UNIFIED ANALYSIS OF DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC PROBLEMS

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Abstract. We provide a framework for the analysis of a large class of discontinuous methods for second-order elliptic problems. It allows for the understanding and comparison of most of the discontinuous Galerkin methods that have been proposed for the numerical treatment of elliptic problems by diverse communities over three decades.

Key words. elliptic problems, discontinuous Galerkin, interior penalty

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1. Introduction. In 1973, Reed and Hill [50] introduced the first *discontinuous Galerkin* (DG) method for hyperbolic equations, and since that time there has been an active development of DG methods for hyperbolic and nearly hyperbolic problems. Recently, these methods have also been applied to purely elliptic problems; examples are the original method of Bassi and Rebay [9], the variations studied in [19] and [18], and a generalization called the local discontinuous Galerkin (LDG) methods introduced in [35] and further studied in [27], [21], and [30]. Also in the 1970's, but independently, Galerkin methods for elliptic and parabolic equations using discontinuous finite elements were proposed and a number of variants introduced and studied; see, for example, [43], [7], [59], [2], and [3]. These DG methods were then usually called *interior penalty* (IP) methods and their development remained independent of the development of the DG methods for hyperbolic equations. In this paper, we present a detailed study of a class of DG methods for second-order elliptic problems which includes all the above mentioned methods.

Next, we introduce the DG methods. For the sake of simplicity and to ease the presentation of the main ideas, we restrict ourselves to the model problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is assumed to be a convex polygonal domain and f a given function in $L^2(\Omega)$. To obtain the DG methods, we first rewrite the problem as a first-order system

$$\sigma = \nabla u, \quad -\nabla\cdot\sigma = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Multiplying these equations by test functions τ and v respectively, and integrating

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formally on a subset K of Ω, we get

$$\int_K \sigma \cdot \tau \, dx = -\int_K u \nabla \cdot \tau \, dx + \int_{\partial K} u \, n_K \cdot \tau \, ds,$$
$$\int_K \sigma \cdot \nabla v \, dx = \int_K f \, v \, dx + \int_{\partial K} \sigma \cdot n_K \, v \, ds,$$

where n_K is the outward normal unit vector to ∂K. This is the weak formulation we shall use to define the DG methods.

We still need to introduce the finite element spaces associated to the triangulation $\mathcal{T}_h = \{K\}$ of Ω. To fix ideas, we assume that Ω can be triangulated by taking K to be triangles. We set

$$V_h := \{v \in L^2(\Omega) \ : \ v|_K \in P(K) \quad \forall K \in \mathcal{T}_h\},$$
$$\Sigma_h := \{\tau \in [L^2(\Omega)]^2 \ : \ \tau|_K \in \Sigma(K) \quad \forall K \in \mathcal{T}_h\},$$

where $P(K) = \mathcal{P}_p(K)$ is the space of polynomial functions of degree at most $p \geq 1$ on K and $\Sigma(K) = [\mathcal{P}_p(K)]^2$. Now, following Cockburn and Shu [35], we consider the following general formulation: find $u_h \in V_h$ and $\sigma_h \in \Sigma_h$ such that for all $K \in \mathcal{T}_h$ we have

$$\int_K \sigma_h \cdot \tau \, dx = -\int_K u_h \nabla \cdot \tau \, dx + \int_{\partial K} \widehat{u}_K \, n_K \cdot \tau \, ds \quad \forall \tau \in \Sigma(K), \tag{1.2}$$
$$\int_K \sigma_h \cdot \nabla v \, dx = \int_K f \, v \, dx + \int_{\partial K} \widehat{\sigma}_K \cdot n_K \, v \, ds \qquad \forall v \in P(K), \tag{1.3}$$

where the *numerical fluxes* $\widehat{\sigma}_K$ and $\widehat{u}_K$ are approximations to $\sigma = \nabla u$ and to u, respectively, on the boundary of K. To complete the specification of a DG method we must express the numerical fluxes $\widehat{\sigma}_K$ and $\widehat{u}_K$ in terms of σ_h and u_h and in terms of the boundary conditions. This is why the above formulation is called the *flux formulation*. As we shall see, the choice of the numerical fluxes is quite delicate as it can affect the stability and the accuracy of the method, as well as the sparsity and symmetry of the stiffness matrix; cf. [35], [20], and [1]. In [1], the present authors showed how to choose these numerical fluxes to recover virtually all the DG methods that have been proposed so far. See also Table 3.1.

In this paper, we continue the work started in [1] in several ways. In order to put our work into proper perspective and give the reader an idea of the origins of the DG methods, we begin in § 2 with a brief overview of the history of the development of the DG methods. Then, in § 3, we introduce a suitable functional setting and show how to go from the flux formulation (1.2)–(1.3) to a typical finite element formulation, called the *primal formulation*, which is obtained by eliminating the auxiliary variable σ_h. Here, we relate the properties of *consistency* and *conservativity* of the numerical fluxes and the properties to the *consistency* and *adjoint consistency* of the bilinear form of the primal formulation. We make the discussion more concrete by looking more carefully at a few typical examples, and then tabulate the flux choices and primal bilinear form for nine different methods which have appeared in the literature.

Next, we perform a unified error analysis of this model class of DG methods. We begin in § 4 by analyzing the classical properties of consistency, boundedness, and stability typically used in finite element error analysis. In § 5 we obtain our error estimates. We begin with the fully stable and consistent methods. We show optimal error estimates in the energy norm, and we show that for adjoint-consistent methods, an optimal L^2-error estimate can also be achieved. Then, we relax the consistency conditions and show that optimal error estimates can be still obtained by forcing the penalty weights to be huge. This super-penalty technique, however, forces the DG method to behave like a classical conforming method. Finally, we relax the stability condition and consider two methods that do not have penalty terms and, as a consequence, are only *weakly* stable; they are the DG method of Baumann and Oden and the original DG method of Bassi and Rebay. The analyses of these methods are *ad hoc*, but illustrate interesting theoretical techniques.

We end in § 6 by summarizing the main features of the various methods, see Table 6.1, and by commenting on extensions in several possible directions. For convenience a brief notation index is provided as an appendix.

2. An historical overview. To put our work into proper perspective, we give a brief account of the development of the DG methods. We begin by considering penalty methods for elliptic equations.

2.1. Enforcing Dirichlet boundary conditions through penalties. The use of a penalty formulation for enforcing the Dirichlet boundary condition can be traced back to the late sixties. Indeed, in 1968 Lions [46] considered the problem of solving elliptic problems with very rough Dirichlet boundary data; for example, $-\Delta w = f$ in Ω and $w = g$ on $\partial\Omega$ where f is taken in $\mathrm{L}^2(\Omega)$ and g in $\mathrm{H}^{-1/2}(\partial\Omega)$. He regularized the above problem by replacing the Dirichlet boundary condition by the approximate boundary condition $u + \mu^{-1}\partial u/\partial n = g$ where was μ a large positive parameter. He proved that for each $\mu > 0$, there exists a unique solution u of the problem, and that, as μ goes to infinity, this solution converges to the solution w of the original problem. The weak form of the regularized problem is to find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \mu(u - g)\, v \, ds = \int_\Omega f v \, dx \quad \forall v \in H^1(\Omega).$$

Note that the trial functions v do not satisfy the boundary conditions and that a penalty term has been added in order to force, in the limit as μ tends to infinity, the satisfaction of the boundary conditions. In 1970, this approach was used by Aubin [4] in the framework of finite difference approximations of non-linear problems. He proved convergence to the exact solution provided the penalty parameter μ goes to infinity as the discretization parameter h goes to zero; in the linear case, convergence is achieved if μ is of the order of $h^{-1+\epsilon}$ for arbitrarily small $\epsilon > 0$. Finally, in 1973, the same approach was used in the finite element context by Babuška [5] for the case $g = 0$. If we use a finite element space using polynomials of degree p, the best error estimate obtained by the author gives a rate of convergence of order $h^{(2p+1)/3}$ in the energy norm, provided that the penalty parameter μ is taken to be of the order of $h^{-(2p+1)/3}$. The lack of optimality in the order of convergence is a direct consequence of the lack of consistency of the weak formulation. Indeed, note that the exact solution w does not satisfy the weak formulation of the regularized problem; instead, it satisfies

$$\int_\Omega \nabla w \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial w}{\partial n} v \, ds + \int_{\partial\Omega} \mu(w - g)\, v \, ds = \int_\Omega f v \, dx,$$

for all $v \in H^1(\Omega)$.

A different approach which still includes a penalty term but does not introduce any consistency error was proposed in 1971 by Nitsche [48]. Nitsche's method determines an approximate solution u_h in a finite element subspace of $H^1(\Omega)$ such that $B(u_h, v) = \int fv$ for all v in the same space. The bilinear form $B(\cdot\,,\,\cdot)$ is given by

$$B(u,v) := \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds - \int_{\partial\Omega} \frac{\partial v}{\partial n} u\, ds + \int_{\partial\Omega} \mu uv\, ds,$$

for any weighting function μ. Note that the second term of the bilinear form B, which arises naturally from an integration by parts, ensures the consistency of the method. On the other hand, the third term renders the discrete problem symmetric and hence ensures the property of adjoint consistency. Finally, the last term penalizes the departure of the trace of the approximate solution from the Dirichlet data $g = 0$ and is necessary to guarantee stability. Nitsche proved that if μ is taken as η/h where h is the element size and η is a sufficiently large constant, then the discrete solution converges to the exact solution with optimal order in H^1 and L^2.

A third approach for weakly imposing the Dirichlet boundary condition is obtained by including all the terms in the bilinear form $B(\cdot\,,\,\cdot)$ but reversing the sign of the third term. The resulting bilinear form B is no longer symmetric, but it has a favorable coercivity property, namely, $B(u,u) \geq \int |\nabla u|^2$, no matter how $\mu \geq 0$ is chosen. However, as we shall see, this method does not enjoy the adjoint consistency property mentioned above, a drawback that will adversely affect its L^2 convergence properties.

2.2. The IP methods for elliptic problems. The IP methods arose from the observation that, just as Dirichlet boundary conditions could be imposed weakly instead of being built into the finite element space, so inter-element continuity could be attained in a similar fashion. This makes it possible to use spaces of discontinuous piecewise polynomials for solving second order problems (which could, for example, facilitate adaptivity). In 1973, Babuška and Zlámal [6] used interior penalties to weakly impose C^1 continuity for fourth order problems. Their bilinear form is analogous to the penalization technique used by Lions [46], Aubin [4], and Babuška [5].

The natural generalization of Nitsche's method to second-order elliptic problems is stated in Wheeler's 1978 paper [59] on IP collocation–finite element methods, and is attributed to a private communication of Douglas and Dupont. That method is analyzed in detail for linear and non-linear elliptic and parabolic problems in the 1979 thesis of Arnold [2] which is summarized in [3]. Interior penalties of this sort were also used in 1977 by Baker [7] for imposing C^1 inter-element continuity on C^0 elements for fourth order problems. In these, of course, it is the jump in the normal derivative that is penalized. In 1976, Douglas and Dupont [43] had penalized the jump in the normal derivative of C^0 elements for second order elliptic and parabolic problems, with the goal of enforcing a degree of continuity in some sense intermediate between C^0 and C^1. This very same technique was applied then to a non-linear *hyperbolic* equation (arising in secondary oil recovery problems) in 1979 by Douglas, Darlow, Kendall, and Wheeler [42]. The interior penalty term they introduced was devised to force the approximate solution to become nearly C^1 away from the shock discontinuity without affecting the already existing artificial diffusion inserted in the method to properly deal with the shock.

Less attention has been paid to IP methods since the early 1980s. This might be due to the fact that they were never proven to be more advantageous or efficient than classical conforming finite element methods; moreover, the difficulty in finding optimal values for the penalty parameters and the corresponding efficient solvers may have also contributed to this situation. Thus, the ease with which the IP methods handle hanging nodes and varying-in-space polynomial degree approximations, which makes them ideally suited for *hp*-adaptivity, has never been fully exploited. Instead, techniques for enforcing continuity of the approximate solution in the framework of *hp*-refinement were developed. Also, new methods that weakly enforce continuity across boundary elements were explored like the classical non-conforming and mixed methods [17], the mortar methods [16], [14], [15], and the cell discretization methods [44], [58].

In spite of this, IP methods have recently found a few new applications. Thus, in 1990, Baker, Jureidini and Karakashian [8] enforced the divergence-free condition pointwise inside each element for the Stokes system and used interior penalties to cope with the resulting lack of continuity in the approximation of the velocity. In the same year, Rusten, Vassilevski, and Winther [54] used an interior penalty method for second order elliptic problems as part of a preconditioner for mixed methods. Finally, recently Becker and Hansbo [13] and do Carmo and Duarte [41] used the IP approach as a way to dealing with non-matching grids for domain decomposition.

2.3. The DG methods for convection-dominated problems. On the other hand, DG methods for the numerical treatment of non-linear hyperbolic systems experienced a vigorous development during the last ten years due to a strong interaction with the ideas of numerical fluxes, approximate Riemann solvers and slope limiters as developed for finite difference and finite volume methods for hyperbolic problems. Due to the non-linear character of the equations, the DG methods had to be carefully crafted to achieve stability, high-order accuracy and convergence to the so-called entropy solution. This is the case of the Runge–Kutta DG (RKDG) methods developed by Cockburn and Shu in [34], [33], [32], [28], and [36]; see the introduction to the subject in [26] and the short essay about the ideas used to devise DG methods for non-linear hyperbolic equations in [25]. Unlike the IP methods for elliptic problems, these DG methods have been proven to be clearly superior to the already existing finite element methods for hyperbolic conservation laws. A review of the development of DG methods up to May 1999 can be found in [31].

But the evolution of the DG methods did not stop there. The necessity of dealing with problems that, together with a dominant convective part, had a non-negligible diffusive part, prompted several authors to extend the DG methods to elliptic problems. Thus, in 1992, Richter [51] proposed a direct extension of the original DG method to linear convection-diffusion equations and proved that if the convection is dominant, that is, if the viscosity coefficients are of the order of the mesh size, the optimal order of convergence is $k + 1/2$ when polynomials of degree k are used.

For problems in which the diffusion might be dominant at least in some regions of the domain, a better handling of the second-order terms is necessary. Since mixed methods can handle very well elliptic operators and use a discontinuous approximation for the potential, they can be easily combined with discontinuous methods for advection. This idea was explored by several authors. In [37], [38], and [39], Dawson combined the Raviart-Thomas mixed method for second-order operators with high-order Godunov methods for convection giving rise to the so-called upwind-mixed methods (UMM) for advection-diffusion problems. See also [40] where he and Aizinger [40] obtain error estimates for arbitrary polynomial degree, and the application papers referenced there. In [22] and [23] use similar ideas for the hydrodynamic and quantum hydrodynamic models of semiconductor device simulation. Specifically, they combined the Raviart-Thomas spaces for handling second-order elliptic operators with the discontinuous approximations of the RKDG method. Finally, Lomtev, Quillen and

Karniadakis [47] used the DG-space discretization method to deal with the convective part of the compressible Navier–Stokes equations combined with a mixed method to approximate the diffusive part of the equations.

All of the above methods used discontinuous approximation only for the convective terms and mixed methods for second-order elliptic operators which enforce the continuity of the normal component of the approximation to $\sigma = \nabla u$ across the elements. It was only in 1997 that completely discontinuous approximations for both u and $\sigma = \nabla u$ were used by Bassi and Rebay [9]. Indeed, these authors used the discretization ideas of the RKDG methods to introduce a new DG method for the compressible Navier–Stokes equations. In 1998 Cockburn and Shu [35] introduced the so-called local discontinuous Galerkin (LDG) methods for transient non-linear convection-diffusion problems by generalizing the original DG method of Bassi and Rebay; see also the extension of the LDG methods to problems with quite general second-order terms by Cockburn and Dawson [27] and the recent review on the different discretizations of second-order terms by means of DG methods by Shu [55].

Around the same time, Baumann and Oden [12], [11] introduced another DG method for diffusion problems. Their approach is the analogue, for enforcing inter-element continuity, to the third approach described earlier for weakly enforcing Dirichlet boundary conditions. This results in a coercive bilinear form even when the penalty parameter vanishes, but, on the other hand, the bilinear form is not symmetric even for symmetric problems, and so the method is not adjoint consistent.

2.4. A first attempt to unify the DG methods. In recent years several authors were struck by the similarities between recently introduced DG methods and the IP methods, and started to apply to the former the techniques of analysis developed earlier for the latter. Thus Brezzi et al. [18], [19] studied several variations of the original method of Bassi and Rebay; Oden, Babuška, and Baumann [49] studied the DG method of Baumann and Oden; Rivière and Wheeler [52] and Rivière, Wheeler, and Girault [53] analyzed several variations of the DG method of Baumann and Oden; in [57], [56] and [45], Süli, Schwab, and Houston synthesized the elliptic, parabolic, and hyperbolic theory by extending the analysis of DG methods to partial differential equations with non-negative characteristic form; and, more recently, Castillo, Cockburn, Perugia, and Schötzau [21] and Cockburn, Kanschat, Perugia, and Schötzau [30] studied the LDG method as applied to purely elliptic problems in arbitrary and Cartesian meshes, respectively.

The presentation of all these methods, however, followed two main styles. Indeed, the methods inspired by the original IP methods were typically presented in their primal formulation, as for instance in [12], [11], [49], [57], [56], and [45], while the methods inspired by the finite volume techniques for hyperbolic problems where presented in terms of suitably chosen numerical fluxes, as for instance [9], [35], [27], even if the analysis was accomplished by shifting to a suitable associated bilinear form.

In a previous paper [1], we made a first attempt to unify these two families, and succeeded in recasting *all* of the above mentioned methods within a single framework in order to better understand the connections among them. In particular, it was shown that the methods of the first family, based on the choice of the bilinear form, could be obtained as special cases of the second family simply by choosing the proper numerical fluxes; see Table 3.1. Since some of the new DG methods (of the second family) have inherited the carefully crafted technique of defining numerical fluxes achieved for non-linear hyperbolic problems, it is plausible that the resulting DG methods for elliptic problems could come out, possibly after some suitable refinement, to be more efficient than the old ones.

In this paper, we complete the work started in [1]. To do that, we start by relating the flux and primal formulations.

3. The flux formulation and the primal formulation. In this section, we relate the flux formulation (1.2)–(1.3) of a DG method to its the primal formulation (3.10). We show that consistency and conservation properties of the numerical fluxes are reflected in consistency and adjoint consistency of the primal formulation. We introduce nine examples of DG methods which have appeared in the literature, some derived originally in a flux formulation, others in a primal formulation, and present the numerical fluxes and primal form for all of them.

3.1. Traces and numerical fluxes. We begin by introducing an appropriate functional setting. We denote by $H^l(\mathcal{T}_h)$ the space of functions on Ω whose restriction to each element K belongs to the Sobolev space $H^l(K)$. Thus, the finite element spaces V_h and Σ_h are subsets of $H^l(\mathcal{T}_h)$ and $[H^l(\mathcal{T}_h)]^2$, respectively, for any l. The *traces* of functions in $H^1(\mathcal{T}_h)$ belong $T(\Gamma) := \Pi_{K\in\mathcal{T}_h} L^2(\partial K)$, where Γ is used to denote the union of the boundaries of the elements K of $\mathcal{T}_h$. Function in $T(\Gamma)$ are thus double-valued on $\Gamma^0 := \Gamma \setminus \partial\Omega$ and single-valued on $\partial\Omega$. The space $L^2(\Gamma)$ can then be identified as the subspace of $T(\Gamma)$ consisting of functions for which the two values coincide on all internal edges.

We take the scalar numerical flux $\widehat{u} = (\widehat{u}_K)_{K\in\mathcal{T}_h}$ and the vector numerical flux $\widehat{\sigma} = (\widehat{\sigma}_K)_{K\in\mathcal{T}_h}$ to be linear functions

$$\widehat{u} : H^1(\mathcal{T}_h) \to T(\Gamma), \qquad \widehat{\sigma} : H^2(\mathcal{T}_h) \times [H^1(\mathcal{T}_h)]^2 \to [T(\Gamma)]^2. \tag{3.1}$$

In fact, only the normal component of $\widehat{\sigma}$ plays a role in the DG method; see (1.3). We could, without loss of generality, insist that $\widehat{\sigma}$ be directed normally on each edge, but for simplicity we do not do so.

The properties of consistency and conservativity of the numerical fluxes are important in the analysis of the DG methods. We say that the numerical fluxes are *consistent* if

$$\widehat{u}(v) = v|_\Gamma, \qquad \widehat{\sigma}(v, \nabla v) = \nabla v|_\Gamma,$$

whenever v is a smooth function satisfying the Dirichlet boundary conditions. We say that the numerical fluxes $\widehat{u}$ and $\widehat{\sigma}$ are *conservative* if $\widehat{u}(\,\cdot\,)$ and $\widehat{\sigma}(\,\cdot\,,\,\cdot\,)$, respectively, are single-valued on Γ. The terminology conservative comes from the following useful property, which holds whenever the vector flux $\widehat{\sigma}$ is single-valued. If S is the union of any collection of elements, then, taking v in (1.3) to be identically one in S and adding over the elements K contained in S, we get

$$\int_S f\,dx + \int_{\partial S} \widehat{\sigma}\cdot n_S\,ds = 0. \tag{3.2}$$

Next, we introduce some trace operators that will help us to manipulate the numerical fluxes and obtain the primal formulation. For $q \in T(\Gamma)$, we define the *average* $\{q\}$ and the *jump* $[\![\,q\,]\!]$ of q on Γ^0 as follows. Let e be an interior edge shared by elements K_1 and K_2. Define the unit normal vectors n_1 and n_2 on e pointing exterior to K_1 and K_2, respectively. With $q_i := q|_{\partial K_i}$ we set

$$\{q\} = \frac{1}{2}(q_1 + q_2), \quad [\![\,q\,]\!] = q_1 n_1 + q_2 n_2 \quad \text{on } e \in \mathcal{E}_h^\circ,$$

where $\mathcal{E}_h^\circ$ is the set of interior edges e. For $\varphi \in [T(\Gamma)]^2$ we define φ_1 and φ_2 analogously, and set

$$\{\varphi\} = \frac{1}{2}(\varphi_1 + \varphi_2), \quad [\![\varphi]\!] = \varphi_1 \cdot n_1 + \varphi_2 \cdot n_2 \quad \text{on } e \in \mathcal{E}_h^\circ.$$

Notice that the jump $[\![q]\!]$ of the scalar function q is a vector parallel to the normal, and the jump $[\![\varphi]\!]$ of the vector function φ is a scalar quantity. The advantage of these definitions is that they do not depend on assigning an ordering to the elements K_i. For $e \in \mathcal{E}_h^\partial$, the set of boundary edges, each $q \in T(\Gamma)$ and $\varphi \in [T(\Gamma)]^2$ has a uniquely defined restriction on e. We set

$$[\![q]\!] = qn, \quad \{\varphi\} = \varphi \quad \text{on } e \in \mathcal{E}_h^\partial,$$

where n is the outward unit normal. We don't require either of the quantities $\{q\}$ or $[\![\varphi]\!]$ on boundary edges, and leave them undefined. In short,

$$\begin{aligned} &\{\cdot\} : T(\Gamma) \to L^2(\Gamma^0), && [\![\cdot]\!] : T(\Gamma) \to [L^2(\Gamma)]^2, \\ &\{\cdot\} : [T(\Gamma)]^2 \to [L^2(\Gamma)]^2, && [\![\cdot]\!] : [T(\Gamma)]^2 \to L^2(\Gamma^0). \end{aligned}$$

3.2. The primal formulation. With the notation introduced above, we are ready to obtain the primal formulation. If, in the equations (1.2)–(1.3), we add over all the elements, we obtain

$$\begin{aligned} &\int_\Omega \sigma_h \cdot \tau \, dx = -\int_\Omega u_h \nabla_h \cdot \tau \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \widehat{u}_K \, n_K \cdot \tau \, ds \quad && \forall \tau \in \Sigma_h, \\ &\int_\Omega \sigma_h \cdot \nabla_h v \, dx = \int_\Omega f \, v \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \widehat{\sigma}_K \cdot n_K \, v \, ds && \forall v \in V_h, \end{aligned}$$

where $\nabla_h v$ and $\nabla_h \cdot \tau$ are the functions whose restriction to each element $K \in \mathcal{T}_h$ are equal to ∇v and $\nabla \cdot \tau$, respectively.

To deal with the sums of the form $\sum_{K \in \mathcal{T}_h} \int_{\partial K} q_K \varphi_K \cdot n_K \, ds$, we use the average and jump operators. A straightforward computation shows that for all $q \in T(\Gamma)$ and for all $\varphi \in [T(\Gamma)]^2$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} q_K \varphi_K \cdot n_K \, ds = \int_\Gamma [\![q]\!] \cdot \{\varphi\} \, ds + \int_{\Gamma^0} \{q\} [\![\varphi]\!] \, ds. \tag{3.3}$$

After a simple application of this identity, we get

$$\int_\Omega \sigma_h \cdot \tau \, dx = -\int_\Omega u_h \nabla_h \cdot \tau \, dx + \int_\Gamma [\![\widehat{u}]\!] \cdot \{\tau\} \, ds + \int_{\Gamma^0} \{\widehat{u}\} [\![\tau]\!] \, ds \quad \forall \tau \in \Sigma_h, \tag{3.4}$$

$$\int_\Omega \sigma_h \cdot \nabla_h v \, dx - \int_\Gamma \{\widehat{\sigma}\} \cdot [\![v]\!] \, ds - \int_{\Gamma^0} [\![\widehat{\sigma}]\!] \{v\} = \int_\Omega f v \, dx \qquad \forall v \in V_h. \tag{3.5}$$

Now, we express σ_h solely in terms of u_h. To do that, we use another identity. If in (3.3), we take q equal to the trace of v and φ equal to the trace of τ we obtain, for all $\tau \in [H^1(\mathcal{T}_h)]^2$ and $v \in H^1(\mathcal{T}_h)$, the integration by parts formula

$$-\int_\Omega \nabla_h \cdot \tau \, v \, dx = \int_\Omega \tau \cdot \nabla_h v \, dx - \int_\Gamma \{\tau\} \cdot [\![v]\!] \, ds - \int_{\Gamma^0} [\![\tau]\!] \{v\} \, ds. \tag{3.6}$$

Taking $v = u_h$ in the above identity and inserting the resulting right hand side into the equation (3.4), we get that, for every $\tau \in \Sigma_h$,

$$\int_\Omega \sigma_h \cdot \tau \, dx = \int_\Omega \nabla_h u_h \cdot \tau \, dx + \int_\Gamma [\![\widehat{u} - u_h]\!] \cdot \{\tau\} \, ds + \int_{\Gamma^0} \{\widehat{u} - u_h\} [\![\tau]\!] \, ds. \quad (3.7)$$

Recalling that $\nabla_h V_h \subset \Sigma_h$ and defining lifting operators $r : [L^2(\Gamma)]^2 \to \Sigma_h$ and $l : L^2(\Gamma^0) \to \Sigma_h$ by

$$\int_\Omega r(\varphi) \cdot \tau \, dx = -\int_\Gamma \varphi \cdot \{\tau\} \, ds, \qquad \int_\Omega l(q) \cdot \tau \, dx = -\int_{\Gamma^0} q [\![\tau]\!] \, ds \quad \forall \tau \in \Sigma_h, \quad (3.8)$$

we may rewrite (3.7) as

$$\sigma_h = \sigma_h(u_h) := \nabla_h u_h - r([\![\widehat{u}(u_h) - u_h]\!]) - l(\{\widehat{u}(u_h) - u_h\}). \quad (3.9)$$

Taking $\tau = \nabla_h v$ in the identity (3.7) we may then rewrite (3.5) as follows:

$$B_h(u_h, v_h) = \int_\Omega f v \, dx \quad \forall v \in V_h, \quad (3.10)$$

where

$$B_h(u_h, v) := \int_\Omega \nabla_h u_h \cdot \nabla_h v \, dx + \int_\Gamma \left([\![\widehat{u} - u_h]\!] \cdot \{\nabla_h v\} - \{\widehat{\sigma}\} \cdot [\![v]\!] \right) ds$$
$$+ \int_{\Gamma^0} \left(\{\widehat{u} - u_h\} [\![\nabla_h v]\!] - [\![\widehat{\sigma}]\!] \{v\} \right) ds. \quad (3.11)$$

For any functions $u_h \in H^2(\mathcal{T}_h)$ and $v \in H^2(\mathcal{T}_h)$, (3.11) defines $B_h(u_h, v)$, with the understanding that $\widehat{u} = \widehat{u}(u_h)$ and $\widehat{\sigma} = \widehat{\sigma}(u_h, \sigma_h(u_h))$ where $\sigma_h : H^2(\mathcal{T}_h) \to [H^1(\mathcal{T}_h)]^2$ is given by (3.9). The form $B_h : H^2(\mathcal{T}_h) \times H^2(\mathcal{T}_h) \to \mathbb{R}$ is bilinear, and if $(u_h, \sigma_h) \in V_h \times \Sigma_h$ solves (1.2)–(1.3), then u_h solves (3.10) and $\sigma_h = \sigma_h(u_h)$ given by (3.9). We call (3.10) the *primal formulation* of the method and the bilinear form $B_h(\cdot, \cdot)$ the *primal form.*

3.3. Consistency and conservation. Let u solve the boundary value problem (1.1). By the integration by parts formula (3.6), we have for any $v \in H^2(\mathcal{T}_h)$ that

$$\int_\Omega \nabla_h u \cdot \nabla_h v \, dx = -\int_\Omega \Delta u \, v \, dx + \int_\Gamma \{\nabla_h u\} \cdot [\![v]\!] \, ds + \int_{\Gamma^0} [\![\nabla_h u]\!] \{v\} \, ds,$$

and since $\{u\} = u$, $[\![u]\!] = 0$, $\{\nabla_h u\} = \nabla u$, $[\![\nabla_h u]\!] = 0$, and $-\Delta u = f$,

$$B_h(u, v) = \int_\Omega f \, v \, dx + \int_\Gamma \left([\![\widehat{u}]\!] \cdot \{\nabla_h v\} + (\nabla u - \{\widehat{\sigma}\}) \cdot [\![v]\!] \right) ds$$
$$+ \int_{\Gamma^0} [(\{\widehat{u}\} - u) [\![\nabla_h v]\!] - [\![\widehat{\sigma}]\!] \{v\}] \, ds, \quad (3.12)$$

where $\widehat{u} = \widehat{u}(u)$, $\widehat{\sigma} = \widehat{\sigma}(u, \sigma_h(u))$. If the numerical flux $\widehat{u}$ is consistent, i.e., $\widehat{u}(u) = u|_\Gamma$, then $[\![\widehat{u}]\!] = 0$ and $\{\widehat{u}\} = u$ on Γ. Then (3.9) implies that $\sigma_h(u) = \nabla u$. If the vector numerical flux $\widehat{\sigma}$ is also consistent, we then get that $[\![\widehat{\sigma}]\!] = 0$, $\{\widehat{\sigma}\} = \nabla u$ on Γ. Inserting these relations in (3.12) we conclude that

$$B_h(u, v) = \int_\Omega f \, v \, dx. \quad (3.13)$$

Thus, if the numerical fluxes are consistent, (3.13) holds for all test functions $v \in H^2(\mathcal{T}_h)$. This implies that the primal formulation is *consistent*, which is defined to mean that (3.13) holds, at least for all $v \in V_h$, or equivalently, in view of (3.10), that Galerkin orthogonality holds:

$$B_h(u-u_h, v) = 0 \quad \forall v \in V_h. \tag{3.14}$$

In fact, using the density of the range of the trace operator, it is not difficult to reverse the above argument, and show that if (3.13) holds for all $v \in H^2(\mathcal{T}_h)$, then the numerical fluxes must be consistent. We do not know of any DG methods for which (3.13) holds for all $v \in V_h$ but not for all $v \in H^2(\mathcal{T}_h)$. Thus consistency of the numerical fluxes is practically equivalent to consistency of the primal formulation.

Now let ψ solve

$$-\Delta\psi = g \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega. \tag{3.15}$$

If

$$B_h(v,\psi) = \int_\Omega v\, g\, dx \tag{3.16}$$

for all $v \in H^2(\mathcal{T}_h)$, then we say that the primal form is adjoint consistent. (The boundary value problem (3.15) is the adjoint of the problem we started with, which in this case is again the Dirichlet problem for the Poisson equation, because that problem is self-adjoint.) Since $\psi \in H^2(\Omega)$, $\{\psi\} = \psi$, $[\![\psi]\!] = 0$, $\{\nabla\psi\} = \nabla\psi$, and $[\![\nabla\psi]\!] = 0$, whence

$$\int_\Omega \nabla_h v \cdot \nabla_h \psi\, dx = \int_\Omega v\, g\, dx + \int_\Gamma [\![v]\!] \cdot \nabla\psi\, ds,$$

and so, with (3.11),

$$B_h(v,\psi) = \int_\Omega v\, g\, dx + \int_\Gamma [\![\widehat{u}(v)]\!] \cdot \nabla\psi\, ds - \int_{\Gamma^0} [\![\widehat{\sigma}(v, \sigma_h(v))]\!]\, \psi\, ds.$$

Now suppose that the numerical fluxes are conservative. This means that $[\![\widehat{u}]\!] = 0$ and $[\![\widehat{\sigma}]\!] = 0$. Thus, conservativity of the numerical fluxes implies adjoint consistency. Conversely, if either $[\![\widehat{u}(v)]\!]$ or $[\![\widehat{\sigma}(v, \sigma_h(v))]\!]$ does not vanish for some v, then there is a smooth function ψ for which (3.16) does not hold.

3.4. Examples of DG methods. A simple and natural choice of numerical fluxes is

$$\widehat{u} = \{u_h\} \text{ on } \Gamma^0, \quad \widehat{u} = 0 \text{ on } \partial\Omega, \quad \text{and} \quad \widehat{\sigma} = \{\sigma_h\} \text{ on } \Gamma.$$

This is the choice proposed by Bassi and Rebay in [9]. With this choice of $\widehat{u}$, we have $\{\widehat{u} - u_h\} = 0$ and $[\![\widehat{u} - u_h]\!] = -[\![u_h]\!]$, so (3.9) gives

$$\sigma_h = \nabla_h u_h + r([\![u_h]\!]). \tag{3.17}$$

Therefore

$$\int_\Gamma \{\widehat{\sigma}\} \cdot [\![v]\!]\, ds = \int_\Gamma \{\nabla_h u\} \cdot [\![v]\!]\, ds - \int_\Omega r([\![u_h]\!]) r([\![v]\!])\, dx, \tag{3.18}$$

where we used the fact that $r([\![u_h]\!]) \in \Sigma_h$ and the definition (3.8) of r in the last step. Substituting in (3.11) we obtain the primal form for the method of Bassi–Rebay [9]:

$$B_h(u_h, v) = \int_\Omega [\nabla_h u_h \cdot \nabla_h v + r([\![u_h]\!]) r([\![v]\!])]\, dx - \int_\Gamma (\{\nabla_h u_h\} \cdot [\![v]\!] + [\![u_h]\!] \cdot \{\nabla_h v\})\, ds.$$

As a second example, we consider the classic IP method. This was originally proposed as a primal formulation, with

$$B_h(u_h, v) = \int_\Omega \nabla_h u_h \cdot \nabla_h v\, dx - \int_\Gamma ([\![u_h]\!] \cdot \{\nabla_h v\} + \{\nabla_h u_h\} \cdot [\![v]\!])\, ds + \alpha^{\mathrm{j}}(u_h, v), \quad (3.19)$$

where

$$\alpha^{\mathrm{j}}(u_h, v) = \int_\Gamma \mu [\![u_h]\!] \cdot [\![v]\!]\, ds \quad (3.20)$$

is the interior penalty or stabilization term with the penalty weighting function $\mu : \Gamma \to \mathbb{R}$ given by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h$ with η_e a positive number. It is easy to see that this method arises as well from a proper choice of fluxes:

$$\widehat{u} = \{u_h\} \text{ on } \Gamma^0, \quad \widehat{u} = 0 \text{ on } \partial\Omega, \quad \text{and} \quad \widehat{\sigma} = \{\nabla_h u_h\} - \alpha_{\mathrm{j}}([\![u_h]\!]) \text{ on } \Gamma, \quad (3.21)$$

where $\alpha_{\mathrm{j}}(\varphi)$ is simply $\mu\varphi$, i.e., $\eta_e h_e^{-1}\varphi$ on e. Again we have σ_h as in (3.17), while instead of (3.18) we get

$$\int_\Gamma \{\widehat{\sigma}\} \cdot [\![v]\!]\, ds = \int_\Gamma \{\nabla_h u_h\} \cdot [\![v]\!]\, ds - \int_\Gamma \alpha_{\mathrm{j}}([\![u_h]\!]) \cdot [\![v]\!]\, ds, \quad (3.22)$$

and (3.19) follows by substituting (3.22) in (3.11).

The vector flux for the IP method contains the jump term $\alpha_{\mathrm{j}}([\![u_h]\!])$ which is equal to $\eta_e h_e^{-1} [\![u_h]\!]$ on e. An alternative jump term is obtained using the lift operator $r_e : [L^1(e)]^2 \to \Sigma_h$ given by

$$\int_\Omega r_e(\varphi) \cdot \tau\, dx = -\int_e \varphi \cdot \{\tau\}\, ds \quad \forall \tau \in \Sigma_h,\ \varphi \in [L^1(e)]^2. \quad (3.23)$$

We then define $\alpha_{\mathrm{r}}(\varphi) = -\eta_e \{r_e(\varphi)\}$ on e. Note that $r_e(\varphi)$ vanishes outside the union of the one or two triangles containing e and that $r(\varphi) = \sum_{e \in \mathcal{E}_h} r_e(\varphi)$ for all $\varphi \in [L^1(\Gamma)]^2$. If we keep the choice of $\widehat{u}$ in (3.21), but change α_{j} to α_{r} in the choice of $\widehat{\sigma}$, we obtain a method of Bassi et al. [10]. In this case the primal form is

$$B_h(u_h, v) = \int_\Omega \nabla_h u_h \cdot \nabla_h v\, dx - \int_\Gamma ([\![u_h]\!] \cdot \{\nabla_h v\} + \{\nabla_h u_h\} \cdot [\![v]\!])\, ds + \alpha^{\mathrm{r}}(u_h, v),$$

where

$$\alpha^{\mathrm{r}}(u_h, v) = \int_\Gamma \alpha_{\mathrm{r}}(u_h) \cdot [\![v]\!]\, ds = \sum_{e \in \mathcal{E}_h} \int_\Omega \eta_e\, r_e([\![u_h]\!]) \cdot r_e([\![v]\!])\, dx. \quad (3.24)$$

As a fourth example we consider the LDG method introduced in [35]. The fluxes are taken as

$$\widehat{u} = \{u_h\} - \beta \cdot [\![u_h]\!] \text{ on } \Gamma^0, \quad \widehat{u} = 0 \text{ on } \partial\Omega, \tag{3.25}$$

and

$$\widehat{\sigma} = \{\sigma_h\} + \beta[\![\sigma_h]\!] - \alpha_{\mathrm{j}}([\![u_h]\!]) \text{ on } \Gamma^0, \quad \widehat{\sigma} = \{\sigma_h\} - \alpha_{\mathrm{j}}([\![u_h]\!]) \text{ on } \partial\Omega. \tag{3.26}$$

Here $\beta \in [L^2(\Gamma^0)]^2$ is a vector-valued function which is constant on each edge. From the scalar flux choice (3.25) we get $\{\widehat{u} - u_h\} = -\beta \cdot [\![u_h]\!]$ on Γ^0 and $[\![\widehat{u} - u_h]\!] = -[\![u_h]\!]$ on Γ, so that (3.9) gives

$$\sigma_h = \nabla_h u_h + \tau$$

where

$$\tau = r([\![u_h]\!]) + l(\beta \cdot [\![u_h]\!]) \in \Sigma_h$$

Then the vector flux choice (3.26) gives

$$\widehat{\sigma} = \{\nabla_h u_h\} + \{\tau\} + \beta[\![\nabla_h u_h]\!] + \beta[\![\tau]\!] - \alpha_{\mathrm{j}}([\![u_h]\!]),$$

(with the term involving β missing on $\partial\Omega$). Using (3.8) we obtain

$$\begin{aligned}\int_\Gamma \{\widehat{\sigma}\} \cdot [\![v]\!]\, ds = \int_\Gamma \{\nabla_h u_h\} \cdot [\![v]\!]\, ds + \int_{\Gamma^0} [\![\nabla_h u_h]\!] \beta \cdot [\![v]\!]\, ds \\ - \int_\Omega [r([\![v]\!]) + (\beta \cdot [\![v]\!])] \cdot \tau\, dx - \alpha^{\mathrm{j}}(u_h, v).\end{aligned}$$

Substituting in (3.11) and recalling the definition of τ, we obtain the bilinear form for the LDG method:

$$\begin{aligned}B_h(u_h, v) = \int_\Omega \nabla_h u_h \cdot \nabla_h v\, dx - \int_\Gamma ([\![u_h]\!] \cdot \{\nabla_h v\} + \{\nabla_h u_h\} \cdot [\![v]\!])\, ds \\ + \int_{\Gamma^0} (\beta \cdot [\![u_h]\!] [\![\nabla_h v]\!] + [\![u_h]\!] \beta \cdot [\![v]\!])\, ds \\ + \int_\Omega [r([\![u_h]\!]) + l(\beta \cdot [\![u_h]\!])] \cdot [r([\![v]\!]) + l(\beta \cdot [\![v]\!])]\, dx + \alpha^{\mathrm{j}}(u_h, v).\end{aligned} \tag{3.27}$$

In Table 3.1 we summarize the interior edge flux choices for the four methods just discussed and a variety of other methods which have appeared previously, and in Table 3.2 we show the primal bilinear forms for these same methods. For convenience, in the table we write g for the gradient term $(\nabla_h w, \nabla_h v)$ and use the shorter notations (a, b) and $\langle a, b\rangle$ instead of $\int_\Omega ab\, dx$ and $\int_\Gamma ab\, ds$.

We close this section with some comments on the tabulated methods. First we note that the scalar flux $\widehat{u}$ is consistent for all the methods. The vector flux $\widehat{\sigma}$ is consistent for the first seven of the nine methods, but not for the last two. This lack of consistency can be seen from the primal forms as well. If w is a smooth function which vanishes on $\partial\Omega$, then the first seven primal forms tabulated reduce to $B_h(w, v) = (\nabla w, \nabla_h v) - \langle\{\nabla w\}, [\![v]\!]\rangle$, which is equal to $(-\Delta w, v)$ for all $v \in H^1(\mathcal{T}_h)$. For the last two methods, the edge terms are missing, and so the primal form is

TABLE 3.1
Some DG methods and their numerical fluxes.

Method	$\widehat{u}_K$	$\widehat{\sigma}_K$
Bassi–Rebay [9]	$\{u_h\}$	$\{\sigma_h\}$
Brezzi et al. [18]	$\{u_h\}$	$\{\sigma_h\} - \alpha_{\mathrm{r}}([\![u_h]\!])$
LDG [35]	$\{u_h\} - \beta \cdot [\![u_h]\!]$	$\{\sigma_h\} + \beta[\![\sigma_h]\!] - \alpha_{\mathrm{j}}([\![u_h]\!])$
IP [43]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_{\mathrm{j}}([\![u_h]\!])$
Bassi et al. [10]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_{\mathrm{r}}([\![u_h]\!])$
Baumann–Oden [12]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\}$
NIPG [53]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\} - \alpha_{\mathrm{j}}([\![u_h]\!])$
Babuška–Zlámal [6]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_{\mathrm{j}}([\![u_h]\!])$
Brezzi et al. [19]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_{\mathrm{r}}([\![u_h]\!])$

TABLE 3.2
Primal forms for the DG methods in Table 3.1.

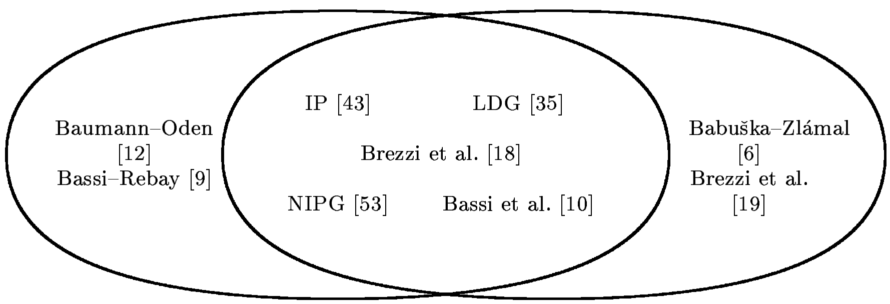
Method	$B_h(w,v)$
BR [9]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \big(r([\![w]\!]), r([\![v]\!])\big)$
BMMPR [18]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \big(r([\![w]\!]), r([\![v]\!])\big) + \alpha^{\mathrm{r}}(w,v)$
LDG [35]	see (3.27)
IP [43]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \alpha^{\mathrm{j}}(w,v)$
BRMPS [10]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \alpha^{\mathrm{r}}(w,v)$
BO [12]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle + \langle[\![w]\!], \{\nabla_h v\}\rangle$
NIPG [53]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle + \langle[\![w]\!], \{\nabla_h v\}\rangle + \alpha^{\mathrm{j}}(w,v)$
BZ [6]	$g + \alpha^{\mathrm{j}}(w,v)$
BMMPR [19]	$g + \alpha^{\mathrm{r}}(w,v)$

inconsistent. The consistent methods are shown in the left oval of the Venn diagram in Figure 3.1.

Next we note that the vector flux $\widehat{\sigma}$ is conservative for all the methods, and thus the conservation property (3.2) is satisfied for all of them. The scalar flux is conservative for the first five methods listed, so they are adjoint consistent, but not for the last four methods. In fact for the method of Baumann–Oden and its stabilized version NIPG, the primal form is not even symmetric. For the methods of Babuška–Zlámal and Brezzi et al. [19], the primal form is symmetric, but since it is not consistent, it is also not adjoint consistent.

Finally, we remark on the sparsity of the stiffness matrix arising from the primal form. Let $w, v \in V_h$ with w supported in a single triangle K_1 and v supported in a triangle K_2. Clearly the term $(\nabla_h w, \nabla_h v)$ entering the primal forms will vanish unless $K_1 = K_2$. The terms given by integrals over Γ and (for LDG) over Γ^0 will vanish unless the K_1 and K_2 share an edge, and the same is true of the penalty terms $\alpha^{\mathrm{j}}(w,v)$ and $\alpha^{\mathrm{r}}(w,v)$. However this is not true of the additional domain integral terms that occur in the first three methods: these terms will in general be nonzero if there is a third triangle K which shares an edge with each of K_1 and K_2. Thus these terms have a big negative impact on the sparsity of the stiffness matrix. In some cases the problem can be made less severe. For example, for the LDG method, if we take $\beta = -n_K/2$ on some edge e of a triangle K, and v is supported in K, then one can check that $r([\![v]\!]) + l(\beta \cdot [\![v]\!])$ vanishes on the triangle across e from K. Cockburn and Shu [35] used this technique to reduce the stencil of the LDG equations in the framework of one-dimensional convection-diffusion problems; see their equation (2.9). Moreover, some superconvergence results can be proved for the LDG methods with such a choice of β; see [20], [30].

FIG. 3.1. *Consistency and stability of some DG methods.*



4. Boundedness, stability and approximation properties. In this section, we discuss separately the boundedness and stability of the bilinear form B_h and the approximation properties of the space V_h with respect to an appropriate norm. We will then be ready to carry out a unified error analysis.

4.1. Boundedness. To consider the boundedness and stability of the primal forms B_h, we let $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega) \subset H^2(\mathcal{T}_h)$ and define the following seminorms and norms for $v \in V(h)$:

$$|v|_{1,h}^2 = \sum_K |v|_{1,K}^2, \quad |v|_*^2 = \sum_{e \in \mathcal{E}_h} \|r_e([\![v]\!])\|_{0,\Omega}^2, \tag{4.1}$$

$$|\!|\!| v |\!|\!|^2 = |v|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2 + |v|_*^2. \tag{4.2}$$

The norm (4.2) is the natural one for obtaining boundedness of the bilinear form B_h. On the other hand, the weaker norm

$$v \mapsto (|v|_{1,h}^2 + |v|_*^2)^{1/2}, \tag{4.3}$$

is the natural one for analyzing the stability of many DG methods. Restricted to $v \in V_h$, these two norms are equivalent, as is evident from a local inverse inequality. We also remark that both (4.2) and (4.3) define norms, not just seminorms, on $V(h)$.

Indeed, the discrete Poincaré inequality given in [3], Lemma 2.1, together with the second inequality of (4.5) below, implies the existence of a constant C for which

$$\|v\|_0 \le C(|v|_{1,h}^2 + |v|_*^2)^{1/2} \quad \forall v \in V(h).$$

We now show that for many DG methods, including all those listed in Table 3.2, the primal bilinear form B_h is bounded with respect to the norm $|||\cdot|||$, that is,

$$B_h(w,v) \le C_b |||w|||\, |||v||| \quad \forall w, v \in V(h). \tag{4.4}$$

We do this by bounding each of the various terms that appear in the table.

Obviously we have $(\nabla_h w, \nabla_h v) \le C|w|_{1,h}|v|_{1,h}$ and, from the definition (3.24), $\alpha^{\mathrm{r}}(w,v) \le (\sup_e \eta_e)|w|_*|v|_*$.

Now, by the definition of r_e, (3.23), and after using inverse inequalitites as in [18], we get

$$C_1\|r_e(\varphi)\|_{0,\Omega}^2 \le h_e^{-1}\|\varphi\|_{0,e}^2 \le C_2\|r_e(\varphi)\|_{0,\Omega}^2 \quad \forall \varphi \in [\mathcal{P}_p(e)]^2,$$

where h_e denotes the length of the edge e and the constants C_1 and C_2 only depend on the minimum angle of the decomposition and the polynomial degree p. Since $[\![v]\!]$ vanishes for $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we can apply the above inequalities with $\varphi = [\![v]\!]$ for any $v \in V(h)$, and then summation over the edges e then gives

$$C_1|v|_*^2 \le \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![v]\!]\|_{0,e}^2 \le C_2|v|_*^2 \quad \forall v \in V(h). \tag{4.5}$$

It thus follows that $\alpha^{\mathrm{j}}(w,v) \le C_2(\sup_e \eta_e)|w|_*|v|_*$ for $w, v \in V(h)$, and also that the term $\int_{\Gamma_0} [\![w]\!] \beta \cdot [\![v]\!]\, ds$, which occurs in the primal form of the LDG method (3.27), is bounded by $C|w|_*|v|_*$ with $C = \sup_e h_e\|\beta\|_{L^\infty(e)}$.

Next we recall that for $\varphi \in [L^2(\Gamma)]^2$, $r_e(\varphi)$ vanishes on the interior of any triangle K unless e is one of the edges of K. Therefore

$$\|r(\varphi)\|_{0,\Omega}^2 = \|\sum_{e\in\mathcal{E}_h} r_e(\varphi)\|_{0,\Omega}^2 \le 3\sum_{e\in\mathcal{E}_h}\|r_e(\varphi)\|_{0,\Omega}^2,$$

whence

$$|r([\![v]\!])|_{0,\Omega}^2 \le 3|v|_*^2 \quad \forall v \in V(h), \tag{4.6}$$

and so, $\int_\Omega r([\![w]\!]) \cdot r([\![v]\!])\, dx \le 3|w|_*|v|_*$ for $w, v \in V(h)$.

Next we bound the terms $\int_\Gamma \{\nabla_h w\} \cdot [\![v]\!]\, ds$ and $\int_\Gamma [\![w]\!] \cdot \{\nabla_h v\}\, ds$ which arise for the first seven of the tabulated methods. We begin by noting that if $u \in H^2(K)$ and e is an edge of K, we have ([3], equation (2.5)),

$$\left\|\frac{\partial w}{\partial n}\right\|_{0,e}^2 \le C(h_e^{-1}|w|_{1,K}^2 + h_e|w|_{2,K}^2), \tag{4.7}$$

where C depends only on the minimum angle of K. It follows that, for every $q \in L^2(e)$,

$$\int_e \left|\frac{\partial w}{\partial n} q\right| ds \le C\left(|w|_{1,K}^2 + h_e^2|w|_{2,K}^2\right)^{1/2} h_e^{-1/2}\|q\|_{0,e},$$

and this implies that

$$\int_\Gamma \{\nabla_h w\} \cdot [\![v]\!] \, ds = \sum_{e \in \mathcal{E}_h} \int_e \{\nabla_h w\} \cdot [\![v]\!] \, ds$$
$$\leq C \left[\sum_K (|w|_{1,K}^2 + h_K^2 |w|_{2,K}^2) \right]^{1/2} \left[\sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e |[\![v]\!]|^2 \, ds \right]^{1/2} \leq C |\!|\!| w |\!|\!| |v|_*.$$

Similarly, $\int_\Gamma [\![w]\!] \cdot \{\nabla_h v\} \, ds \leq C|w|_* |\!|\!| v |\!|\!|$ and $\int_{\Gamma^0} [\![\nabla_h w]\!] \beta \cdot [\![v]\!] \, ds \leq C |\!|\!| w |\!|\!| |v|_*$ for $w, v \in V(h)$ (provided that $|\beta|$ is bounded on Γ^0).

It remains only to bound the second integral over Ω in (3.27). For $e \in \mathcal{E}_h^\circ$, define $l_e : L^1(e) \to \Sigma_h$ by

$$\int_\Omega l_e(q) \cdot \tau \, dx = - \int_e q [\![\tau]\!] \, ds \quad \forall \tau \in \Sigma_h, \ q \in L^1(e). \tag{4.8}$$

Then $l_e(q)$ vanishes outside the union the two triangles containing e and $l(q) = \sum_{e \in \mathcal{E}_h^\circ} l_e(q)$. Moreover, from the definitions of l_e, r_e, and the jump and average operators, we find that, if e is an edge of the element K,

$$l_e(q) = 2 \, r_e(q \, n_K) \text{ on } K.$$

It follows that

$$\|l(q)\|_{0,\Omega}^2 \leq 3 \sum_{e \in \mathcal{E}_h^\circ} \|l_e(q)\|_{0,\Omega}^2 \leq 12 \sum_{e \in \mathcal{E}_h^\circ} \|r_e(q \, n_e)\|_{0,\Omega}^2$$

(where we can choose either normal n_e to the edge e), and then that

$$\|l(\beta \cdot [\![v]\!])\|_{0,\Omega}^2 \leq 12 |\beta|_{L^\infty(\Gamma^0)} |v|_*^2 \quad \forall v \in V(h). \tag{4.9}$$

This enables us to bound the remaining term:

$$\int_\Omega [r([\![w]\!]) + l(\beta \cdot [\![w]\!])] \cdot [r([\![v]\!]) + l(\beta \cdot [\![v]\!])] \, dx \leq C |w|_* |v|_*, \quad \forall w, v \in V(h).$$

We have thus established the bound (4.4) for all the methods in Table 3.2 (and any other methods involving the same sorts of terms). The constant C_b depends only on the minimum angle of the decomposition $\mathcal{T}_h$, the polynomial degree p, an upper bound on the edge dependent penalty parameter η for those methods which include the penalty term α^{j} or α^{r}, and, for the LDG method, an upper bound on the function β which enters into the formulas (3.26), (3.25) for the fluxes.

4.2. Stability. Now we show that many DG methods satisfy the stability condition

$$B_h(v, v) \geq C_s |\!|\!| v |\!|\!|^2 \quad \forall v \in V_h \tag{4.10}$$

with C_s a positive constant.

With reference to Tables 3.2 and 3.1 we may write for all nine tabulated methods

$$B_h(v, v) = \|\nabla_h b\|_{0,\Omega}^2 + \alpha(v, v) + b(v, v),$$

where α is either α^{r}, α^{j}, or zero depending on whether the flux $\widehat{\sigma}_K$ for the method contains α_{r}, α_{j}, or neither, and b gathers up all the remaining terms of the primal form. From the bounds we obtained in the last section, we know that $|b(v,v)| \le C|\!|\!|v|\!|\!||v|_*$ for some constant C and all $v \in V(h)$.

When present, the penalty term α^{r} or α^{j}, contributes to the stability of the method. For α^{r}, we obtain immediately from (3.24) and the definition of $|\cdot|_*$ in (4.1), that

$$\alpha^{\mathrm{r}}(v,v) \ge \eta_0 |v|_*^2 \quad \forall v \in V_h, \tag{4.11}$$

where $\eta_0 \equiv \inf_e \eta_e$. In view of (4.5) we have for α^{j} that

$$\alpha^{\mathrm{j}}(v,v) \ge C_1 \eta_0 |v|_*^2 \quad \forall v \in V_h. \tag{4.12}$$

Thus, for methods involving a penalty term,

$$B_h(v,v) \ge |v|_{1,h}^2 + C_0\eta_0|v|_*^2 - C|\!|\!|v|\!|\!||v|_* \quad \forall v \in V_h,$$

where C_0 equals 1 or C_1, depending on whether the penalty term is α^{r} or α^{j}, and C depends on the angle condition, polynomial degree, and in the case of the LDG method, a bound for the coefficient β. We may then use the arithmetic-geometric mean inequality $(2ab \le a^2\epsilon + b^2/\epsilon)$ on the last term and then the equivalence of the norms (4.2) and (4.3), to show that (4.10) holds for large enough η_0. Thus all the methods considered which include a penalty term α^{r} or α^{j} are stable supposing that the stabilizing coefficients η_e are chosen sufficiently large. This includes seven of the nine methods listed in table. The remaining two methods, that of Baumann and Oden [12] and that of Bassi and Rebay [9] are not stable, as will be discussed below. The stable methods are indicated in the right oval of the Venn diagram in Figure 3.1.

While the argument just presented establishes stability when the η_e are chosen sufficiently large, it does not make clear just how large they must be taken. For some methods, a precise sufficient condition can be obtained by a sharper analysis. To this end we define $R(v) = r([\![v]\!])$ and $L_\beta(v) = l(\beta \cdot [\![v]\!])$, so

$$\int_\Omega R(v)\cdot\tau\,dx = -\int_\Gamma [\![v]\!]\cdot\{\tau\}\,ds, \qquad \int_\Omega L_\beta(v)\cdot\tau\,dx = -\int_{\Gamma^0} \beta\cdot[\![v]\!]\,[\![\tau]\!]\,ds, \tag{4.13}$$

for all $\tau \in \Sigma_h$. We may use these notations to give simpler expressions of the primal form when both arguments belong to V_h. See Table 4.1.

From the table we see that five of the methods—all but the unstable methods (Bassi–Rebay [9] and Baumann–Oden [12]), the IP method, and the method of Bassi et al. [10]—satisfy

$$B_h(v,v) \ge \int_\Omega |\nabla_h v + S(v)|^2\,dx + \alpha(v,v) \quad \forall v \in V_h, \tag{4.14}$$

where $S(v)$ stands for a linear combination of the terms $R(v)$ and $L_\beta(v)$ and α is either α^{r} or α^{j}. From (4.6) and (4.9) we know that

$$\|S(v)\|_{0,\Omega} \le C|v|_*. \tag{4.15}$$

If (4.14) holds, then we deduce, using for instance $\alpha = \alpha^{\mathrm{j}}$ and hence (4.12),

$$B_h(v,v) \ge |v|_{1,h}^2 + 2\int_\Omega \nabla_h v\cdot S(v)\,dx + \|S(v)\|_0^2 + C_1\eta_0|v|_*^2 \quad \forall v \in V_h,$$

TABLE 4.1
Bilinear forms restricted to $V_h \times V_h$ for some DG methods.

Method	$B_h(u,v)$
Bassi–Rebay [9]	$(\nabla_h u + R(u), \nabla_h v + R(v))$
Brezzi et al. [18]	$(\nabla_h u + R(u), \nabla_h v + R(v)) + \alpha^{\rm r}(u,v)$
LDG [35]	$(\nabla_h u + R(u) + L_\beta(u), \nabla_h v + R(v) + L_\beta(v)) + \alpha^{\rm j}(u,v)$
IP [43]	$(\nabla_h u, \nabla_h v) + (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^{\rm j}(u,v)$
Bassi et al. [10]	$(\nabla_h u, \nabla_h v) + (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^{\rm r}(u,v)$
Baumann–Oden [12]	$(\nabla_h u, \nabla_h v) - (R(u), \nabla_h v) + (\nabla_h u, R(v))$
NIPG [53]	$(\nabla_h u, \nabla_h v) - (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^{\rm j}(u,v)$
Babuška–Zlámal [6]	$(\nabla_h u, \nabla_h v) + \alpha^{\rm j}(u,v)$
Brezzi et al. [19]	$(\nabla_h u, \nabla_h v) + \alpha^{\rm r}(u,v)$

and applying the arithmetic-geometric mean inequality we have for every $\varepsilon > 0$

$$B_h(v,v) \geq |v|_{1,h}^2(1-\varepsilon) + (1-1/\varepsilon)\|S(v)\|_0^2 + C_1\eta_0|v|_*^2 \quad \forall v \in V_h. \tag{4.16}$$

Inserting (4.15) into (4.16) we easily get

$$B_h(v,v) \geq |v|_{1,h}^2(1-\varepsilon) + (C(1-1/\varepsilon) + C_1\eta_0)|v|_*^2 \quad \forall v \in V_h,$$

for any $\varepsilon < 1$. Since we may choose ε as close to 1 as we please, we can demonstrate stability for *any* $\eta_0 > 0$.

For the method of Bassi et al. [10], we have instead of (4.14)

$$B_h(v,v) = \int_\Omega (|\nabla_h v + R(v)|^2 - |R(v)|^2)\,dx + \alpha^{\rm r}(v,v) \quad \forall v \in V_h.$$

Using (4.6), (4.11), and the above argument we can deduce the result of [19] that stability holds whenever $\eta_0 > 3$.

4.3. Approximation. The last ingredient in the error analysis is a bound on the approximation error $|\!|\!|u-u_I|\!|\!|$ when $u_I \in V_h$ is a suitable interpolant of the exact solution u. If u_I is chosen to be the usual *continuous* interpolant, then the jumps of $u-u_I$ will be zero at the inter-element boundaries, so that (4.2) immediately gives:

$$|\!|\!|u-u_I|\!|\!|^2 = |u-u_I|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2|u-u_I|_{2,K}^2 \leq C_a^2 h^{2p}|u|_{p+1,\Omega}^2. \tag{4.17}$$

For some purposes, for example for the analysis of the method of Baumann and Oden below, or to extend the analysis to meshes with hanging nodes, it is convenient to take an interpolant u_I which is *discontinuous* across the inter-element boundaries. We just require the local approximation property:

$$|u-u_I|_{s,K} \leq Ch_K^{p+1-s}|u|_{p+1,K} \quad \forall K \in \mathcal{T}_h, \quad s = 0,1,2, \tag{4.18}$$

where C depends only on p and the minimum angle of K. Using a discontinuous u_I forces us to take into account the term $|u - u_I|_*^2$ in (4.2). For this, we first use (4.5) to obtain

$$|||u - u_I|||^2 \leq |u - u_I|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |u - u_I|_{2,K}^2 + C_1^{-1} \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [\![u - u_I]\!] \|_{0,e}^2. \quad (4.19)$$

Next, we recall that (see, e.g., equation (2.4) of [3])

$$\|v\|_{0,e}^2 \leq C(h_e^{-1} \|v\|_{0,K}^2 + h_e |v|_{1,K}^2) \quad \forall v \in H^1(K), \quad (4.20)$$

where K is a generic triangle having e as an edge, and C is a constant depending only on the minimum angle of K. Finally, using (4.19) and (4.20), we obtain

$$|||u - u_I|||^2 \leq C(|u - u_I|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |u - u_I|_{2,K}^2 + \sum_{K \in \mathcal{T}_h} h_K^{-2} \|u - u_I\|_{0,K}^2), \quad (4.21)$$

and from (4.18) and (4.21) we have again

$$|||u - u_I||| \leq C_a h^p |u|_{p+1,\Omega}. \quad (4.22)$$

From now on, when speaking about interpolants, we shall always assume that they satisfy (4.18), and hence (4.22).

5. Error estimates. We now prove error estimates for the DG methods by using the properties of consistency, boundedness, stability, and approximation just discussed. First, we consider methods that are consistent, adjoint consistent, and stable; in this case, optimal error estimates follow in the standard way. Then, we study methods that are not consistent and show how to overcome their lack of consistency (and obtain optimal error estimates) by means of a super-penalty procedure. This is the case for the two pure penalty methods at the bottom of Table 3.1, as well as the variant of the NIPG method (consistent but lacking adjoint consistency) which is considered in [53]. Finally, we consider the two unstable methods, Bassi–Rebay [9] and Baumann–Oden [12]. These methods do not have a penalty term and their more subtle convergence behavior requires a finer analysis.

5.1. Stable and completely consistent methods. Methods which are completely consistent and stable can be shown to converge with optimal order with respect to the norm $||| \cdot |||$ in the standard way. Indeed, let u_I be a piecewise $\mathcal{P}_p$ interpolant of u which satisfies (4.18). Then, from the stability condition (4.10), the consistency condition (3.14), the boundedness condition (4.4), and the approximation property (4.22), we have

$$\begin{aligned} C_s |||u_I - u_h|||^2 &\leq B_h(u_I - u_h, u_I - u_h) = B_h(u_I - u, u_I - u_h) \\ &\leq C_b |||u_I - u||| \, |||u_I - u_h||| \leq C h^p |u|_{p+1,\Omega} |||u_I - u_h|||. \end{aligned}$$

Hence, the triangle inequality gives the optimal estimate

$$|||u - u_h||| \leq C h^p |u|_{p+1,\Omega}.$$

Next, we show that when the adjoint consistency condition (3.16) holds, we can obtain optimal order L^2−error estimates by using the standard duality argument. As usual, we define the auxiliary function ψ as the solution of the adjoint problem

$$-\Delta \psi = u - u_h \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega, \quad (5.1)$$

and write, in view of the adjoint consistency condition (3.16),

$$B_h(v, \psi) = (u - u_h, v) \quad \forall v \in V(h). \tag{5.2}$$

We take ψ_I to be a piecewise linear interpolant of ψ. Then, taking $v = u - u_h$ in (5.2), and using the consistency condition (3.14), we obtain

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= B_h(u - u_h, \psi) = B_h(u - u_h, \psi - \psi_I) \\ &\leq C_b |||u - u_h||| \; |||\psi - \psi_I||| \leq Ch|\psi|_{2,\Omega} \, |||u - u_h|||. \end{aligned}$$

As Ω is convex, elliptic regularity gives, $|\psi|_{2,\Omega} \leq C_r \|u - u_h\|_{0,\Omega}$, with C_r depending only on the domain Ω. Hence we get the optimal estimate

$$\|u - u_h\|_{0,\Omega} \leq Ch^{p+1} |u|_{p+1,\Omega}.$$

For the NIPG method, the above argument fails because the method does not satisfy the adjoint consistency condition (3.16). In fact, this method does not achieve optimal order convergence in L^2. However, as we shall see in a moment, the optimal rate of convergence in the L^2-norm can be recovered if we use a penalty term similar to the one used for pure penalty methods.

5.2. Inconsistent methods and super-penalties. As pointed out before, the two pure penalty methods shown on the bottom of Table 3.1 are inconsistent. That is, instead of satisfying the consistency condition (3.13), they satisfy

$$B_h(u, v) = \int_\Omega f \, v \, dx + \int_\Gamma \{\nabla u\} \cdot [\![v]\!] \, ds,$$

whenever u is the exact solution of (1.1) and $v \in H^2(\mathcal{T}_h)$. (This follows immediately from the identity (3.6) with $\tau = \nabla u$.) These methods are, of course, adjoint inconsistent as well: instead of (3.16) they satisfy

$$B_h(v, \psi) = \int_\Omega v \, g \, dx + \int_\Gamma [\![v]\!] \cdot \{\nabla \psi\} \, ds$$

for ψ the solution to (3.15) and $v \in H^2(\mathcal{T}_h)$. The method of Baumann–Oden and its stabilized version NIPG, though consistent, are adjoint inconsistent. For them

$$B_h(v, \psi) = \int_\Omega v \, g \, dx + 2 \int_\Gamma [\![v]\!] \cdot \{\nabla \psi\} \, ds.$$

In this section we show that for the pure penalty methods and NIPG we can choose the penalty large enough to drive down the consistency error to the point where it does not interfere with optimal order convergence. We achieve this by choosing the penalty parameter η_e proportional to a negative power of h_e instead of keeping it bounded as for the consistent methods. However, this super-penalty procedure tends to make the DG method behave like a standard conforming method and significantly increases the condition number of the stiffness matrix.

We take the penalty term as

$$\alpha(u, v) = \sum_{e \in \mathcal{E}_h} \int_e \eta_e h_e^{-2p-1} [\![u]\!] \cdot [\![v]\!] \, ds, \tag{5.3}$$

where the η_e are bounded uniformly above and below by positive constants, and so the analysis of this section applies to the method Babuška and Zlámal and the NIPG method. Similar choices based on α_{r} give similar results; see the analysis of the method of Brezzi et al. given in [19], which is essentially the one we display next.

Having increased the penalty term, if we want to maintain boundedness we have now to take the norm in $V(h)$ as

$$|||v|||_h^2 = |v|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \alpha(v,v). \tag{5.4}$$

Note that the last term is now more heavily weighted than in (4.1) and (4.2).

We begin with an estimate that will be useful in the sequel. Using the new definition of the penalty term (5.3) and the definition of the new norm (5.4), we get, for all $u, v \in V(h)$,

$$\begin{aligned}\sum_{e\in\mathcal{E}_h}\int_e \{\nabla u\}\cdot[\![\, v\,]\!]\,ds &= \sum_{e\in\mathcal{E}_h}\int_e \left({h_e}^{2p+1}\right)^{1/2}\{\nabla u\}\cdot[\![\, v\,]\!]\left({h_e}^{-2p-1}\right)^{1/2} ds\\ &\le C|||v|||_h\left(\sum_{e\in\mathcal{E}_h} {h_e}^{2p+1}\int_e |\{\nabla u\}\cdot n_e|^2\,ds\right)^{1/2} \le Ch^p|||v|||_h\|u\|_{2,h}, \end{aligned}\tag{5.5}$$

where the last inequality follows from the trace inequality (4.7) and, as usual, $\|u\|_{2,h}^2 = \sum_K \|u\|_{2,K}^2$.

We are now ready to obtain the estimate. As in § 4.2, we have stability in the norm (5.4) provided that the lower bound for the η_e is sufficiently large. Hence, in such a case, we can write

$$C_s|||u_I - u_h|||_h^2 \le B_h(u_I - u, u_I - u_h) + B_h(u - u_h, u_I - u_h) =: T_1 + T_2, \tag{5.6}$$

where u_I is the usual *continuous* interpolant of u in V_h. Now, using the continuity of $u - u_I$, the estimate of T_1 is the usual one:

$$T_1 \le C|||u_I - u|||_h|||u_I - u_h|||_h \le Ch^p|||u_I - u_h|||_h|u|_{p+1,\Omega}. \tag{5.7}$$

The term T_2 arises the inconsistency of the method, so it vanishes for NIPG, while for the method of Babuška and Zlámal it can be estimated by using our auxiliary inequality (5.5) as follows:

$$T_2 = \int_\Gamma \{\nabla u\}\cdot[\![\, u_I - u_h\,]\!]\,ds \le Ch^p|||u_I - u_h|||_h\|u\|_{2,\Omega}. \tag{5.8}$$

Hence, inserting (5.7) and (5.8) into (5.6), we get

$$|||u_I - u_h|||_h \le Ch^p\|u\|_{p+1,\Omega},$$

and the optimal order estimate

$$|||u - u_h|||_h \le Ch^p\|u\|_{p+1,\Omega}, \tag{5.9}$$

follows by the triangle inequality.

For the L^2-error estimate of either the Babuška–Zlámal method or NIPG, we proceed in the usual way. If ψ is again the solution of the adjoint problem (5.1), we have

$$\|u - u_h\|^2_{0,\Omega} = B_h(u - u_h, \psi) - c \int_\Gamma \{\nabla\psi\} \cdot [\![u - u_h]\!] \, ds =: T_1 + T_2, \tag{5.10}$$

where c is either 1 or 2 depending on the method. The estimate of the term T_1 is quite easy. Indeed, if ψ_I is the continuous interpolant of ψ in V_h, then $B_h(u, \psi_I) = (f, \psi_I)$ and therefore

$$\begin{aligned} T_1 &= B_h(u - u_h, \psi) = B_h(u - u_h, \psi - \psi_I) \\ &\leq C|||u - u_h|||_h |||\psi - \psi_I|||_h \leq Ch|||u - u_h|||_h \|u - u_h\|_{0,\Omega}. \end{aligned} \tag{5.11}$$

The term T_2 arises from the adjoint inconsistency and can be estimated by means of the auxiliary inequality (5.5) as follows:

$$\begin{aligned} T_2 &= -c \int_\Gamma \{\nabla\psi\} \cdot [\![u - u_h]\!] \, ds \leq Ch^p |||u - u_h|||_h \|\psi\|_{2,\Omega} \\ &\leq Ch^p |||u - u_h|||_h \|u - u_h\|_{0,\Omega}, \end{aligned} \tag{5.12}$$

where again we used elliptic regularity. Inserting (5.11) and (5.12) into (5.10), and using (5.9) we obtain the desired optimal estimate:

$$\|u - u_h\|_{0,\Omega} \leq Ch^{p+1} \|u\|_{p+1,\Omega}.$$

Note that being able to use continuous interpolant u_I and ψ_I, with optimal approximation properties, is a crucial ingredient in the above analysis since (4.21), and hence (4.22), do not hold for the new norm $|||\cdot|||_h$. This is clearly due to the heavier influence of the jump term. For more general decompositions and settings, a continuous interpolant might not be available, and the penalty term $\alpha(u - u_I, u - u_I)$ would have to be estimated. In these more general cases, the analysis would be more difficult and optimal estimates might be unachievable. See, for instance, the analysis in [53] for the NIPG with super-penalty.

5.3. Weakly stable methods. We now briefly present an analysis of the convergence properties of the two remaining methods, namely, the method of Baumann and Oden and the original method of Bassi and Rebay. A common feature of these unstable but consistent methods is that they do not use penalty terms but enjoy the following *weak* stability property:

$$B_h(v, v) \geq C|v|^2_\# \quad \forall v \in V_h, \tag{5.13}$$

where $|\cdot|_\#$ is only a seminorm. In a situation like this, in order to get estimates for the discrete error $u_I - u_h$ in the seminorm, it is reasonable to first write, using (5.13) and consistency,

$$C|u_I - u_h|^2_\# \leq B_h(u_I - u_h, u_I - u_h) = B_h(u_I - u, u_I - u_h), \tag{5.14}$$

and then try to obtain the inequality

$$B_h(u_I - u, u_I - u_h) \leq Ch^p |u|_{p+1,\Omega} |u_I - u_h|_\#, \tag{5.15}$$

which would immediately give the estimate

$$|u_I - u_h|_{\#} \leq Ch^p|u|_{p+1,\Omega}. \tag{5.16}$$

Note that an estimate of the type (5.15) is quite delicate to obtain since, in general, the bilinear form B_h is *not* bounded with respect to the seminorm $|\cdot|_{\#}$. Once this has been achieved, a similar approach can be taken to obtain an L^2−estimate. In the analysis of the two aforementioned methods, we shall follow this approach.

5.3.1. The method of Baumann and Oden. Since the bilinear form B_h for this method is

$$B_h(w,v) = \int_\Omega \nabla_h w \cdot \nabla_h v \, dx + \int_\Gamma \left([\![w]\!] \cdot \{\nabla_h v\} - \{\nabla_h w\} \cdot [\![v]\!] \right) ds, \tag{5.17}$$

we obtain

$$B_h(v,v) = |v|_{1,h}^2 \quad \forall v \in V(h),$$

which implies a weak stability property of the form (5.13). Note that the quantity on the right-hand side of this equation is just a seminorm since it vanishes for all piecewise constant functions, and that the bilinear form B_h cannot be bounded in terms of it. It is therefore not clear how to obtain error estimates for the method by using the standard analysis; in fact, for linear elements, it appears that the method is not convergent. However, in [53] Rivière, Wheeler, and Girault showed how to obtain optimal order error estimates in the $H^1(\mathcal{T}_h)$-norm under the assumption that the polynomial degree $p \geq 2$.

The key idea in the analysis carried out in [53] is the use of an interpolant $u_I \in V_h$ for which the mean value of $\{\nabla_h(u-u_I)\}$ vanishes on each edge. This property, which is also satisfied by a straightforward modification of the Morley interpolant for $p=2$ and by the Fraeijs de Veubeke interpolant for $p=3$, cf. [24, pp. 374–375], is only possible for $p \geq 2$. In view of (5.17) it implies immediately that

$$B_h(u-u_I, v) = 0 \quad \forall\, v \text{ piecewise constant with respect to } \mathcal{T}_h, \tag{5.18}$$

and hence, if P_0 is the orthogonal projection of $L^2(\Omega)$ onto the space of piecewise constant functions, for $v \in V(h)$,

$$B_h(u-u_I, v) \leq C_b |||u-u_I|||\, |||v - P_0 v|||. \tag{5.19}$$

On the other hand, it is easy to see that $|||v - P_0 v||| \leq C|v|_{1,h}$ for $v \in V_h$, so $B_h(u - u_I, v) \leq C|||u-u_I|||\,|v|_{1,h}$ for $v \in V_h$, and finally, using $v = u_I - u_h$ as in (5.14)–(5.16) and then (4.22)

$$|u - u_h|_{1,h} \leq Ch^p|u|_{p+1,\Omega}. \tag{5.20}$$

Notice that, for discontinuous elements, this estimate is rather weak. It is therefore important to obtain a bound on the L^2-norm of the error as well. Special care has to be taken because of the lack of adjoint consistency of the method. Let again ψ be the solution of the problem (5.1), and let ψ_I be an interpolant satisfying a property of the type (5.18). As usual, from elliptic regularity we have $\|\psi\|_{2,\Omega} \leq C\|u-u_h\|_{0,\Omega}$, and so $|||\psi - \psi_I||| \le Ch\|u - u_h\|_{0,\Omega}$. Now, using (3.13), then (5.17), and finally (3.14) we have

$$\begin{aligned}\|u - u_h\|_{0,\Omega}^2 &= B_h(\psi, u - u_h) = B_h(\psi, u - u_h) + B_h(u - u_h, \psi) - B_h(u - u_h, \psi) \\ &= 2\int_\Omega \nabla\psi \cdot \nabla_h(u - u_h)\,dx - B_h(u - u_h, \psi - \psi_I).\end{aligned}$$

A suboptimal estimate for the first term can be easily obtained using (5.20):

$$\int_\Omega \nabla\psi \cdot \nabla_h(u - u_h)\,dx \le |\psi|_{1,\Omega}|u - u_h|_{1,h} \le C\|u - u_h\|_{0,\Omega}h^p|u|_{p+1,\Omega}.$$

To deal with the second term, we first notice that, although B_h is not symmetric, properties (5.18) and (5.19) do hold for $B_h^t(u, v) \equiv B_h(v, u)$. Using this fact and proceeding as above, we get

$$\begin{aligned}B_h(u - u_h, \psi - \psi_I) &\le C_b|||(u - u_h) - P_0(u - u_h)|||\;|||\psi - \psi_I||| \\ &\le Ch^{p+1}|u|_{p+1,\Omega}\|u - u_h\|_{0,\Omega}.\end{aligned}$$

Combining the last four estimates we obtain a suboptimal estimate in L^2 (to be expected due to the lack of adjoint consistency) but an optimal estimate in $H^1(\mathcal{T}_h)$:

$$\|u - u_h\|_{0,\Omega} \le Ch^p|u|_{p+1,\Omega}, \quad |||u - u_h||| \le Ch^p|u|_{p+1,\Omega}.$$

For a more detailed and comprehensive analysis of this method, see [53]. Let us point out again that the addition of a strong penalty term can compensate for the loss of optimality in L^2; see also [53] for similar results.

5.3.2. The original method of Bassi and Rebay. To conclude our analysis, we consider now the DG method introduced by Bassi and Rebay in [9]. Like the method of Baumann and Oden, this method violates the stability condition (4.10) and is only weakly stable. However, for this method the violation is more delicate since the set of functions for which the corresponding seminorm is zero has a more complex structure. Indeed, for w and v in V_h, the bilinear form for this method is

$$B_h(w, v) = \int_\Omega (\nabla_h w + R(w)) \cdot (\nabla_h v + R(v))\,dx, \tag{5.21}$$

which implies the weak stability property, as in (5.13),

$$\|\nabla_h v + R(v)\|_0^2 = B_h(v, v) \quad \forall v \in V_h. \tag{5.22}$$

We see that $B_h(v, v)$ vanishes on the set

$$Z := \{v \in V_h :\ \nabla_h v + R(v) = 0\}, \tag{5.23}$$

which, in general, can be non-empty; see, for instance, [18].

In spite of this unfortunate situation, we show that the existence of a solution to the discrete problem, together with optimal rates of convergence, can still be obtained for suitable functions f. We proceed as follows. First, we find under what conditions on f an approximate solution u_h exits. Integrating by parts and recalling the definition $R(v)$, (4.13), and (3.3), we obtain that for every $v \in V_h$, and for every $\tau \in \Sigma_h$,

$$\int_\Omega (\nabla_h v + R(v)) \cdot \tau\,dx = -\int_\Omega v\,\nabla_h \cdot \tau\,dx + \int_{\Gamma^0} \{v\}[\![\tau]\!]\,ds.$$

Suitable choices for τ give that the condition $v \in Z$ is then equivalent to have both

$$\int_K vq\,dx = 0 \quad \forall q \in P_{p-1}(K)\ \forall K \qquad \text{and} \qquad \{v\}|_e = 0 \quad \forall e \in \mathcal{E}_h^\circ.$$

Thus, if f is a piecewise polynomial of degree $p-1$, then $(f,v) = 0$ for all $v \in Z$, defined in (5.23). Hence, for such f, the solution u_h exists and is unique up to an element of Z.

To get error estimates, we need a special interpolation operator acting on gradients. We observe that if g is a piecewise polynomial of degree $p-1$, and $w \in H_0^1(\Omega)$ is the solution of $-\Delta w = g$ in Ω, then we can find $\sigma_I = \sigma_I(w)$ in $\Sigma_h \cap H(\mathrm{div};\Omega)$ such that

$$-\nabla \cdot \sigma_I(w) = g \qquad \text{and} \qquad \|\nabla w - \sigma_I(w)\|_{0,\Omega} \le Ch^k|w|_{k+1,\Omega} \quad k \le p. \tag{5.24}$$

Indeed, thanks to the fact that g is locally in $\mathcal{P}_{p-1}$, such a construction is possible; for instance, a Brezzi–Douglas–Marini element of degree p or a Raviart–Thomas element of degree between $p-1$ and p could be used; see [17]. Thus, we easily get

$$\int_\Omega [\nabla w - \sigma_I(w)] \cdot \nabla_h v\,dx - \int_\Gamma [\nabla w - \sigma_I(w)] [\![v]\!]\,ds = 0 \quad \forall v \in V_h,$$

which, using (5.21) and (4.13) can be rewritten as follows:

$$B_h(w,v) = \int_\Omega \sigma_I(w) \cdot [\nabla_h v + R(v)]\,dx \quad \forall v \in V_h. \tag{5.25}$$

We are now ready to obtain our error estimates. Let u_I be again the interpolant of u in $V_h \cap C^0(\Omega)$. We begin by obtaining an estimate of the L^2-norm of $\nabla_h\chi_h + R(\chi_h)$, where $\chi_h := u_I - u_h$. Using (5.22), consistency, (5.25), and (5.24), we obtain

$$\begin{aligned} \|\nabla_h\chi_h + R(\chi_h)\|_{0,h}^2 &= B_h(\chi_h,\chi_h) = B_h(u_I - u, \chi_h) \\ &= \int_\Omega [\nabla u_I - \sigma_I(u)] \cdot [\nabla_h\chi_h + R(\chi_h)]\,dx \le Ch^p|u|_{p+1,\Omega}\|\nabla_h\chi_h + R(\chi_h)\|_{0,h}, \end{aligned}$$

which implies our first estimate

$$\|\nabla_h\chi_h + R(\chi_h)\|_{0,h} \le Ch^p|u|_{p+1,\Omega}. \tag{5.26}$$

Since, by (3.9), $\sigma_h(u_h) = \nabla_h u_h + R(u_h)$, we have from the above estimate that

$$\begin{aligned} \|\nabla u - \sigma_h(u_h)\|_{0,h} &\le \|\nabla u - \nabla u_I\|_{0,h} + \|\nabla u_I - \sigma_h(u_h)\|_{0,h} \\ &= \|\nabla u - \nabla u_I\|_{0,h} + \|\nabla_h\chi_h + R(\chi_h)\|_{0,h} \le Ch^p|u|_{p+1,\Omega}. \end{aligned} \tag{5.27}$$

Finally, to estimate the approximation to u in the L^2-norm, we must filter out the spurious oscillatory modes in Z that the approximate solution u_h might have. This filtering can be done simply by L^2-projecting the error into the space of functions that are piecewise polynomials of degree at most $p-1$. In other words, if we denote by P_{p-1} such a projection, we simply estimate $P_{p-1}(u-u_h)$ instead of $u - u_h$. We then take ψ to be the solution of

$$-\Delta\psi = P_{p-1}(u-u_h) \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega,$$

and ψ_I its continuous piecewise linear interpolant. Using adjoint consistency, then (5.25) and (5.21), and finally (5.26) and interpolation estimates we obtain

$$\begin{aligned}\|P_{p-1}(u-u_h)\|_{0,\Omega}^2 &= (P_{p-1}(u-u_h), u-u_h) = B_h(u-u_h,\psi)\\ &= B_h(u-u_h,\psi-\psi_I) = B_h(\chi_h,\psi-\psi_I) + B_h(u-u_I,\psi-\psi_I)\\ &= \int_\Omega [\sigma_I(\psi)-\nabla\psi_I]\cdot[\nabla_h\chi_h + R(\chi_h)]\,dx + \int_\Omega \nabla(u-u_I)\cdot\nabla(\psi-\psi_I)\,dx\\ &\le Ch^{p+1}|u|_{p+1,\Omega}|\psi|_{2,\Omega},\end{aligned}$$

and, by elliptic regularity

$$\|P_{p-1}(u-u_h)\|_{0,\Omega} \le Ch^{p+1}|u|_{p+1,\Omega}. \tag{5.28}$$

In other words, the L^2-projection of the error into the space of piecewise polynomials of degree $p-1$ superconverges.

6. Summary and conclusions. In this paper, we propose a general framework that allows us to obtain a unified analysis of virtually all the methods found in the literature for dealing with linear elliptic problems by means of DG methods.

We have shown that all these DG methods can be obtained by suitably choosing the numerical fluxes in the flux formulation (1.2)–(1.3). We also made clear the connection between the flux and primal formulations and between consistency and conservativity of the numerical fluxes and consistency and adjoint consistency, respectivity, of the primal formulation.

We have shown that DG methods that are completely consistent and stable achieve optimal error estimates. Inconsistent DG methods, like the pure penalty methods, can still achieve optimal error estimates provided they are super-penalized which, however, makes the methods similar to the standard conforming methods. The same holds true for methods that lack adjoint consistency like the super-penalized version of the NIPG method.

The method of Baumann and Oden and its stabilized version, NIPG are both consistent but, since they use non-conservative numerical fluxes $\widehat{u}$, they are not adjoint-consistent. The lack of adjoint consistency of these two methods is reflected in a suboptimal rate of convergence in the L^2-norm.

The stabilization of DG methods via the inclusion of a penalty term is crucial: without its convergence is degraded or lost. In terms of fluxes, stability is related to the suitable choice of the stabilizing term α in the vector flux. Fortunately, there are no apparent drawbacks to the inclusion of such terms. We considered here two forms of stabilization terms, one arising in the IP methods and the other in the method of Bassi et al. These are equivalent within a constant multiple as far as the analysis is concerned, although in some cases the condition on the coefficient η required for stability can be made more explicit for the second form. A comparison of the relative efficacy of the two approaches to stabilization remains to be made.

Two methods without stabilizing terms were considered: the method of Baumann and Oden and the original method of Bassi and Rebay. The former method is unstable for $p = 1$, but recovers what we have called weak stability for $p \ge 2$. In this case, optimal error bounds can be proved in the H^1-norm. The least stable method seems to be the first method of Bassi and Rebay, which might have a singular matrix on certain grids. However, the use of a right-hand side locally polynomial of degree $p-1$ makes the system compatible, and then stability and optimal error bounds are achieved in a suitable seminorm (essentially obtained by projecting the error onto the space of piecewise polynomials of degree $p-1$).

These results are summarized in the Table 6.1, which reports, for the various methods: consistency, adjoint consistency, stability, type of stabilization term, theoretical requirement on $\eta_0 = \inf_e \eta_e$ for stability, and rates of convergence in $H^1(\mathcal{T}_h)$ and in L^2 . In the last row the brackets around the convergence rate are to remind us that the estimates (5.27) and (5.28) are bounds only on certain seminorms of the error.

TABLE 6.1
Properties of the DG methods

Method	cons.	a.c.	stab.	type	cond.	H^1	L^2
Brezzi et al. [18]	✓	✓	✓	α^{r}	$\eta_0 > 0$	h^p	h^{p+1}
LDG [35]	✓	✓	✓	α^{j}	$\eta_0 > 0$	h^p	h^{p+1}
IP [43]	✓	✓	✓	α^{j}	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [10]	✓	✓	✓	α^{r}	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [53]	✓	×	✓	α^{j}	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [6]	×	×	✓	α^{j}	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [19]	×	×	✓	α^{r}	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann–Oden ($p=1$)	✓	×	×	-	-	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [9]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

Although we have only considered the model problem of the Laplacian with homogeneous Dirichlet boundary conditions on a convex polygon, extension of our framework and analysis to more general scalar elliptic operators and more general boundary conditions can be easily carried out. There can, however, be several variants of this extension since the definition of the auxiliary variable σ_h can take several forms, see, for example, [35] and [27]. Of course, DG methods can also be easily defined for numerically approximating the solutions of more complex problems like, for example, the system of linear elasticity, the Stokes system, and plate problems. Much of the approach proposed in this paper can be carried over to those situations.

Finally, we conclude by pointing out that the theoretical comparison carried out in this paper ought to be complemented by a numerical study of the effects that the different choices of the numerical fluxes have on the quality of the approximation and the efficiency of its computation.

Notation index. For convenience we list here some of the most used notations and the equation number nearest their definition.

Γ, Γ^0, $\mathcal{T}(\Gamma)$, $[\![\cdot]\!]$, $\{\cdot\}$: (3.1)	$\lvert\cdot\rvert_{1,h}$, $\lvert\cdot\rvert_*$: (4.1)	$\lvert\lvert\lvert \cdot \rvert\rvert\rvert$: (4.2)	
r, l: (3.8)	r_e: (3.23)	l_e: (4.8)	R, L_β: (4.13)
α^{j}: (3.20)	α_{j}: (4.12)	α^{r}: (3.24)	α_{r}: (4.11)

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