution relies on the following argument. By linearity, it will be enough to prove that if $\psi$, $g$ and $f$ are zero, then the function $u \equiv 0$ is the unique solution to the problem (4.5). So, let us assume, for the time being, that $u$ solves (4.5) with $\psi = 0$, $g = 0$ and $f = 0$.

We have that

$$\int_{B_R(0)\setminus\Sigma} A\nabla u \cdot \nabla \overline{u} - kb|u|^2 = \int_{\partial B_R(0)} u\frac{\partial \overline{u}}{\partial \nu}.$$

Hence the imaginary part of $\int_{\partial B_R(0)} u\frac{\partial \overline{u}}{\partial \nu} = 0$ and so, see for instance[8], we have that $u \equiv 0$ outside $B_R(0)$. Unique continuation implies that $u$ is identically equal to zero in $\mathbb{R}^2$ and so the uniqueness is established.

For what concerns existence we need to introduce the following notation. Let $\Phi(z, w)$ be the fundamental solution to the Helmholtz equation $\Delta u + ku = 0$, which is defined as

$$\Phi(z, w) = \frac{\mathrm{i}}{4} H_0^{(1)}(\sqrt{k}|z - w|), \quad z \neq w$$

where $H_0^{(1)}$ denotes the *Hankel function* of first kind of order 0, see [8] for details. For any fixed $w \in \mathbb{R}^2$, $\Phi(\cdot, w)$ satisfies the Helmholtz equation in $\mathbb{R}^2\setminus\{w\}$. Given the fundamental solution we define the *acoustic single-layer potential* as follows. For details and its basic properties we refer again to [8].

Given a continuous function $\varphi$ on $\partial B_R(0)$, the function

$$S\varphi(z) = \int_{\partial B_R(0)} \varphi(w)\Phi(z, w)\mathrm{d}s(w), \quad z \in \mathbb{R}^2\setminus\partial B_R(0)$$

is called the single-layer potential with density $\varphi$.

We look for a solution constructed in the following way. We take a Hölder continuous function $\varphi$ on $\partial B_R(0)$, that is $\varphi \in C^\alpha(\partial \underline{B_R(0)})$ for a given $\alpha$, $0 < \alpha < 1$, and we define $v^+(z) = S\varphi(z)$ for every $z \in \mathbb{R}^2\setminus\overline{B_R(0)}$ and $v^-(z) = S\varphi(z)$ for every $z \in B_R(0)$.

We introduce the function $v$ as the (weak) solution to

$$\begin{cases} \mathrm{div}(A\nabla v) + kbv = \mathrm{div}(f) + g & \text{in } B_R(0)\setminus\Sigma \\ v = \psi & \text{on } \partial\Sigma \\ v = v^- & \text{on } \partial B_R(0) \end{cases} . \tag{4.6}$$

Then we look for a density $\varphi$ in such a way that the function $u$ defined as follows

$$u(z) = \begin{cases} v^+(z) & \text{for every } z \in \mathbb{R}^2\setminus B_R(0) \\ v(z) & \text{for every } z \in B_R(0) \end{cases} \tag{4.7}$$

solves the problem (4.5). The properties of the single-layer potential, its regularity and the regularity of the solution to the boundary value problem (4.6) imply that $u$ is a solution to (4.5) provided that the following transmission condition holds

$$\frac{\partial v^+}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial B_R(0). \tag{4.8}$$

We pick the auxiliary function $\tilde{v} = v - (1-h)v^- - \psi$, where $h$ is the cut-off function introduced before, and we rewrite the transmission condition (4.8) in the following form

$$\frac{\partial v^+}{\partial \nu} = \frac{\partial \tilde{v}}{\partial \nu} + \frac{\partial v^-}{\partial \nu}.$$

The behaviour of the single-layer potential on $\partial B_R(0)$ implies that $\varphi$ must verify

$$\varphi = -\frac{\partial \tilde{v}}{\partial \nu}.$$

We have that $\tilde{v}$ belongs to $H_0^1(B_R(0)\backslash\Sigma)$ and solves in $B_R(0)\backslash\Sigma$

$$\operatorname{div}(A\nabla\tilde{v}) + kb\tilde{v} = \operatorname{div}(f) + g - (\Delta(1-h))v^- - \operatorname{div}(A\nabla\psi) - kb\psi.$$

Letting $\mathcal{L}$ be the elliptic operator defined on $H_0^1(B_R(0)\backslash\Sigma)$ such that $\mathcal{L}[u] = \operatorname{div}(A\nabla u) + kbu$ in $B_R(0)\backslash\Sigma$, we have that this operator, by the assumption on $k$, is invertible. If we denote its inverse by $\mathcal{L}^{-1}$, then

$$\tilde{v} = \mathcal{L}^{-1}[\operatorname{div}(f) + g - (\Delta(1-h))v^- - \operatorname{div}(A\nabla\psi) - kb\psi].$$

So, we have that our construction provides a solution if $\varphi$ solves the following equation

$$\varphi + \frac{\partial}{\partial \nu}(\mathcal{L}^{-1}[(\Delta(h-1))S\varphi]) = -\frac{\partial}{\partial \nu}(\mathcal{L}^{-1}[\operatorname{div}(f) + g - \operatorname{div}(A\nabla\psi) - kb\psi]). \quad (4.9)$$

In order to obtain an existence result for our problem (4.5) it is enough to show that (4.9) admits a solution. Standard regularity theory for elliptic equations implies that the right-hand side belongs to $C^\alpha(\partial B_R(0))$ and the operator $K : C^\alpha(\partial B_R(0)) \mapsto C^\alpha(\partial B_R(0))$ defined by

$$K\varphi = \frac{\partial}{\partial \nu}(\mathcal{L}^{-1}[(\Delta(h-1))S\varphi])$$

is compact, therefore, by Fredholm theory for compact operators, it remains to prove that the equation $\varphi + K\varphi = 0$ has a unique solution given by $\varphi = 0$.

We observe that if $\varphi$ solves $\varphi + K\varphi = 0$, then the function $u$ which is constructed as before solves (4.5) with $\psi$, $g$ and $f$ equal to zero. The uniqueness previously obtained implies that $u$ would be identically equal to zero. Therefore the single-layer potential with density $\varphi$ will be identically equal to zero outside $B_R(0)$. From the continuity property of the single-layer potential we deduce that $S\varphi$ is equal to zero on $\partial B_R(0)$ and satisfies the Helmholtz equation in $B_R(0)$. Our assumption on $k$ therefore implies that the single-layer potential with density $\varphi$ is zero everywhere in $\mathbb{R}^2$ and from this it is immediate to deduce that also $\varphi$ must be zero. □

Given existence and uniquess of solutions to the direct problem (4.2)-(4.3), we can define the *far-field operator* $\mathcal{F}$ which associates to $d$, the direction of propagation of the incident field, the far-field pattern $u^s_\infty(\cdot; d)$, that is

$$\mathcal{F}[d] = u^s_\infty(\cdot; d) \quad \text{for every } d \text{ in the unit circle.}$$

It is clear that the far-field operator depends on the frequency (characterized by $k$), on the inhomogeneity (characterized by the coefficients of the Helmholtz type equation $A$ and $b$) and by the multiple defect $\Sigma$. Since we assume that the inhomogeneity and the frequency are given, whereas the defect is unknown, we shall explicitly state the dependence of the far-field operator from $\Sigma$, denoting it with $\mathcal{F}(\Sigma)$. Therefore the inverse scattering problem we shall investigate is of the following kind. We want to determine the shape and the location of $\Sigma$ by suitable information on the operator $\mathcal{F}(\Sigma)$.

First of all we show that the far-field pattern determines the near field, therefore the measurement of the far-field pattern provides us with corresponding boundary measurements on the boundary of $B_R(0)$. Before establishing this relation, let us introduce the operator which represents boundary measurements.

Given $A$, $b$, $k$ and $\Sigma$ satisfying the previously described assumptions, we know that for every $\psi \in H^{1/2}(\partial B_R(0))$ the boundary value problem (2.27) (with $\Omega$ replaced by $B_R(0)$), admits a unique solution $u$. We recall that the conormal derivative of $u$ on $\partial B_R(0)$, $A\nabla u\cdot\nu|_{\partial B_R(0)}$, is a well-defined element of $H^{-1/2}(\partial B_R(0))$. The operator $\Lambda : H^{1/2}(\partial B_R(0)) \mapsto H^{-1/2}(\partial B_R(0))$ such that

$$\Lambda[\psi] = A\nabla u\cdot\nu|_{\partial B_R(0)} \quad \text{for every } \psi \in H^{1/2}(\partial B_R(0))$$

where $u$ solves (2.27), will be called *Dirichlet-to-Neumann map*. Again this operator depends on $A$, $b$, $k$, $\Sigma$ and, obviously, $R$, but since we consider all the data except $\Sigma$ as fixed, we shall explicitly state only this dependence and we shall denote the operator by $\Lambda(\Sigma)$.

**Proposition 4.2** *Let the previously stated assumptions on $A$, $b$ and $k$ be satisfied for some constants $\lambda$ and $R$ and let $\Sigma$ and $\Sigma'$ be two multiple defects in $B_R(0)$. Fixed a direction $d$, let $u$ be the solution to* (4.2)-(4.3) *and $u'$ be the solution to the same problem where $\Sigma$ is replaced by $\Sigma'$. Then, if $\mathcal{F}(\Sigma)[d] = \mathcal{F}(\Sigma')[d]$ we have that $u = u'$ on $\mathbb{R}^2\backslash B_R(0)$ and, denoting $\psi = u|_{\partial B_R(0)} = u'|_{\partial B_R(0)}$, also*

$$\Lambda(\Sigma)[\psi] = \Lambda(\Sigma')[\psi].$$

PROOF. The scattered fields related to $\Sigma$ and $\Sigma'$ respectively satisfy the same Helmholtz equation in $\mathbb{R}^2\backslash\overline{B_R(0)}$ and they also share the same far-field pattern. Therefore they coincide in $\mathbb{R}^2\backslash\overline{B_R(0)}$, see [8]. This, in turn, immediately implies that $u = u'$ in $\mathbb{R}^2\backslash\overline{B_R(0)}$. Unique continuation allows us to prove that $u = u'$ in an open neighbourhood of $\mathbb{R}^2\backslash B_R(0)$ and this concludes the proof of the proposition. □

By using Proposition 3.1, it is immediate to notice that, under the assumptions of Proposition 4.2, $\mathcal{F}(\Sigma)[d] = \mathcal{F}(\Sigma')[d]$ implies also that $u = u'$ everywhere in $\mathbb{R}^2$. Furthermore, we remark that $\psi = u|_{\partial B_R(0)} = u'|_{\partial B_R(0)}$ is a nontrivial function, since otherwise the total field would be identically equal to zero and this would imply that $u^s = -u^i$ which is a contradiction to the Sommerfeld condition. Then as a corollary of Proposition 4.2 we obtain the following uniqueness result for the determination of a multiple obstacle.

**Theorem 4.3** *Let $A$, $b$ and $k$ verify the previously described assumptions with constants $\lambda$, $R$.*

*Let $\Sigma$ and $\Sigma'$ be two multiple defects in $B_R(0)$ such that $\Sigma = \overline{\overset{\circ}{\Sigma}}$ and $\Sigma' = \overline{\overset{\circ}{\Sigma}{}'}$. Then if $\mathcal{F}(\Sigma)[d] = \mathcal{F}(\Sigma')[d]$ for a fixed direction $d$ we have that $\Sigma = \Sigma'$.*

The relation between the far-field operator and the Dirichlet-to-Neumann map is indeed quite stronger and we describe it in the following proposition.

**Proposition 4.4** *Given two positive constants $\lambda$ and $R$, let the previous assumptions on $A$, $b$ and $k$ be verified and let $\Sigma$ and $\Sigma'$ be two multiple defects in $B_R(0)$.*

*Then we have that $\mathcal{F}(\Sigma) = \mathcal{F}(\Sigma')$ if and only if $\Lambda(\Sigma) = \Lambda(\Sigma')$.*

PROOF. We shall follow the lines of the proof of an analogous result contained in [14].

One of the two directions is easy. Namely it is immediate to show that if the Dirichlet-to-Neumann maps are equal so are the far-field operators. In fact, fixed a direction $d$, let $u$ be the solution to the direct scattering problem (4.2)-(4.3) and let $\psi = u|_{\partial B_R(0)}$. The fact that $\Lambda(\Sigma) = \Lambda(\Sigma')$, through unique continuation and the kind of reasoning used in the proof of Proposition 3.1, implies that $u|_{B_R(0)}$, which is a solution in $B_R(0)$ of (2.27) with Dirichlet datum $\psi$, is also a solution of the same boundary value problem when $\Sigma$ is replaced by $\Sigma'$. Therefore, we obtain, in turn, that $u$ solves also (4.2)-(4.3) when $\Sigma$ is replaced by $\Sigma'$. By uniqueness of the solutions to the scattering problems, we have that the solutions to the two scattering problems, with $\Sigma$ and $\Sigma'$ respectively, are equal and therefore also their far-field patterns are the same.

The proof that $\mathcal{F}(\Sigma) = \mathcal{F}(\Sigma')$ implies $\Lambda(\Sigma) = \Lambda(\Sigma')$ relies on a density argument. For any direction of propagation $d$, $|d| = 1$, of the incident field, let $u(\cdot;d)$ be the solution to (4.2)-(4.3). It is clear from our previous results (in particular from Proposition 4.2) that if $\psi = u(\cdot;d)|_{\partial B_R(0)}$, for a fixed direction $d$, then $\Lambda(\Sigma)[\psi] = \Lambda(\Sigma')[\psi]$. Therefore it is enough to prove that the subspace of $H^{1/2}(\partial B_R(0))$ which is generated by the set $\{u(\cdot;d)|_{\partial B_R(0)} : \ |d| = 1\}$, which we shall denote by $\mathrm{span}\{u(\cdot;d)|_{\partial B_R(0)} : \ |d| = 1\}$, is dense in $H^{1/2}(\partial B_R(0))$.

We fix a positive constant $\varepsilon_0$ such that $k$ verifies also $0 < k < (\sqrt{2}\lambda^2)/(R+\varepsilon_0)$ and $\Sigma$ and the supports of $A - Id$ and $b - 1$ are contained in $B_{R-\varepsilon_0}(0)$.

Then we claim that for every $\varepsilon$, $0 < \varepsilon < \varepsilon_0$, if $u$ is a solution to

$$\begin{cases} \mathrm{div}(A\nabla u) + kbu = 0 & \text{in } B_{R+\varepsilon}(0)\backslash\Sigma \\ u = 0 & \text{on } \Sigma \end{cases} \tag{4.10}$$

then there exists a sequence of elements of $\mathrm{span}\{u(\cdot;d) : \ |d| = 1\}$ converging to $u$ in $L^2(B_{R+\varepsilon/2}(0)\backslash\overline{B_{R-\varepsilon/2}(0)})$.

First of all we show that the claim allows us immediately to prove the density result needed. We take $\psi$ a real-analytic function on $\partial B_R(0)$ and we consider the solution $u$ to (2.27) with $\psi$ as boundary datum and $\Omega$ replaced by $B_R(0)$. By the analyticity of the boundary, the boundary datum and the coefficients of the Helmholtz type equation near the boundary of $B_R(0)$, $u$ can be extended on a neighbourhood of $\overline{B_R(0)}$ in such a way that $u$ satisfies $\operatorname{div}(A\nabla u)+kbu=0$ in $B_{R+\varepsilon}(0)\backslash\Sigma$ for a positive $\varepsilon$. Therefore, in a neighbourhood of $\partial B_R(0)$ we can approximate $u$ in the $L_2$ norm by a sequence of elements of $\operatorname{span}\{u(\cdot;d):\ |d|=1\}$. By standard regularity estimates in the interior for elliptic equations, we can find a, possibly smaller, neighbourhood of $\partial B_R(0)$ where $u$ can be approximated by a sequence of elements of $\operatorname{span}\{u(\cdot;d):\ |d|=1\}$ in the $H^1$ norm. Therefore every real-analytic function on $\partial B_R(0)$ can be approximated, in $H^{1/2}(\partial B_R(0))$, by elements of the set $\operatorname{span}\{u(\cdot;d)|_{\partial B_R(0)}:\ |d|=1\}$ which, consequently, is dense in $H^{1/2}(\partial B_R(0))$.

It remains to prove our claim and we shall proceed as in the proof of Lemma 6.1.6 in [14]. We argue by contradiction. If the claim is false, by the Hahn-Banach theorem, we can find a function $g\in L^2(B_{R+\varepsilon/2}(0)\backslash\overline{B_{R-\varepsilon/2}(0)})$ such that $g$ is orthogonal to every element of $\operatorname{span}\{u(\cdot;d):\ |d|=1\}$ but not to a solution $u$ to the problem (4.10). Let $v$ be the solution to (4.5) with $g$ as above, $f=0$ and $\psi=0$.

Then, if we set $r=R+\varepsilon$, we have that for every $d$, $|d|=1$,

$$0=-\int_{B_r(0)\backslash\Sigma}g\overline{u}(\cdot;d)=\int_{B_r(0)\backslash\Sigma}(A\nabla v\cdot\nabla\overline{u}(\cdot;d)-kbv\overline{u}(\cdot;d))-\int_{\partial B_r(0)}\frac{\partial v}{\partial\nu}\overline{u}(\cdot;d).$$

Since also $u(\cdot;d)$ solves the Helmholtz type equation in $B_r(0)\backslash\Sigma$ we obtain that

$$\int_{\partial B_r(0)}\frac{\partial\overline{u}(\cdot;d)}{\partial\nu}v-\frac{\partial v}{\partial\nu}\overline{u}(\cdot;d)=0.$$

For any $\rho$, $\rho>r$, the functions $v$, $u^i(\cdot;d)$ and $u^s(\cdot;d)$ satisfy the Helmholtz equation in $B_\rho(0)\backslash\overline{B_r(0)}$, and $v$ and $u^s(\cdot;d)$ satisfy also the Sommerfeld condition. Therefore we have that

$$\int_{\partial B_r(0)}\frac{\partial\overline{u^s}(\cdot;d)}{\partial\nu}v-\frac{\partial v}{\partial\nu}\overline{u^s}(\cdot;d)=\int_{\partial B_\rho(0)}\frac{\partial\overline{u^s}(\cdot;d)}{\partial\nu}v-\frac{\partial v}{\partial\nu}\overline{u^s}(\cdot;d)$$

and, since by the Sommerfeld condition we deduce that

$$\lim_{\rho\to\infty}\int_{\partial B_\rho(0)}\frac{\partial\overline{u^s}(\cdot;d)}{\partial\nu}v-\frac{\partial v}{\partial\nu}\overline{u^s}(\cdot;d)=0,$$

we obtain that

$$\int_{\partial B_r(0)}\frac{\partial\overline{u^i}(\cdot;d)}{\partial\nu}v-\frac{\partial v}{\partial\nu}\overline{u^i}(\cdot;d)=0.$$

If we take $v_0$ as the solution in $B_r(0)$ of

$$\begin{cases}\Delta v_0+kv_0=0 & \text{in } B_r(0)\\ v_0=v & \text{on } \partial B_r(0)\end{cases}$$

we immediately have that, since $u^i(\cdot;d)$ solves the same Helmholtz equation in $B_R(0)$,

$$\int_{\partial B_r(0)}\frac{\partial\overline{u^i}(\cdot;d)}{\partial\nu}v_0-\frac{\partial v_0}{\partial\nu}\overline{u^i}(\cdot;d)=0,$$

and hence we deduce that

$$\int_{\partial B_r(0)}\left(\frac{\partial v}{\partial \nu}-\frac{\partial v_0}{\partial \nu}\right)\overline{u^i}(\cdot;d)=0.$$

Since the exponential solutions $\{u^i(\cdot;d):\ |d|=1\}$ are dense in $L^2(\partial B_r(0))$, see [13], we obtain that $v$ and $v_0$ share the same Cauchy data on $\partial B_r(0)$ and this implies that $v$ can be extended to a solution to the Helmholtz equation in the whole plane. By the fact that $v$ satisfies the radiation condition we can deduce that $v$ is identically equal to zero outside $\overline{B_{R+\varepsilon/2}(0)}$. With a reasoning similar to the one used before we obtain that

$$-\int_{B_{R+\varepsilon/2}(0)\backslash\overline{B_{R-\varepsilon/2}(0)}} g\overline{u}=-\int_{B_r(0)\backslash\Sigma} g\overline{u}=\int_{\partial B_r(0)}\frac{\partial \overline{u}}{\partial \nu}v-\frac{\partial v}{\partial \nu}\overline{u}=0,$$

since $v$ is identically equal to zero in a neighbourhood of $\partial B_r(0)$. Therefore $g$ should be orthogonal even to $u$ and this provides the contradiction required. □

Proposition 4.4, with the aid either of Theorem 3.3 or of Theorem 3.4, implies immediately the following uniqueness result for the determination of a multiple defect by scattering data.

**Theorem 4.5** *Fixed two positive constants $\lambda$ and $R$, let the previously stated assumptions on $A$, $b$ and $k$ be satisfied.*

*Then if $\Sigma$ and $\Sigma'$ are two multiple defects in $B_R(0)$ verifying $\mathcal{F}(\Sigma)=\mathcal{F}(\Sigma')$ we have that $\Sigma=\Sigma'$.*

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