

Existence and Partial Regularity of Static Liquid Crystal Configurations

ROBERT HARDT (*) - DAVID KINDERLEHRER (*) - FANG-HUA LIN (**)

0. - Introduction.

The equilibrium configuration of a liquid crystal may be described in terms of its optical axis, a unit vector field n defined on the region Ω in \mathbb{R}^3 occupied by the material (see [E]). For a nematic liquid crystal, the Oseen-Frank free energy density W is given by

$$2W(\nabla n, n) = \kappa_1(\operatorname{div} n)^2 + \kappa_2(n \cdot \operatorname{curl} n)^2 + \kappa_3 |n \times \operatorname{curl} n|^2 + (\kappa_2 + \kappa_4)[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2], \quad (0.1)$$

where the constants $\kappa_1, \kappa_2, \kappa_3,$ and κ_4 are generally assumed to satisfy

$$\kappa_1 > 0, \quad \kappa_2 > 0, \quad \kappa_3 > 0, \quad \kappa_2 \geq |\kappa_4|, \quad \text{and} \quad 2\kappa_1 \geq \kappa_2 + \kappa_4.$$

(Here we will assume only that $\kappa_1, \kappa_2,$ and κ_3 are positive.)

The principal questions we shall discuss are the existence and partial regularity of a vectorfield n with the property that

$$\mathcal{W}(n) = \inf \mathcal{W}(u)$$

where

$$\mathcal{W}(u) = \int_{\Omega} W(\nabla u, u) dx \quad (0.2)$$

and where the infimum is taken over all $u : \Omega \rightarrow S^2$ having prescribed boundary values n_0 on $\partial\Omega$.

(*) Research partially supported by the National Science Foundation
(**) Research supported by an Alfred P. Sloan Graduate Fellowship

The existence of a minimizer $n \in H^1(\Omega, S^2)$ by direct methods is presented in §1. The first ingredient of the proof is to establish that the class of competing functions is nonempty. The second involves certain coerciveness estimates for the functional \mathcal{W} . Of relevance here is the observation by C. Oseen, and later independently by J. L. Ericksen, that the last term in \mathcal{W} is a surface energy in the sense that it depends only on the restriction of n and its tangential gradient to the boundary $\partial\Omega$. In §1.5, additional terms are added to \mathcal{W} which allow, in particular, treatment of cholesteric liquid crystals and applied magnetic fields.

The partial regularity result we prove in §2 is that a minimizer n of \mathcal{W} is real analytic on $\Omega \setminus Z$ for some relatively closed subset Z of one dimensional Hausdorff measure zero. Continuity of n on all of $\bar{\Omega}$ may be impeded by topological considerations; as for example, if Ω is the unit ball B and n_0 has nonzero degree as a mapping from $\partial B = S^2$ to S^2 . Here the set Z (of singularities) is defined as the set of points a in $\bar{\Omega}$ for which the normalized Dirichlet integral on the ball $B_r(a)$,

$$r^{-1} \int_{\Omega \cap B_r(a)} |\nabla u|^2 dx \quad (0.3)$$

fails to approach zero as $r \rightarrow 0$. Our main work involves establishing that, near points $a \in \Omega \setminus Z$, this normalized integral decays, as $r \rightarrow 0$, like a positive power of r . Local Hölder continuity of n on $\Omega \setminus Z$ then follows by Morrey's lemma, and the higher regularity is established in 2.6.

A few words about the proof of Hölder continuity may be appropriate. In 2.3, an integral estimate is used to reduce the question of energy decay to estimating a normalized L^2 norm, a quantity more readily controlled under the "blowing-up" process. An analogous inequality was employed similarly for excess decay in [HL]. Our proof of this integral estimate involves a construction of R. Schoen and K. Uhlenbeck, [SU, 4.3].

It is interesting to note that, in our proof by contradiction of energy decay, the constraint that $|n| = 1$ implies that the image of any blow-up limit function v lies in a two dimensional plane. (This observation and the use of 2.3 seem to simplify somewhat the regularity theory of harmonic maps as well [SU], [SU₂], [GG].) Moreover, this v satisfies an elliptic

system with constant coefficients, even though, unlike the situation with harmonic maps, the original minimizing problem is not necessarily elliptic.

In the presence of an electric field, one generally assumes a dielectric effect, unlike the common assumption with magnetic fields. As a consequence, the electrostatic energy depends on an unknown electric field potential which competes with the bulk energy \mathcal{W} . In §4 we show, in a typical problem, how one obtains not only the optical axis n but also the electric field potential.

In all of these problems, the set Z defined above will be a compact subset of $\bar{\Omega}$ having one dimensional measure zero. The solution is regular on $\Omega \setminus Z$ (and on $\bar{\Omega} \setminus Z$ for smooth boundary data (§5)). For minimizing harmonic maps from Ω to S^2 (the case $\kappa_1 = \kappa_2 = \kappa_3$, $\kappa_4 = 0$) the work of R. Schoen and K. Uhlenbeck ([SU], [SU₂]), using the monotonicity in r of the normalized Dirichlet integral (0.3), shows that Z is just a finite subset of Ω . Moreover, here the asymptotic behavior of a minimizer near a point of Z is describable [GW] by the work of [S] and the elementary classification of harmonic maps from S^2 to S^2 . However the precise nature of Z is not well understood. Brezis and Coron [BC] have shown that if Ω is convex and n_0 is constant, then any stationary harmonic map with finitely many singularities has none at all. For the liquid crystal problem, E. MacMillan obtained an existence result in [Ma] by assuming the κ_i satisfy certain inequalities to guarantee coercitivity of \mathcal{W} . He also treated regularity of planar solutions.

Our partial regularity results carry over to higher dimensions (where $\mathcal{H}^{\dim \Omega - 2}(Z) = 0$) for mappings between manifolds which minimize, e.g., the elliptic integrands treated in [GG]. This topic is not pursued here.

The authors are pleased to acknowledge many stimulating and fruitful discussions with J. L. Ericksen. We also thank H. Brezis for many interesting conversations. Moreover we benefited from our contact with the participants of the I. M. A. workshop on the Theory and Applications of Liquid Crystals, January, 1985. Many open mathematical questions are described in the Abstracts and Proceedings of this conference.

1. - An Existence Theory.

To simplify technical aspects of our presentation we shall assume some smoothness of the domain and the boundary data. Henceforth, let Ω be a bounded region in \mathbb{R}^3 with smooth boundary $\partial\Omega$ and outward unit normal ν . To demonstrate the existence of a solution by direct methods, we are obliged to show that the Oseen-Frank energy functional (0.1), (0.2) has some lower semi-continuity and coerciveness properties.

To begin, we consider, in $H^1(\Omega, \mathbb{R}^3)$, the closed subset

$$H^1(\Omega, S^2) = \{ u \in H^1(\Omega, \mathbb{R}^3) : |u| = 1 \text{ almost everywhere in } \Omega \}.$$

Note, in particular, that the expression $x/|x|$ defines a function in $H^1(\Omega, S^2)$. This example may be used to illustrate how $H^1(\Omega, S^2)$ may be larger than the strong H^1 closure of smooth functions mapping Ω into S^2 [SU₂, p.267].

1.1. LEMMA. *If $n_0 : \partial\Omega \rightarrow S^2$ is Lipschitz, then the family*

$$\mathcal{A}(n_0) = \{ u \in H^1(\Omega, S^2) : n_0 = \text{trace of } u \text{ on } \partial\Omega \}$$

is nonempty.

PROOF. If $\bar{\Omega}$ has any handles, then we can choose a smooth embedded closed disk $B \subset \bar{\Omega}$ so that $B \cap \partial\Omega = \partial B$, B is orthogonal to $\partial\Omega$, and B crosses transversely at one point some generator of $\Pi_1(\Omega)$. Using two distinct copies B_- and B_+ of B , we may form a new Lipschitz 3 manifold with boundary, $(\Omega \sim B) \cup B_- \cup B_+$, which has one less handle than $\bar{\Omega}$. Since $n_0|_{\partial B}$ is null-homotopic in S^2 we may extend n_0 to B_- and B_+ . Continuing, we eventually reduce to the case where $\bar{\Omega}$ is bilipschitz homeomorphic to a closed ball. Then an elementary calculation shows that homogeneous degree 0 extension gives the desired finite energy extension. \square

This is only a special case of a general result of B. White [W] on the existence of finite energy extensions. It is also interesting that the strong H^1 closure of the continuous functions in $\mathcal{A}(n_0)$ may be strictly smaller than $\mathcal{A}(n_0)$ even when n_0 has degree 0 [HL₂].

In most discussions of the liquid crystal equations, the last term in W is set to zero, namely $\kappa_4 = -\kappa_2$. This is because it is (in a formal sense) a divergence,

$$\text{tr}(\nabla u)^2 - (\text{div } u)^2 = \text{div}[(\nabla u)u - (\text{div } u)u] ,$$

and thus does not contribute to the equilibrium equations. It is sometimes called a surface energy. As Oseen, and independently, Ericksen [E p.238], cf. also [E₂], have observed, it is a surface energy in the strictest sense, for the expression $[(\nabla u)u - (\text{div } u)u] \cdot \nu$ depends only on $u|_{\partial\Omega}$ and its tangential derivatives.

1.2. LEMMA. *For any Lipschitz function $n_0 : \partial\Omega \rightarrow S^2$, there is a number $\mathfrak{L}(n_0)$ such that*

$$\mathfrak{L}(n_0) = \frac{1}{2} \int_{\Omega} [\text{tr}(\nabla u)^2 - (\text{div } u)^2] dx \quad \text{for all } u \in \mathcal{A}(n_0) .$$

PROOF. First suppose $u \in \mathcal{A}(n_0) \cap C^1(\bar{\Omega})$. On $\partial\Omega$ we define

$$\nabla_{\text{tan}} u = \nabla u - (\nabla u) \nu \otimes \nu ,$$

and easily verify that $\nabla_{\text{tan}} u$ depends only on n_0 . Noting that

$$\begin{aligned} & [(\nabla u)u - (\text{div } u)u] \cdot \nu - [(\nabla_{\text{tan}} u)u - \text{tr}(\nabla_{\text{tan}} u)u] \cdot \nu \\ &= [(\nabla u)(\nu \otimes \nu)u - \text{tr}(\nabla u \nu \otimes \nu)u] \cdot \nu \\ &= \sum_{i,j,k} u^i \frac{\nu^j \nu^k}{x_j} u^k \nu^i - u^i \frac{\nu^j \nu^i}{x_j} u^k \nu^k = 0 , \end{aligned}$$

we deduce from the divergence theorem that

$$\begin{aligned} 2\mathfrak{L}(n_0) &= \int_{\partial\Omega} [(\nabla u)u - (\text{div } u)u] \cdot \nu \, d\mathcal{H}^2 \\ &= \int_{\partial\Omega} [(\nabla_{\text{tan}} u)u - \text{tr}(\nabla_{\text{tan}} u)u] \cdot \nu \, d\mathcal{H}^2 , \end{aligned}$$

which depends only on $u|_{\partial\Omega} = n_0$.

For an arbitrary $u \in \mathcal{A}(n_0)$, one may check by the standard device of straightening $\partial\Omega$ locally and applying Fourier transforms, that the latter expression for $\mathfrak{L}(n_0)$ is well-defined and depends only on n_0 . \square

For a given choice of positive constants κ_1, κ_2 , and κ_3 , let

$$\alpha = \min\{\kappa_1, \kappa_2, \kappa_3\}, \quad \beta = 3(\kappa_1 + \kappa_2 + \kappa_3),$$

$$\tilde{\mathcal{W}}(u) = \int_{\Omega} \tilde{W}(\nabla u, u) dx$$

where

$$\begin{aligned} 2\tilde{W}(\nabla u, u) &= 2W(\nabla u, u) + (\alpha - \kappa_2 - \kappa_4)[\text{tr}(\nabla u)^2 - (\text{div } u)^2] \\ &= \kappa_1(\text{div } u)^2 + \kappa_2(u \cdot \text{curl } u)^2 + \kappa_3|u \times \text{curl } u|^2 \\ &\quad + \alpha[\text{tr}(\nabla u)^2 - (\text{div } u)^2] \quad \text{for } u \in H^1(\Omega, S^2). \end{aligned}$$

1.3 COROLLARY. For any Lipschitz function $n_0 : \partial\Omega \rightarrow S^2$ and $n \in \mathcal{A}(n_0)$,

n minimizes \mathcal{W} in $\mathcal{A}(n_0)$ if and only if n minimizes $\tilde{\mathcal{W}}$ in $\mathcal{A}(n_0)$.

PROOF. For any $u \in \mathcal{A}(n_0)$, one sees from 1.2 that

$$\begin{aligned} \tilde{\mathcal{W}}(u) - \tilde{\mathcal{W}}(n) &= \mathcal{W}(u) + (\alpha - \kappa_2 - \kappa_4)\mathfrak{L}(n_0) - [\mathcal{W}(n) + (\alpha - \kappa_2 - \kappa_4)\mathfrak{L}(n_0)] \\ &= \mathcal{W}(u) - \mathcal{W}(n). \quad \square \end{aligned}$$

The following indicates the reason for the particular choice of the coefficient $\frac{1}{2}(\alpha - \kappa_2 - \kappa_4)$ in the definition of \tilde{W} .

1.4. LEMMA. $\frac{1}{2}\alpha|\nabla u|^2 \leq \tilde{W}(\nabla u, u) \leq \beta|\nabla u|^2$ for $u \in H^1(\Omega, S^2)$.

Moreover, $\tilde{\mathcal{W}}$ is lower semicontinuous with respect to weak convergence in $H^1(\Omega, S^2)$.

PROOF. For $u \in H^1(\Omega, S^3)$, we have the identity

$$|\nabla u|^2 = \text{tr}(\nabla u)^2 + |\text{curl } u|^2.$$

To verify this, note that for any square matrix A ,

$$|A|^2 = \text{tr}(AA^t) = \text{tr}(A^2) + \frac{1}{2}|A - A^t|^2.$$

By the above identity and the definitions of α and β ,

$$\begin{aligned} \alpha|\nabla u|^2 &\leq 2\tilde{W}(\nabla u, u) \\ &= (\kappa_1 - \alpha)(\text{div}u)^2 + \kappa_2(u - \text{curl}u)^2 + \kappa_3|u \times \text{curl}u|^2 + \alpha \text{tr}(\nabla u)^2 \\ &\leq 2\beta|\nabla u|^2. \end{aligned}$$

Letting $\gamma = \min\{\kappa_2, \kappa_3\}$, we may write

$$\begin{aligned} \tilde{W}(\nabla u, u) &= \frac{1}{2}(\kappa_1 - \alpha)(\text{div}u)^2 + \frac{1}{2}(\gamma - \alpha)|\text{curl}u|^2 + \frac{1}{2}(\kappa_2 - \gamma)(u - \text{curl}u)^2 \\ &\quad + \frac{1}{2}(\kappa_3 - \gamma)|u \times \text{curl}u|^2 + \frac{1}{2}\alpha|\nabla u|^2 \quad \text{for } u \in H^1(\Omega, S^2). \end{aligned}$$

Since each term has a non-negative coefficient, the lower semicontinuity of \tilde{W} on $\mathcal{A}(n_0)$ follows from the strong convergence in L^2 of any weakly convergent sequence in H^1 . \square

1.5. THEOREM. *For any Lipschitz function $n_0 : \partial\Omega \rightarrow S^2$, there exists an $n \in \mathcal{A}(n_0)$ such that $\mathcal{W}(n) = \inf_{u \in \mathcal{A}(n_0)} \mathcal{W}(u)$.*

PROOF. Let n_k be a minimizing sequence in $\mathcal{A}(n_0)$ for the functional \mathcal{W} defined above. By 1.4, this sequence has bounded H^1 norm and so possesses a subsequence, weakly convergent to a limit $n \in H^1(\Omega, \mathbb{R}^3)$. From the strong convergence in L^2 and H^1 trace theory, $n \in \mathcal{A}(n_0)$. Finally by the lower semicontinuity in 1.4, $\tilde{\mathcal{W}}(n) = \inf_{u \in \mathcal{A}(n_0)} \mathcal{W}(u)$, and the theorem follows from 1.3. \square

A cholesteric liquid crystal [E, p.246] has an energy density of the form

$$\begin{aligned} 2W_{\text{cholesteric}}(\nabla n, n) &= \kappa_1(\text{div}n)^2 + \kappa_2[(n \cdot \text{curl}n) + \tau]^2 + \kappa_3|n \times \text{curl}n|^2 \\ &\quad + (\kappa_2 + \kappa_4)(\text{tr}(\nabla n)^2 - (\text{div}n)^2) \\ &= 2W(\nabla n, n) + 2\kappa_2\tau(n \cdot \text{curl}n) + \kappa_2\tau^2 \end{aligned}$$

for some real constant τ . Unlike with nematic liquid crystals, a constant vector field is not a minimizer of the cholesteric free energy. When $\kappa_2 + \kappa_4 = 0$, this role is played, for example, by a mapping of the form

$$n(x) = (\cos \tau x_3, \sin \tau x_3, 0) .$$

For a given smooth divergence free vector field H , representing a magnetic field, the contribution to energy may be described by adding to W or $W_{\text{cholesteric}}$ a term of the form

$$F(\nabla u, u) = \sum_{j,k,l} [a_{jkl} u^j x_k u^l + b_{jk} u^j u^k + c_j u^j]$$

where a_{jkl} , b_{jk} , and c_j are bounded (or sufficiently integrable) functions on Ω . In general, let

$$\mathfrak{F}(u) = \int_{\Omega} F(\nabla u, u) dx \text{ for } u \in H^1(\Omega, S^2) .$$

1.6. THEOREM. For F as above and $n_0 : \partial\Omega \rightarrow S^2$ Lipschitz, there exists an $n \in \mathcal{A}(n_0)$ such that $(\mathcal{W} + \mathfrak{F})(n) = \inf_{u \in \mathcal{A}(n_0)} (\mathcal{W} + \mathfrak{F})(u)$.

PROOF. The proof is virtually identical to the previous one. With $\tilde{\mathcal{W}}$ defined as before, note that

$$(\alpha/2) \int_{\Omega} |\nabla u|^2 dx \leq (\tilde{\mathcal{W}} + \mathfrak{F})(u) + \int_{\Omega} |F(\nabla u, u)| dx .$$

By Cauchy's inequality and the L^∞ bound on u , a minimizing sequence for $\mathcal{W} + \mathfrak{F}$ remains bounded in H^1 . Moreover \mathfrak{F} is continuous under H^1 bounded weak convergence. \square

We complete this section with a discussion of the Euler-Lagrange equations satisfied by a stationary point of \mathcal{W} . These will be helpful in studying the blow-up limits of the next section.

Consider a stationary point n of \mathcal{W} in $\mathcal{A}(n_0)$,

$$\delta \int_{\Omega} W(\nabla n, n) dx = 0.$$

To simplify the explanation, we shall proceed formally, introducing a multiplier $-\frac{1}{2}\lambda(|u|^2 - 1)$. Then,

$$\delta \int_{\Omega} \{W(\nabla n, n) - \frac{1}{2}\lambda(|n|^2 - 1)\} dx = 0$$

for $n = n_0$ on $\partial\Omega$, but otherwise unconstrained, or

$$(d/dt) \Big|_{t=0} \int_{\Omega} \{W(\nabla(n+t\zeta), n+t\zeta) - \frac{1}{2}\lambda(|n+t\zeta|^2 - 1)\} dx = 0$$

for $\zeta \in H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty$. Thus

$$\int_{\Omega} \{W_p(\nabla n, n) \cdot \nabla \zeta + W_u(\nabla n, n) \cdot \zeta - \lambda n \cdot \zeta\} dx = 0 \quad (1.1)$$

where λ is an unknown function. The (weak) equation

$$\begin{aligned} -\operatorname{div}(W_p(\nabla n, n)) + W_u(\nabla n, n) &= \lambda n \text{ in } \Omega, \\ n &= n_0 \text{ on } \partial\Omega, \end{aligned}$$

with the unknown multiplier λ is useless to us as it stands. The multiplier may be found by choosing $\zeta = \eta n$ in (1.1) where η is a smooth cut-off function. This gives that

$$\int_{\Omega} \{\eta (W_p(\nabla n, n) \cdot \nabla n + W_u(\nabla n, n) \cdot n - \lambda) + W_p(\nabla n, n) \cdot n \otimes \nabla \eta\} dx = 0.$$

We now write

$$W(\nabla n, n) = \frac{1}{2}\alpha |\nabla n|^2 + V(\nabla n, n),$$

where $\alpha = \min\{\kappa_1, \kappa_2, \kappa_3\}$, so

$$W_p(\nabla n, n) = \alpha \nabla n + V_p(\nabla n, n) .$$

Then, since $|n| = 1$ and $n \nabla n = 0$ a. e., we have that

$$n W_p(\nabla n, n) = n V_p(\nabla n, n) ,$$

hence,

$$W_p(\nabla n, n) \cdot n \otimes \nabla \eta = n W_p(\nabla n, n) \cdot \nabla \eta = n V_p(\nabla n, n) \cdot \nabla \eta .$$

Deleting now the arguments in the symbols $W_p(\nabla n, n)$, $V_p(\nabla n, n)$, $W_u(\nabla n, n)$, and integrating by parts, we find that (in a weak sense)

$$\lambda = -\operatorname{div}(n V_p) + W_p \cdot \nabla n + W_u \cdot n ,$$

hence,

$$-\operatorname{div} W_p + W_u = (-\operatorname{div}(n V_p))n + (W_p \cdot \nabla n)n + (W_u \cdot n)n .$$

It is convenient to compare the first term on the right with $\operatorname{div}(n \otimes n V_p)$,

$$(-\operatorname{div}(n V_p))n = \operatorname{div}(n \otimes n V_p) - \nabla n (n V_p) .$$

We finally arrive at the weak equation

$$-\operatorname{div}(W_p - n \otimes n V_p) + (1 - n \otimes n)W_u - \nabla n (n V_p) - (W_p \cdot \nabla n)n = 0 \text{ in } \Omega . \quad (1.2)$$

This system may be abbreviated by writing

$$-\operatorname{div}(W_p - n \otimes n V_p) + Y(\nabla n, n) = 0 \text{ in } \Omega , \quad (1.3)$$

where

$$|Y(p, u)| \leq c |p|^2 .$$

For example, in the special case $W(\nabla u, u) = \frac{1}{2} |\nabla u|^2$, the equilibrium equation is

$$\Delta u + |\nabla u|^2 u = 0 \text{ in } \Omega .$$

But in general, λ involves the influence of second derivatives, and it is not clear that (1.2) is elliptic at every n .

2. - Partial Regularity of Minimizers.

We first discuss *scaling*. For any n that minimizes \mathcal{W} in the class $\mathcal{A}(n_0)$ and any ball $B_R(a) \subset \Omega$, the formula,

$$n_{R,a}(x) = n(Rx+a) \text{ for } x \in B = B_1(0) ,$$

defines a function in $H^1(B, S^2)$ which is \mathcal{W} minimizing in B with respect to its trace on ∂B . We shall study "small energy" minimizers of \mathcal{W} in $H^1(B, S^2)$. For $u \in H^1(B, S^2)$ and $0 < r < 1$, we define the normalized Dirichlet energy in $B_r = B_r(0)$ by

$$E_r(u) = r^{-1} \int_{B_r} |\nabla u|^2 dx .$$

Note that

$$E_r(u_{R,a}) = (rR)^{-1} \int_{B_{rR}(a)} |\nabla u|^2 dx .$$

Our regularity theorem is based on the behavior of blow-up sequences, obtained through rotation and scaling, and a "hybrid" integral inequality involving the L^2 norms of a minimizer and its gradient.

First we describe a few properties of interest concerning *blowing-up*. For any sequence $u_i \in H^1(B, S^2)$, the *associated normalized sequence*

$$v_i = E_1(u_i)^{-1/2} (u_i - \bar{u}_i) \quad \left(\text{where } \bar{u}_i = \int_B u_i dx = |B|^{-1} \int_B u_i dx \right)$$

satisfies $\|v_i\|_{H^1} \leq 1 + c_B$. Here c_B is the best constant for the Poincaré inequality for B ,

$$\int_B |\nabla u|^2 dx \leq c_B \int_B |u - \bar{u}|^2 dx \quad \text{for } u \in H^1(B, \mathbb{R}^3) .$$

A subsequence of v_i converges weakly in $H^1(\mathbf{B})$, strongly in $L^2(\mathbf{B})$, and pointwise almost everywhere to a function in $H^1(\mathbf{B}, \mathbb{S}^2)$. We will say that a sequence $u_i \in H^1(\mathbf{B}, \mathbb{S}^2)$ is a *special blow-up sequence* if

$$\begin{aligned} \bar{u}_i &= (0, 0, \lambda_i) \text{ for some } \lambda_i \geq 0, \text{ and, as } i \rightarrow \infty, \\ \varepsilon_i^2 = E_1(u_i) &\rightarrow 0 \text{ and } v_i = \varepsilon_i^{-1}(u_i - \bar{u}_i) \text{ converges weakly in } H^1(\mathbf{B}). \end{aligned}$$

Then $v = \lim_{i \rightarrow \infty} v_i$ is called the *blow-up limit function* for u_i .

2.1. LEMMA. (blow-up of constraint) *The image of any blow-up limit function v lies essentially in the X-Y plane.*

PROOF. Note that,

$$(1 - |\bar{u}_i|)^2 = \int_{\mathbf{B}} (1 - |\bar{u}_i|)^2 dx \leq \int_{\mathbf{B}} |u_i - \bar{u}_i|^2 dx \leq c_{\mathbf{B}} \varepsilon_i^2.$$

by the Poincaré inequality. Thus, as $i \rightarrow \infty$,

$$|1 - |\bar{u}_i|| \leq c_{\mathbf{B}}^{1/2} \varepsilon_i \rightarrow 0,$$

and for a subsequence of $\{i\}$,

$$\varepsilon_i^{-1}(1 - |\bar{u}_i|) \rightarrow d \geq 0 \text{ and } \bar{u}_i \rightarrow \mathbf{e} = (0, 0, 1).$$

Observing that, almost everywhere,

$$\begin{aligned} |v_i|^2 + 2\varepsilon_i^{-1}v_i \cdot \bar{u}_i + \varepsilon_i^{-2}|\bar{u}_i|^2 &= |v_i + \varepsilon_i^{-1}\bar{u}_i|^2 = |\varepsilon_i^{-1}u_i|^2 = \varepsilon_i^{-2}, \\ \varepsilon_i|v_i|^2 + 2v_i \cdot \bar{u}_i &= \varepsilon_i^{-1}(1 - |\bar{u}_i|^2) = (1 + |\bar{u}_i|)\varepsilon_i^{-1}(1 - |\bar{u}_i|), \end{aligned}$$

we pass to the limit as $i \rightarrow \infty$, strongly in L^1 , to conclude that

$$0 + 2v \cdot \mathbf{e} = 2d \text{ almost everywhere in } \mathbf{B}.$$

Moreover, $d = \int_{\mathbf{B}} v \cdot \mathbf{e} dx = \lim_{i \rightarrow \infty} \int_{\mathbf{B}} \bar{v}_i \cdot \mathbf{e} = 0$. \square

2.2. LEMMA. (blow-up equation) *For any blow-up limit $v = (v^1, v^2, 0)$ of a special blow-up sequence of \mathcal{W} minimizers, $v = (v^1, v^2)$, is a solution of the constant coefficient elliptic system*

$$-\operatorname{div} \tilde{W}_p'(\nabla v, \mathbf{e}) = 0 \quad \text{in } \mathbf{B} ,$$

where \tilde{W}_p' denotes the first two rows of the matrix \tilde{W}_p . In particular, there is a positive constant c_0 (depending only on $\kappa_1, \kappa_2,$ and κ_3) such that

$$\int_{\mathbf{B}_r} |v|^2 dx \leq c_0 r^2 \int_{\mathbf{B}} |v|^2 dx \quad \text{for } 0 \leq r \leq 1 . \quad (2.1)$$

PROOF. In view of 1.3 and equation (1.3),

$$\int_{\mathbf{B}} \{ [\tilde{W}_p(\nabla u_i, u_i) - u_i \otimes u_i \tilde{V}_p(\nabla u_i, u_i)] \cdot \nabla \zeta + \tilde{Y}(\nabla u_i, u_i) \cdot \zeta \} dx = 0$$

for any $\zeta \in H^1_0(\mathbf{B}, \mathbb{R}^3) \cap L^\infty$. Substituting $\nabla u_i = \varepsilon_i \nabla v_i$, dividing by ε_i , and letting $i \rightarrow \infty$, we obtain

$$\int_{\mathbf{B}} [\tilde{W}_p(\nabla v, \mathbf{e}) - \mathbf{e} \otimes \mathbf{e} \tilde{V}_p(\nabla v, \mathbf{e})] \cdot \nabla \zeta dx = 0$$

because $\tilde{Y}_n(\nabla u_i, u_i)$ is quadratic in ∇u_i ,

$$\begin{aligned} u_i &\rightarrow \mathbf{e} \text{ strongly in } L^2, \quad |u_i| = 1 \text{ almost everywhere,} \\ \nabla v_i &\rightarrow \nabla v \text{ weakly in } L^2, \quad \text{and } \sup_i \|\nabla v_i\|_{L^2} < \infty . \end{aligned}$$

Moreover, by choosing ζ with $\zeta \cdot \mathbf{e} = 0$ (i.e., $\zeta = (\zeta^1, \zeta^2, 0)$), the equation simplifies to

$$\int_{\mathbf{B}} \tilde{W}_p(\nabla v, \mathbf{e}) \cdot \zeta dx = 0$$

because

$$(\mathbf{e} \otimes \mathbf{e}) \tilde{V}_p(\nabla v, \mathbf{e}) \cdot \nabla \zeta = [\tilde{V}_p(\nabla v, \mathbf{e}) - \mathbf{e} \otimes \mathbf{e}] \nabla \zeta = [\tilde{V}_p(\nabla v, \mathbf{e}) - \nabla(\mathbf{e} \cdot \zeta)] \mathbf{e} = 0 .$$

Thus $v = (v_1, v_2)$ is a weak solution of the system of two equations in two unknowns

$$- \operatorname{div} \tilde{W}_p(\nabla v, e) = 0 .$$

This is elliptic because, writing

$$\tilde{W}_{p_{ij}}(\xi, e) = \sum_{h,k} A_{ijhk} \xi_{hk} ,$$

we have the inequality

$$\tilde{W}_p(\xi, e) \cdot \xi \geq \alpha |\xi|^2 \text{ for all } \xi = (\xi_{ij}) ,$$

which implies, in the special case of ξ with vanishing third column, that

$$\sum_{i,k=1,2} \sum_{j,k=1,2,3} A_{hkij} \xi_{hk} \xi_{ij} \geq \alpha |\xi|^2 .$$

Finally since $\bar{v} = 0$, the L^2 estimate follows from standard linear elliptic theory (see e.g. [F, 5.2.5]). \square

2.3. LEMMA. (Hybrid inequality) *There exist positive constants c and q (depending only on κ_1, κ_2 , and κ_3) so that if $0 < \lambda < 1$ and if u is a minimizer of \mathcal{W} in $H^1(\mathbf{B}, \mathbf{S}^2)$ with $E_1(u) \leq c^{-1} \lambda^{q/2}$, then*

$$E_{1/2}(u) \leq \lambda E_1(u) + c \lambda^{-q} \int_{\mathbf{B}} |u - \bar{u}|^2 dx .$$

PROOF. According to Lemma 4.3 of [SU], there are positive numbers δ, q , and c_1 so that if $\varepsilon > 0$, $0 < r < 1$, $\xi \in \mathbf{S}^2$, and $\omega \in H^1(\partial \mathbf{B}_r, \mathbf{S}^2)$ satisfy the small oscillation condition

$$r^{-2} \left[\int_{\partial \mathbf{B}_r} |\nabla_{\tan} \omega|^2 d\mathcal{H}^2 \right] \left[\int_{\partial \mathbf{B}_r} |\omega - \xi|^2 d\mathcal{H}^2 \right] \leq \delta^2 \varepsilon^q ,$$

then there exists a function $w \in H^1(\mathbf{B}_r, \mathbf{S}^2)$ with $w|_{\partial \mathbf{B}_r} = \omega$ satisfying

$$\int_{B_r} |\nabla w|^2 dx \leq \varepsilon \int_{\partial B_r} |\nabla_{\tan} \omega|^2 d\mathcal{H}^2 + c_1 \varepsilon^{-q} \int_{\partial B_r} |\omega - \xi|^2 d\mathcal{H}^2 .$$

For the reader's convenience we mention some slight corrections that are needed in the proof of [SU, 4.3]. There should be an additional term $|u(\theta_0) - u^*|^2$ on the right hand sides of the inequalities on line 6 of page 323 in case $u^* \notin \text{image of } u$. Taking θ_0 as a minimum point for $|u - u^*|^2$, one may readily bound this term by $(4\pi)^{-1} W(u)$. One may check that, in either of the two cases $E(u) \leq \delta$ or $W(u) \leq \delta$, the quantity $\max |u - u^*|$ (not $\min |u - u^*|$) is small, and then complete the proof of [SU, 4.2] as given. Note also that in the statement of [SU, 4.4], α should replace the first σ on line 9 of page 324.

Now to apply this to our situation, first note that, for an increasing function η on $[0,1]$,

$$\{s : \eta'(s) \geq 8[\eta(1) - \eta(0)]\}$$

has Lebesgue measure $\leq 1/8$. Thus, there is an $r \in [1/2, 1]$ such that $\omega = u|_{\partial B_r} \in H^1(\partial B_r, S^2)$,

$$\int_{\partial B_r} |\nabla_{\tan} \omega|^2 d\mathcal{H}^2 \leq 8 \int_B |\nabla u|^2 dx \quad \text{and} \quad \int_{\partial B_r} |\omega - \bar{u}|^2 d\mathcal{H}^2 \leq 8 \int_B |u - \bar{u}|^2 dx .$$

Using these inequalities along with Poincaré's inequality, we will satisfy the desired small oscillation condition by taking

$$\xi = \bar{u}, \quad \varepsilon = \mu\lambda, \quad \text{and} \quad c \geq 8\delta^{-1}\mu^{-q}c_B^{1/2}$$

for any positive μ .

Next we use 1.4 and, by 1.3, the \tilde{W} minimality of $u|_{B_r}$ to infer that

$$E_{1/2}(u) \leq 2E_r(u) \leq 4\alpha^{-1}\tilde{W}(u|_{B_r}) \leq 4\alpha^{-1}\tilde{W}(w|_{B_r}) \leq 4\beta\alpha^{-1} \int_{B_r} |\nabla w|^2 dx .$$

We conclude that

$$\begin{aligned}
E_{1/2}(u) &\leq 4\beta\alpha^{-1} \left[\varepsilon \int_{\partial B_r} |\nabla u|^2 d\mathcal{H}^2 + c_1 \varepsilon^{-q} \int_{\partial B_r} |u - \bar{u}|^2 d\mathcal{H}^2 \right] \\
&\leq 32\beta\alpha^{-1} \varepsilon \int_B |\nabla u|^2 dx + 32\beta\alpha^{-1} c_1 \varepsilon^{-q} \int_B |u - \bar{u}|^2 dx,
\end{aligned}$$

and the lemma is proven by taking $\mu = \alpha/(32\beta)$ and further requiring that

$$c \geq (32\beta/\alpha)^{q+1} c_1. \quad \square$$

2.4. THEOREM. (Energy improvement) *There are positive constants ε and $\theta < 1$ (depending only on $\kappa_1, \kappa_2,$ and κ_3) so that if u is a minimizer of \mathcal{W} in $H^1(B, S^2)$ with $E_1(u) < \varepsilon^2$, then*

$$E_\theta(u) \leq \theta E_1(u).$$

PROOF. Were the theorem false, there would be, for each θ with $0 < \theta < 1$, a sequence u_1, u_2, \dots , of \mathcal{W} minimizers so that

$$\varepsilon_i^2 = E_1(u_i) \rightarrow 0 \text{ as } i \rightarrow \infty \text{ and } E_\theta(u_i) > \theta \varepsilon_i^2 \text{ for each } i.$$

Passing to a subsequence, we may assume that $v_i = \varepsilon_i^{-1}(u_i - \bar{u}_i)$ converges weakly in H^1 to a function $v \in H^1(B, \mathbb{R}^3)$. Moreover, by choosing rotations Q_i of \mathbb{R}^3 so that the vectors $\overline{Q_i u_i}$ are proportional to \mathbf{e} and by replacing u_i by $Q_i u_i$, we see that u_i now becomes a special blow-up sequence (as considered in 2.1 and 2.2).

For $\theta \leq r \leq 1$, and i sufficiently large (depending on θ),

$$\left| \int_{B_r} |v_i|^2 dx - \int_{B_r} |v|^2 dx \right| \leq \int_{B_r} |v_i - v|^2 dx \leq c_0 \theta^2 \leq c_0 r^2.$$

For such i , it follows from (2.1) that

$$\int_{B_r} |u_i - \bar{u}_i|^2 dx \leq \varepsilon_i^2 \int_{B_r} |v_i|^2 dx \leq \varepsilon_i^2 [c_0 r^2 + \int_{B_r} |v|^2 dx] \leq 2c_0 r^2 \varepsilon_i^2 \quad (2.2)$$

whenever $\theta \leq r \leq 1$. Noting that, for such r ,

$$E_r(u_i) \leq r^{-1} E_1(u_i) \leq \theta^{-1} \varepsilon_i^2 ,$$

we see that, for any $0 < \lambda < 1$, and i sufficiently large (depending on θ and λ),

$$E_r(u_i) \leq \theta^{-1} \varepsilon_i^2 \leq c^{-1} \lambda^{q/2} .$$

For such i , we may apply the hybrid inequality 2.3, at $r = \theta$ (i.e. to the normalized function $(u_i)_{2\theta,0}$) to obtain

$$E_\theta(u_i) \leq \lambda E_{2\theta}(u_i) + c\lambda^{-q} \int_{B_{2\theta}} |u_i - \bar{u}_i|^2 dx .$$

Choosing the positive integer $k = k(\theta)$ for which $\frac{1}{2} < 2^k \theta \leq 1$, we iterate $k-1$ more times and apply estimate (2.2) to get

$$\begin{aligned} E_\theta(u_i) &\leq \lambda^k E_{2^k \theta}(u_i) + \sum_{j=1}^k \lambda^{j-1} c \lambda^{-q} \int_{B_{2^j \theta}} |u_i - \bar{u}_i|^2 dx \\ &\leq \dots \leq 2 \lambda^k \varepsilon_i^2 + 2c_0 c \lambda^{-q} \sum_{j=1}^k \lambda^{j-1} (2^j \theta)^2 \varepsilon_i^2 \\ &\leq 2 \left[\lambda^k + (1-4\lambda)^{-1} c_0 c \lambda^{-q-1} \theta^2 \right] \varepsilon_i^2 , \end{aligned}$$

for i sufficiently large (depending on θ and λ). Letting $\lambda = \theta^{3/k}$, we see that $\lambda \leq 1/8$. Finally, since $k \rightarrow \infty$ as $\theta \rightarrow 0$, we may fix $\theta < 1/4$ small enough to guarantee that

$$16c_0 c \theta < (\theta^{3/k})^{q+1} ,$$

and conclude that, for i sufficiently large,

$$E_\theta(u_i) < [2\theta^3 + 2 \cdot 2 \cdot (1/16)\theta] \varepsilon_i^2 \leq (\frac{1}{4}\theta + \frac{1}{4}\theta) \varepsilon_i^2 < \theta \varepsilon_i^2 ,$$

contradicting the choice of u_i . \square

2.5. COROLLARY. (Energy decay) *If $n \in H^1(\Omega, S^2)$ is a minimizer of \mathcal{W} (as in §1), if $B_R(a) \subset \Omega$, and if $\int_{B_R(a)} |\nabla n|^2 dx \leq \epsilon^2 R$, then*

$$\int_{B_r(a)} |\nabla n|^2 dx \leq \theta^{-2} R^{-1} \epsilon^2 r^2 \quad \text{for } 0 \leq r \leq R$$

where ϵ and θ are as in 2.4.

PROOF. Apply 2.4 with $u(x) = n_{R,a}(x) = n(Rx+a)$, then with

$$u(x) = n_{\theta R,a}(x), n_{\theta^2 R,a}(x), n_{\theta^3 R,a}(x), \dots$$

to infer inductively that

$$\begin{aligned} (\theta^k R)^{-1} \int_{B_{\theta^k R}(a)} |\nabla n|^2 dx &= E_1(n_{\theta^k R,a}) = E_\theta(n_{\theta^{k-1} R,a}) \\ &\leq \theta E_1(n_{\theta^k R,a}) \leq \theta \cdot \theta^{k-1} E_1(n_{R,a}) = \theta^k \epsilon^2 \end{aligned}$$

for $k = 1, 2, 3, \dots$. Given $0 < r \leq R$, choose k so that $\theta^{k+1} R < r \leq \theta^k R$ to conclude that

$$\begin{aligned} r^{-1} \int_{B_r(a)} |\nabla n|^2 dx &\leq \theta^{-1} (\theta^k R)^{-1} \int_{B_{\theta^k R}(a)} |\nabla n|^2 dx \\ &\leq \theta^{-1} \theta^k \epsilon^2 \leq \theta^{-2} R^{-1} \epsilon^2 r \quad . \quad \square \end{aligned}$$

2.6. THEOREM. (Interior partial regularity) *If $n \in H^1(\Omega, S^2)$ is a minimizer of \mathcal{W} (as in §1), then n is analytic on $\Omega - Z$ for some relatively closed subset Z of Ω which has one dimensional Hausdorff measure zero.*

PROOF. Let

$$Z = \{ a \in \Omega : \limsup_{r \downarrow 0} r^{-1} \int_{B_r(a)} |\nabla n|^2 dx > 0 \} .$$

Since $\int_{\Omega} |\nabla n|^2 dx < \infty$, Lebesgue's density theorem and an elementary covering argument [F, 2.10.19(3)] show that Z has one dimensional Hausdorff measure zero.

Fix a point $a \in \Omega \sim Z$ and chose $R > 0$ so that $B_{2R}(a) \subset \Omega \sim Z$ and

$$R^{-1} \int_{B_{2R}(a)} |\nabla n|^2 dx \leq \varepsilon^2 .$$

Then for any $b \in B_R(a)$,

$$R^{-1} \int_{B_R(b)} |\nabla n|^2 dx \leq \varepsilon^2 ,$$

and so, by 2.3,

$$\int_{B_r(b)} |\nabla n|^2 dx \leq \theta^{-2} R^{-1} \varepsilon^2 r^2 \quad \text{for } 0 \leq r \leq R .$$

Thus $B_R(a) \subset \Omega \sim Z$. We conclude that Z is relatively closed in Ω , and, by Morrey's Lemma [M, 3.5.2], that

$$n|_{B_R(a)} \in C^{0,1/2}(B_R(a)) .$$

To infer the higher regularity of n near a , we assume that $n(a) = \mathbf{e} = (0,0,1)$ and choose $0 < s < R$ so that $n(B_s(a))$ is contained in $S^2 \cap B_{1/2}(\mathbf{e})$. Writing $n = (n^1, n^2, n^3)$, we see that, in $B_s(a)$, we may make the substitutions

$$n^3 = \sqrt{1 - (n^1)^2 - (n^2)^2}, \quad \nabla n^3 = (-n^1 \nabla n^1 - n^2 \nabla n^2) / \sqrt{1 - (n^1)^2 - (n^2)^2} ,$$

in the weak equation (1.3) with $\zeta = (\zeta^1, \zeta^2, 0)$ to infer that $(n^1, n^2)|_{B_s(a)}$ is a critical point for $\int_{B_s(a)} J(\nabla u, u) dx$ where, for $p = (p^1, p^2) \in (\mathbb{R}^2)^2$ and $u = (u^1, u^2) \in \mathbb{R}^2$,

$$J(p,u) = \tilde{W}[(p^1, p^2, (-u^1 p^1 - u^2 p^2) / \sqrt{1 - (u^1)^2 - (u^2)^2}), (u^1, u^2, \sqrt{1 - (u^1)^2 - (u^2)^2})].$$

Note that J is analytic on $\{(p,u) : |u| < 1/2\}$ and that $J(\cdot, u)$ is a quadratic polynomial for each u . Moreover, by 1.3,

$$J(p,0) = \tilde{W}((p,0),e) \geq 1/2 \alpha |p|^2 \quad \text{for all } p \in (\mathbb{R}^2)^2.$$

Then, for some positive $\delta < 1/2$,

$$J(p,u) \geq 1/4 \alpha |p|^2 \quad \text{on } \{(p,u) : |u| < \delta\}.$$

Choosing, by the continuity of n at a , a positive t so that $n(B_t(a))$ is contained in $B_\delta(e)$, we conclude that $(n^1, n^2)|_{B_t(a)}$ satisfies a strongly elliptic system with analytic coefficients. By [M, 6.7], $(n^1, n^2)|_{B_t(a)}$, and hence $n|_{B_t(a)}$, is analytic. \square

3. - Partial Regularity with the Modified Functional.

Let F and \mathfrak{F} be as in 1.5, and let

$$\|F\| = \sup_{\Omega} \sum_{j,k,l} |a_{jkl}| + \sum_{j,k} |b_{jk}| + \sum_j |c_j|.$$

Our discussion of partial regularity will closely follow §2.

3.1. *Scaling.* If n is a minimizer for the functional $\mathcal{W} + \mathfrak{F}$ in $H^1(\Omega, S^2)$, then, for any ball $B_r(a) \subset \Omega$, the scaled function $n_{r,a}$ (considered in 2.1) now minimizes $\mathcal{W} + \mathfrak{F}_{r,a}$, where the coefficients of the corresponding integrand $F_{r,a}$ are defined, for $x \in \bar{B}$, by the expressions

$$r a_{jkl}(rx+a), \quad r^2 b_{jk}(rx+a), \quad r^2 c_j(rx+a).$$

Note that $\|F_{r,a}\| \leq r \|F\|$. We shall study "small energy" minimizers of $\mathcal{W} + \mathfrak{F}$ in $H^1(B, S^2)$ where $\|F\|$ is small.

3.2. LEMMA. (blow-up equation) *Suppose $v = (v^1, v^2, 0) \in H^1(\mathbf{B}, \mathbf{S}^2)$ is a blow-up limit for a special blow-up sequence u_i (as in 2.1) where each u_i minimizes some functional $\mathcal{W} + \mathfrak{F}_i$ so that the integrand F_i corresponding to \mathfrak{F}_i satisfies*

$$\lim_{i \rightarrow \infty} \mathbb{[F_i]} / \varepsilon_i = 0 .$$

Then, as in 2.2, $v = (v^1, v^2)$ again satisfies the elliptic system

$$- \operatorname{div} \tilde{W}_p'(\nabla v, \mathbf{e}) = 0 \text{ in } \mathbf{B} ,$$

(and hence the L^2 estimate of 2.2).

PROOF. With $\zeta = (\zeta^1, \zeta^2, 0)$ as in 2.2, we again substitute $\nabla u_i = \varepsilon_i \nabla v_i$ in the weak equation for u_i and divide by ε_i . The new terms are

$$\begin{aligned} & \sum_{j,k,l} a_{jkl}^i [(v_i)^j \zeta_i^l + \varepsilon_i^{-1} (u_i)^l (\zeta_i^j)_{x_k}] \\ & + \varepsilon_i^{-1} \sum_{j,k} b_{jk}^i [(u_i)^j \zeta_i^k + (u_i)^k \zeta_i^j] + \varepsilon_i^{-1} \sum_j c_j^i \zeta_i^j \end{aligned}$$

where $\zeta_i = (1 - u_i \otimes u_i) \zeta$, $(\nabla \zeta)_i = (1 - u_i \otimes u_i) \nabla \zeta$, and a_{jkl}^i , b_{jk}^i , and c_j^i are the coefficients of F_i . The integral of these terms approaches 0 as $i \rightarrow \infty$ because of the assumption on $\mathbb{[F_i]}$. \square

One may check that Lemma 3.2 remains true without the hypothesis $\lim_{i \rightarrow \infty} \mathbb{[F_i]} / \varepsilon_i = 0$ in case the F_i all come from a single cholesteric energy function.

3.3. LEMMA. (Hybrid inequality) *There exist positive constants c and q (depending only on κ_1, κ_2 , and κ_3) so that if $0 < \lambda < 1$, F is as in 1.5 with $\Omega = \mathbf{B}$, and u is a minimizer of $\mathcal{W} + \mathfrak{F}$ in $H^1(\mathbf{B}, \mathbf{S}^2)$ with $E_1(u) \leq c^{-1} \lambda^{q/2}$, then*

$$E_{1/2}(u) \leq \lambda E_1(u) + c \lambda^{-q} \int_{\mathbf{B}} |u - \bar{u}|^2 dx + c \lambda^{-q-1} \mathbb{[F]}^2 .$$

PROOF. To verify this, we argue as in 2.2 (with λ replaced by $\lambda/2$) and use again the function w as a comparison function. We now have various additional terms arising from $\mathcal{F}(u)$ and $\mathcal{F}(w)$. To handle these, note that $E_r(w) \leq c\lambda^{-q}E_1(u)$. Thus we may employ the inequality

$$\left| \sum_{j,k,l} a_{jkl} n^j n^k n^l \right| \leq \mu |\nabla n|^2 + (2\mu)^{-1} \|\mathcal{F}\|^2$$

with $n = u$ or $n = w$ and with μ being a suitable multiple of λ^{q+1} to guarantee that the square gradient terms may be absorbed as contributions to $\lambda E_r(u)$. The remaining terms coming from $\mathcal{F}(u)$ and $\mathcal{F}(w)$ all can be bounded by $c\lambda^{-q-1}\|\mathcal{F}\|^2$ because $|u| = |w| = 1$ almost everywhere. \square

3.4. THEOREM. (Energy improvement) *There are positive constants ε , η , and $\theta < 1$ (depending only on κ_1, κ_2 , and κ_3) so that if u is a minimizer of $\mathcal{W} + \mathcal{F}$ in $H^1(B, S^2)$ with $E_1(u) < \varepsilon^2$, then*

$$E_\theta(u) \leq \theta \max\{E_1(u), \eta \|\mathcal{F}\|^2\}$$

PROOF. We argue as in 2.4. If the theorem were false, then, for any fixed $0 < \theta < 1/2$, there would exist F_i as in 1.5 as well as minimizers u_i of $\mathcal{W} + F_i$ for which

$$\varepsilon_i^2 = E_1(u_i) \rightarrow 0 \text{ and } E_\theta(u_i)/\theta\|F_i\|^2 \rightarrow \infty \text{ as } i \rightarrow \infty \text{ while } E_\theta(u_i) > \theta\varepsilon_i^2 \text{ for each } i.$$

In particular,

$$\lim_{i \rightarrow \infty} \|F_i\|/\varepsilon_i = 0$$

because $\varepsilon_i^2 = E_1(u_i) \geq \theta E_\theta(u_i)$. A blow-up limit function v , chosen as in 2.4, now satisfies the conclusions of both 2.1 and 3.2. Using the same L^2 estimate for v and choosing k , θ , and λ as before, we deduce that

$$E_\theta(u_i) \leq \frac{1}{2}\theta\varepsilon_i^2 + c\lambda^{-q-1}\|F_i\|^2 \sum_{j=1}^k \lambda^{j-1}(2j\theta)^2 < \theta\varepsilon_i^2,$$

for i sufficiently large, contradicting the choice of u_i . \square

3.5. COROLLARY. (Energy decay) *If $n \in H^1(\Omega, S^2)$ is a minimizer of $\mathcal{W} + \mathcal{F}$ (as in 1.6), if $B_R(a) \subset \Omega$, and if $\int_{B_R(a)} |\nabla n|^2 dx \leq \varepsilon^2 R$, then*

$$\int_{B_r(a)} |\nabla n|^2 dx \leq \theta^{-2} \max\{\varepsilon^2, \eta \|F\|^2 R^2\} R^{-1} r^2 \quad \text{for } 0 < r < R.$$

PROOF. Recalling from 3.1 the scaling estimate for $\|F\|$, we iterate 3.4 as in 2.5 with ε^2 replaced by $\max\{\varepsilon^2, \eta \|F\|^2 R^2\}$. \square

3.6. THEOREM. (Interior partial regularity) *If the coefficients of F belong to $C^{k, \mu}_{loc}(\Omega)$ where $k \in \{0, 1, \dots, \infty, \omega\}$ and $0 < \mu < 1$, then each minimizer n of $\mathcal{W} + \mathcal{F}$ belongs to $C^{k, \mu}_{loc}(\Omega \setminus Z, S^2)$ for some relatively closed subset Z of Ω which has one dimensional Hausdorff measure zero.*

PROOF. Taking Z as in 2.6, Hölder continuity on $\Omega \setminus Z$ again follows from Morrey's Lemma. The argument for higher regularity using elliptic theory continues to hold in the presence of the additional lower order terms whose coefficients are locally bounded in $C^{k, \mu}$ norm. \square

4. - Electric Fields.

An electric field E impressed on the liquid crystal gives rise to a polarization density

$$P = \varepsilon_{\perp} E + \varepsilon_{\parallel} (E \cdot u)u \quad \text{in } \Omega$$

for some constants ε_{\perp} and ε_{\parallel} . For a positive constant ε_0 , the displacement vector is given by

$$D = \varepsilon_0 E + P \quad \text{in } \Omega,$$

and contributes the term $-\frac{1}{2}D \cdot E$ to the energy of the system. In the absence of free charge and in static equilibrium, Maxwell's equations for D and E are

$$\operatorname{div} D = 0 \quad \text{and} \quad \operatorname{curl} E = 0 .$$

Because of the second equation, it is reasonable to consider an electric potential function ϕ for E , $E = -\nabla\phi$. The field E , or, for our purposes, the potential ϕ , will be regarded as another dependent variable in the problem, unlike the case of a magnetic field, cf. [DeG, p.99] or [HK].

In order to simplify notation and to avoid confusion, we set

$$\alpha_0 = \epsilon_0 + \epsilon_a \quad \text{and} \quad \alpha_a = \epsilon_a .$$

Then

$$\begin{aligned} D &= D(\nabla\phi, n) = -(\alpha_0 \mathbf{1} + \alpha_a n \otimes n) \nabla\phi , \\ A(\nabla\phi, n) &= \frac{1}{2} D \cdot E = -\frac{1}{2} (\alpha_0 \mathbf{1} + \alpha_a n \otimes n) \nabla\phi \cdot \nabla\phi . \end{aligned}$$

Assuming further that $|\alpha_a| < \alpha_0$, we obtain the coerciveness condition

$$A(\xi, u) \geq \lambda |\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^3 ,$$

where λ depends only on α_a and α_0 . The total energy of a virtual director configuration u and field potential ψ is

$$\mathcal{E}^*(u, \psi) = \int_{\Omega} [W(\nabla u, u) - A(\nabla\psi, u)] dx .$$

This functional is not bounded below because the two energies compete. Nevertheless, we can obtain critical points by imposing Gauss's law as a constraint. We may then extend our partial regularity theorem to this case.

As a typical problem, we shall consider given fixed functions

$$n_0 : \partial\Omega \rightarrow \mathbb{S}^2 \quad \text{and} \quad \phi_0 : \partial\Omega \rightarrow \mathbb{R}$$

where n_0 is Lipschitz and $\phi_0 \in H^{1/2}(\partial\Omega)$. (Other boundary values and boundary value problems may be treated.) Let

$$\mathcal{A}^*(n_0) = \mathcal{A}(n_0) \times \{ \psi \in H^1(\Omega) : \psi = \varphi_0 \text{ on } \partial\Omega \} .$$

We wish to find a pair $(n, \varphi) \in \mathcal{A}^*(n_0)$ such that

$$- \delta \mathcal{E}^*(n, \varphi) = 0 .$$

For any $u \in H^1(\Omega, S^2)$, the Dirichlet problem

$$\begin{aligned} - \operatorname{div}(\alpha_0 \mathbf{1} + \alpha_g u \otimes u) \nabla \psi &= 0 \quad \text{in } \Omega , \\ \psi &= \varphi_0 \quad \text{on } \partial\Omega , \end{aligned}$$

has a unique solution which we denote by $\Phi(u)$ (or by $\Phi_{\varphi_0}(u)$ to indicate the dependence on the boundary values φ_0). Thus $\Phi(u)$ is the unique minimizer of

$$\int_{\Omega} A(\nabla \psi, u) dx$$

among $\psi \in H^1(\Omega)$ with $\psi = \varphi_0$ on $\partial\Omega$. Then, if $\tilde{\varphi}$ is some fixed H^1 extension of φ_0 to Ω , we have the obvious estimate

$$\begin{aligned} \lambda \int_{\Omega} |\nabla \Phi(u)|^2 dx &\leq \int_{\Omega} A(\nabla \Phi(u), u) dx \leq \int_{\Omega} A(\nabla \tilde{\varphi}, u) dx \\ &\leq c \int_{\Omega} |\nabla \tilde{\varphi}|^2 dx \leq c(\Omega, \tilde{\varphi}) . \end{aligned} \quad (4.1)$$

For $u \in H^1(\Omega, S^2)$ we now let

$$\mathcal{E}(u) \quad (\text{or } \mathcal{E}_{\varphi_0}(u) \text{ to indicate the dependence on } \varphi_0)$$

denote the energy

$$\mathcal{E}(u) = \mathcal{E}^*(u, \Phi(u)) \quad (\text{i.e. } \mathcal{E}_{\varphi_0}(u) = \mathcal{E}^*(u, \Phi_{\varphi_0}(u))) . \quad (4.2)$$

It is worthwhile noting immediately that the potential contribution $\varphi = \Phi(n)$ will not impede the partial regularity of n (obtained below) because this contribution occurs through $\nabla \varphi$ and, owing to the De Giorgi - Nash theorem,

$$\int_{B_r(a)} |\nabla \phi|^2 dx \leq Cr^{1+\mu} \quad \text{for } a \in \Omega \text{ and } r \text{ sufficiently small.} \quad (4.3)$$

For our existence theory, we shall obviously minimize $\mathcal{E}(u)$ in $\mathcal{Q}(n_0)$, but first let us verify that this gives the correct result.

4.1. THEOREM. *A pair (n, ϕ) is a critical point of $\mathcal{E}^*(u, \psi)$,*

$$\delta \mathcal{E}^*(n, \phi) = 0 \quad \text{on } \mathcal{Q}^*(n_0), \quad (4.4)$$

if and only if $\phi = \phi(n)$ and

$$\delta \mathcal{E}(n) = 0 \quad \text{on } \mathcal{Q}(n_0). \quad (4.5)$$

In this case, (n, ϕ) is a (weak) solution of the field equations (see (1.11))

$$-\operatorname{div}\{W_p - n \otimes n V_p\} + \Upsilon(\nabla n, n) + (1 - n \otimes n) A_{ij}(\nabla \phi, n) = 0, \quad (4.6)$$

$$\operatorname{div} D(\nabla \phi, n) = 0, \quad (4.7)$$

subject to $n = n_0$ and $\phi = \phi_0$ on $\partial\Omega$.

First we prove the following lemma.

4.2. LEMMA. *Let $\gamma \in H^{1/2}(\partial\Omega)$ and let $a(x, t) = (a_{ij}(x, t))$ be a square matrix-valued function on $\Omega \times \mathbb{R}$ which is uniformly positive definite,*

$$a(x, t) \xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{for } (x, t) \in \Omega \times \mathbb{R} \text{ and all } \xi,$$

measurable for $x \in \Omega$, and Lipschitz for $t \in \mathbb{R}$. Let ψ_t and Ψ denote the solutions of the problems

$$\begin{aligned} -\operatorname{div}[a(\cdot, t) \nabla \psi] &= 0 \quad \text{in } \Omega, \\ \psi &= \gamma \quad \text{on } \partial\Omega, \end{aligned} \quad \text{and}$$

$$\begin{aligned} -\operatorname{div}[a(\cdot, 0) \nabla \Psi] &= -\operatorname{div}[(\partial a / \partial t)(\cdot, 0) \nabla \phi_0] \quad \text{in } \Omega, \\ \Psi &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (4.8)$$

Then,

$$[\psi_t - \Psi_0]/t \rightarrow \Psi \quad \text{in } H^1(\Omega) \quad \text{as } t \rightarrow 0.$$

PROOF. For any $\eta \in H^1_0(\Omega)$,

$$\int_{\Omega} [a(x,t) \nabla(\psi_t - \psi_0)] \cdot \nabla \eta \, dx = - \int_{\Omega} [a(x,t) - a(x,0)] \nabla \psi_0 \cdot \nabla \eta \, dx .$$

Choosing $\eta = \psi_t - \psi_0$ and applying Schwarz's inequality, we find that

$$\lambda \int_{\Omega} |\nabla(\psi_t - \psi_0)|^2 \, dx \leq \| \partial a / \partial t \|_{L^\infty} \cdot \| \nabla \psi_0 \|_{L^2} \cdot \| \nabla(\psi_t - \psi_0) \|_{L^2} \cdot |t| .$$

hence,

$$\sup_t \| \nabla[(\psi_t - \psi_0)/t] \|_{L^2} < \infty .$$

We conclude that $(\psi_t - \psi_0)/t \rightarrow \Psi$ weakly in $H^1(\Omega)$ as $t \rightarrow 0$ because any limit of an H^1 weakly convergent subsequence $(\psi_{t_i} - \psi_0)/t_i$ must satisfy (4.8) and hence equal Ψ .

To obtain strong convergence in $H^1(\Omega)$, we note that

$$\begin{aligned} I(t) &= \int_{\Omega} [a(x,t) \nabla(\psi_t - \psi_0 - t\Psi)] \cdot \nabla(\psi_t - \psi_0 - t\Psi) \, dx \\ &= \int_{\Omega} \{ [a(x,t) \nabla(\psi_t - \psi_0)] \cdot \nabla(\psi_t - \psi_0) \\ &\quad - 2t [a(x,t) \nabla(\psi_t - \psi_0)] \cdot \nabla \Psi + t^2 [a(x,t) \nabla \Psi] \cdot \nabla \Psi \} \, dx \\ &= \int_{\Omega} \{ [a(x,t) - a(x,0)] \nabla \psi_0 \cdot \nabla(\psi_t - \psi_0) \\ &\quad - 2t [a(x,t) \nabla(\psi_t - \psi_0)] \cdot \nabla \Psi + t^2 [a(x,t) \nabla \Psi] \cdot \nabla \Psi \} \, dx \end{aligned}$$

We conclude from the weak convergence and (4.8) that, as $t \rightarrow 0$,

$$\begin{aligned} \lambda (\| \nabla[\psi_t - \psi_0]/t - \nabla \Psi \|_{L^2})^2 &\leq t^{-2} I(t) \\ &\rightarrow \int_{\Omega} [(\partial a / \partial t)(x,0) \nabla \psi_0 - 2a(x,0) \nabla \Psi + a(x,0) \nabla \Psi] \cdot \nabla \Psi \, dx = 0 . \quad \square \end{aligned}$$

PROOF OF THEOREM 4.1. The argument for this is standard. One need only account for the constraint $|n| = 1$ and the variation of the field as a function of n , which is the motivation for Lemma 4.2. For example, let

$v \in H^1(\Omega, \mathbb{R}^3) \cap L^\infty$ be given and set

$$n_t = (n + tv)/|n + tv| \quad \text{and} \quad \psi_t = \Phi(n_t) \quad \text{for } |t| < \|v\|_{L^\infty} .$$

Then

$$\begin{aligned} n_0 &= n , & \psi_0 &= \varphi = \Phi(n) , \\ \zeta &= [d/dt]_{t=0} n_t = (1 - n \otimes n)v , & \Psi &= [d/dt]_{t=0} \psi_t \in H^1_0(\Omega) , \end{aligned}$$

$$[d/dt]_{t=0} \mathfrak{E}(n_t) = \int_{\Omega} \{W_p(\nabla n, n) \nabla \zeta + W_u(\nabla n, n) \zeta + A_u(\nabla \varphi, n) \zeta + D(\nabla \varphi, n) \nabla \Psi\} dx$$

So (4.5) implies (4.6). Likewise (4.6) and (4.7) imply (4.5).

Similarly, one shows that (4.4) is equivalent to (4.6) and (4.7). \square

The existence of a critical point of \mathfrak{E}^* on $\mathcal{A}^*(n_0)$ is now a consequence of 4.2 and the following.

4.3. THEOREM. *For n_0 and φ_0 as above, there exists an $n \in \mathcal{A}(n_0)$ such that*

$$\mathfrak{E}(n) = \inf_{u \in \mathcal{A}(n_0)} \mathfrak{E}(u) .$$

PROOF. Let $n_i \in \mathcal{A}(n_0)$ be an \mathfrak{E} minimizing sequence, and set $\varphi_i = \Phi(n_i)$. Using the energy bound (4.1) and arguing as in 1.5 and 1.6, we obtain bounds

$$\begin{aligned} \int_{\Omega} |\nabla \varphi_i|^2 dx &\leq \lambda^{-1} c(\Omega, \varphi_0) , \\ \int_{\Omega} |\nabla n_i|^2 dx &\leq 2\alpha^{-1} [\sup_i \mathfrak{E}(n_i) + \mathfrak{E}(n_0) + c(\Omega, \delta) + c_1 |\Omega|] . \end{aligned}$$

(for some choice of constant c_1 in the cholesteric case). Passing to subsequences, we may suppose that

$$n_i \rightharpoonup n \in \mathcal{A}(n_0) \quad \text{and} \quad \varphi_i \rightharpoonup \varphi \in H^1(\Omega) ,$$

weakly in H^1 , strongly in L^2 , and pointwise almost everywhere. Now, as

before,

$$\int_{\Omega} W(\nabla n, n) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} W(\nabla n_j, n_j) dx .$$

On the other hand, since ϕ_j is the solution of a minimum problem,

$$\int_{\Omega} A(\nabla \phi_j, n_j) dx \leq \int_{\Omega} A(\nabla \phi, n) dx .$$

By the uniform bound, $|n_j| \leq 1$, and Lebesgue's theorem,

$$\lim_{j \rightarrow \infty} \int_{\Omega} A(\nabla \phi, n_j) dx = \int_{\Omega} A(\nabla \phi, n) dx .$$

Thus

$$\limsup_{j \rightarrow \infty} \int_{\Omega} A(\nabla \phi_j, n_j) dx \leq \int_{\Omega} A(\nabla \phi, n) dx ,$$

and

$$\mathfrak{E}(n) = \inf_{u \in \mathcal{A}(n_0)} \mathfrak{E}(u) . \quad \square$$

We now turn to the partial regularity of a critical point (n, ϕ) , obtained by minimizing \mathfrak{E} . To simplify technical aspects, our attention is confined to the nematic case. We first discuss *scaling*. Suppose that $\phi = \Phi(n)$ where n is, as above, a minimizer of \mathfrak{E} in $\mathcal{A}(n_0)$. For any ball $B_r(a) \subset \Omega$, one may consider the scaled functions $\phi_{r,a}$ and $n_{r,a}$, defined by

$$\phi_{r,a}(x) = \phi(rx+a) \quad \text{and} \quad n_{r,a}(x) = n(rx+a) \quad \text{for } x \in B .$$

Then, with respect to their own boundary values on ∂B ,

$$\phi_{r,a} = \Phi(n_{r,a}) \quad \text{and} \quad n_{r,a} \text{ minimizes } \mathfrak{E} .$$

Note that

$$\int_B |\nabla \phi_{r,a}|^2 dx \leq r^{-1} \int_{B_r(a)} |\nabla \phi|^2 dx .$$

We shall obtain estimates for small energy minimizers $u \in H^1(B, S^2)$ of \mathfrak{E}

$$\|\nabla\phi(u)\|^2 = \int_{\mathbf{B}} |\nabla\phi(u)|^2 dx ,$$

as well as $\|\nabla u\|_{L^2}$.

4.4. LEMMA. (blow-up equation) *Suppose $v \in H^1(\mathbf{B}, \mathbf{S}^2)$ is a blow-up limit for a special blow-up sequence u_i where each u_i minimizes some functional $\mathfrak{E}_i = \mathfrak{E}_{\gamma_i}$ (see (4.2)) with $\gamma_i \in H^{1/2}(\partial\mathbf{B})$ and where*

$$\lim_{i \rightarrow \infty} \|\nabla\phi(u_i)\|^2 / \varepsilon_i = 0 .$$

Then, as in 2.2, $v = (v^1, v^2)$ again satisfies the elliptic system

$$- \operatorname{div} \tilde{W}_p'(\nabla v, \mathbf{e}) = 0 \text{ in } \mathbf{B} ,$$

(and hence the L^2 estimate of 2.2).

PROOF. The function u_i satisfies the weak form of equation (4.5) with ϕ replaced by $\phi_i = \Phi(u_i)$. Using a test function $\zeta = (\zeta^1, \zeta^2, 0)$ as in 2.2, we again substitute $\nabla u_i = \varepsilon_i \nabla v_i$ and divide by ε_i . The integral of the new term

$$\varepsilon_i^{-1} A_u(\nabla\phi_i, \bar{u}_i + \varepsilon_i v_i) \cdot \zeta$$

approaches 0 as $i \rightarrow \infty$ because of the assumption on $\|\nabla\phi_i\|^2$. \square

4.5. LEMMA. (Hybrid inequality) *There exist positive constants c and q (depending only on κ_1, κ_2 , and κ_3) so that if $0 < \lambda < 1$ and u is a minimizer of $\mathfrak{E} = \mathfrak{E}_{\gamma}$ in $H^1(\mathbf{B}, \mathbf{S}^2)$ with $\gamma \in H^{1/2}(\partial\mathbf{B})$ and $E_1(u) \leq c^{-1} \lambda^{q/2}$, then*

$$E_{1/2}(u) \leq \lambda E_1(u) + c \lambda^{-q} \int_{\mathbf{B}} |u - \bar{u}|^2 dx + c \|\nabla\phi(u)\|^2 .$$

PROOF. First note that u is, by the argument of 1.3, a minimizer for the corresponding adjusted energy $\tilde{\mathfrak{E}} = \mathfrak{E} - \mathcal{W} + \tilde{\mathcal{W}}$,

$$\mathfrak{E}(u) = \int_{\Omega} [\tilde{W}(\nabla u, u) - A(\nabla \Phi(u), u)] dx .$$

Then choosing r , w , and λ as in 2.2 and changing c suitably, we may use the $\tilde{\mathfrak{E}}$ minimality of $u|_{B_r}$ to infer that

$$\begin{aligned} E_{1/2}(u) &\leq 2E_r(u) \leq 4\alpha^{-1}\tilde{W}(u|_{B_r}) \leq 4\alpha^{-1}\tilde{\mathfrak{E}}(u|_{B_r}) + c\|\nabla\Phi(u)\|^2 \\ &\leq 4\alpha^{-1}\tilde{\mathfrak{E}}(w|_{B_r}) + c\|\nabla\Phi(u)\|^2 \leq 4\beta\alpha^{-1}\int_{B_r} |\nabla w|^2 dx + c\|\Phi(u)\|^2 \\ &\leq \lambda E_1(u) + c\lambda^{-q}\int_B |u - \bar{u}|^2 dx + c\|\nabla\Phi(u)\|^2 . \quad \square \end{aligned}$$

4.6. THEOREM. (Energy improvement) *There are positive constants ε , η , and $\theta < 1$ (depending only on κ_1 , κ_2 , and κ_3) so that if u is a minimizer of $\mathfrak{E} = \mathfrak{E}_\gamma$ with $\gamma \in H^{1/2}(\partial B)$ and $E_1(u) < \varepsilon^2$, then*

$$E_\theta(u) \leq \theta \max\{E_1(u), \eta\|\nabla\Phi(u)\|^2\} .$$

PROOF. We argue just as in 3.4 with $\|F\|^2$ replaced by $\|\nabla\Phi(u)\|^2$. \square

4.7. COROLLARY. (Energy decay) *If n is a minimizer of $\mathfrak{E} = \mathfrak{E}_\gamma$ in $H^1(\Omega, S^2)$ with $\gamma \in H^{1/2}(\partial\Omega)$, if $B_R(a) \subset \Omega$, and if*

$$\int_{B_R(a)} |\nabla n|^2 dx \leq \varepsilon^2 R , \text{ then}$$

$$\int_{B_r(a)} |\nabla n|^2 dx \leq c\theta^{-2} \max\{\varepsilon^2, \eta\|\nabla\Phi(n)\|^2 R^{-1}\} R^{-1} r^{1+\mu} \quad \text{for } 0 < r < R/2 ,$$

for some positive constants c and μ , depending only on the electric field constants ε_0 , ε_\perp , and ε_\parallel .

PROOF. From the inequality (4.3) and scaling, we obtain the estimate

$$r^{-1} \int_{B_r(a)} |\Phi(n)|^2 dx \leq c \|\Phi(n)\|^2 R^{-1} r^\mu \text{ for } 0 < r < R/2.$$

Recalling from 4.3 the scaling estimate for $\|\nabla\Phi(n)\|^2$, we now iterate 4.6 as in 3.5. \square

4.8. THEOREM. (Interior partial regularity) *If n is a minimizer of $\mathcal{E} = \mathcal{E}_{\phi_0}$ in $H^1(\Omega, S^2)$ with $\phi_0 \in H^{1/2}(\partial\Omega)$, then $\psi = \Phi(n)$ is locally Hölder continuous on Ω , and both n and ψ are analytic on $\Omega \sim Z$ for some relatively closed subset Z of Ω which has one dimensional Hausdorff measure zero.*

PROOF. The Hölder continuity of ψ follows from De Giorgi's theorem [D]. Taking Z as in 2.6, the Hölder continuity of n on $\Omega \sim Z$ follows from Morrey's Lemma and 4.7.

To verify the higher regularity of n and ψ near a point $a \in \Omega \sim Z$, we assume, for convenience, that $n(a) = e = (0,0,1)$. Recalling the argument for higher regularity in 2.5 and using the Euler equations for (n, ψ) obtained in 4.2, we readily verify that on a small neighborhood of a , the triple (n^1, n^2, ψ) satisfies a strongly elliptic system with analytic coefficients. Thus n and ψ are analytic near a by [M, 6.7]. \square

5. - Partial Regularity at the Boundary.

Suppose n minimizes \mathcal{W} in the family $\mathcal{A}(n_0)$ as in §1. Since n extends to an H^1 function defined in a neighborhood of $\bar{\Omega}$, the set

$$Y = \{ a \in \partial\Omega : \limsup_{r \downarrow 0} r^{-1} \int_{B_r(a)} |\nabla n|^2 dx > 0 \}$$

has, as in 2.6, one dimensional measure zero. Here, assuming that $a \in \partial\Omega \sim Y$, $k \in \{1, 2, \dots, \infty, \omega\}$, $0 < \mu < 1$, and both $\partial\Omega$ and n_0 are $C^{k, \mu}$ near a , we show that n is $C^{k, \mu}$ near a . Our discussion below, which involves modifying §2, can easily be adapted to handle boundary regularity

for the problem of minimizing $\mathcal{W} + \mathcal{F}$ (§1.5, §3) or for the electric field problem (§4).

5.1. *Scaling.* For any C^1 function $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\psi(0) = 0 = |\nabla\psi(0)|$, let

$$\Omega_\psi = \{(x_1, x_2, x_3) \in \mathbf{B}_2 : x_3 < \psi(x_1, x_2)\}.$$

For Ω , a , and n_0 as above, there exists a positive number R , a rotation $h \in \mathbf{SO}(3)$, and a function $\psi_{R,a} \in C^1(\mathbb{R}^2)$ so that

$$\psi_{R,a}(0) = 0 = |\nabla\psi_{R,a}(0)|, \quad \text{Lip}(\psi_{R,a}) \leq 1,$$

$$\Omega_{\psi_{R,a}} = \{h^{-1}[(y-a)/R] : y \in \mathbf{B}_{2R}(a) \cap \Omega\},$$

and n_0 is C^1 on $\mathbf{B}_{2R}(a) \cap \partial\Omega$. For $0 < r \leq R$, let $\psi_{r,a}(x) = \psi_{R,a}(rx/R)$. Then, for n as above, the expression $n_{r,a}(x) = n(rh(x)+a)$ defines a function in $H^1(\Omega_{\psi_{r,a}}, \mathbf{S}^2)$ which is \mathcal{W} minimizing. Its trace on $\mathbf{B}_2 \cap \partial\Omega_{\psi_{r,a}}$ is given by $(n_0)_{r,a}(x) = n_0(rh(x)+a)$. Note that

$$E_1(n_{r,a}) = r^{-1} \int_{\mathbf{B}_r(a)} |\nabla n|^2 dx,$$

$$\text{Lip}(\psi_{r,a}) = \text{Lip}(\psi_{R,a})R^{-1}r, \quad \text{Lip}((n_0)_{r,a}) = \text{Lip}((n_0)_{R,a})R^{-1}r.$$

We will study the behavior of small energy minimizers of \mathcal{W} whose traces on $\mathbf{B}_2 \cap \partial\Omega_\psi$ have small Lipschitz norms.

To treat *blowing-up* at a boundary point, suppose that, for $i = 1, 2, \dots$,

$\psi_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^1 with $\psi_i(0) = 0 = |\nabla\psi_i(0)|$ and $\text{Lip}(\psi_i) \leq 1$,

u_i belongs to $H^1(\Omega_{\psi_i}, \mathbf{S}^2)$,

$g_i = u_i|_{\mathbf{B}_2 \cap \partial\Omega_{\psi_i}}$ is Lipschitz,

and, as $i \rightarrow \infty$,

$$E_1(u_i) = \int_{\mathbf{B} \cap \Omega_{\psi_i}} |\nabla u_i|^2 dx \rightarrow 0 \quad \text{and} \quad [\text{Lip}(\psi_i) + \text{Lip}(g_i)]^2 / E_1(u_i) \rightarrow 0 .$$

It is convenient to have u_i defined on the whole ball $\mathbf{B} = \mathbf{B}_1$ by letting

$$u_i(x) = -u_i(x_1, x_2, -x_3 + 2\psi_i(x_1, x_2)) \quad \text{for } x \in \mathbf{B} \sim \Omega_{\psi_i} .$$

Then

$$\lim_{i \rightarrow \infty} \varepsilon_i^{-2} \int_{\mathbf{B} \cap \Omega_{\psi_i}} |\nabla u_i|^2 dx = \frac{1}{2} \quad \text{where} \quad \varepsilon_i^2 = \int_{\mathbf{B}} |\nabla u_i|^2 dx .$$

As in §2, we can compose with rotations and then pass to a subsequence to insure that

$$\begin{aligned} \mathbf{e} \cdot \bar{u}_i &= |\bar{u}_i| \quad \text{for all } i \quad (\text{where } \bar{u}_i = \int_{\mathbf{B}} u_i dx) \quad \text{and} \\ v_i &= \varepsilon_i^{-1} (u_i - \bar{u}_i) \Big|_{\mathbf{B}} \quad \text{converges weakly in } H^1 . \end{aligned}$$

Under all these conditions, we say that u_i is a *special boundary blow-up sequence*, and we again call $v = \lim_{i \rightarrow \infty} v_i$ the *blow-up limit*.

5.2. LEMMA. (blow-up equation) *Suppose v is a blow-up limit for a special boundary blow-up sequence u_i of \mathcal{W} minimizers, as above. Then, for almost all $x \in \mathbf{B}$,*

$$v(x) \cdot \mathbf{e} = 0 \quad \text{and} \quad v(x_1, x_2, -x_3) = -v(x_1, x_2, x_3) .$$

Moreover, $v = (v^1, v^2)$ is a solution of the elliptic system

$$-\text{div } \tilde{W}_p'(\nabla v, \mathbf{e}) = 0 \quad \text{on } \{(x_1, x_2, x_3) \in \mathbf{B} : x_3 > 0\} ,$$

and satisfies the L^2 estimate of 2.2.

PROOF. The first conclusion was established in 2.1. To obtain the

second, we note that $\lim_{i \rightarrow \infty} \epsilon_i^{-2} \text{Lip}(\psi_i) = 0$ and use the strong L^1 convergence of v_i to v to verify that

$$\int_{\mathbf{B}} v \cdot \zeta \, dx = \lim_{i \rightarrow \infty} \int_{\mathbf{B}} v_i \cdot \zeta \, dx = 0$$

for any $\zeta \in C_0(\mathbf{B}, \mathbf{R}^3)$ satisfying $\zeta(x_1, x_2, -x_3) = \zeta(x_1, x_2, x_3)$.

Next we note that a function with support in $\mathbf{B} \cap \{x_3 > 0\}$ has, for i sufficiently large, support in $\mathbf{B} \cap \Omega_{\psi_i}$. Using such a function as a variation, we find that, in $\{(x_1, x_2, x_3) \in \mathbf{B} : x_3 > 0\}$, $v' = (v^1, v^2)$ satisfies (in a weak sense) the above system. Moreover, since $v' \in H^1(\mathbf{B}, \mathbf{R}^2)$ and is odd in x_3 , v' has zero trace on $\{(x_1, x_2, x_3) \in \mathbf{B} : x_3 = 0\}$ and so satisfies the L^2 estimate of 2.2 by the linear elliptic boundary estimate [M1, 6.3]. \square

5.3. LEMMA. (Hybrid inequality) *There exist positive constants c and q (depending only on $\kappa_1, \kappa_2,$ and κ_3) so that if $0 < \lambda < 1$, ψ is as in 5.1, $\text{Lip}(\psi) \leq 1$, u is a minimizer of \mathcal{W} in $H^1(\Omega_\psi, \mathbf{S}^2)$, g is the trace of u on $\mathbf{B} \cap \partial\Omega_\psi$, and $\text{Lip}(\psi) + E_1(u) \leq c^{-1} \lambda^{q/2}$, then*

$$E_{1/2}(u) \leq \lambda E_1(u) + c \lambda^{-q} \int_{\Omega_\psi \cap \mathbf{B}} |u - \bar{u}|^2 \, dx + c \lambda^{-q} (\text{Lip } g)^2.$$

(Here $E_r(u) = r^{-1} \int_{\Omega_\psi \cap \mathbf{B}_r} |\nabla u|^2 \, dx$.)

PROOF. First one may find a universal constant Λ so that, for any $r \in [1/2, 1]$ and any $\psi : \mathbf{R}^2 \rightarrow \mathbf{R}$ with $\text{Lip}(\psi) \leq 1$, there exists a bilipschitz map $\Upsilon = \Upsilon(r, \psi) : \overline{\mathbf{B}_r \cap \Omega_\psi} \rightarrow \overline{\mathbf{B}_r}$ with $\sup\{\text{Lip } \Upsilon, \text{Lip } \Upsilon^{-1}\} \leq \Lambda$. Next we choose $r \in [1/2, 1]$ exactly as in the proof of 2.3. Using the extension $G : \overline{\Omega_\psi} \rightarrow \mathbf{S}^2$ of g given by $G(x_1, x_2, x_3) = g(x_1, x_2, \psi(x_1, x_2))$, we define the map $\omega : \partial\mathbf{B}_r \rightarrow \mathbf{S}^2$ by

$$\omega = (u - G) \circ \Upsilon^{-1} \text{ on } \Upsilon(\Omega_\psi \cap \partial\mathbf{B}_r), \quad \omega = 0 \text{ on } \Upsilon(\mathbf{B}_r \cap \partial\Omega_\psi).$$

Also let $\varepsilon = \lambda\alpha/(32\Lambda^{10}\beta)$, and let δ , q , and c_1 be as in 2.3. For c sufficiently large, we have the small oscillation condition

$$r^{-2} \left[\int_{\partial B_r} |\nabla_{\tan} \omega|^2 d\mathcal{H}^2 \right] \left[\int_{\partial B_r} |\omega - \bar{u} - \bar{G}|^2 d\mathcal{H}^2 \right] \leq \delta^2 \varepsilon^q,$$

where $\bar{G} = \int_{\mathbf{B}} G dx$ and where we have used that

$$\int_{\partial B_r} |G - \bar{G}|^2 d\mathcal{H}^2 \leq 16\pi(\text{Lip } g)^2.$$

With this small oscillation condition, we may, as before, apply [S-U, 4.3] to obtain a function $w \in H^1(B_r, S^2)$ which has trace ω on ∂B_r and which satisfies

$$\int_{B_r} |\nabla w|^2 dx \leq \varepsilon \int_{\partial B_r} |\nabla_{\tan} \omega|^2 d\mathcal{H}^2 + c_1 \varepsilon^{-q} \int_{\partial B_r} |\omega - \bar{u} - \bar{G}|^2 d\mathcal{H}^2.$$

Noting that $w \circ \Upsilon + G$ and $u|_{\Omega_\psi \cap B_r}$ have the same trace on $\partial(\Omega_\psi \cap B_r)$, we conclude from the $\tilde{\mathcal{W}}$ minimality of $u|_{\Omega_\psi \cap B_r}$ that,

$$\begin{aligned} E_{1/2}(u) &\leq 2E_r(u) \leq 4\alpha^{-1} \tilde{\mathcal{W}}(u|_{\Omega_\psi \cap B_r}) \leq 4\alpha^{-1} \tilde{\mathcal{W}}(w \circ \Upsilon + G) \\ &\leq 4\beta\alpha^{-1} \int_{B_r} |\nabla(w \circ \Upsilon + G)|^2 dx \leq 4\beta\alpha^{-1} \Lambda^5 \int_{\Omega_\psi \cap B_r} |\nabla(w + G \circ \Upsilon^{-1})|^2 dx \\ &\leq 4\beta\alpha^{-1} \Lambda^{10} \left[\varepsilon \int_{\partial B_r} |\nabla_{\tan} \omega|^2 d\mathcal{H}^2 + c_1 \varepsilon^{-q} \int_{\partial B_r} |\omega - \bar{u} - \bar{G}|^2 d\mathcal{H}^2 + c \int_{B_r} |\nabla G|^2 dx \right] \\ &\leq \lambda E_1(u) + c\lambda^{-q} \int_{\Omega_\psi \cap \mathbf{B}} |u - \bar{u}|^2 dx + c\lambda^{-q} (\text{Lip } g)^2, \end{aligned}$$

for an appropriate choice of c . \square

5.4. THEOREM. (Energy improvement) *There are positive constants ε , η , and $\theta < 1$ depending only on κ_1 , κ_2 , and κ_3 so that if ψ is as in 5.1, u is a minimizer of \mathcal{W} in $H^1(\Omega_\psi, S^2)$, g is the trace of u on $\mathbf{B} \cap \partial\Omega_\psi$, and $E_1(u) < \varepsilon^2$, then*

$$E_\theta(u) \leq \theta \max\{E_1(u), \eta(\text{Lip } g + \text{Lip } \psi)^2\}.$$

PROOF. If the theorem were false, then, for any $0 < \theta < 1$, there would exist C^1 functions $\psi_i : \mathbf{R}^2 \rightarrow \mathbf{R}$ with $\psi_i(0) = 0 = |\nabla\psi_i(0)|$ and \mathcal{W} minimizers u_i in $H^1(\Omega_{\psi_i}, \mathbf{S}^2)$ having Lipschitz traces g_i on $\mathbf{B} \cap \partial\Omega_{\psi_i}$ so that

$$E_\theta(u_i) > \theta E_1(u_i) \text{ for all } i \text{ and, as } i \rightarrow \infty, \\ E_1(u_i) \rightarrow 0 \text{ and } [\text{Lip}(\psi_i) + \text{Lip}(g_i)]^2 / E_1(u_i) \rightarrow 0.$$

As in 2.4, we may, after composing with rotations and passing to a subsequence, assume that u_i is a special boundary blow-up sequence. We define u_i on the whole ball \mathbf{B} as in 5.1, and let ε_i , v_i , and v be as in 5.1 and 5.2. Using 5.2 and repeated application of 5.3, the argument now proceeds as in 3.3 (with $\|F_i\|^2$ replaced by $\text{Lip}(\psi_i) + \text{Lip}(g_i)$). Since

$$\varepsilon_i^2 / E_1(u_i) \rightarrow 2 \text{ as } i \rightarrow \infty,$$

we now find that, for i sufficiently large,

$$E_\theta(u_i) \leq \frac{1}{4}\theta E_1(u_i) + \frac{1}{4}\theta \varepsilon_i^2 < \theta E_1(u_i),$$

the desired contradiction. \square

5.5. COROLLARY. (Energy decay) *Suppose Ω is a domain in \mathbf{R}^3 , $a \in \partial\Omega$, $n_0 : \partial\Omega \rightarrow \mathbf{S}^2$, and both $\partial\Omega$ and n_0 are both C^1 near a . Then, if $R > 0$ is sufficiently small (depending on the C^1 norms of $\partial\Omega$ and n_0 near a) and if $n \in H^1(\Omega, \mathbf{S}^2)$ is a minimizer of \mathcal{W} with*

$$n|_{\partial\Omega} = n_0 \text{ and } \int_{\mathbf{B}_R(a)} |\nabla n|^2 dx \leq \varepsilon^2 R,$$

then

$$\int_{\mathbf{B}_r(a)} |\nabla n|^2 dx \leq \theta^{-2} \max\{\varepsilon^2, \eta R^2\} R^{-1} r^2 \text{ for } 0 < r < R.$$

PROOF. Recalling 5.1, one uses 5.4 and argues as in 2.5 or 3.5. \square

5.6. THEOREM. (Partial regularity at the boundary) *Suppose Ω , n_0 , n , and Y are as in §1 and 5.0, Z is as in 2.6, and X is a relatively closed subset of $\partial\Omega$ such that n_0 and $\partial\Omega$ are $C^{k,\mu}$ off of X for some $k \in \{1, 2, \dots, \infty, \omega\}$ and $0 < \mu < 1$. Then n belongs to $C^{k,\mu}_{loc}(\overline{\Omega} \sim X \sim Y \sim Z)$.*

PROOF. To obtain local Hölder continuity on $\overline{\Omega} \sim X \sim Y \sim Z$, we note that trivially

$$\int_{B_r(b) \cap \Omega} |\nabla n|^2 dx \leq \int_{B_r(a) \cap \Omega} |\nabla n|^2 dx \text{ whenever } B_r(b) \subset B_r(a),$$

and apply 5.0, 5.1, 5.5, and 2.5 as in the proof of 2.6.

To prove higher regularity near a point $a \in \partial\Omega \sim X \sim Y$, we assume that $n(a) = (0, 0, 1)$, and observe as in the proof of 2.6 that, near a , the pair (n^1, n^2) satisfies an elliptic system with analytic coefficients on a $C^{k,\mu}$ domain and has $C^{k,\mu}$ boundary values near a . The theorem now follows from [M, §6]. \square

REFERENCES

- [BC] H. BREZIS and J. M. CORON. Preprint.
- [D] E. DE GIORGI, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Acad. Ser. Torino, **143** (1957), pp.25-43.
- [E] J. L. ERICKSEN, *Equilibrium theory of liquid crystals*, in *Advances in Liquid Crystals* **2**, ed. G. H. Brown, Academic Press, 1976, pp.233-299.
- [E₂] _____, *Nilpotent energies in liquid crystal theory*, Arch. Rat. Mech. Anal., **10** (1962), pp.189-196.
- [F] H. FEDERER, *Geometric measure theory*, Springer-Verlag, Heidelberg and New York, 1969.
- [GG] M. GIAQUINTA and E. GIUSTI, *The singular set of the minima of certain quadratic functionals*, Ann. Sc. Norm. Sup. Pisa (IV), **XI**, **1** (1984), pp. 45-55.

- [GW]** R. GULLIVER and B. WHITE, In preparation.
- [HK]** R. HARDT and D. KINDERLEHRER, *Mathematical questions of liquid crystal theory*, To appear in the Proceedings- I.M.A. Workshop on the Theory and Applications of Liquid Crystals, January, 1985, (ed., J. Ericksen and D. Kinderlehrer).
- [HL]** R. HARDT and F. H. LIN, *Tangential regularity near the C^1 boundary*, Geometric measure theory, Amer. Math. Soc. Proc. Sym. Pure Math., **43**.
- [HL₂]** _____, *A remark on H^1 mappings*, In preparation.
- [M]** C. B. MORREY, JR., *Multiple integrals in the calculus of variations*, Springer-Verlag, Heidelberg and New York, 1966.
- [M₁]** _____, *Second order elliptic systems of differential equations*, Ann. of Math. Studies **33**(1954), pp.101-159.
- [Ma]** E. MACMILLAN, *The statics of liquid crystals*, Thesis, Master of Science, Johns Hopkins University, 1982.
- [S]** L. SIMON, *Isolated singularities of extrema of geometric variational problems*, preprint, Australian National University, 1984.
- [SU]** R. SCHOEN and K. UHLENBECK, *A regularity theory for harmonic maps*, J. Diff. Geom., **17**(1982), pp. 307-335.
- [SU₂]** _____, *Boundary regularity and the Dirichlet problem of harmonic maps*, J. Diff. Geom., **18**(1983), pp. 253-268.
- [W]** B. WHITE, In preparation.

School of Mathematics
 University of Minnesota
 Minneapolis, MN 55455
 USA