MOMENT ATTRACTIVITY, STABILITY AND CONTRACTIVITY EXPONENTS OF STOCHASTIC DYNAMICAL SYSTEMS

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(Communicated by Shouchuan Hu)

Abstract. Nonlinear stochastic dynamical systems as ordinary stochastic differential equations and stochastic difference equations are in the center of this presentation in view of the asymptotic behavior of their moments. We study the exponential p-th mean growth behavior of their solutions as integration time tends to infinity. For this purpose, the concepts of attractivity, stability and contractivity exponents for moments are introduced as generalizations of well-known moment Lyapunov exponents of linear systems. Under appropriate monotonicity assumptions we gain uniform estimates of these exponents from above and below. Eventually, these concepts are generalized to describe the exponential growth behavior along certain Lyapunov-type functionals.

1. INTRODUCTION

The analysis of stochastic dynamical systems with respect to their asymptotic behavior has attracted many researchers, see e.g. Arnold [1] - [6], Baxendale [7] - [9], Freidlin and Wentzell [12], Imkeller and Scheutzow [16], Khas'minskij [21], Kifer [22], Mao [27] or, for a recent and comprehensive treatment, see Arnold [6]. Among them systems which have a 'finite asymptotic structure' when integration time tends to infinity are given by the class of dissipative ones (see Hale [15]). Roughly speaking, these systems have some compact attracting sets such that their trajectories at least in the vicinity of these sets will approach to these sets and stay there afterwards. Under randomness we can observe a similar behavior, mainly classified by the behavior of their moments and trajectories.

We will examine the case of 'moment dissipativity' here, since this approach permits to carry over the deterministic analysis to stochastic one in a fairly straight forward manner. We are also inspired by the results from Hale [15], Krylov [25] and Schurz [32], [33] (Note: The terminology 'moment dissipativity' becomes clear from latter two works.). Especially, the interest of this contribution lies in the estimation of maximum and minimum exponential growth rates of 'moment dissipative stochastic systems'. We will distinguish between the exponents characterizing the exponential growth behavior of the distance of the state process to deterministic sets (a property also called attractivity) and those exponents characterizing the

1991 *Mathematics Subject Classification.* 34F05, 37H10, 60H10, 65C30, 93D20.

Key words and phrases. Stochastic dynamical systems, asymptotic behavior, characteristic exponents, moment attractivity exponents, moment stability exponents, moment contractivity exponents, moment Lyapunov exponents, sample Lyapunov exponents, (ordinary) stochastic differential equations, stochastic numerical methods, explicit-implicit methods.

Research partially supported by the University of Minnesota, University of Kaiserslautern, University of Los Andes and DFG.

exponential growth behavior of initial perturbations (a property here called contractivity). In the linear situation with trivial steady state these exponents coincide with the generally common moment Lyapunov exponents (see [3], [4]). It is worth noting that there is a well–known relation between moment and sample Lyapunov exponents (for a formula, see [2]). The existence of sample Lyapunov exponents can be justified by the fundamental multiplicative ergodic theorem of Oseledec [30] which induces a decomposition of the original domain of definition into random Oseledec spaces. We will not go into details of the structure of these spaces. In the nonlinear situation sample Lyapunov exponents replicate the behavior of the corresponding linearized dynamics, hence in general the local behavior in a small neighborhood of equilibria is determined in general. We are rather interested in the global exponential growth behavior of moments characterized by some significant deterministic numbers without using the idea of linearization and without using anticipative calculus. The presented concepts turn out to be very appropriate for the study of asymptotic moment behavior of stochastic dynamical systems both in continuous and discrete time as integration time tends to infinity.

The paper is organized as follows. In section 2 we introduce the definition of moment attractivity and stability exponents and give first uniform estimations. Section 3 presents the concept of moment contractivity exponents together with some uniform estimates. Both section 2 and 3 treat stochastic differential equations (SDEs) and the simplest, discrete time, iterative, random mappings (such as numerical methods for SDEs) using monotonicity conditions as in Krylov [25] or Schurz [32], [33]. The paper continues with section 4 containing examples of linear systems illustrating that some estimates are sharp and illustrating relations between the concepts of attractivity (stability) and contractivity. Sections 5 and 6 deal with a generalization to V-exponents, illustrations of their usefulness and almost sure V-attractivity. Eventually, final remarks finish this paper by section 7.

2. Moment attractivity and moment stability exponents

Throughout this paper let $< .,. >$ denote the Euclidean scalar product defined by $< x, y >= \sum_{i=1}^{d} x_i y_i$ for vectors x, y in $\mathbb{R}^d$, $d \in \mathbb{N}_+ \setminus \{0\}$, and $\|.\|$ the Euclidean vector norm in $\mathbb{R}^d$. Furthermore, $(.)_+$ or $[.]_+$ will denote the positive part of inscribed expression, $(.)_-$ or $[.]_-$ the negative part. Fix a $p \in \mathbb{R}_+, p > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and suppose $\mathcal{T}$ to be a discrete or continuous, deterministic time scale, respectively. Consider a stochastic process $(X(t))_{(t \in \mathcal{T})}$ defined for all $t \in \mathcal{T} \subset [t_0, +\infty)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$ (a.s.), started with values in the domain $\mathbb{D}_1 \subseteq \mathbb{R}^d$ and with finite p-th absolute moments for all finite times $t \in \mathcal{T}$. Throughout the paper we shall use the word 'domain' $\mathbb{D}$ with knowledge that the stochastic process $(X(t))_{(t \in \mathcal{T})}$ can be viewed as a measurable family of multi-valued functions mapping from the domain $\mathbb{D}_1 \subseteq \mathbb{R}^d$ into its range contained in $\mathbb{R}^d$ for each element $\omega \in \Omega$.

Definition 1. The *upper (forward p–th moment) attractivity exponent* of the given stochastic process $(X(t))_{(t \in \mathcal{T})}$ with respect to the nonempty, bounded, deterministic set $\mathbb{D}_2 \subset \mathbb{R}^d$ is defined to be

$$\overline{\lambda}_p(x) := \limsup_{t \to +\infty} \frac{1}{t} \ln \Big(\inf_{y \in \mathbb{D}_2} \mathbb{E} \, \|X(t) - y\|^p \Big) \tag{2.1}$$

for $X(t_0) = x \in \mathbb{D}_1$ (a.s.). The *lower (forward p–th moment) attractivity exponent* of the given stochastic process $(X(t))_{(t \in \mathcal{T})}$ with respect to the nonempty, bounded, deterministic set $\mathbb{D}_2 \subset \mathbb{R}^d$ is defined to be

$$\underline{\lambda}_p(x) \ := \ \liminf_{t\to+\infty} \frac{1}{t} \ln \left(\inf_{y\in\mathbb{D}_2} \ \mathbb{E} \ \|X(t)-y\|^p \right) \tag{2.2}$$

for $X(t_0) = x \in \mathbb{D}_1$ (a.s.).

Remark. A similar definition one could introduce for stochastic fields where the time scale is a partially ordered set. In the case of $\mathbb{D}_2 = \{x_*\}$ with $X(t) = x_*$ for all $t \in \mathcal{T}$ (i.e. x_* is a steady state) we call the $\mathbb{D}_2$-attractivity exponent also *stability exponent*. For simplicity, we will carry out a first analysis for the case when the set $\mathbb{D}_2 = \{0\}$ and we may drop off the phrase 'with respect to the set $\mathbb{D}_2$' then.

2.1. **$\{0\}$-Attractivity (stability) exponents of SDEs.** Let us look at nonlinear stochastic differential equations (SDEs) with monotone coefficients. For general theory of SDEs, see Arnold [1, 6], Dynkin [11], Friedman [13], Freidlin and Wentzell [12], Gard [14], Khas'minskij [21], Krylov [26] or Protter [31]. For simplicity, we restrict us to SDEs driven by scalar, independent Wiener processes $W^j = (W^j_t)_{t\geq t_0}$.

Theorem 2.1. *Let the stochastic process $(X(t))_{(t\geq t_0)}$ satisfy the Itô SDE*

$$dX(t) \ = \ a(t,X(t))\,dt + \sum_{j=1}^{m} b^j(t,X(t))\,dW^j_t\,, \tag{2.3}$$

exclusively started with values in the deterministic set $\mathbb{D}_1 \subseteq \mathbb{R}^d$, where the deterministic coefficients a, b^j are such that a strong solution of this SDE with finite p–th absolute moments exists. Assume that

$$< a(t,x),x > + \frac{1}{2}\sum_{j=1}^{m} \|b^j(t,x)\|^2 + \frac{(p-2)}{2}\sum_{j=1}^{m} \frac{< b^j(t,x),x >^2}{\|x\|^2}$$

$$\leq \ \overline{K}_p(t)\|x\|^2 \tag{2.4}$$

for all $t \in [t_0,+\infty)$, for all $x \in \mathbb{D}_1$, where the deterministic function $\overline{K}_p(t)$ is L^1–integrable on $[t_0,+\infty)$ with respect to the Lebesgue measure, and

$$\mathbb{P}\{\omega \in \Omega : X(t)(\omega) \in \mathbb{D}_1, \forall t \in [t_0,+\infty) | X(t_0) = x \in \mathbb{D}_1\} \ = \ 1\,. \tag{2.5}$$

Then we have

$$\sup_{x\in\mathbb{D}_1} \ \overline{\lambda}_p(x) \ \leq \ p \limsup_{t\to+\infty} \frac{\int_{t_0}^t \overline{K}_p(s)ds}{t}\,. \tag{2.6}$$

Furthermore, if

$$< a(t,x),x > + \frac{1}{2}\sum_{j=1}^{m} \|b^j(t,x)\|^2 + \frac{(p-2)}{2}\sum_{j=1}^{m} \frac{< b^j(t,x),x >^2}{\|x\|^2} \ \geq \ \underline{K}_p(t)\|x\|^2 \tag{2.7}$$

for all $t \in [t_0,+\infty)$, for all $x \in \mathbb{D}_1$, where the deterministic function $\underline{K}_p(t)$ is L^1–integrable on $[t_0,+\infty)$ with respect to the Lebesgue measure, then we have

$$\inf_{x\in\mathbb{D}_1\setminus\{x_*\neq 0\}} \underline{\lambda}_p(x) \ \geq \ p \liminf_{t\to+\infty} \frac{\int_{t_0}^t \underline{K}_p(s)ds}{t}\,, \underline{\lambda}_p(x_* = 0) \ = \ -\infty\,. \tag{2.8}$$

Proof. The main idea is to apply Dynkin's formula [11], to evaluate the arising differential operators under the required monotonicity and to apply a generalized Gronwall–Bellman Lemma (see [27], [32]). The required L^1–integrability ensures us that the expressions in the definition of attractivity exponents exist, and together with the existence of finite p–th absolute moments, that we can apply Dynkin's formula at any time $t \in [t_0, +\infty)$. The application of linear partial differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + < a(t,x), \nabla_x > + \frac{1}{2} \sum_{j=1}^{m} \sum_{k,l=1}^{d} b_k^j(t,x) b_l^j(t,x) \frac{\partial^2}{\partial x_k \partial x_l}$$

to $\|x\|^p, p > 0$ for corresponding diffusion process $X(t)$ exactly gives $\mathcal{L}\|x\|^p =$

$$p \left(< a(t,x), x > + \frac{1}{2} \sum_{j=1}^{m} \|b^j(t,x)\|^2 + \frac{(p-2)}{2} \sum_{j=1}^{m} \frac{< b^j(t,x), x >^2}{\|x\|^2} \right) \|x\|^{p-2}$$

after some laborious calculations, hence

$$\mathcal{L}\|x\|^p \ \leq \ p\overline{K}_p(t)\|x\|^p \quad \text{and} \quad \mathcal{L}\|x\|^p \ \geq \ p\underline{K}_p(t)\|x\|^p \, ,$$

respectively, presuming the validity of inequalities (2.4) and (2.7). Note that

$$\nabla_x \ = \ \Big(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d} \Big)^T$$

represents the d-dimensional gradient vector in $x = (x_1, ..., x_d)^T$–direction. By the formula of Dynkin we know that

$$\mathbb{E} \, \|X(t)\|^p \ = \ \mathbb{E} \, \|X(s)\|^p + \mathbb{E} \int_s^t \mathcal{L}\|X(u)\|^p du$$

for all s, t with $t \geq s; s, t \in [t_0, +\infty)$. Under the monotonicity assumptions (2.4) and (2.7), this implies

$$\overline{v}(t) := \mathbb{E} \, \|X(t)\|^p \ \leq \ \mathbb{E} \, \|X(s)\|^p + p \int_s^t \overline{K}_p(u) \mathbb{E} \, \|X(u)\|^p du \, ,$$

$$\underline{v}(t) := \mathbb{E} \, \|X(t)\|^p \ \geq \ \mathbb{E} \, \|X(s)\|^p + p \int_s^t \underline{K}_p(u) \mathbb{E} \, \|X(u)\|^p du \, ,$$

respectively. Now one applies the generalized Gronwall–Bellman Lemma (see [27], [32]) to $\overline{v}(t)$ and $\underline{v}(t)$, respectively, takes the logarithm and the limit as time t tends to $+\infty$ and encounters with desired result which completes the proof. ⋄

Remark. The estimates of Theorem 2.1 are 'worst case estimates' (but sharp ones, see linear systems as in subsection 4.1). While requiring L^1-integrability throughout this paper, we refer to the spaces $L^1_{loc}([t_0, +\infty), \mathcal{B}([t_0, +\infty)), \mu)$ with respect to the Lebesgue measure μ, equipped with the Borel σ-field $\mathcal{B}([t_0, +\infty))$.

2.2. $\{0\}$-**Attractivity (stability) exponents of stochastic iterative mappings.** Let us now look at nonlinear stochastic difference equations with monotone coefficients. Such difference equations arise in numerical methods for SDEs or in time series. For stochastic-numerical methods, see Kloeden, Platen and Schurz [24], Mil'shtein [29], Schurz [32], [36] and Talay [38], among many others. Set $X_0 = x$. Suppose that $x \neq 0$ if $X_n(0) = 0$ for all $n \in \mathbb{N}$. Consider d-dimensional iterations

$$\begin{aligned} X_{n+1} \ &= \ X_n \ + \ \Phi_0^I(X_i : i \leq n+1)\Delta_n \ + \Phi_0^E(X_i : i \leq n)\Delta_n \\ &\quad + \sum_{j=1}^{m} \Phi_j(X_i : i \leq n)\xi_n^j \sqrt{\Delta_n} \end{aligned} \tag{2.9}$$

where $\Delta_n := t_{n+1} - t_n$ can be interpreted as a sequence of step sizes with monotonically increasing time-instants $(t_i)_{i \in \mathbb{N}}$ and $\lim_{i \to +\infty} t_i = +\infty$; $\Phi_0^I, \Phi_0^E, \Phi_j$ where

$j = 1, 2, ..., m$ represent deterministic mappings from all so far generated values into $\mathbb{R}^d$, and ξ_n^j are real-valued, independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E}\, \xi_n^j = 0 \quad \text{and} \quad \mathbb{E}\, |\xi_n^j|^2 = (\sigma_n^j)^2 < +\infty\,.$$

Theorem 2.2. *Let process* $(X_n)_{n \in \mathbb{N}}$ *satisfy the stochastic difference equation (2.9) under the above mentioned conditions for all* $n \in \mathbb{N}$, *whereas all* ξ_n^j *are independent of* X_0 *as well. Assume that* $\forall n \in \mathbb{N}\, \forall x^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\ \forall j = 1, 2, ..., m$:

$$\begin{array}{rcl} < \Phi_0^I(x^{(l)} : l \le n+1), x^{(n+1)} > & \le & \overline{k}_I(n) \|x^{(n+1)}\|^2 \\ < \Phi_0^E(x^{(l)} : l \le n), x^{(n)} > & \le & \overline{k}_E(n) \|x^{(n)}\|^2 \\ \|\Phi_0^I(x^{(l)} : l \le n+1)\|^2 & \ge & \overline{k}_0^I(n) \|x^{(n+1)}\|^2 \\ \|\Phi_0^E(x^{(l)} : l \le n)\|^2 & \le & \overline{k}_0^E(n) \|x^{(n)}\|^2 \\ \|\Phi_j(x^{(l)} : l \le n)\|^2 & \le & \overline{k}_j(n) \|x^{(n)}\|^2 \\ 2\overline{k}_I(n)\Delta_n & < & 1 + \overline{k}_0^I(n)\Delta_n^2 \end{array} \tag{2.10}$$

where $\overline{k}_I, \overline{k}_E(n), \overline{k}_0^I(n), \overline{k}_0^E(n), \overline{k}_j(n)$ *are finite, deterministic, real numbers. Then*

$$\overline{\lambda}_2(x) \;\le\; \limsup_{n \to +\infty} \frac{\displaystyle\sum_{i=0}^{n} \Big(\frac{2\overline{k}_E(i) + 2\overline{k}_I(i) + (\overline{k}_0^E(i) - \overline{k}_0^I(i))\Delta_i + \sum_{j=1}^{m} (\sigma_i^j)^2 \overline{k}_j(i)}{1 - 2\overline{k}_I(i)\Delta_i + \overline{k}_0^I(i)\Delta_i^2} \Big) \Delta_i}{t_{n+1}}\,. \tag{2.11}$$

Furthermore, if $\forall n \in \mathbb{N}\, \forall x^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\ \forall j = 1, 2, ..., m$:

$$\begin{array}{rcl} < \Phi_0^I(x^{(l)} : l \le n+1), x^{(n+1)} > & \ge & \underline{k}_I(n) \|x^{(n+1)}\|^2 \\ < \Phi_0^E(x^{(l)} : l \le n), x^{(n)} > & \ge & \underline{k}_E(n) \|x^{(n)}\|^2 \\ \|\Phi_0^I(x^{(l)} : l \le n+1)\|^2 & \le & \underline{k}_0^I(n) \|x^{(n+1)}\|^2 \\ \|\Phi_0^E(x^{(l)} : l \le n)\|^2 & \ge & \underline{k}_0^E(n) \|x^{(n)}\|^2 \\ \|\Phi_j(x^{(l)} : l \le n)\|^2 & \ge & \underline{k}_j(n) \|x^{(n)}\|^2 \\ -2\underline{k}_E(n)\Delta_n & < & 1 + \underline{k}_0^E(n)\Delta_n^2 + \sum_{j=1}^{m} (\sigma_n^j)^2 \underline{k}_j(n)\Delta_n \end{array} \tag{2.12}$$

where $\underline{k}_I, \underline{k}_E(n), \underline{k}_0^I(n), \underline{k}_0^E(n), \underline{k}_j(n)$ *are finite, deterministic, real numbers, then*

$$\underline{\lambda}_2(x) \;\ge\; \liminf_{n \to +\infty} \frac{\displaystyle\sum_{i=0}^{n} \Big(\frac{2\underline{k}_E(i) + 2\underline{k}_I(i) + (\underline{k}_0^E(i) - \underline{k}_0^I(i))\Delta_i + \sum_{j=1}^{m} (\sigma_i^j)^2 \underline{k}_j(i)}{1 + 2\underline{k}_E(i)\Delta_i + \underline{k}_0^E(i)\Delta_i^2 + \sum_{j=1}^{m} (\sigma_i^j)^2 \underline{k}_j(i)\Delta_i} \Big) \Delta_i}{t_{n+1}}\,. \tag{2.13}$$

Proof. Define $v_n := \mathbb{E}\,\|X_n\|^2$. Taking the square of the Euclidean vector norm of random vector $X_{n+1} - \Phi_0^I(X_i : i \le n+1)\Delta_n$ and its expectation value gives

$$\begin{aligned}
& v_{n+1}\left(1 - 2\Delta_n \overline{k}_I(n) + \Delta_n^2 \overline{k}_0^I(n)\right) \\
\le\ & v_{n+1} - 2\Delta_n \mathbb{E} < \Phi_0^I(X_i : i \le n+1), X_{n+1} > + \Delta_n^2 \mathbb{E}\,\|\Phi_0^I(X_i : i \le n+1)\|^2 \\
=\ & v_n + 2\Delta_n \mathbb{E} < \Phi_0^E(X_i : i \le n), X_n > + \Delta_n^2 \mathbb{E}\,\|\Phi_0^E(X_i : i \le n)\|^2 \\
& \quad + \Delta_n \sum_{j=1}^m (\sigma_n^j)^2 \mathbb{E}\,\|\Phi_j(X_i : i \le n)\|^2 \\
\le\ & v_n\left(1 + 2\Delta_n \overline{k}_E(n) + \Delta_n^2 \overline{k}_0^E(n) + \Delta_n \sum_{j=1}^m (\sigma_n^j)^2 \overline{k}_j(n)\right)
\end{aligned}$$

using the mutual independence of random variables ξ_n^j, that $\mathbb{E}\,(\xi_n^j)^2 = (\sigma_n^j)^2$ and conditions (2.10). A similar estimate is derived under conditions (2.12). We obtain

$$\begin{aligned}
& v_n\left(1 + 2\Delta_n \underline{k}_E(n) + \Delta_n^2 \underline{k}_0^E(n) + \Delta_n \sum_{j=1}^m (\sigma_n^j)^2 \underline{k}_j(n)\right) \\
\le\ & v_n + 2\Delta_n \mathbb{E} < \Phi_0^E(X_i : i \le n), X_n > + \Delta_n^2 \mathbb{E}\,\|\Phi_0^E(X_i : i \le n)\|^2 \\
& \quad + \Delta_n \sum_{j=1}^m (\sigma_n^j)^2 \mathbb{E}\,\|\Phi_j(X_i : i \le n)\|^2 \\
=\ & v_{n+1} - 2\Delta_n \mathbb{E} < \Phi_0^I(X_i : i \le n+1), X_{n+1} > + \Delta_n^2 \mathbb{E}\,\|\Phi_0^I(X_i : i \le n+1)\|^2 \\
\le\ & v_{n+1}\left(1 - 2\Delta_n \underline{k}_I(n) + \Delta_n^2 \underline{k}_0^I(n)\right)
\end{aligned}$$

Thus

$$\begin{aligned}
v_{n+1} \ \le\ & v_n \cdot \left(\frac{1 + 2\overline{k}_E(n)\Delta_n + \overline{k}_0^E(n)\Delta_n^2 + \sum_{j=1}^m (\sigma_n^j)^2 \overline{k}_j(n)\Delta_n}{1 - 2\overline{k}_I(n)\Delta_n + \overline{k}_0^I(n)\Delta_n^2}\right) \\
\le\ & v_0 \cdot \prod_{i=0}^n \left(\frac{1 + \overline{k}_E(i)\Delta_i + \overline{k}_0^E(i)\Delta_i^2 + \sum_{j=1}^m (\sigma_i^j)^2 \overline{k}_j(i)\Delta_i}{1 - 2\overline{k}_I(i)\Delta_i + \overline{k}_0^I(i)\Delta_i^2}\right)_+ \\
\le\ & v_0 \cdot \exp\left(\sum_{i=0}^n \Big(\frac{2\overline{k}_E(i) + 2\overline{k}_I(i) + (\overline{k}_0^E(i) - \overline{k}_0^I(i))\Delta_i + \sum_{j=1}^m (\sigma_i^j)^2 \overline{k}_j(i)}{1 - 2\overline{k}_I(i)\Delta_i + \overline{k}_0^I(i)\Delta_i^2}\Big)\Delta_i\right)
\end{aligned}$$

and in the other direction

$$v_n \ \le\ v_{n+1} \cdot \left(\frac{1 - 2\underline{k}_I(n)\Delta_n + \underline{k}_0^I(n)\Delta_n^2}{1 + 2\underline{k}_E(n)\Delta_n + \underline{k}_0^E(n)\Delta_n^2 + \sum_{j=1}^m (\sigma_n^j)^2 \underline{k}_j(n)\Delta_n}\right)$$

$$\leq\ v_{n+1}\cdot\exp\left(\frac{-2\underline{k}_E(n)-2\underline{k}_I(n)-(\underline{k}_0^E(n)-\underline{k}_0^I(n))\Delta_n-\sum_{j=1}^{m}(\sigma_n^j)^2\underline{k}_j(n)}{1+2\underline{k}_E(n)\Delta_n+\underline{k}_0^E(n)\Delta_n^2+\sum_{j=1}^{m}(\sigma_n^j)^2\underline{k}_j(n)\Delta_n}\Delta_n\right),$$

and therefore

$$v_{n+1}\geq v_0\cdot\exp\left(\sum_{i=0}^{n}\frac{2\underline{k}_E(i)+2\underline{k}_I(i)+(\underline{k}_0^E(i)-\underline{k}_0^I(i))\Delta_i+\sum_{j=1}^{m}(\sigma_i^j)^2\underline{k}_j(i)}{1+2\underline{k}_E(i)\Delta_i+\underline{k}_0^E(i)\Delta_i^2+\sum_{j=1}^{m}(\sigma_i^j)^2\underline{k}_j(i)\Delta_i}\Delta_i\right),$$

respectively, using twice the elementary inequality

$$\frac{1+y}{1+x}\ \leq\ \exp\left(\frac{y-x}{1+x}\right).$$

Taking the logarithm and the limit as n tends to $+\infty$ yield the desired estimates. Consequently, the proof is completed. $\diamond$

2.3. **A nonlinear example.** An illustrative example is given by the discretization of the nonlinear one-dimensional Itô SDE

$$dX(t)\ =\ [(\alpha X(t)-\beta X^3(t)]dt+\sigma X(t)dW_t$$

where α,β,σ^2 are nonnegative real constants - an equation which is met in physical field theory (there a Stratonovich interpretation is studied, but this is included in our model class by application of transformation rules between calculi, see [1]). For this equation one can show that

$$\overline{\lambda}_p(x)\leq p\left(\alpha+\frac{p-1}{2}\sigma^2\right),\quad \overline{\lambda}_2(x)=\underline{\lambda}_2(x)=2\alpha+\sigma^2\ (x\neq 0)\,.$$

Now, the problem is how to simulate and discretize this SDE in a nonanticipative and efficient way such that a control on the stability behavior of used discretization method can be achieved under the presence of nonlinearities (i.e. $\beta>0$). Applying our result, we suggest to take the following explicit-implicit method

$$\begin{aligned}Y_{n+1}\ &=\ Y_n+(\alpha)_+Y_n-(\alpha)_-Y_{n+1}-\beta Y_n^2Y_{n+1})\Delta_n+\sigma Y_n\Delta W_n\\ &\qquad +|c|(Y_n-Y_{n+1})|\Delta W_n|\\ &=\ Y_n\left(\frac{1+(\alpha)_+\Delta_n+\sigma\Delta W_n+|c\Delta W_n|}{1+(\alpha)_-\Delta_n+\beta Y_n^2\Delta_n+|c\Delta W_n|}\right)\end{aligned}$$

where c is a further control parameter. Obviously, it can be made explicitly by simple algebraic rearrangements – an advantage from practical implementation point of view. This scheme also establishes a numerical method with the same numerical L^2-convergence order towards the exact solution as the well-known and most-used Euler method does. However, our explicit-implicit method is able to achieve a complete control on the asymptotic stability behavior in contrast to that of the explicit Euler method, where (random) step size restrictions are needed. The assumptions of Theorem 2.2 are fulfilled by taking

$$\overline{k}_I(n)=-(\alpha)_-,\overline{k}_0^I(n)=0,\overline{k}_E(n)=(\alpha)_+,\overline{k}_0^E(n)=((\alpha)_+)^2,\overline{k}_j(n)=\sigma^2\,.$$

Thus, the upper stability exponent λ_2 of considered numerical method is under control for all initial values (for simplicity, consider the cases $\alpha<0$ and $\alpha>0$ separated). For example, when $2\alpha+\sigma^2<0$ one knows about the asymptotic mean square stability of trivial solution of that SDE. The same is true for the suggested discretization method using any step size. Namely, for equidistant step size Δ, $c = 0$, and for all $x \in \mathbb{R}^1$, one estimates

$$\overline{\lambda}_2(x) \;\leq\; \frac{2\alpha + \sigma^2}{1 + 2(\alpha)_- \Delta} + ((\alpha)_+)^2 \Delta \,.$$

Thus, the asymptotic stability behavior of underlying SDE is replicated by our explicit-implicit discretization, and it can be used to approximate the upper stability exponents. It is not hard to find a situation where the explicit Euler method fails in this respect (e.g. take a sufficiently large equidistant step size and a bilinear equation). Moreover, by an appropriate choice of parameter c, one even gains control on the boundedness of sample paths (a.s.). Note that the chosen splitting into explicit and implicit treatment is very important for an asymptotically adequate numerical integration. Of course, a control on λ_2 could also be obtained by the use of fully drift-implicit Euler method (for the definition, see [24]) under the condition $2\alpha + \sigma^2 < 0$. However, the fully drift-implicit Euler method requires the local algebraic resolution of implicit equations at each step, hence more computational effort and additional local errors. Consequently, we may prefer our explicit-implicit technique in view of adequate stability control with discretization of SDEs.

3. Moment contractivity exponents

Fix a $p \in \mathbb{R}_+, p > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{T}$ a discrete or continuous, deterministic time scale, respectively. Consider a stochastic process $(X(t,z))_{(t\in\mathcal{T})}$ defined for all $t \in \mathcal{T} \subset [t_0, +\infty]$ on $(\Omega, \mathcal{F}, \mathbb{P})$, started at values z in domain $\mathbb{D}$ at time $t_0 \in \mathcal{T}$, with values in domain $\mathbb{D} \subset \mathbb{R}^d$ for all times $t \in \mathcal{T}$ (a.s.) and with finite p-th absolute moments for all finite times $t \in \mathcal{T}$.

Definition 2. The *upper (forward p–th moment) contractivity exponent* of the given stochastic process $(X(t,z))_{(t\in\mathcal{T})}$ in domain $\mathbb{D}$ is defined to be

$$\overline{\kappa}_p(x,y) \;:=\; \limsup_{t\to+\infty} \frac{1}{t} \ln\left(\mathbb{E}\,\|X(t,x) - X(t,y)\|^p\right) \tag{3.1}$$

for $X(t_0,x) = x \in \mathbb{D}$ (a.s.), $X(t_0,y) = y \in \mathbb{D}$ (a.s.). The *lower (forward p–th moment) contractivity exponent* of the given stochastic process $(X(t,z))_{(t\in\mathcal{T})}$ in domain $\mathbb{D}$ is defined to be

$$\underline{\kappa}_p(x,y) \;:=\; \liminf_{t\to+\infty} \frac{1}{t} \ln\left(\mathbb{E}\,\|X(t,x) - X(t,y)\|^p\right) \tag{3.2}$$

for $X(t_0,x) = x \in \mathbb{D}$ (a.s.), $X(t_0,y) = y \in \mathbb{D}$ (a.s.).

3.1. Contractivity exponents of SDEs with monotone coefficients.

Theorem 3.1. *Let process $(X(t,z))_{t\geq t_0}$ satisfy the Itô SDE*

$$dX(t,z) \;=\; a(t,X(t,z))\,dt + \sum_{j=1}^{m} b^j(t,X(t,z))\,dW_t^j\,, \tag{3.3}$$

exclusively started with values in deterministic domain $\mathbb{D} \subseteq \mathbb{R}^d$ (a.s.), where deterministic coefficients a, b^j are such that strong solution of this SDE with finite *p–th absolute moments exists. Assume that*

$$
\begin{aligned}
& < a(t,x) - a(t,y), x-y > + \frac{1}{2}\sum_{j=1}^{m} \|b^j(t,x) - b^j(t,y)\|^2 \\
(3.4) \qquad & + \frac{(p-2)}{2}\sum_{j=1}^{m} \frac{< b^j(t,x) - b^j(t,y), x-y >^2}{\|x-y\|^2} \quad \leq \quad \overline{C}_p(t)\|x-y\|^2
\end{aligned}
$$

for all $t \in [t_0, +\infty)$, for all $x, y \in I\!D$, where deterministic function $\overline{C}_p(t)$ is L^1–integrable on $[t_0, +\infty)$ with respect to the Lebesgue measure, and

$$
(3.5) \qquad I\!P\{\omega \in \Omega : X(t,z)(\omega) \in I\!D, \forall t \in [t_0, +\infty) | X(t_0, z) = z \in I\!D\} \; = \; 1 \, .
$$

Then we have

$$
(3.6) \qquad \sup_{x,y \in I\!D} \overline{\kappa}_p(x,y) \; \leq \; p \limsup_{t \to +\infty} \frac{\int_{t_0}^{t} \overline{C}_p(s)ds}{t} \, .
$$

Furthermore, if

$$
\begin{aligned}
& < a(t,x) - a(t,y), x-y > + \frac{1}{2}\sum_{j=1}^{m} \|b^j(t,x) - b^j(t,y)\|^2 \\
(3.7) \qquad & + \frac{(p-2)}{2}\sum_{j=1}^{m} \frac{< b^j(t,x) - b^j(t,y), x-y >^2}{\|x-y\|^2} \quad \geq \quad \underline{C}_p(t)\|x-y\|^2
\end{aligned}
$$

for all $t \in [t_0, +\infty)$, for all $x, y \in I\!D$, where deterministic function $\underline{C}_p(t)$ is L^1–integrable on $[t_0, +\infty)$ with respect to the Lebesgue measure, then, for all $x \neq y$ for which there exists $t \in [t_0, +\infty)$ such that $X(t,x) \neq X(t,y)$ with positive probability, we have

$$
(3.8) \qquad \inf_{x,y \in I\!D, x \neq y} \underline{\kappa}_p(x,y) \; \geq \; p \liminf_{t \to +\infty} \frac{\int_{t_0}^{t} \underline{C}_p(s)ds}{t} \, .
$$

Proof. The main idea is to enlarge the dynamical system on $I\!D \times I\!D \subseteq I\!R^{2d}$, apply Dynkin's formula [11] to this new system, to evaluate the arising linear partial differential operators under the required monotonicity conditions and finally to apply a generalized Gronwall–Bellman Lemma (see [27] and [32]). The required L^1–integrability ensures us that the expressions in the definition of contractivity exponents exist, and together with the existence of finite p–th absolute moments, that we can apply (the unstopped form of) Dynkin's formula at any time $t \in [t_0, +\infty)$. We need to refer to the 2-point generator since the property of contractivity relates to the behavior of the difference of two solutions in the course of time. Now, apply the 2-point generator

$$
\hat{\mathcal{L}} = \frac{\partial}{\partial t} + < a(t,x), \nabla_x > + < a(t,y), \nabla_y > + \frac{1}{2}\sum_{j=1}^{m}\sum_{k,l=1}^{2d} b_k^j(t,x,y) b_l^j(t,x,y) \frac{\partial^2}{\partial x_k \partial x_l}
$$

to $\|x-y\|^p, p > 0$, where

$$
b_k^j(t,x,y) := b_k^j(t,x) \quad \text{and} \quad b_{k+d}^j(t,x,y) := b_k^j(t,y), k = 1, 2, ..., d
$$

for corresponding $2d$-dimensional diffusion process $\hat{X}(t,z)$ on $\mathbb{D} \times \mathbb{D}$. After some tedious calculations, this procedure exactly gives

$$\hat{\mathcal{L}}\|x-y\|^p \;=\; p\Bigg(< a(t,x)-a(t,y), x-y > + \frac{1}{2}\sum_{j=1}^{m} \|b^j(t,x)-b^j(t,y)\|^2 + \frac{(p-2)}{2}\sum_{j=1}^{m} \frac{< b^j(t,x)-b^j(t,y), x-y >^2}{\|x-y\|^2}\Bigg) \|x-y\|^{p-2}$$

for all $x,y \in \mathbb{D}$ with $\|x-y\| > 0$, hence

$$\hat{\mathcal{L}}\|x-y\|^p \;\leq\; p\overline{C}_p(t)\|x-y\|^p \quad \text{and} \quad \hat{\mathcal{L}}\|x-y\|^p \;\geq\; p\underline{C}_p(t)\|x-y\|^p\,,$$

respectively, presuming the validity of inequalities (3.4) and (3.7). Note that

$$\nabla_y \;=\; \Big(\frac{\partial}{\partial x_{d+1}}, ..., \frac{\partial}{\partial x_{2d}}\Big)^T$$

represents the d-dimensional gradient vector in $y = (x_{d+1}, ..., x_{2d})^T$–direction, and ∇_x the gradient in x-direction as in previous section. By the formula of Dynkin we know that

$$\mathbb{E}\,\|X(t,x)-X(t,y)\|^p \;=\; \mathbb{E}\,\|X(s,x)-X(s,y)\|^p + \mathbb{E}\int_s^t \hat{\mathcal{L}}\|X(u,x)-X(u,y)\|^p du$$

for all s,t with $t \geq s; s,t \in [t_0,+\infty)$. Define

$$v(t) \;:=\; \mathbb{E}\,\|X(t,x)-X(t,y)\|^p$$

for all $x,y \in \mathbb{D}$. Now, fix initial values x,y with $x \neq y$. Under the monotonicity assumptions (3.4) and (3.7), this implies

$$v(t) \;\leq\; v(s) + p\int_s^t \overline{C}_p(u)v(u)du$$

and

$$v(t) \;\geq\; v(s) + p\int_s^t \underline{C}_p(u)v(u)du\,,$$

respectively. Now one applies the generalized Gronwall–Bellman lemma (see [27], [32]) to both estimates of $v(t)$, respectively, takes the logarithm and the limit as time t tends to $+\infty$ and encounters with desired result which completes the proof. ⋄

3.2. Contractivity exponents of stochastic iterative mappings. Set $X_0(z) = z$ for $z \in \mathbb{D} \subset \mathbb{R}^d$. Consider the d-dimensional iterative mappings

$$X_{n+1}(z) \;=\; X_n(z) + \Phi_0^I(X_i(z) : i \leq n+1)\Delta_n + \Phi_0^E(X_i(z) : i \leq n)\Delta_n + \sum_{j=1}^{m} \Phi_j(X_i(z) : i \leq n)\xi_n^j\sqrt{\Delta_n} \tag{3.9}$$

on deterministic domain $\mathbb{D}$ (a.s.), started at any $z \in \mathbb{D}$, where $\Delta_n := t_{n+1} - t_n$ is a sequence of step sizes with monotonically increasing time-instants $(t_i)_{i\in\mathbb{N}}$ and $\lim_{i\to+\infty} t_i = +\infty$, and ξ_n^j are real-valued, independent random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with moments

$$\mathbb{E}\,\xi_n^j = 0 \quad \text{and} \quad \mathbb{E}\,|\xi_n^j|^2 = (\sigma_n^j)^2 < +\infty\,.$$

For convenience of statement, define $\delta_n(x,y) := x^{(n)} - y^{(n)}$.

Theorem 3.2. *Let process $(X_n(z))_{n\in\mathbb{N}}$ satisfy the stochastic difference equation (3.9) started at value $z \in \mathbb{D}$ under the above mentioned conditions for all $n \in \mathbb{N}$, whereas all ξ_n^j are independent of $X_0(z)$ as well. Assume that $\forall n \in \mathbb{N}\,\forall x^{(l)}, y^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\ \forall j = 1, 2, ..., m$:*

$$
\begin{aligned}
< \Phi_0^I(x^{(l)} : l \le n+1) - \Phi_0^I(y^{(l)} : l \le n+1), \delta_{n+1}(x,y) > &\le \overline{c}_I(n)\|\delta_{n+1}(x,y)\|^2\\
< \Phi_0^E(x^{(l)} : l \le n) - \Phi_0^E(y^{(l)} : l \le n), \delta_n(x,y) > &\le \overline{c}_E(n)\|\delta_n(x,y)\|^2\\
\|\Phi_0^I(x^{(l)} : l \le n+1) - \Phi_0^I(y^{(l)} : l \le n+1)\|^2 &\ge \overline{c}_0^I(n)\|\delta_{n+1}(x,y)\|^2\\
\|\Phi_0^E(x^{(l)} : l \le n) - \Phi_0^E(y^{(l)} : l \le n)\|^2 &\le \overline{c}_0^E(n)\|\delta_n(x,y)\|^2\\
\|\Phi_j(x^{(l)} : l \le n) - \Phi_j(y^{(l)} : l \le n)\|^2 &\le \overline{c}_j(n)\|\delta_n(x,y)\|^2\\
2\overline{c}_I(n)\Delta_n &< 1 + \overline{c}_0^I(n)\Delta_n^2
\end{aligned}
\tag{3.10}
$$

where $\overline{c}_I, \overline{c}_E(n), \overline{c}_0^I(n), \overline{c}_0^E(n), \overline{c}_j(n)$ are finite, deterministic, real numbers. Then, for all $x, y \in \mathbb{D}$, $x \ne y$ ($\exists t \in [t_0, +\infty) : X(t,x) \ne X(t,y)$ with positive probability), we have

(3.11)

$$
\overline{\kappa}_2(x,y) \le \limsup_{n\to+\infty} \frac{\sum_{i=0}^{n}\Big(\frac{2\overline{c}_E(i) + 2\overline{c}_I(i) + (\overline{c}_0^E(i) - \overline{c}_0^I(i))\Delta_i + \sum_{j=1}^{m}(\sigma_i^j)^2\overline{c}_j(i)}{1 - 2\overline{c}_I(i)\Delta_i + \overline{c}_0^I(i)\Delta_i^2}\Big)\Delta_i}{t_{n+1}}.
$$

Furthermore, if $\forall n \in \mathbb{N}\,\forall x^{(l)}, y^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\ \forall j = 1, 2, ..., m$:

$$
\begin{aligned}
< \Phi_0^I(x^{(l)} : l \le n+1) - \Phi_0^I(y^{(l)} : l \le n+1), \delta_{n+1}(x,y) > &\ge \underline{c}_I(n)\|\delta_{n+1}(x,y)\|^2\\
< \Phi_0^E(x^{(l)} : l \le n) - \Phi_0^E(y^{(l)} : l \le n), \delta_n(x,y) > &\ge \underline{c}_E(n)\|\delta_n(x,y)\|^2\\
\|\Phi_0^I(x^{(l)} : l \le n+1) - \Phi_0^I(y^{(l)} : l \le n+1)\|^2 &\le \underline{c}_0^I(n)\|\delta_{n+1}(x,y)\|^2\\
\|\Phi_0^E(x^{(l)} : l \le n) - \Phi_0^E(y^{(l)} : l \le n)\|^2 &\ge \underline{c}_0^E(n)\|\delta_n(x,y)\|^2\\
\|\Phi_j(x^{(l)} : l \le n) - \Phi_j(y^{(l)} : l \le n)\|^2 &\ge \underline{c}_j(n)\|\delta_n(x,y)\|^2\\
1 + \underline{c}_0^E(n)\Delta_n^2 + \sum_{j=1}^{m}(\sigma_n^j)^2\underline{c}_j(n)\Delta_n &> -2\underline{c}_E(n)\Delta_n
\end{aligned}
\tag{3.12}
$$

where $\underline{c}_I, \underline{c}_E(n), \underline{c}_0^I(n), \underline{c}_0^E(n), \underline{c}_j(n)$ are finite, deterministic, real numbers, then, for all $x, y \in \mathbb{D}$, we have

(3.13)

$$
\underline{\kappa}_2(x,y) \ge \liminf_{n\to+\infty} \frac{\sum_{i=0}^{n}\Big(\frac{2\underline{c}_E(i) + 2\underline{c}_I(i) + (\underline{c}_0^E(i) - \underline{c}_0^I(i))\Delta_i + \sum_{j=1}^{m}(\sigma_i^j)^2\underline{c}_j(i)}{1 + 2\underline{c}_E(i)\Delta_i + \underline{c}_0^E(i)\Delta_i^2 + \sum_{j=1}^{m}(\sigma_i^j)^2\underline{c}_j(i)\Delta_i}\Big)\Delta_i}{t_{n+1}}.
$$

Proof. Fix any $x, y \in \mathbb{D}$. Define $v_n := \mathbb{E}\,\|X_n(x) - X_n(y)\|^2$. Taking the square of Euclidean vector norm of

$$
X_{n+1}(x) - \Phi_0^I(X_l(x) : l \le n+1)\Delta_n - X_{n+1}(y) + \Phi_0^I(X_l(x) : l \le n+1)\Delta_n
$$

and its expectation value afterwards one receives

$$\begin{aligned} & v_{n+1}\left(1-2\Delta_n\overline{c}_I(n)+\Delta_n^2\overline{c}_0^I(n)\right) \\ \le\ & v_n+2\Delta_n\mathbb{E}\ <\Phi_0^E(X_i: i\le n)-\Phi_0^E(X_i(y): i\le n), X_n(x)-X_n(y)> \\ & +\Delta_n^2\mathbb{E}\ \|\Phi_0^E(X_i(x): i\le n)-\Phi_0^E(X_i(y): i\le n)\|^2 \\ & +\Delta_n\sum_{j=1}^m(\sigma_n^j)^2\mathbb{E}\ \|\Phi_j(X_i(x): i\le n)-\Phi_j(X_i(y): i\le n)\|^2 \\ \le\ & v_n\left(1+2\Delta_n\overline{c}_E(n)+\Delta_n^2\overline{c}_0^E(n)+\Delta_n\sum_{j=1}^m(\sigma_n^j)^2\overline{c}_j(n)\right) \end{aligned}$$

using the independence of ξ_n^j, that $\mathbb{E}\ (\xi_n^j)^2=(\sigma_n^j)^2$ and conditions (3.10). A similar estimate is derived under conditions (3.12). In the other direction we obtain

$$\begin{aligned} & v_n\left(1+2\Delta_n\underline{c}_E(n)+\Delta_n^2\underline{c}_0^E(n)+\Delta_n\sum_{j=1}^m(\sigma_n^j)^2\underline{c}_j(n)\right) \\ \le\ & v_n+2\Delta_n\mathbb{E}\ <\Phi_0^E(X_i(x): i\le n)-\Phi_0^E(X_i(y): i\le n), X_n(x)-X_n(y)> \\ & +\Delta_n^2\mathbb{E}\ \|\Phi_0^E(X_i(x): i\le n)-\Phi_0^E(X_i(y): i\le n)\|^2 \\ & +\Delta_n\sum_{j=1}^m(\sigma_n^j)^2\mathbb{E}\ \|\Phi_j(X_i: i\le n)-\Phi_j(X_i(y): i\le n)\|^2 \\ \le\ & v_{n+1}\left(1-2\Delta_n\underline{c}_I(n)+\Delta_n^2\underline{c}_0^I(n)\right) \end{aligned}$$

Obviously, under (3.10) and (3.12), one gains the estimates

$$\begin{aligned} v_{n+1}\ \le\ & v_n\cdot\left(\frac{1+2\overline{c}_E(n)\Delta_n+\overline{c}_0^E(n)\Delta_n^2+\sum_{j=1}^m(\sigma_n^j)^2\overline{c}_j(n)\Delta_n}{1-2\overline{c}_I(n)\Delta_n+\overline{c}_0^I(n)\Delta_n^2}\right) \\ \le\ & v_0\cdot\prod_{i=0}^n\left(\frac{1+\overline{c}_E(i)\Delta_i+\overline{c}_0^E(i)\Delta_i^2+\sum_{j=1}^m(\sigma_i^j)^2\overline{c}_j(i)\Delta_i}{1-2\overline{c}_I(i)\Delta_i+\overline{c}_0^I(i)\Delta_i^2}\right)_+ \\ \le\ & v_0\cdot\exp\left(\sum_{i=0}^n\Big(\frac{2\overline{c}_E(i)+2\overline{c}_I(i)+(\overline{c}_0^E(i)-\overline{c}_0^I(i))\Delta_i+\sum_{j=1}^m(\sigma_i^j)^2\overline{c}_j(i)}{1-2\overline{c}_I(i)\Delta_i+\overline{c}_0^I(i)\Delta_i^2}\Big)\Delta_i\right) \end{aligned}$$

and in the other direction

$$v_n\ \le\ v_{n+1}\cdot\left(\frac{1-2\underline{c}_I(n)\Delta_n+\underline{c}_0^I(n)\Delta_n^2}{1+2\underline{c}_E(n)\Delta_n+\underline{c}_0^E(n)\Delta_n^2+\sum_{j=1}^m(\sigma_n^j)^2\underline{c}_j(n)\Delta_n}\right)$$

$$\leq\ v_{n+1}\cdot\exp\left(\frac{-2\underline{c}_E(n)-2\underline{c}_I(n)-(\underline{c}_0^E(n)-\underline{c}_0^I(n))\Delta_n-\sum_{j=1}^{m}(\sigma_n^j)^2\underline{c}_j(n)}{1+2\underline{c}_E(n)\Delta_n+\underline{c}_0^E(n)\Delta_n^2+\sum_{j=1}^{m}(\sigma_n^j)^2\underline{c}_j(n)\Delta_n}\Delta_n\right),$$

and therefore

$$v_{n+1}\geq v_0\cdot\exp\left(\sum_{i=0}^{n}\frac{2\underline{c}_E(i)+2\underline{c}_I(i)+(\underline{c}_0^E(i)-\underline{c}_0^I(i))\Delta_i+\sum_{j=1}^{m}(\sigma_i^j)^2\underline{c}_j(i)}{1+2\underline{c}_E(i)\Delta_i+\underline{c}_0^E(i)\Delta_i^2+\sum_{j=1}^{m}(\sigma_i^j)^2\underline{c}_j(i)\Delta_i}\Delta_i\right),$$

respectively, using twice the elementary inequality

$$\frac{1+y}{1+x}\ \leq\ \exp\left(\frac{y-x}{1+x}\right).$$

Taking the logarithm and the limit as n tends to $+\infty$ yield the desired estimates. Consequently, the proof is completed. $\diamond$

4. The example of linear systems and correlations

Consider nonautonomous linear stochastic systems

$$dX(t)\ =\ \Big(A(t)\,X(t)\ +\ b^0(t)\Big)\,dt\ +\ \sum_{j=1}^{m}\Big(B^j(t)\,X(t)\ +\ b^j(t)\Big)\,dW^j(t) \tag{4.1}$$

where $X(t)$ denotes their d–dimensional solution; $A, B^j (j=1,2,...,m)$ deterministic, real–valued matrices, b^j deterministic, real–valued vectors and W^j are uncorrelated, one-dimensional standard Wiener processes. Take $\mathbb{D}=\mathbb{R}^d$.

4.1. **A corollary for linear systems.** As a consequence of presented analysis, we get the following corollary which can be proven by the application of continuous variation-of-constants inequalities (see Lemma 8.10.2 in [32]) as a generalization of the well-known Gronwall-Bellman Lemma. For a proof, see [32].

Corollary 4.1. *Consider $X(t)$ satisfying Itô SDE (4.1) for $t\in[t_0,+\infty)$. Assume that $\mathbb{E}\,\|X(t_0)\|^2<+\infty$ and $\forall x\in\mathbb{R}^d, \forall t\geq t_0$*

$$2<x, A(t)x+b^0(t)>+\sum_{j=1}^{m}\|B^j(t)x+b^j(t)\|^2\ \leq\ \overline{K}_1(t)+\overline{K}_2(t)\|x\|^2$$

where the deterministic functions $\overline{K}_1,\overline{K}_2$ are L^1-integrable with respect to Lebesgue measure, and assume

$$\forall t>t_0\ :\ \mathbb{E}\,\|X(t_0)\|^2+\int_{t_0}^{t}\overline{K}_1(s)\exp(-\int_{t_0}^{s}\overline{K}_2(z)dz)\,ds>0\,.$$

Then

$$\overline{\lambda_2}\ \leq\ \limsup_{t\to+\infty}\frac{\ln\Big(\mathbb{E}\,\|X(t_0)\|^2+\int_{t_0}^{t}\overline{K}_1(s)\exp(-\int_{t_0}^{s}\overline{K}_2(z)dz\,ds)\Big)+\int_{t_0}^{t}\overline{K}_2(s)ds}{t}.$$

Furthermore, assume that $\forall x\in\mathbb{R}^d, \forall t\geq t_0$

$$2<x, A(t)x+b^0(t)>+\sum_{j=1}^{m}\|B^j(t)x+b^j(t)\|^2\ \geq\ \underline{K}_1(t)+\underline{K}_2(t)\|x\|^2$$

where the deterministic functions $\underline{K}_1, \underline{K}_2$ are L^1-integrable with respect to Lebesgue measure, and assume

$$\forall t > t_0 \; : \; \mathbb{E}\,\|X(t_0)\|^2 + \int_{t_0}^{t} \underline{K}_1(s) \exp(-\int_{t_0}^{s} \underline{K}_2(z)dz)\, ds > 0\,.$$

Then

$$\underline{\lambda_2} \;\geq\; \liminf_{t\to+\infty} \frac{\ln\left(\mathbb{E}\,\|X(t_0)\|^2 + \int_{t_0}^{t} \underline{K}_1(s)\exp(-\int_{t_0}^{s}\underline{K}_2(z)dz)\,ds\right) + \int_{t_0}^{t}\underline{K}_2(s)ds}{t}\,.$$

Remark. Here, both the exponents and the limits on the right hand side do not depend on initial values really. The above assumptions on $\overline{K}_i, \underline{K}_i$ are only made to exclude meaningless extreme cases. If $b^j \equiv 0$ (i.e. the case of fundamental solution), matrices A and all B^j have complete basis systems of eigenvectors, then $\overline{\lambda}_2$ and $\underline{\lambda}_2$ are exclusively controlled by the interaction of eigenvalues of $A, B^1, ..., B^m$. For example, if the matrices A, B^j are time-independent, then

$$\begin{aligned} \overline{\lambda}_2 &\leq 2\,\max_i\{\mu_i\} + \sum_{j=1}^{m} \max_i\{\rho_{i,j}^2\}, \\ \underline{\lambda}_2 &\geq 2\,\min_i\{\mu_i\} + \sum_{j=1}^{m} \min_i\{\rho_{i,j}^2\} \end{aligned}$$

where μ_i are the eigenvalues of matrix A, $\rho_{i,j}^2$ the eigenvalues of positive semi-definite $B^j(B^j)^T$. Note, in the linear autonomous case with pure multiplicative noise, the concept of moment stability exponents coincides with that of moment Lyapunov exponents. In the one-dimensional case we can obtain the equality in these estimates, hence sharp estimates have been found by Corollary 4.1.

A similar corollary holds for the contractivity exponents $\underline{\kappa}, \overline{\kappa}$ of the affine equation as above. For example, under

$$\underline{C}_2(t)\|x-y\|^2 \;\leq\; 2 < x-y, A(t)(x-y) > + \sum_{j=1}^{m} \|B^j(t)(x-y)\|^2 \;\leq\; \overline{C}_2(t)\|x-y\|^2$$

for all $x, y \in \mathbb{R}^d$ and all $t \geq t_0$, where the deterministic functions $\overline{C}_2, \underline{C}_2$ are L^1-integrable with respect to Lebesgue measure, one can show that

$$\liminf_{t\to+\infty} \frac{\int_{t_0}^{t}\underline{C}_2(s)ds}{t} \;\leq\; \underline{\kappa_2} \;\leq\; \overline{\kappa_2} \;\leq\; \limsup_{t\to+\infty} \frac{\int_{t_0}^{t}\overline{C}_2(s)ds}{t}\,.$$

Trivially, the behavior of inhomogeneouity $b^j = b^j(t)$ does not play any role within the concept of contractivity, in contrast to the concept of attractivity (stability).

4.2. **An example where $\{0\}$-attractivity (stability) and contractivity differ.** The following one-dimensional, linear, nonautonomous SDE shows that the concepts of stability ($\{0\}$-attractivity) and contractivity exponents do not coincide in general! Moreover, these concepts will provide assertions of totally different qualitative behavior. The concept of contractivity is more appropriate under the presence of inhomogeneouities (as in the case of additive noise). Let $W = (W_t)_{t\geq 0}$ represent a standard Wiener process. Consider

$$dX(t) \;=\; \mu\, X(t)\, dt \,+\, \sigma \exp(\alpha t)\, dW_t \tag{4.2}$$

where μ, σ, α are deterministic, real parameters.

Theorem 4.2. *Assume that $p \geq 1$ and $\sigma^2 > 0$. Then the stochastic processes $(X(t))_{(t \geq t_0)}$ governed by SDE (4.2) possess characteristic exponents with*

$$\overline{\lambda}_p(x) \geq \underline{\lambda}_p(x) \geq p\,\mu$$

and

$$\overline{\kappa}_p(x,y) = \underline{\kappa}_p(x,y) = p\,\mu$$

whenever $X(t_0,x) = x \neq y = X(t_0,y)$, where $x, y \in R^1$. Moreover, if $p \geq 2$ then

$$\underline{\lambda}_p(x) \leq \overline{\lambda}_p(x) \leq p\,\max(\mu,\alpha)$$

If $p = 2$, then

$$\overline{\lambda}_2(x) = \underline{\lambda}_2(x) = 2\,\max(\mu,\alpha)\,.$$

Proof. Without loss of generality, we may set set $t_0 = 0$. First, consider an estimation of the $\{0\}$-attractivity (stability) exponents. Define $v(t) := \mathbb{E}\,|X(t)|^p$. A calculation gives

$$\begin{aligned} v(t) &= v(0) + p\int_0^t \left(\sigma^2 \frac{p-1}{2}\exp(2\alpha s)(\mathbb{E}\,[|X(s)|^{p-2}]) + \mu v(s)\right) ds \\ &\geq v(0) + p\mu\int_0^t v(s)ds \end{aligned}$$

by the use of Dynkin's formula and simple monotonicity arguments. Consequently, $\overline{\lambda}_p \geq \underline{\lambda}_p \geq p\mu$ by the application of Gronwall-Bellman inequality with constant kernel. If $p \geq 2$, the evolution of $v(t)$ satisfies

$$\begin{aligned} v(t) &\leq v(0) + \int_0^t \left(\mathbb{E}\,[p(p-1)\frac{\sigma^2}{2}\exp(2\alpha s)|X(s)|^{p-2}] + p\mu v(s)\right) ds \\ &\leq v(0) + \int_0^t \left(\varepsilon\sigma^2\left[\frac{p-1}{\varepsilon}\right]^{p/2}\exp(p\alpha s) + (p\mu + \varepsilon(p-2)\frac{\sigma^2}{2})v(s)\right) ds \end{aligned}$$

for all $\varepsilon > 0$, by the use of Young's inequality applied to

$$c(s)|y(s)|^{p-2} \;\leq\; \frac{2}{p}|c(s)|^{p/2} + \frac{p-2}{p}|y(s)|^p$$

where

$$c(s) = \frac{p-1}{\varepsilon}\exp(2\alpha s),\; y(s) = |X(s)|\,.$$

Now, apply the continuous variation-of-constants inequality from [32] (see Lemma 8.10.2) to obtained inequality for $v(t)$. Thereby one arrives at the estimate

$$v(t) \leq \left(v(0) + \int_0^t K_1(u)\exp\left(-\int_0^u K_2(z)dz\right)du\right)\cdot\exp\left(\int_0^t K_2(u)du\right)$$

where

$$K_1(u) = \sigma^2\left[\frac{p-1}{\varepsilon}\right]^{p/2}\exp(p\alpha u),\; K_2(u) = p\mu + \varepsilon(p-2)\frac{\sigma^2}{2}\,.$$

By taking the logarithm and the limit as time t tends to $+\infty$ one arrives at

$$\overline{\lambda}_p \leq p\max(\mu + \varepsilon\frac{\sigma^2(p-2)}{2},\alpha)\,.$$

This observation holds for all $\varepsilon > 0$, hence the claimed result for stability exponents

$$\overline{\lambda}_p \leq p\max(\mu,\alpha)$$

can be established. In case of $p = 2$, one may explicitly calculate the exact value of the stability exponent. It is clear that upper and lower exponents must coincide in this case. One encounters with

$$v(t) \quad = \quad v(0) + \int_0^t [\sigma^2 \exp(2\alpha s) + 2\mu v(s)] ds \,.$$

This equation can be solved by the method of variation-of-constants. It yields

$$\begin{aligned} \frac{\ln[v(t)]}{t} &= 2\mu + \frac{\ln\left[v(0) + \sigma^2 \int_0^t \exp(2(\alpha-\mu)s)ds\right]}{t} \\ &= 2\mu + \frac{\ln\left[v(0) + \sigma^2 \dfrac{\exp(2(\alpha-\mu)t) - 1}{2(\alpha-\mu)}\right]}{t} \,. \end{aligned}$$

Therefore, by taking the limit $t \to +\infty$, one arrives at

$$\overline{\lambda}_2 = \underline{\lambda}_2 = 2\mu + 2(\alpha - \mu)_+ = 2\max(\alpha, \mu)$$

provided that $\sigma^2 > 0$. The argumentation for the contractivity exponents is much easier, since

$$\begin{aligned} u(t,x,y) &= \mathbb{E}\,|X(t,x) - X(t,y)|^p = u(0,x,y) + p\mu \int_0^t u(s,x,y) ds \\ &= u(0,x,y) \exp\Big(p\mu t\Big) \end{aligned}$$

for any fixed, finite $x, y \in \mathbb{R}^1$. After the application of Gronwall-Bellman Lemma, one finds the claimed estimate for $x \neq y$. Thus, the proof can be completed. ⋄

Remark. The proof can also be carried out by using the pathwise expression of exact solution

$$X(t) = \exp(\mu t) \left(X(0) + \sigma \int_0^t \exp((\alpha - \mu)s) dW_s \right)$$

and well-known moment martingale inequalities. However, our suggested approach combining variation-of-constants inequalities and monotonicity arguments can be applied to much more general classes of SDEs.

4.3. **When contractivity and attractivity (stability) coincide.** There is a certain specific situation when the estimates for moment contractivity and attractivity (stability) coincide. This will be the case for systems with an equilibrium x_*. For simplicity, consider $\mathbb{D}_1 = \mathbb{R}^d$.

Proposition 4.3. *Let process $X(t,z)_{(t \geq t_0)}$ be a stochastic dynamical system with*

$$\exists x_* \in \mathbb{D}_1 \; \forall t \geq t_0 \;:\; X(t, x_*) = x_* \; (a.s.) \,.$$

Then, moment contractivity and moment attractivity (stability) exponents coincide with respect to the single set $\mathbb{D}_2 = \{x_*\}$ of the equilibrium x_*, i.e.

$$\forall z \in \mathbb{D} \;:\; \overline{\lambda}_p(z) = \overline{\kappa}_p(z, x_*), \underline{\lambda}_p(z) = \underline{\kappa}_p(z, x_*) \,.$$

Moreover, in the situation of Itô SDEs with infinitesimal generator $\mathcal{L}$ when there are nonnegative real constants $K_{1,p}, K_{2,p}$ such that for all $x \in \mathbb{D}$

$$\mathcal{L}\|x\|^p \;\leq\; K_{1,p} + K_{2,p}\|x\|^p$$

and initial values which are independent of all σ–fields $\mathcal{F}_\infty^j = \sigma\{W_t^j : t \geq t_0\}$, then one finds uniform estimates of upper exponents which do not depend on the choice *of initial value z. Additionally, for Itô SDEs with infinitesimal generator $\mathcal{L}$ and nonnegative real constants $C_{1,p}, C_{2,p}$ such that for all $x \in \mathbb{D}$*

$$\mathcal{L}\|x\|^p \ \geq \ C_{1,p} - C_{2,p}\|x\|^p$$

and initial values which are independent of all σ–fields $\mathcal{F}^j_\infty = \sigma\{W^j_t : t \geq t_0\}$, there are uniform estimates of lower exponents which do not depend on the choice of initial value z.

Proof. Suppose that $\mathbb{D}_2 = \{x_*\}$. For the first assertion, one only has to note

$$\mathbb{E}\,\|X(t,x) - X(t,x_*)\|^p \ = \ \mathbb{E}\,\|X(t,x) - x_*\|^p$$

with x_* satisfying $X(t,x_*) = x_*$ for all $t \geq t_0$. The second assertion for Itô SDEs becomes clear from the following. From the inequality of Minkowski one knows

$$(\mathbb{E}\,\|X(t,x) - X(t,y)\|^p)^{1/p} \leq (\mathbb{E}\,\|X(t,x) - x_*\|^p)^{1/p} + (\mathbb{E}\,\|X(t,y) - x_*\|^p)^{1/p}\,.$$

After taking this inequality to the power p one recognizes that the stability evolution can be used to dominate the evolution of initial perturbations (hence that of contractivity, see also next subsection). Under the condition above, by Dynkin's formula, we also arrive at

$$\mathbb{E}\,\|X(t,z)\|^p \ \leq \ (\|z\|^p + K_{1,p}(t-t_0))\exp(K_{2,p}(t-t_0))\,,$$

hence $\overline{\lambda}_p \leq K_{2,p}$ and $\overline{\kappa}_p \leq K_{2,p}$. The third assertion can be seen after application of the inverse triangle inequality for vector norms, namely

$$|(\mathbb{E}\,\|X(t,x) - x_*\|^p)^{1/p} - (\mathbb{E}\,\|X(t,y) - x_*\|^p)^{1/p}| \ \leq \ (\mathbb{E}\,\|X(t,x) - X(t,y)\|^p)^{1/p}\,.$$

Analogously to steps for second assertion, one confirms the third assertion and obtains $\min(\underline{\lambda}_p, \underline{\kappa}_p) \geq -C_{2,p}\chi_{\{C_{1,p}=0\}}$ where $\chi_{\{.\}}$ is the characteristic function of inscribed set. Thus, proof is complete. ⋄

Remark. The coincidence of $\{x_*\}$-attractivity (stability) and contractivity concept for fixed $x_* = 0$ can also be motivated from the simple fact

$$< f(t,x_*) - f(t,y), x_* - y > = < f(t,y), y >$$

provided that $x_* = 0, f(t,0) = 0$. The operator $\mathcal{L}$ applied to $\|x\|^p$ can always be bounded from below and above if all coefficients a, b^j of the considered SDE are globally Lipschitz continuous (uniformly with respect to time t). Thus, for the class of SDEs with that property, we obtain an estimate for the spectrum of stability and contractivity exponents which does not depend on the initial values - as a consequence of our proposition (spectrum of attractivity (stability) and contractivity exponents with respect to the deterministic, non-singleton, open set $\mathbb{D} \subseteq \mathbb{R}^d$ is the distance

$$\sup_{x\in\mathbb{D}} \overline{\lambda}_p(x) - \inf_{x\in\mathbb{D}} \underline{\lambda}_p(x) + \sup_{x,y\in\mathbb{D}} \overline{\kappa}_p(x,y) - \inf_{x,y\in\mathbb{D}:x\neq y} \underline{\kappa}_p(x,y)\,.$$

4.4. **Upper attractivity (stability) exponents dominate contractivity exponents.** As already noted in proof of Proposition 4.3, the attractivity (stability) concept is dominating that of contractivity in general.

Proposition 4.4. *Let process $X(t,z)_{(t\geq t_0)}$ be a stochastic dynamical system with finite p-th moment ($p\geq 1$) for all finite times t and with finite upper attractivity (stability) exponents $\overline{\lambda}_p(x)$ and $\overline{\lambda}_p(y)$ with respect to a bounded, deterministic, nonempty set $\mathbb{D}_2\subset\mathbb{R}^d$, where $x,y\in\mathbb{D}_1$. Then, we have*

$$\overline{\kappa}_p(x,y) \ \leq\ \max(\overline{\lambda}_p(x),\overline{\lambda}_p(y))\,.$$

Proof. By application of the inequalities of Minkowski and Hoelder one knows

$$u(t,x,y) := \mathbb{E}\,\|X(t,x)-X(t,y)\|^p \leq 2^{p-1}\left(\mathbb{E}\,\|X(t,x)-z\|^p+\mathbb{E}\,\|X(t,y)-z\|^p\right)$$

for any $z\in\mathbb{D}_2$. This implies that

$$\ln[u(t,x,y)] \ \leq\ \ln[2^{p-1}]+\max(\overline{\lambda}_p(x),\overline{\lambda}_p(y))t\,+$$
$$+\ln\Big[\exp\left(-\max(\overline{\lambda}_p(x),\overline{\lambda}_p(y))t\right)\sup_{z\in\mathbb{D}_2}\left(\mathbb{E}\,\|X(t,x)-z\|^p+\mathbb{E}\,\|X(t,y)-z\|^p\right)\Big].$$

The latter logarithm must converge to a finite number as time t tends to infinity because of the existence of upper attractivity (stability) exponents with respect to $\mathbb{D}_2$. Now, one takes the limit with respect to time and finds

$$\limsup_{t\to+\infty}\frac{\ln[u(t,x,y)]}{t} \ \leq\ \max(\overline{\lambda}_p(x),\overline{\lambda}_p(y))\,.$$

This completes the proof. ⋄

5. Generalized V-exponents and examples

More general than in previous sections, the following definition provides with concepts for the qualitative description of asymptotic growth behavior along certain functionals of underlying stochastic process. This definition is particularly introduced to describe and investigate the dissipativity, attractivity (stability) and contractivity properties of nonlinear stochastic systems. In passing, the concepts of V-dissipativity (as well as V-stability and V-contractivity) have been introduced in [32]. Here we only continue with a refinement concerning those concepts (i.e. in order to find appropriate *nonlinear speed measures of exponential growth*).

5.1. **Definition of V-exponents.** Our aim is to incorporate a larger class of nonlinear stochastic systems in its qualitative analysis.

Definition 3. The *upper (forward moment) V-exponent* of a given stochastic process $(X_t)_{t\in\mathcal{T}}$ on domain $\mathbb{D}$ is defined to be

$$\overline{\lambda}_V(x) \ :=\ \limsup_{t\to+\infty}\ \ln\Big(\mathbb{E}\,V(t,X(t))\Big) \tag{5.1}$$

for $X(t_0)=x\in\mathbb{D}$ and a fixed deterministic function $V(t,x):[t_0,+\infty)\times\mathbb{D}\longrightarrow\mathbb{R}_+$ (or positive functional). The *lower (forward moment) V-exponent* of a given stochastic process $(X_t)_{t\in\mathcal{T}}$ on domain $\mathbb{D}$ is defined to be

$$\underline{\lambda}_V(x) \ :=\ \liminf_{t\to+\infty}\ \ln\Big(\mathbb{E}\,V(t,X(t))\Big) \tag{5.2}$$

for $X(t_0)=x\in\mathbb{D}$ and a fixed deterministic function $V(t,x):[t_0,+\infty)\times\mathbb{D}\longrightarrow\mathbb{R}_+$ (or positive functional).

Obviously, the art consists in finding appropriate functionals V. We will demonstrate with two illustrative, nonlinear examples how such functions could look like.

5.2. **Asymptotics of a nonlinear stochastic oscillator.** In Mechanical and Electronical Engineering one encounters with nonlinear oscillators perturbed by noise and excited by periodic forces (e.g. in modeling of stabilizing electric circuits or of mechanical structures). For simplicity, consider the one-degree of freedom oscillator

$$\ddot{x} + \alpha^2 x + \frac{\beta^2 r^2}{\dot{x}} - \gamma^2 (r^2 - \alpha^2 x^2 - (\dot{x})^2)\dot{x} \;=\; \sigma \xi_t \tag{5.3}$$

where $r, \alpha, \beta, \gamma, \sigma \in \mathbb{R}$ and ξ_t is white or colored noise. $x = x(t)$ represents the displacement of oscillations from the rest-point, whereas $\dot{x} = \dot{x}(t)$ the velocity of the oscillations. Let us restrict to the white noise case only. Here $r > 0$ is a real parameter controlling the asymptotic behavior. We will see that $|r|$ and $|\alpha|$ determine the shape of a limit ellipsoid where all the ellipsoid-interior trajectories converge in probability to. Additionally, we are interested in a measure for the convergence speed of the interior trajectories, without considering the possibility of 'crossing' trajectories or leaves after first entrance. Define

$$\mathbb{D} := \left\{(x,y) \in \mathbb{R}^2 \setminus (\mathbb{R}^1 \times \{0\}) : r^2 - \alpha^2 x^2 - y^2 > 0\right\}.$$

Theorem 5.1. *Consider the stochastic process $(x(t), \dot{x}(t))$, $t \in \mathbb{R}_+$, governed by SDE (5.3), started at any values $(x(0), \dot{x}(0)) \in \mathbb{D}$. Define*

$$V(x(t), \dot{x}(t)) := (r^2 - \alpha^2 x^2(t) - \dot{x}^2(t))_+ .$$

Then the V-exponents satisfy

$$\overline{\lambda}_V(r,\beta,\gamma,\sigma) \begin{cases} = & -\infty \quad \text{if} \quad 2\beta^2 r^2 - \sigma^2 < 0 \\ \leq & 0 \quad \text{if} \quad 2\beta^2 r^2 - \sigma^2 \geq 0 \end{cases}$$

and

$$\underline{\lambda}_V(r,\beta,\gamma,\sigma) \begin{cases} \geq & -2r^2\gamma^2 \quad \text{if} \quad 2\beta^2 r^2 - \sigma^2 \geq 0 \\ = & -\infty \quad \text{if} \quad 2\beta^2 r^2 - \sigma^2 < 0 \end{cases} .$$

Furthermore, assume that $2\beta^2 r^2 - \sigma^2 < 0$. Then we have

$$\forall \alpha, \gamma \in \mathbb{R}^1 \; : \; \lim_{t \to +\infty} V(x(t), \dot{x}(t)) \;=\; 0 \; (a.s.) \; .$$

Proof. Consider Lyapunov functional

$$V(x,y) := (r^2 - \alpha^2 x^2 - y^2)_+$$

for fixed parameters $r, \alpha, \beta, \gamma, \sigma$. Then, by Dynkin's formula (stopped at first hitting the boundary of deterministic domain $\mathbb{D}$) one arrives at

$$\begin{aligned} &\left(v(0) + (2\beta^2 r^2 - \sigma^2)t\right)_+ \;\geq \\ v(t) \;&:=\; \mathbb{E}\, V(x(t), \dot{x}(t)) \;=\; \mathbb{E}\, V(x(0), \dot{x}(0)) + \mathbb{E} \int_0^t \mathcal{L}V(x(s), \dot{x}(s))ds \\ &\geq\; \left(\mathbb{E}\, V(x(0), \dot{x}(0)) + (2\beta^2 r^2 - \sigma^2)t - 2\gamma^2 r^2 \int_0^t \mathbb{E}\, V(x(s), \dot{x}(s))ds\right)_+ \\ &\geq\; \left(\mathbb{E}\, V(x(0), \dot{x}(0)) + (2\beta^2 r^2 - \sigma^2)t\right)_+ \exp(-2\gamma^2 r^2 t) \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}V(x,y) \;&=\; \left(y\frac{\partial}{\partial x} + [\gamma^2(r^2 - \alpha^2 x^2 - y^2)y - \frac{r^2\beta^2}{y} - \alpha^2 x]\frac{\partial}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\right) V(x,y) \\ &= (-\alpha^2 2xy + 2\beta^2 r^2 + \alpha^2 2xy - 2\gamma^2 y^2 V(x,y) - \sigma^2)\chi_{\{r^2 - \alpha^2 x^2 - y^2 > 0\}}(x,y) \\ &=\; (2\beta^2 r^2 - \sigma^2 - 2\gamma^2 y^2 V(x,y))\chi_{\{r^2 - \alpha^2 x^2 - y^2 > 0\}}(x,y) \, . \end{aligned}$$

If $2\beta^2 r^2 < \sigma^2$ then there is a finite critical time $\hat{t} > 0$ such that for all $t > \hat{t}$ one finds $v(t) = 0$, hence $\overline{\lambda}_V = \underline{\lambda}_V = -\infty$. Now, assume $2\beta^2 r^2 \geq \sigma^2$. Then we can immediately conclude from the above application of Dynkin's formula that

$$-2\gamma^2 r^2 \leq \underline{\lambda}_V \leq \overline{\lambda}_V \leq 0 .$$

Now suppose that $2\beta^2 r^2 < \sigma^2$. Then the convergence of $V(x(t), \dot{x}(t))$ towards 0 (a.s.) can be established by the application of Theorem 6.1 due to the polynomially fast decline of $\mathbb{E}\, V(x(t), \dot{x}(t))$ to 0 at a finite time, hence the proof is complete. ⋄

Remark. If $2\beta^2 r^2 - \sigma^2 < 0$ then one may establish almost sure convergence of $V(x(t), \dot{x}(t))$ to 0, using the Theorem 6.1 from section 6.

5.3. **How noise may stabilize nonlinear oscillations.** A slight change of our model from previous subsection brings out an interesting effect. Consider Itô SDE

$$\ddot{x} + \alpha^2 x - \gamma^2 (r^2 - \alpha^2 x^2 - (\dot{x})^2)\dot{x} = \sigma \sqrt{|r^2 - \alpha^2 x^2 - (\dot{x})^2|}\, \xi_t \tag{5.4}$$

where $r, \alpha, \gamma, \sigma \in \mathbb{R}$ and ξ_t is white noise. Again $x = x(t)$ represents the displacement from the rest-point, whereas $\dot{x} = \dot{x}(t)$ the velocity of the oscillations. Define

$$\mathbb{D} := \left\{ (x, y) \in \mathbb{R}^2 : r^2 - \alpha^2 x^2 - y^2 > 0 \right\}$$

as the interior of an ellipsoid with radius $|r|$ and scale parameter $|\alpha|$.

Theorem 5.2. *Assume that the stochastic process $(x(t), \dot{x}(t))_{(t \in \mathbb{R}_+)}$ is generated by Itô SDE (5.4), started at any $(x(0), \dot{x}(0)) \in \mathbb{D}$ with parameters $r^2, \sigma^2 > 0$. Then*

$$\forall \alpha, \gamma \in \mathbb{R}^1 : \lim_{t \to +\infty} V(x(t), \dot{x}(t)) = 0 \text{ (a.s.)},$$

where $V(x, y) := (r^2 - \alpha^2 x^2 - y^2)_+$. This process possesses V-exponents satisfying

$$-2\gamma^2 r^2 - \sigma^2 \leq \underline{\lambda}_V(r, \gamma, \sigma) \leq \overline{\lambda}_V(r, \gamma, \sigma) \leq -\sigma^2 .$$

Proof. Consider Lyapunov functional

$$V(x, y) := (r^2 - \alpha^2 x^2 - y^2)_+$$

for fixed parameters $r, \alpha, \gamma, \sigma$. Define $v(t) := \mathbb{E}\, V(x(t), \dot{x}(t))$. Then, by Dynkin's formula (stopped at first hitting the boundary of deterministic domain $\mathbb{D}$) one arrives at

$$\left(v(0) - \sigma^2 \int_0^t v(s) ds \right)_+ \geq v(t) = v(0) + \mathbb{E} \int_0^t \mathcal{L}V(x(s), \dot{x}(s)) ds$$
$$\geq \left(\mathbb{E}\, V(x(0), \dot{x}(0)) - (2\gamma^2 r^2 + \sigma^2) \int_0^t \mathbb{E}\, V(x(s), \dot{x}(s)) ds \right)_+$$

where

$$\begin{aligned} \mathcal{L}V(x, y) &= \left(y \frac{\partial}{\partial x} + [\gamma^2 V(x, y) y - \alpha^2 x] \frac{\partial}{\partial y} + \frac{\sigma^2}{2} V(x, y) \frac{\partial^2}{\partial y^2} \right) V(x, y) \\ &= (-2\alpha^2 xy + 2\alpha^2 xy - (2\gamma^2 y^2 + \sigma^2) V(x, y)) \chi_{\{r^2 - \alpha^2 x^2 - y^2 > 0\}}(x, y) \\ &= -(2\gamma^2 y^2 + \sigma^2) V(x, y) \chi_{\{r^2 - \alpha^2 x^2 - y^2 > 0\}}(x, y) , \end{aligned}$$

where $\chi_{\{.\}}$ represents the characteristic function of inscribed set. After application of the Gronwall-Bellman inequality with nonpositive kernels, taking the logarithm and the limit as time t tends to infinity, one arrives at the claimed estimation of V-exponents. Now assume $r^2, \sigma^2 > 0$. Then the convergence of $V(x(t), \dot{x}(t))$ towards 0 (a.s.) by the application of Theorem 6.1 due to the exponentially fast decline of $\mathbb{E}\, V(x(t), \dot{x}(t))$ to 0 as integration time advances, hence the proof is complete. ⋄

5.4. **Asymptotics of a randomly excited generalized Duffing-type oscillator.** Models with Duffing-type oscillations (i.e. with cubic type of dissipative nonlinearities) are found in Mechanical Engineering fairly often in order to describe the qualitative behavior of structures under external and random loads. There, in particular, the problem of reliability of mechanical structures arises. For simplicity, let us here consider the one-degree of freedom, randomly perturbed *generalized Duffing-type oscillator* (thought as one component of a multi-degree of freedom system with independent noise sources)

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 f(x) \;=\; \sigma_1\,\xi_t^1 \;+\; \sigma_2\sqrt{h^2(\dot{x})}\,\xi_t^2 \tag{5.5}$$

where $\sigma_1, \sigma_2 \in \mathbb{R}$ and ξ_t^1, ξ_t^2 represent independent white or colored noises. Thus, the classical model of *Duffing's oscillator* is included by the case $f(x) = x + \gamma x^3$ with real parameter $\gamma > 0$. Also the *Ueda's oscillator* is contained by the choice $f(x) = -x + \gamma x^3$ where $\gamma > 0$, and the damped harmonic oscillator as well. $x = x(t)$ again represents the displacement of oscillations from the rest-point, whereas $\dot{x} = \dot{x}(t)$ the velocity of the oscillations. The parameter $\zeta \in \mathbb{R}_+$ controls the intensity of damping part, and parameter $\omega \in \mathbb{R}_+$ is the eigenfrequency and determines the stiffness of the system. $h(.)$ is a further control function on noise influence. Let us restrict to the white noise case only. Here $\gamma, \zeta, \omega, \sigma_2$ essentially control the asymptotic behavior. We will see that the interplay of ζ, ω, σ_2 establishes the exponential growth behavior of trajectories in a decisive manner. It is clear that the requirements f is integrable and $\int^x f(z)dz \geq k_1 - k_2\|x\|^2$ with constants $k_1, k_2 \geq 0$ (or $\gamma \geq 0$) together with that of the finiteness of certain initial moments are essential for existence of nonblowing up solutions at all time-instants t. Take $\mathbb{D} = \mathbb{R}^2$.

Theorem 5.3. *Consider the stochastic process $(x(t), \dot{x}(t))_{t \in \mathbb{R}_+}$, governed by Itô SDE (5.5) with locally Lipschitz-continuous f, h, started at any values $(x(0), \dot{x}(0))$ such that $\mathbb{E}\, V(x(0), \dot{x}(0)) \;<\; +\infty$ along the Lyapunov functional*

$$V(x, \dot{x}) \;=\; \frac{\dot{x}^2}{2} \;+\; \omega^2 \int^x f(z)\, dz\,.$$

Assume that there are real, deterministic constants c_1, c_2 such that for all $(x, y) \in \mathbb{D}$

$$\int^x f(z)dz \geq 0,\; h^2(y) \leq c_1^2 + c_2^2 y^2\,.$$

Then, if $c_2^2\sigma_2^2 < 4\zeta\omega, c_1 = 0$ and $\sigma_1 = 0$, we have $\lim_{t\to+\infty} \dot{x}(t) \;=\; 0$ (a.s.) and

$$\underline{\lambda}_{(\dot{x})^2}(c_2, \zeta, \omega, \sigma_2) \;\leq\; \overline{\lambda}_{(\dot{x})^2}(c_2, \zeta, \omega, \sigma_2) \;\leq\; c_2^2\sigma_2^2 - 4\zeta\omega\,.$$

Moreover, the stochastic process $(x(t), \dot{x}(t))$ has V-exponents

$$\overline{\lambda}_V(c_2, \zeta, \omega, \sigma_2) \begin{cases} \leq & \dfrac{(c_2^2\sigma_2^2 - 4\zeta\omega)_+}{2} \qquad \textit{else} \\ = & 0 \qquad \textit{if} \quad c_2^2\sigma_2^2 = \zeta\omega = 0 \end{cases}$$

and

$$\underline{\lambda}_V(c_2, \zeta, \omega, \sigma_2) \begin{cases} \geq & -2(\zeta\omega)_+ \qquad \textit{else} \\ = & 0 \qquad \textit{if} \quad c_2^2\sigma_2^2 = \zeta\omega = 0 \end{cases}.$$

Proof. Consider Lyapunov functional

$$V(x, y) \;:=\; \frac{y^2}{2} + \omega^2 \int^x f(z)dz$$

where parameter ω is fixed. Define $v(t) := \mathbb{E}\, V(x(t),\dot{x}(t))$. Then, by Dynkin's formula one arrives at

$$\begin{aligned} \frac{1}{2}\mathbb{E}\,\Big(\dot{x}(t)\Big)^2 \;\le\; v(t) &= v(0) + \mathbb{E}\int_0^t \mathcal{L}V(x(s),\dot{x}(s))ds \\ &= v(0) + \frac{\sigma_1^2}{2}t + \mathbb{E}\int_0^t \left(\frac{\sigma_2^2}{2}h^2(\dot{x}(s)) - 2\zeta\omega(\dot{x}(s))^2\right) ds \\ &\le v(0) + \frac{c_1^2\sigma_2^2+\sigma_1^2}{2}t + \mathbb{E}\int_0^t \left(\frac{c_2^2\sigma_2^2}{2} - 2\zeta\omega\right)(\dot{x}(s))^2\, ds \\ &\le v(0) + \frac{c_1^2\sigma_2^2+\sigma_1^2}{2}t + \int_0^t \left(\frac{c_2^2\sigma_2^2}{2} - 2\zeta\omega\right)_+ v(s)\, ds\,. \end{aligned}$$

After taking the logarithm and limit as t tends to $+\infty$, this immediately implies

$$-2(\zeta\omega)_+ \le \underline{\lambda}_V \le \overline{\lambda}_V \le \frac{(c_2^2\sigma_2^2 - 4\zeta\omega)_+}{2}\,.$$

If $c_2^2\sigma_2^2 \le 4\zeta\omega$ then one finds $\overline{\lambda}_V \le 0$, as well as, if $\zeta = 0$ or $\omega = 0$ then $\underline{\lambda}_V \ge 0$. The equality $\overline{\lambda}_V = \underline{\lambda}_V = 0$ if $c_2^2\sigma_2^2 = \zeta\omega = 0$ is obvious after a careful view on the application of Dynkin's formula (see above). In a similar way we can conclude estimates for $(\dot{x})^2$-exponents from above. Now, suppose that $c_2^2\sigma_2^2 < 4\zeta\omega$, $c_1 = 0$ and $\sigma_1 = 0$. Then the convergence of $(\dot{x}(t))^2$ towards 0 (a.s.) can be established by the application of Theorem 6.1 due to the exponentially fast decline of $\mathbb{E}\,(\dot{x}(t))^2$ towards zero, hence the proof is complete. $\diamond$

5.5. **On general V-asymptotics of iterative random mappings.** In analogy to deterministic analysis, we receive the following discrete inequality. Let $(t_n)_{n\in\mathbb{N}}$ be a monotonically nondecreasing sequence of deterministic time-instants with t_n diverging to $+\infty$ as n tends to $+\infty$, and, for a discrete time $\mathbb{D}$-valued stochastic process $X = (X_n)_{n\in\mathbb{N}}$ on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$, define

$$\Delta\mathbb{E}\,V_n := \mathbb{E}\,V(n+1, X_{n+1}) - \mathbb{E}\,V(n, X_n)\,.$$

Theorem 5.4. *Assume that $\mathbb{E}\,V(0,X_0) < +\infty$ for a Borel-measurable function $V : \mathbb{N}\times\mathbb{D} \longrightarrow \mathbb{R}^1_+$ (or a $\mathcal{F}_n$-adapted, nonnegative functional) with*

$$\underline{k}_n \mathbb{E}\,V(n,X_n) \le \Delta\mathbb{E}\,V_n \le \overline{k}_n\mathbb{E}\,V(n,X_n)$$

for all $n\in\mathbb{N}$, where $\underline{k}_i, \overline{k}_i$ are deterministic, real constants along the dynamics of process $(X_n)_{n\in\mathbb{N}}$, and for all $n\in\mathbb{N}$

$$1 + \underline{k}_n > 0\,.$$

Then, for all $n\in\mathbb{N}$, we have

$$\exp\left(\sum_{i=0}^n \frac{\underline{k}_i}{1+\underline{k}_i}\right)\mathbb{E}\,V(0,X_0) \le \mathbb{E}\,V(n+1,X_{n+1}) \le \exp\left(\sum_{i=0}^n \overline{k}_i\right)\mathbb{E}\,V(0,X_0)$$

and, if the limits exist, then

$$\liminf_{n\to+\infty}\frac{\sum_{i=0}^{n-1}\frac{\underline{k}_i}{1+\underline{k}_i}}{t_n} \le \underline{\lambda}_V \le \overline{\lambda}_V \le \limsup_{n\to+\infty}\frac{\sum_{i=0}^{n-1}\overline{k}_i}{t_n}\,.$$

Moreover, if $\overline{\lambda}_V < 0$ then we have

$$\lim_{n\to+\infty} V(n,X_n) \;=\; 0 \text{ (a.s.)}.$$

Proof. First, assume $\Delta \mathbb{E}\, V_n \leq \overline{k}_n \mathbb{E}\, V(n, X_n)$ (for all $n \in \mathbb{N}$). Making use of elementary splitting

$$z(n+1) = z(n) + z(n+1) - z(n)$$

with $z(n+1) := \mathbb{E}\, V(n+1, X_{n+1})$, one concludes

$$z(n+1) \leq z(n)(1+\overline{k}_n) \leq z(0) \prod_{i=0}^{n} (1+\overline{k}_i)_+ \leq z(0) \exp\left(\sum_{i=0}^{n} \overline{k}_i\right) .$$

On the other hand, when $\Delta \mathbb{E}\, V_n \leq \underline{k}_n \mathbb{E}\, V(n, X_n)$ and $1 + \underline{k}_n > 0$ (for all $n \in \mathbb{N}$), one recognizes the validity of

$$z(n) \leq \frac{z(n+1)}{1+\underline{k}_n} \leq z(n+1) \exp\left(\frac{-\underline{k}_n}{1+\underline{k}_n}\right)$$

which implies

$$z(n+1) \geq z(n) \exp\left(\frac{\underline{k}_n}{1+\underline{k}_n}\right) \geq z(0) \exp\left(\sum_{i=0}^{n} \frac{\underline{k}_i}{1+\underline{k}_i}\right) ,$$

using elementary inequality

$$\frac{1}{1+x} \leq \exp(-\frac{x}{1+x}) .$$

Now one arrives at the second result by taking the exponential logarithm and limit when integration time t_n advances. The almost sure convergence of $V(n, X_n)$ to 0 is established by the direct application of Theorem 6.1. Therefore, the proof is complete. ⋄

5.6. **An example for V-asymptotics in discrete time.** Consider

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x \;=\; \sigma \dot{x}\, \xi_t \tag{5.6}$$

where $\zeta, \omega > 0$ and the stochastic integration is understood in the sense of Itô. Then the corresponding deterministic equation has an asymptotically stable zero solution if $0 < \zeta < 1$, and does not exponentially grow if $0 \leq \zeta \leq 1$. Thanks to Theorem 5.3, we know about the stochastic version that the upper V-exponent with $V(x, y) = y^2 + \omega^2 x^2$ is not larger than zero if $0 \leq \sigma^2 \leq 4\zeta\omega$. Define

$$V(n+1, x, y) := \omega^2 x^2 + (1 + 2\zeta\omega\Delta_n) y^2$$

where $\Delta_n = t_{n+1} - t_n$ is current step size, and $v_{n+1} := \mathbb{E}\, V(n+1, X_{n+1}, Y_{n+1})$.

Theorem 5.5. *Assume that the stochastic oscillator (5.6) is discretized by the fully drift-implicit Euler method given by*

$$\begin{aligned} X_{n+1} &= X_n + Y_{n+1}\Delta_n \\ Y_{n+1} &= Y_n - (2\zeta\omega Y_{n+1} + \omega^2 X_{n+1})\Delta_n + \sigma Y_n \Delta W_n \end{aligned} \tag{5.7}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ along a time-discretization $(t_n)_{n \in \mathbb{N}}$, and

$$\mathbb{E}\,[X_0^2 + Y_0^2] < +\infty.$$

Then, for all $n \in \mathbb{N}$, all $l \in \mathbb{N}$ with $1 \leq l < n$, we have

$$v_l \exp\left(\sum_{i=l}^{n} \frac{\underline{k}_i}{1+\underline{k}_i}\right) \;\leq\; v_{n+1} = \mathbb{E}\, V(n+1, X_{n+1}, Y_{n+1}) \;\leq\; v_l \exp\left(\sum_{i=l}^{n} \overline{k}_i\right)$$

where

$$\overline{k}_i = \frac{-\omega^2 \Delta_i^2 (1 + 2\zeta\omega\Delta_{i-1}) + [(\sigma^2 - 2\zeta\omega)\Delta_i - 2\zeta\omega\Delta_{i-1}(1 + 2\zeta\omega\Delta_i)]_+}{(1 + 2\zeta\omega\Delta_{i-1})(1 + 2\zeta\omega\Delta_i + \omega^2\Delta_i^2)} ,$$

$$\underline{k}_i = \frac{-\omega^2\Delta_i^2(1+2\zeta\omega\Delta_{i-1}) - [(\sigma^2-2\zeta\omega)\Delta_i - 2\zeta\omega\Delta_{i-1}(1+2\zeta\omega\Delta_i)]_-}{(1+2\zeta\omega\Delta_{i-1})(1+2\zeta\omega\Delta_i+\omega^2\Delta_i^2)}.$$

Furthermore, if $(\Delta_n)_{n\in I\!N}$ is a deterministic sequence then the V-exponents can be estimated by

$$\liminf_{n\to+\infty}\frac{1}{t_n}\sum_{i=1}^{n-1}\frac{\underline{k}_i}{1+\underline{k}_i} \;\le\; \underline{\lambda}_V \;\le\; \overline{\lambda}_V \;\le\; \limsup_{n\to+\infty}\frac{1}{t_n}\sum_{i=1}^{n-1}\overline{k}_i\,.$$

Additionally, in the following assume that

$$\exists\Delta_a,\Delta_b\in I\!R_+ \,:\, \forall n\in I\!N \quad 0<\Delta_b\le\Delta_n\le\Delta_a<+\infty\,. \tag{5.8}$$

If

$$(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n) \;\le\; 0 \tag{5.9}$$

for all $n\in I\!N$ then $\lim_{n\to+\infty}V(n+1,X_n,Y_n) \;=\; 0$ (a.s.) and

$$\overline{\lambda}_V \le -\frac{\omega^2\Delta_b}{1+2\zeta\omega\Delta_a+\omega^2\Delta_a^2}\,.$$

If

$$(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n) \;\ge\; 0 \tag{5.10}$$

for all $n\in I\!N$ then

$$\underline{\lambda}_V \ge -\frac{\omega^2\Delta_a}{1+2\zeta\omega\Delta_b}\,.$$

Proof. First, we equivalently rearrange the scheme (5.7) to an explicit one. Thus

$$\begin{aligned} X_{n+1} &= \frac{1+2\zeta\omega\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n + \frac{(1+\Delta W_n)\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n \\ Y_{n+1} &= -\frac{\omega^2\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n + \frac{(1+\Delta W_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n\,. \end{aligned} \tag{5.11}$$

After some calculations this relation implies

$$v_{n+1} \;=\; \omega^2 I\!E\left[\frac{1+2\zeta\omega\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n^2\right] + I\!E\left[\frac{1+\sigma^2\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right],$$

hence

$$-\frac{\omega^2\Delta_a\Delta_n}{1+2\zeta\omega\Delta_b+\omega^2\Delta_b^2}v_n - I\!E\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n-2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right]_- \;\le$$

$$\begin{aligned} \Delta I\!E\,V_n &= -I\!E\left[\frac{\omega^2\Delta_n^2}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}\omega^2X_n^2\right] \\ &\quad + I\!E\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n-\omega^2\Delta_n^2-2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right] \\ &= -\frac{\omega^2\Delta_n^2}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}v_n + I\!E\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n-2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right] \\ &\le -\frac{\omega^2\Delta_b\Delta_n}{1+2\zeta\omega\Delta_a+\omega^2\Delta_a^2}v_n + I\!E\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n-2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right]_+ \end{aligned}$$

Now, we may choose $\overline{k}_n, \underline{k}_n$ as indicated above, and apply the Theorem 5.4 with $\overline{k}_n, \underline{k}_n$ in order to complete the proof in an obvious manner. ⋄

Remark. Most of the clever variable step size algorithms have implemented conditions on the step size selection like that of (5.8). We can conclude from our assertion that the fully drift-implicit Euler method (5.7) applied to stochastic oscillator (5.6) produces overdamped approximations compared to the asymptotics of exact solution. This can be seen particularly in the critical case (the energy-conservative case) when $\sigma^2 = 4\zeta\omega$ under the condition (5.8). However, the observed effect of numerical stabilization also explains that the requirement (5.8) is meaningful in variable step size algorithms in order to achieve asymptotically stable approximations (i.e. with 'sure side argumentation'). Asymptotically considered, when maximum step size Δ_a tends to zero, the V-exponents of the continuous time dynamics are correctly replicated by the discretization method (5.7) with a convergence of order Δ_a.

6. Almost sure V-attractivity of sets (an invariance principle)

The attractivity concept can be useful for the detection of almost sure attractivity of sets. Suppose that $\mathbb{E}\, V(t,X) < +\infty$ for all $t \geq t_0$, where $\mathcal{F}_t$-adapted, nonnegative functional $V(t,X)$ of the stochastic process $X = (X(t))_{t\geq t_0}$ is fixed.

Definition 4. The *upper (forward p–th moment) weak V-attractivity exponent* of the given stochastic process $(X(t))_{(t\in\mathcal{T})}$ with respect to the nonempty, bounded, deterministic set $\mathbb{D}_2 \subset \mathbb{R}^d$ is defined to be

$$\overline{\lambda}_V^w(x) \;:=\; \limsup_{t\to+\infty} \frac{1}{t}\, \ln \Big(\mathbb{E} \inf_{z\in\mathbb{D}_2} V(t, X-z)\Big) \tag{6.1}$$

for $X(t_0) = x \in \mathbb{D}_1$ (a.s.). The *lower (forward p–th moment) weak V-attractivity exponent* of the given stochastic process $(X(t))_{(t\in\mathcal{T})}$ with respect to the nonempty, bounded, deterministic set $\mathbb{D}_2 \subset \mathbb{R}^d$ is defined to be

$$\underline{\lambda}_V^w(x) \;:=\; \liminf_{t\to+\infty} \frac{1}{t}\, \ln \Big(\mathbb{E} \inf_{z\in\mathbb{D}_2} V(t, X-z)\Big) \tag{6.2}$$

These exponents trivially coincide with those occurring at definition 1 in the case of singleton sets $\mathbb{D}_2$ and $V(t,X-z) = ||X(t)-z||^p$. It is hard to compute such exponents for non-singleton sets $\mathbb{D}_2$. However, this quite general result is found.

Theorem 6.1. *Assume that $\mathbb{E}\, V(t,X(t)-z) < +\infty$ for all $t \geq t_0$, for all $z \in \mathbb{D}_2$, where $V : [t_0,+\infty) \times \mathbb{R}^d \to \mathbb{R}_+$ is a Borel-measurable, deterministic function, and $\overline{\lambda}_V^w(x) < 0$ for all $x \in \mathbb{D}_1$, where $(X(t))_{t\geq t_0}$ is a stochastic process started at the nonempty set $\mathbb{D}_1 \subseteq \mathbb{R}^d$ contained in its domain.*
Then, for all $X(t_0) = x \in \mathbb{D}_1$, we have

$$\lim_{t\to+\infty} \inf_{z\in\mathbb{D}_2} V(t, X(t)-z) = 0 \text{ (a.s.)}.$$

Proof. If $\overline{\lambda}_V^w(x) = -\infty$ the assertion is obvious by indirect proof. Suppose that $-\infty < \overline{\lambda}_V^w(x) < 0$ for all $x \in \mathbb{D}_1$. There exists a finite real constant C such that

$$\mathbb{E} \inf_{z\in\mathbb{D}_2} V(t, X(t)-z) \;\leq\; C \exp(\overline{\lambda}_V^w t)\,.$$

Thanks to corollary 2 from Shiryaev [37, p. 256] it suffices to find $\varepsilon_i \to 0$ such that

$$v \;:=\; \sum_{i=1}^{+\infty} \mathbb{P}\Big\{ \inf_{z\in\mathbb{D}_2} V(t_i, X(t_i)-z) > \varepsilon_i \Big\} \;<\; +\infty$$

for all sequences $(t_i)_{i\in\mathbb{N}}$ with increasing t_i converging to $+\infty$. Now, take

$$\varepsilon_i \;:=\; \frac{\exp((\overline{\lambda}_V^w + \varepsilon_0)t_i)}{t_i - t_{i-1}}$$

with any $\varepsilon_0 > 0$ satisfying $\overline{\lambda}_V^w + \varepsilon_0 < 0$. Using the Chebyshev inequality we get

$$\begin{aligned} v \;&\leq\; \sum_{i=1}^{+\infty} \frac{\mathbb{E}\,\inf_{z\in\mathbb{D}_2} V(t_i, X(t_i) - z)}{\varepsilon_i} \;\leq\; C\sum_{i=1}^{+\infty} \exp(-\varepsilon_0 t_i)(t_i - t_{i-1}) \\ &\leq\; C\int_{t_0}^{+\infty} \exp(-\varepsilon_0 t_0)\,ds \;\leq\; C\frac{\exp(-\varepsilon t_0)}{\varepsilon_0} \;<\; +\infty. \end{aligned}$$

Therefore we have $\lim_{t\to+\infty}\inf_{z\in\mathbb{D}_2} V(t, X(t) - z) = 0$ almost surely for all $X_{t_0} = x \in \mathbb{D}_1$ with finite p-th moment, and this completes the proof. $\diamond$

As a consequence, we can explain almost sure convergence to attracting sets. In particular, take $V(u) = ||u||^p$. Then, under the assumptions of Theorem 6.1, we have $\lim_{t\to+\infty}\inf_{z\in\mathbb{D}_2} ||X(t) - z|| = 0$ (a.s.), i.e. $\lim_{t\to+\infty}\inf_{z\in\mathbb{D}_2} X(t) \in \overline{\mathbb{D}}_2$ (a.s.). In another words, $\mathbb{D}_2$ is a 'forward attracting set' (a.s.) for the process $X = (X(t))_{t\geq t_0}$. In the case $V = V(x)$, $\mathbb{D}_2 = \{0\}$ a nonempty set $\{x \in \mathbb{R}^d : V(x) = 0\}$ forms a forward V-attracting set (a.s.) for the a.s. dissipative stochastic mapping $x \in \mathbb{D}_1 \to X(t, x)$ with $\overline{\lambda}_V^w < 0$. It may trivially be noted that

$$\overline{\lambda}_V^w(x) \;\leq\; \overline{\lambda}_V(x) \;\leq\; \overline{\lambda}_p(x)$$

for nondecreasing V, whenever $V(u) \leq ||u||^p$ for $u \in \mathbb{R}^d$ and $x \in \mathbb{D}_1$ (Recall the definition of $\overline{\lambda}_V$ from section 5). This fact along with the Theorem 6.1 explains that some of our results from the previous sections could be stated with respect to almost sure convergence while presuming $\overline{\lambda}_V(x) < 0$ or $\overline{\lambda}_p(x) < 0$.

7. Conclusions and remarks

An asymptotic moment characterization by deterministic numbers could be made for nonlinear stochastic dynamical systems. The sign of these characteristics decides about a possible existence of finite 'asymptotic structures'. Only if these numbers are zero one can expect the existence of nontrivial limit laws of corresponding moments. In this latter case one needs further investigations to refine the asymptotic behavior of examined systems, for example with different rescaled functionals in the definition (i.e. in order to distinguish between V-dissipative and (polynomially) blowing up solutions). However, meeting an 'optimal' scaling turns out to be very difficult and very case-sensitive. We also suggest to study the new characteristic exponents as functions of the parameters involved in the function(-al) V as done in [3, 4]. All estimates here are based on the generalized Gronwall–Bellman Lemma with nonautonomous kernels suggested by [27], [32] in the context of stochastic systems (We also sometimes called its generalization as variation-of-constants inequality in this contribution.). Other types of integral inequalities can be used to find more delicate estimates of characteristic exponents. The introduced exponents fully determine the global exponential attractivity of solutions of stochastic differential equations and of stochastic difference equations in forward direction. A similar analysis can be carried out in backward direction (i.e. when time t tends to $-\infty$). Also, the convergence proofs of numerical methods should be revised under the knowledge on finiteness of characteristic exponents of underlying continuous time differential systems. A careful look at our uniform estimates of both contractivity and attractivity (stability) exponents yields the conclusion that the drift part $a(t, x)$

in the continuous time case and $\Phi_0^I(x^{(i)} : i \leq n)$ in the discrete time case are sometimes able to compensate the diffusion behavior given by $b^j(t,x)$ and $\Phi_j(x^{(i)} : i \leq n)$ through appropriate terms, respectively. In contrast to that fact, diffusion terms always lead to an increase of moment contractivity and stability exponents within the Itô calculus (provided that $p \geq 2$). The latter fact can dramatically change while studying the attractivity of sets or under other stochastic calculi like that of Stratonovich due to different correction terms in their stochastic chain rules. Besides, it would be interesting to develop a similar concept in the 'almost sure sense' instead of exclusive consideration of moments (i.e. under assumptions like 'almost sure monotonicity' of involved stochastic terms). Then, some stochastic versions of LaSalle's invariance principle and asymptotic properties of exponential martingales are certainly helpful. In the homogeneous, linear framework (as with multiplicative noise) the presented concepts of moment contractivity and stability exponents coincide with the well-known concept of moment Lyapunov exponents (connected by Arnold's formula [2] with sample Lyapunov exponents arising from the multiplicative ergodic theorem of Oseledec [30]). Thus, the main purpose of presented paper is aiming at an analysis of nonlinear and nonautonomous stochastic systems where nontrivial equilibria may occur. Moreover, in any framework where the zero solution is an equilibrium position (a.s.), the concepts of contractivity and stability are identical. These concepts may differ when stochasticity (or inhomogeneouities) comes into play as it can be seen by the trivial example of Ornstein–Uhlenbeck processes with exponentially blowing up nonautonomous diffusion part. The concepts also differ in the deterministic situation as seen by the one-dimensional equation

$$\dot{x} \;=\; -\alpha^2 x + \beta^2$$

where parameters $\alpha^2 > 0$ and $\beta^2 > 0$. This equation satisfies

$$\underline{\kappa}_p = \overline{\kappa}_p = -p\alpha^2 < 0 = \underline{\lambda}_p = \overline{\lambda}_p$$

for all start values $x(0)$, and trivially for all exponents $p > 0$. As seen by the latter example, it may be noted that the concept of contractivity replicates more the asymptotic-qualitative behavior of fundamental solutions in comparison to that of stability where inhomogeneous parts have to take into account in addition. The fairly new concept of stochastic moment contractivity permits to treat stochastic systems with both additive and multiplicative noise by an unified approach. This is a fact in which the major advantage of contractivity analysis can be seen. It is particularly appropriate to describe the propagation of initial perturbations.

Acknowledgement. We thank Shouchuan Hu for an invitation to submit this paper. Special thanks deserve the anonymous referees for some of their comments.

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Received for publication at journal in August 1999 / Revised in December 2000.
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