Some Qualitative Properties for the Total Variational Flow

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Abstract

We prove the existence of a finite extinction time for the solutions of the Dirichlet problem for the total variational flow. For the Neumann problem, we prove that the solutions reach the average of its initial datum in finite time. The asymptotic profile of the solutions of the Dirichlet problem is also studied. It is shown that the profiles are non zero solutions of an eigenvalue type problem which seems to be unexplored in the previous literature. The propagation of the support is analyzed in the radial case showing a behaviour enterely different to the case of the problem associated to the p-Laplacian operator. Finally, the study of the radially symmetric case allows us to point out other qualitative properties which are peculiar of this special class of quasilinear equations.

Key words: Total variation flow, nonlinear parabolic equations, asymptotic behaviour, eigenvalue type problem, propagation of the support.

AMS (MOS) subject classification: 35K65, 35K55.

1 Introduction

Let Ω be a bounded set in $I\!R^N$ with Lipschitz continuous boundary $\partial\Omega$. We are interested in some qualitative properties of the solutions of the Dirichlet problem

$$P_D \begin{cases} \dfrac{\partial u}{\partial t} = div\left(\dfrac{Du}{|Du|}\right) & \text{in} \quad Q = (0,\infty)\times\Omega \\[2ex] u(t,x) = 0 & \text{on} \quad S = (0,\infty)\times\partial\Omega \\[2ex] u(0,x) = u_0(x) & \text{in} \quad x\in\Omega \end{cases}$$

and of the Neumann problem

$$P_N \begin{cases} \dfrac{\partial u}{\partial t} = div\left(\dfrac{Du}{|Du|}\right) & \text{in} \quad Q = (0,\infty)\times\Omega \\[2ex] \dfrac{\partial u}{\partial \eta} = 0 & \text{on} \quad S = (0,\infty)\times\partial\Omega \\[2ex] u(0,x) = u_0(x) & \text{in} \quad x\in\Omega \end{cases}$$

for the total variational flow. Motivated by problems in Image Processing ([24]) existence and uniqueness of solutions for the problems (P_D) and (P_N) have been obtained in [4] and [3], respectively (see also [20] for the Dirichlet problem). We point out that related formulations arise in other different contexts: faceted crystal growth ([19]), Continuum Mechanics ([23]), etc.

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The main goal of this paper is to describe the behaviour of solutions of (P_D) and (P_N) near the extinction time (we shall prove that is finite). We shall prove that this behaviour is described by a function which is a solution of an eigenvalue problem for the operator $-div\left(\frac{Du}{|Du|}\right)$ and we shall describe the solutions of this eigenvalue problem in the radial case. Moreover, the explicit solution found for the case in which $u_0 = k\chi_{B(0,r)}$ with $B(0,r) \subset\subset \Omega$ (see Lemma1) allows us to point out other qualitative properties which are peculiar of this special class of quasilinear equations. For instance, there is a infinite "waiting time", i.e. there is no propagation of the support of the initial datum and, which is more relevant, the solution is discontinuous and has a spatial minimal regularity: $u(t,.) \in BV(\Omega) \setminus W^{1,1}(\Omega)$ for any $t \in [0,+\infty)$ (i.e. the solution do not wins any spatial differentiability, in contrast to what happens for the linear heat equation and many other quasilinear parabolic equations).

Our plan is as follows. First, in Section 2, we recall some results about functions of bounded variation that we shall need in the sequel and we recall the existence and uniqueness results for (P_D) and (P_N) that were proved in [4] and [3], respectively. In Section 3, we prove that the solutions of the Dirichlet problem (P_D) vanish in finite time and we study the asymptotic profile of the Dirichlet problem near the extinction time. This asymptotic profile is a solution of an eigenvalue problem for the operator $-div\left(\frac{Du}{|Du|}\right)$. In Section 4, we study this eigenvalue problem in the radial case. Finally, in Section 5 we sketch analogous results for the Neumann problem.

2 Preliminaries

Due to the linear growth of the energy functional associated with problems (P_N) and (P_D), the natural energy space for these problems is the space of functions of bounded variation. Recall that a function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \dots, \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i \varphi dx = -\int_\Omega \varphi d\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \dots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$|Du|(\Omega) = \sup\{\int_\Omega u \, div\, \varphi \, dx : \varphi \in C_0^\infty(\Omega, I\!R^n), |\varphi(x)| \leq 1 \text{ for } x \in \Omega\}.$$

Let E be a measurable set in $I\!R^N$ and let Ω be an open set in $I\!R^N$. It is said that E has *finite perimeter* in Ω if its characteristic function $\chi_E \in BV(\Omega)$. The *perimeter* of E in Ω is defined as

$$P(E, \Omega) := |D\chi_E|(\Omega).$$

We shall use the notation $Per(E) := P(E, I\!R^N)$. If E has smooth boundary, by the classical Gauss-Green formula, we have

$$P(E, \Omega) := H^{N-1}(\partial E \cap \Omega),$$

being H^{N-1} the Hausdorff $(N-1)$-dimensional measure in $I\!R^N$. For further information concerning functions of bounded variation we refer to [2], [17] and [26].

Also several results from [7] (see also [23]) are needed. Following [7], let

$$X(\Omega) = \{z \in L^\infty(\Omega, I\!R^N) : div(z) \in L^1(\Omega)\}.$$

If $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ the functional $(z, Dw) : C_0^\infty(\Omega) \to I\!R$ is defined by the formula

$$< (z, Dw), \varphi >= -\int_\Omega w \, \varphi \, div(z) \, dx - \int_\Omega w \, z \cdot \nabla\varphi \, dx.$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx$$

for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and

$$\left| \int_B (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B |Dw| \tag{2.1}$$

for any Borel set $B \subseteq \Omega$. Moreover, (z, Dw) is absolutely continuous with respect to $|Dw|$ with Radon-Nikodym derivative $\theta(z, Dw, x)$ which is a $|Dw|$ measurable function from Ω to $I\!R$ such that

$$\int_B (z, Dw) = \int_B \theta(z, Dw, x)|Dw|(\Omega) \tag{2.2}$$

for any Borel set $B \subseteq \Omega$. Moreover

$$\|\theta(z, Dw, .)\|_{L^\infty(\Omega, \|Dw\|)} \leq \|z\|_{L^\infty(\Omega, I\!R^N)}. \tag{2.3}$$

In [7], a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty$$

$$\gamma(z)(x) = z(x) \cdot \nu(x) \quad \text{ for all } x \in \partial\Omega \text{ if } z \in C^1(\overline{\Omega}, I\!R^N).$$

We shall denote $\gamma(z)(x)$ by $[z, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[z, \nu]$ and the measure (z, Dw), for $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, is established:

$$\int_\Omega w \; div(z) \; dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, \nu] w \; dH^{N-1}. \tag{2.4}$$

Existence and uniqueness for (P_D) and (P_N) where obtained in [4] and [3] for general initial conditions in $L^1(\Omega)$ (and general boundary condition in $L^1(\partial\Omega)$ in the case of Dirichlet problem). Since our main results in this paper will concern only initial conditions in $L^\infty(\Omega)$, to avoid some technicalities, we shall restrict our statements concerning existence and uniqueness to initial conditions in $L^2(\Omega)$.

Let us recall the notion of solution for which existence and uniqueness are obtained. For that, we denote by $L^1_w(0, T, BV(\Omega))$ the space of functions $w : [0, T] \to BV(\Omega)$ such that $w \in L^1((0, T) \times \Omega)$, the maps $t \in [0, T] \to < Dw(t), \phi >$ are measurable for every $\phi \in C^1_0(\Omega, I\!R^N)$ and such that $\int_0^T \|Dw(t)\| < \infty$.

Definition 1 Let $u_0 \in L^2(\Omega)$. A measurable function $u : (0, T) \times \Omega \to I\!R$ is a *weak solution* of (P_D) (respectively, P_N) in $(0, T) \times \Omega$ if $u \in C([0, T], L^2(\Omega)) \cap W^{1,2}_{loc}(0, T; L^2(\Omega))$, $u \in L^1_w(0, T; BV(\Omega))$, and there exists $z \in L^\infty((0, T) \times \Omega)$ with $\|z\|_\infty \leq 1$, $u_t = div(z)$ in $\mathcal{D}'((0, T) \times \Omega)$ such that

$$\int_\Omega (u(t) - w) u_t(t) = \int_\Omega (z(t), Dw) - |Du(t)| - \int_{\partial\Omega} [z(t), \nu] w - \int_{\partial\Omega} |u(t)| \tag{2.5}$$

(respectively,

$$\int_\Omega (u(t) - w) u_t(t) = \int_\Omega (z(t), Dw) - |Du(t)|(\Omega) \tag{2.6}$$

in case of the Neumann problem) for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and a.e. on $[0, T]$.

Theorem 1 *([4]) Let $u_0 \in L^2(\Omega)$. Then for every $T > 0$ there exists a unique weak solution of (P_D) in $(0,T) \times \Omega$ such that $u(0) = u_0$. Moreover, the solution $u(t)$ of (P_D) is also characterized as follows: $u \in C([0,T], L^2(\Omega)) \cap W^{1,2}_{loc}(0,T; L^2(\Omega))$, $u \in L^1_w(0,T; BV(\Omega))$, and there exists $z(t) \in X(\Omega)$, such that $\|z(t)\|_\infty \le 1$, $u'(t) = div(z(t))$ in $\mathcal{D}'(\Omega)$ a.e. $t \in [0, +\infty[$ and*

$$\int_\Omega (z(t), Du(t)) = |Du(t)(\Omega)| \tag{2.7}$$

$$[z(t), \nu] \in sign(-u(t)) \qquad H^{N-1} - a.e. \text{ on } \partial\Omega. \tag{2.8}$$

Finally, we have the following comparison principle: if $u(t), \hat{u}(t)$ are solutions corresponding to initial data $u_0, \hat{u}_0$, respectively, then

$$\|(u(t) - \hat{u}(t))^+\|_1 \le \|(u_0 - \hat{u}_0)^+\|_1 \quad \text{and} \quad \|u(t) - \hat{u}(t)\|_1 \le \|u_0 - \hat{u}_0\|_1, \tag{2.9}$$

for all $t \ge 0$.

Theorem 2 *([3]) Let $u_0 \in L^2(\Omega)$. Then for every $T > 0$ there exists a unique weak solution of (P_N) in $(0,T) \times \Omega$ such that $u(0) = u_0$. Moreover, if $u(t), \hat{u}(t)$ are weak solutions corresponding to initial data $u_0, \hat{u}_0$, respectively, then*

$$\|(u(t) - \hat{u}(t))^+\|_1 \le \|(u_0 - \hat{u}_0)^+\|_1 \quad \text{and} \quad \|u(t) - \hat{u}(t)\|_1 \le \|u_0 - \hat{u}_0\|_1, \tag{2.10}$$

for all $t \ge 0$.

Theorems 1 and 2 were proved (in the more general case of data in $L^1(\Omega)$) using the techniques of completely accretive operators [10] and the Crandall-Liggett's semigroup generation Theorem [13]. Let us recall the notion of completely accretive operator introduced in [10]. Let $\mathcal{M}(\Omega)$ be the space of measurable functions in Ω. Given $u, v \in \mathcal{M}(\Omega)$, we shall write

$$u \ll v \quad \text{if and only if} \quad \int_\Omega j(u)dx \le \int_\Omega j(v)dx \tag{2.11}$$

for all $j \in J_0$, where

$$J_0 = \{j : I\!R \to [0, \infty], \text{ convex, l.s.c., } j(0) = 0\} \tag{2.12}$$

(l.s.c. is an abbreviation for lower semicontinuous function). Let A be an operator (possibly multivalued) in $\mathcal{M}(\Omega)$, i.e., $A \subseteq \mathcal{M}(\Omega) \times \mathcal{M}(\Omega)$. We shall say that A is completely accretive if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \quad \text{for all } \lambda > 0 \text{ and all } (u,v), (\hat{u}, \hat{v}) \in A. \tag{2.13}$$

In [4], the m-completely accretive operator $\mathcal{B}$ in $L^2(\Omega)$ associated with problem (P_D) is introduced as:

$$(u,v) \in \mathcal{B} \quad \text{if and only if } u, v \in L^2(\Omega) \text{ and}$$

$$\text{there exists } z \in X(\Omega) \text{ with } \|z\|_\infty \le 1,\ v = -div(z) \text{ in } \mathcal{D}'(\Omega) \text{ such that}$$

$$\int_\Omega (w - u)v \le \int_\Omega (z, Dw) - |Du|(\Omega) - \int_{\partial\Omega} [z, \nu]w - \int_{\partial\Omega} |u|,$$

$$\text{for all } w \in BV(\Omega) \cap L^\infty(\Omega).$$

Now, if $\Phi : L^2(\Omega) \to]-\infty, +\infty]$, is defined by

$$\Phi(u) = \begin{cases} |Du|(\Omega) + \displaystyle\int_{\partial\Omega} |u| & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega) \cap L^2(\Omega) \end{cases} \tag{2.14}$$

then we have that $\mathcal{B} \cap (L^2(\Omega) \times L^2(\Omega)) = \partial\Phi$ ([4]). Since the functional Φ is convex and lower semicontinuous in $L^2(\Omega)$, we have that $\partial\Phi$ is a maximal monotone operator in $L^2(\Omega)$ and, consequently (see [12]), if $\{T(t)\}_{t\geq 0}$ is the semigroup in $L^2(\Omega)$ generated by $\partial\Phi$, then for every $u_0 \in L^2(\Omega)$, the strong solution $T(t)u_0$ of the problem

$$\begin{cases} \dfrac{du}{dt} + \partial\Phi u(t) \ni 0 \\ \\ u(0) = u_0. \end{cases} \tag{2.15}$$

coincides with the weak solution of (P_D).

The m-completely accretive operator $\mathcal{A}$ in $L^2(\Omega)$ associated with problem (P_N) is introduced as:

$$(u, v) \in \mathcal{A} \quad \text{if and only if } u, v \in L^2(\Omega) \text{ and}$$

$$\text{there exists } z \in X(\Omega) \text{ with } \|z\|_\infty \leq 1,\ v = -div(z) \text{ in } \mathcal{D}'(\Omega) \text{ such that}$$
$$\int_\Omega (w - u)v \leq \int_\Omega (z, Dw) - |Du|(\Omega), \quad \forall w \in BV(\Omega) \cap L^\infty(\Omega).$$

If we consider the functional $\Psi : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\Psi(u) = \begin{cases} |Du|(\Omega) & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega) \cap L^2(\Omega). \end{cases} \tag{2.16}$$

then we have $\mathcal{A} \cap (L^2(\Omega) \times (L^2(\Omega)) = \partial\Psi$. Again the semigroup solution and the weak solution of (P_N) coincide ([3]).

3 The Dirichlet problem

This Section is devoted to prove the following result.

Theorem 3 *Let $u_0 \in L^\infty(\Omega)$ and let $u(t,x)$ be the unique solution of problem (P_D). Let $d(\Omega)$ be the smallest radius of a ball containing Ω. If $T^*(u_0) = \inf\{t > 0 : \ u(t) = 0\}$, then*

$$T^*(u_0) \leq \frac{d(\Omega)\|u_0\|_\infty}{N}. \tag{3.1}$$

Let

$$w(t,x) := \begin{cases} \dfrac{u(t,x)}{T^*(u_0) - t} & \text{if } \ 0 \leq t < T^*(u_0), \\ \\ 0 & \text{if } \ t \geq T^*(u_0). \end{cases}$$

Then, there exists an increasing sequence $t_n \to T^*(u_0)$ and a solution $v^* \neq 0$ of the stationary problem

$$S_D \begin{cases} -div\left(\dfrac{Dv}{|Dv|}\right) = v & in \quad \Omega \\ \\ v = 0 & on \quad \partial\Omega \end{cases}$$

such that

$$\lim_{n\to\infty} w(t_n) = v^* \quad \text{in} \quad L^p(\Omega)$$

for all $1 \leq p < \infty$. Moreover v^* is a minimizer of $\Phi(\cdot) - < \cdot, v^* >$ in $BV(\Omega) \cap L^2(\Omega)$.

Notice that Theorem 3 improves a previous result proved in [20] showing that the solutions of problem (P_D) stabilize as $t \to \infty$ by converging in the L^1-norm to zero. We also point out that, in the best of our knowledge, the consideration of the eigenvalue type problem (S_D) is new in the literature.

We start the proof of Theorem 3 by proving a comparison principle between solutions and subsolutions/supersolutions of (P_D) which are independent of the space variable, and, as a consequence, we deduce that the solutions of (P_D) vanish in finite time.

Theorem 4 *Let $u_0 \in L^\infty(\Omega)$ and let $u_1(t,x)$ be the unique solution of problem (P_D). Let $d(\Omega)$ be the smallest radius of a ball containing Ω. Let $u_2(t,x) = \alpha(t)$, satisfying*

$$|\alpha'(t)| \leq \frac{N}{d(\Omega)}. \tag{3.2}$$

Then,
(i) if $\alpha(t) \geq 0$ and $u_0 \leq \alpha(0)$, we have

$$u_1(t) \leq u_2(t) \qquad \text{a.e. on} \quad \Omega,$$

(ii) if $\alpha(t) \leq 0$ and $u_0 \geq \alpha(0)$, we have

$$u_1(t) \geq u_2(t) \qquad \text{a.e. on} \quad \Omega.$$

Proof: We shall only prove (i), the proof of (ii) being similar. Without loss of generality we may assume that $\Omega \subseteq B(0, d(\Omega))$. By Theorem 1 there exists $z_1(t) \in X(\Omega)$ such that $\|z_1(t)\|_\infty \leq 1$, $u_1'(t) = div(z_1(t))$ in $\mathcal{D}'(\Omega)$ a.e. $t \in [0, +\infty[$ and satisfying

$$\int_\Omega (z_1(t), Du_1(t)) = |Du_1(t)|(\Omega) \tag{3.3}$$

$$[z_1(t), \nu] \in sign(-u_1(t)) \qquad H^{N-1} - a.e. \text{ on } \partial\Omega. \tag{3.4}$$

If we take $z_2(t)(x) := \dfrac{\alpha'(t)x}{N}$, since $div(z_2(t)) = \alpha'(t) = u_2'(t)$, applying Green's formula (2.4), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u_1(t) - u_2(t))^+]^2 = \int_\Omega (u_1'(t) - u_2'(t))(u_1(t) - u_2(t))^+$$

$$= \int_\Omega (div((z_1(t)) - div((z_2(t)))(u_1(t) - u_2(t))^+ =$$

$$- \int_\Omega ((z_1(t) - z_2(t)), D[(u_1(t) - u_2(t))^+]) + \int_{\partial\Omega} [z_1(t) - z_2(t), \nu](u_1(t) - u_2(t))^+ \, dH^{N-1}.$$

If $R_t(r) := (r - \alpha(t))^+$, then

$$\int_\Omega ((z_1(t) - z_2(t)), D[(u_1(t) - u_2(t))^+]) = \int_\Omega ((z_1(t) - z_2(t)), DR_t(u_1(t)))$$

$$= \int_\Omega (z_1(t), DR_t(u_1(t))) - \int_\Omega (z_2(t), DR_t(u_1(t))).$$

Now, by Proposition 2.7 in [7], we have

$$\int_\Omega (z_1(t), DR_t(u_1(t))) = \int_\Omega \theta(z_1(t), DR_t(u_1(t)), x)|DR_t(u_1(t))|$$

$$= \int_\Omega \theta(z_1(t), Du_1(t), x)|DR_t(u_1(t))|.$$

From (3.3), we have $\theta(z_1(t), Du_1(t), x) = 1$ a.e. with respect to the measure $|Du_1(t)|$. Now, since the measure $|DR_t(u_1(t))|$ is absolutely continuous respect to the measure $|Du_1(t)|$, we also have $\theta(z_1(t), Du_1(t), x) = 1$ a.e. with respect to the measure $|DR_t(u_1(t))|$. Consequently

$$\int_\Omega (z_1(t), DR_t(u_1(t))) = \int_\Omega |DR_t(u_1(t))|.$$

Moreover, by (3.2) we have that $\|z_2(t)\|_\infty \leq 1$. Hence we have

$$\int_\Omega ((z_1(t) - z_2(t)), D[(u_1(t) - u_2(t))^+]) \geq 0.$$

On the other hand, since $|[z_2(t), \nu]| \leq 1$, $[z_1(t), \nu] \in sign(-u_1(t))$ and $u_2(t) \geq 0$, it is easy to see that

$$\int_{\partial\Omega} [z_1(t) - z_2(t), \nu](u_1(t) - u_2(t))^+ \, dH^{N-1} \leq 0.$$

Consequently, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u_1(t) - u_2(t))^+]^2 \leq 0.$$

Hence the condition $u_1(0) \leq u_2(0)$ gives us $u_1 \leq u_2$, and the proof concludes.

□

Remark 1 Theorem 4 could be compared with what happens in the study of the parabolic problem associated to the p-Laplacian operator. Consider the Dirichlet problem for the p-Laplacian:

$$P_D^p \begin{cases} \dfrac{\partial u}{\partial t} = div\Big(|Du|^{p-2}Du\Big) & \text{in} \quad Q = (0,\infty) \times \Omega \\ u(t,x) = 0 & \text{on} \quad S = (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{in} \quad x \in \Omega, \end{cases}$$

with $1 < p < \infty$. The conditions on $\alpha(t)$ to generate a supersolution are

$$\alpha(t) \geq 0 \text{ and } u_0(x) \leq \alpha(0) \text{ a.e } .x \in \Omega, \tag{3.5}$$

$$\alpha'(t) \geq 0, \tag{3.6}$$

and, in fact, those conditions are also sufficient for the total variation flow. Nevertheless, in the limit case $p = 1$, condition (3.6) can be substituted by the new one given in the above result (which is not the case of problem P_D^p).

As a consequence of the above result we get the following upper bound of the L^∞-norm of the solutions and the existence of the finite extinction time.

Corollary 1 *Let $u_0 \in L^\infty(\Omega)$ and let $u(t,x)$ be the unique solution of problem (P_D). Then, we have*

$$\|u(t)\|_\infty \leq \frac{N}{d(\Omega)}\left(\frac{d(\Omega)\|u_0\|_\infty}{N} - t\right)^+. \tag{3.7}$$

Thus, if $T^*(u_0) = \inf\{t > 0 : \ u(t) = 0\}$, then

$$T^*(u_0) \leq \frac{d(\Omega)\|u_0\|_\infty}{N}. \tag{3.8}$$

Proof: Take

$$\alpha(t) := \frac{N}{d(\Omega)}\left(\frac{d(\Omega)\|u_0\|_\infty}{N} - t\right)^+,$$

since

$$|\alpha'(t)| = \frac{N}{d(\Omega)} \quad \text{and} \quad \alpha(0) = \|u_0\|_\infty,$$

from Theorem 4, it follows that

$$-\alpha(t) \le u(t) \le \alpha(t),$$

and (3.7) follows. □

We observe that the previous estimate can be refined if the support of u_0 is contained in a ball $B(0,r) \subset\subset \Omega$. For that, we compute explicitely the evolution of the characteristic function of a ball.

Lemma 1 *Assume that* $B(0,r) \subset\subset \Omega$ *and let* $u_0 = k\chi_{B(0,r)}$. *Then the unique solution* $u(t,x)$ *of problem* (P_D) *is given by*

$$u(t,x) = \mathrm{sign}(k)\frac{N}{r}\left(\frac{|k|r}{N} - t\right)^+ \chi_{B(0,r)}(x).$$

Proof: Suppose that $k > 0$, the solution for $k < 0$ being constructed in a similar way. We look for a solution of (P_D) of the form $u(t,x) = \alpha(t)\chi_{B(0,r)}(x)$ on some time interval $(0,T)$. Then, we shall look for some $z(t) \in X(\Omega)$ with $\|z\|_\infty \le 1$, such that

$$u'(t) = div(z(t)) \qquad \text{in } \mathcal{D}'(\Omega) \tag{3.9}$$

$$\int_\Omega (z(t), Du(t)) = |Du(t)|(\Omega) \tag{3.10}$$

$$[z(t),\nu] \in \mathrm{sign}(-u(t)) \qquad H^{N-1} - a.e. \tag{3.11}$$

If we take $z(t)(x) = -\frac{x}{r}$ for $x \in \partial B(0,r)$, integrating equation (3.9) we obtain

$$\alpha'(t)|B(0,r)| = \int_{B(0,r)} div(z(t))dx = \int_{\partial B(0,r)} z(t)\cdot\nu = -H^{N-1}(\partial B(0,r)).$$

Thus

$$\alpha'(t) = -\frac{N}{r},$$

therefore,

$$\alpha(t) = k - \frac{N}{r}t,$$

and T is given by $T = \frac{kr}{N}$.

To construct z in $(0,T) \times (\Omega \setminus B(0,r))$ we shall look for z of the form $z = \rho(\|x\|)\frac{x}{\|x\|}$ such that $div(z(t)) = 0$, $\rho(r) = -1$. Since

$$div(z(t)) = \nabla\rho(\|x\|)\cdot\frac{x}{\|x\|} + \rho(\|x\|)div(\frac{x}{\|x\|}) = \rho'(\|x\|) + \rho(\|x\|)\frac{N-1}{\|x\|},$$

we must have

$$\rho'(s) + \rho(s)\frac{N-1}{s} = 0 \qquad \text{for} \quad s > r. \tag{3.12}$$

The solution of (3.12) such that $\rho(r) = -1$ is

$$\rho(s) = -r^{N-1}s^{1-N}.$$

Thus, in $\Omega \setminus B(0,r)$,

$$z(t) = -r^{N-1}\frac{x}{\|x\|^N}.$$

Consequently, the candidate to $z(t)$ is the vector field

$$z(t) := \begin{cases} -\dfrac{x}{r} & \text{if } x \in B(0,r) \text{ and } 0 \le t \le T \\ -r^{N-1}\dfrac{x}{\|x\|^N} & \text{if } x \in \Omega \setminus \overline{B(0,r)}, \text{ and } 0 \le t \le T \\ 0 & \text{if } x \in \Omega \text{ and } t > T, \end{cases}$$

and $u(t,x)$ is the function

$$u(t,x) = \Big(k - \frac{N}{r}t\Big)\chi_{B(0,r)}(x)\chi_{[0,T]}(t),$$

where $T = \dfrac{kr}{N}$. Let us see that $u(t,x)$ satisfies (3.9), (3.10) and (3.11). Since $u(t,x) = 0$ if $x \in \partial\Omega$, it is easy to check that (3.11) holds. On the other hand, if $\varphi \in \mathcal{D}(\Omega)$ and $0 \le t \le T$, we have

$$\int_\Omega \frac{\partial z_i(t)}{\partial x_i}\varphi \, dx = -\frac{1}{r}\int_{B(0,r)} \varphi \, dx + \int_{\partial B(0,r)} \frac{x_i}{r}\frac{x_i}{r}\varphi \, dH^{N-1}$$

$$-\int_{\Omega\setminus B(0,r)} \frac{\partial}{\partial x_i}\Big(\frac{r^{N-1}x_i}{\|x\|^N}\Big)\varphi \, dx - \int_{\partial B(0,r)} \frac{r^{N-1}}{r^N}x_i\frac{x_i}{r}\varphi \, dH^{N-1}.$$

Hence

$$\int_\Omega div \; z(t)\varphi \, dx = -\frac{N}{r}\int_{B(0,r)} \varphi \, dx,$$

and consequently, (3.9) holds. Finally, if $0 \le t \le T$, by Green's formula (2.4), we have

$$\int_\Omega (z(t), Du(t)) = -\int_\Omega div \; z(t)u(t) \, dx + \int_{\partial\Omega} [z(t),\nu]u(t) \, dH^{N-1} =$$

$$-\int_{B(0,r)} \Big(k - \frac{N}{r}t\Big) \; div \; z(t) \, dx = \int_{B(0,r)} \Big(k - \frac{N}{r}t\Big)\frac{N}{r} \, dx =$$

$$\Big(k - \frac{N}{r}t\Big)\frac{N}{r}|B(0,r)| = \Big(k - \frac{N}{r}t\Big)H^{N-1}(\partial B(0,r)) = |Du(t)|(\Omega).$$

Therefore (3.10) holds, and consequently $u(t,x)$ is the solution of (P_D) with initial datum $u_0(x)$. □

Remark 2 The above result shows that there is no spatial smoothing effect, for $t > 0$, similar to the case of the linear heat equation and many other quasilinear parabolic equations. In our case, the solution is discontinuous and has the minimal required spatial regularity: $u(t,.) \in BV(\Omega) \setminus W^{1,1}(\Omega)$.

Remark 3 If $\Omega = B(0,R)$ and $u_0 = k$, then the unique solution $u(t,x)$ of the problem (P_D) is given by

$$u(t,x) = \text{sign}(k)\frac{N}{R}\Big(\frac{|k|R}{N} - t\Big)^+.$$

Indeed, it suffices to take

$$z(t,x) := \begin{cases} -\dfrac{x}{R} & \text{if } x \in B(0,R) \text{ and } 0 \le t \le T, \\ 0 & \text{if } x \in B(0,R) \text{ and } t > T, \end{cases}$$

with $T = \frac{|k|R}{N}$. Then it is easy to check that $u(t,x) = \text{sign}(k)\frac{N}{R}\Big(\frac{|k|R}{N} - t\Big)^+$ and $z(t,x)$ satisfies (3.9), (3.10) and (3.11). .

Corollary 2 *Let $u_0 \in L^\infty(\Omega)$ and denote by $u(t)$ the solution of problem (P_D)(at time t) with initial datum u_0. Then we have that*

$$\|u(t)\|_\infty \geq \frac{N}{d(\Omega)}(T^*(u_0) - t) \qquad \text{for } \ 0 \leq t \leq T^*(u_0). \tag{3.13}$$

Moreover, if $\mathrm{supp}(u_0) \subset B(0,r) \subset\subset \Omega$, *then* $\mathrm{supp}(u(t)) \subset B(0,r)$ *and*

$$T^*(u_0) \leq \frac{\|u_0\|_\infty r}{N}.$$

Proof: Without loss of generality we can assume that $\Omega \subseteq B(0, d(\Omega))$. Take $k > 0$, such that $\frac{kd(\Omega)}{N} = T^*(u_0)$. By Lemma 1, we know that

$$v(t,x) = \frac{N}{d(\Omega)}\left(\frac{kd(\Omega)}{N} - t\right)^+$$

is the solution of problem the (P_D) on $B(0, d(\Omega))$ with initial datum $v_0 = k\chi_{B(0,d(\Omega))}$. The proof of (3.13) will follow from the inequality

$$\|u(t)\|_{L^\infty(\Omega)} \geq \|v(t)\|_{L^\infty(\Omega)}.$$

By contradiction, suppose that there exists $0 < t_0 < T^*(u_0)$ such that

$$\|u(t_0)\|_{L^\infty(\Omega)} < \|v(t_0)\|_{L^\infty(\Omega)}$$

and let $\epsilon > 0$ be such that

$$\|u(t_0)\|_{L^\infty(\Omega)} < \|v(t_0)\|_{L^\infty(\Omega)} - \epsilon = k - \frac{t_0 N}{d(\Omega)} - \epsilon = k_1. \tag{3.14}$$

Consider now the functions in Ω

$$v_1(t,x) := \frac{N}{d(\Omega)}\left(\frac{k_1 d(\Omega)}{N} - t\right)^+, \quad v_2(t,x) := -\frac{N}{d(\Omega)}\left(\frac{k_1 d(\Omega)}{N} - t\right)^+.$$

By (3.14), we have that $v_2(0) \leq u(t_0) \leq v_1(0)$. Hence, by Theorem 4, it follows that $v_2(t) \leq u(t_0+t) \leq v_1(t)$. Hence,

$$T^*(u_0) - t_0 = T^*(u(t_0)) \leq \frac{k_1 d(\Omega)}{N} = \frac{d(\Omega)}{N}(k - \frac{t_0 N}{d(\Omega)} - \epsilon) = T^*(u_0) - t_0 - \frac{\epsilon d(\Omega)}{N},$$

which is a contradiction, and the proof of the first statement concludes.

Suppose now that $\mathrm{supp}(u_0) \subset B(0,r) \subset\subset \Omega$, Let $m := \|u_0\|_\infty$. By Lemma 1 we have that

$$v_1(t,x) := -\frac{N}{r}\left(\frac{mr}{N} - t\right)^+ \chi_{B(0,r)}(x)$$

is the solution of problem (P_D) with initial datum $-m\chi_{B(0,r)}$, and

$$v_2(t,x) := \frac{N}{r}\left(\frac{mr}{N} - t\right)^+ \chi_{B(0,r)}(x)$$

is the solution of problem (P_D) with initial datum $m\chi_{B(0,r)}$. Then, by the comparison principle (2.9), we have $v_1(t,x) \leq u(t,x) \leq v_2(t,x)$ for all $t \geq 0$, $x \in \Omega$. Hence, $\mathrm{supp}(u(t)) \subset B(0,r)$ for all $t \geq 0$, and $u(t) = 0$ for all $t \geq \frac{mr}{N}$. □

Remark 4 It is well known (see [14], [15], [22]) that if $p > 2$ then there is *finite speed of propagation* (i.e., if $\mathrm{supp}(u_0) \subset B(0,r) \subset\subset \Omega$, then the solution of problem (P_D^p) satisfies that $\mathrm{supp}(u(t))$ is a compact set for any $t > 0$, but, if $1 < p \leq 2$ and $u_0 \geq 0$, $u_0 \neq 0$, then $u(t) > 0$ or $u(t) = 0$ in Ω for all $t > 0$ ([15], [22]). Observe that (P_D) can be considered as the limit case $p = 1$ of problem (P_D^p) and the above result shows that there is no propagation of the support of the initial datum (or equivalently, there is an infinite waiting time). Finite time extinction of the solutions of (P_D^p) when $\frac{2N}{N+2} \leq p < 2$, $N \geq 2$ was proved in [8], and, for $1 < p < \frac{2N}{N+1}$, in [21] (see also [25], [6]). The same approach also proves the finite time extinction of solutions of (P_D) (see inequality (3.29) in the proof of Lemma 3).

To study the behaviour of $u(t)$ near the *finite extinction time* $T^*(u_0)$, we follow the method introduced in [11] (see also [16]) . Before giving the proof of Theorem 3, we establish lower and upper bounds on the rate of decay of $\|u(t)\|_N$ and $\|u(t)\|_\infty$, respectively. In order to get the upper bound, let us see first the following regularizing effect due to the homogeneity of the operator $\mathcal{B}$ defined in Section 2 ([9]).

Lemma 2 *Let $u(t) = T(t)u_0$ be the solution of the Dirichlet problem (2.15). Then*

$$|u'(t)| \leq \frac{2}{t}|u_0| \quad \text{for almost all} \quad t > 0. \tag{3.15}$$

Proof. Since

$$\text{if } (u,v) \in\mathcal{B} \text{ and } \lambda > 0, \text{ then } (\lambda u, v) \in\mathcal{B}, \tag{3.16}$$

it follows immediately that

$$\frac{1}{\lambda}(I + \lambda\mu\mathcal{B})^{-1}u_0 = (I + \mu\mathcal{B})^{-1}(\frac{1}{\lambda}u_0) \tag{3.17}$$

for any $\lambda, \mu > 0$. Iterating (3.17) and taking $\mu = \frac{t}{n}$ we obtain

$$(I + \frac{t}{n}\mathcal{B})^{-n}(\frac{1}{\lambda}u_0) = \frac{1}{\lambda}(I + \lambda\frac{t}{n}\mathcal{B})^{-n}u_0 \tag{3.18}$$

for any $\lambda > 0$, $n \in I\!N$. Writing $T(t) = e^{-t\mathcal{B}}$ and letting $n \to \infty$ in (3.18) we may write

$$T(t)(\frac{1}{\lambda}u_0) = \frac{1}{\lambda}T(\lambda t)u_0, \tag{3.19}$$

for any $\lambda > 0$. Fix $t > 0$ and let $h > 0$, $\lambda = 1 + \frac{h}{t}$. Using (3.19) we have that

$$\begin{aligned} T(t+h)u_0 - T(t)u_0 &= T(\lambda t)u_0 - T(t)u_0 = \lambda T(t)(\frac{1}{\lambda}u_0) - T(t)u_0 \\ &= \lambda\Big[T(t)(\frac{1}{\lambda}u_0) - T(t)u_0\Big] + (\lambda - 1)T(t)u_0. \end{aligned}$$

From this, it follows that

$$|T(t+h)u_0 - T(t)u_0| \leq \lambda|T(t)(\frac{1}{\lambda}u_0) - T(t)u_0| + |\lambda - 1||T(t)u_0|. \tag{3.20}$$

The complete accretivity of $\mathcal{B}$ implies that

$$\begin{aligned} &T(t)(\frac{1}{\lambda}u_0) - T(t)u_0 \ll \frac{1}{\lambda}u_0 - u_0, \\ &T(t)u_0 \ll u_0. \end{aligned}$$

Since $u \ll v$, $u, v \in \mathcal{M}(\Omega)$ implies that $\alpha u \ll \alpha v$, $\alpha > 0$, and $|u| \ll |v|$, the previous relations in turn imply that

$$\begin{aligned} &\lambda |T(t)(\frac{1}{\lambda} u_0) - T(t) u_0| \ll (\lambda - 1)|u_0|, \\ &(\lambda - 1)|T(t) u_0| \ll (\lambda - 1)|u_0|. \end{aligned} \tag{3.21}$$

Since the set $\{f \in \mathcal{M}(\Omega) : f \ll (\lambda - 1)|u_0|\}$ is convex we deduce from (3.20) and (3.21) that

$$|T(t+h)u_0 - T(t)u_0| \ll 2(\lambda - 1)|u_0| = 2\frac{h}{t}|u_0|,$$

hence,

$$\frac{|T(t+h)u_0 - T(t)u_0|}{h} \ll \frac{2}{t}|u_0|. \tag{3.22}$$

Now, since $u(t) = T(t)u_0$ is a strong solution, from (3.22) we obtain (3.15).

□

Lemma 3 *Let $u_0 \in L^\infty(\Omega)$ and $u(t,x)$ the unique solution of problem (P_D). Then we have:*
(i) There exists a constant C independent of the initial datum, such that

$$\|u(t)\|_N \geq C(T^*(u_0) - t) \qquad \text{for } \ 0 \leq t \leq T^*(u_0). \tag{3.23}$$

(ii) Given $0 < \tau < T^*(u_0)$, we have

$$\|u(t)\|_\infty \leq \frac{2\|u_0\|_\infty}{\tau}(T^*(u_0) - t) \qquad \text{for } \ \tau \leq t \leq T^*(u_0). \tag{3.24}$$

Proof: (i): By Theorem 1 there exists $z(t) \in X(\Omega)$, $\|z(t)\|_\infty \leq 1$, satisfying

$$\int_\Omega (z(t), Du(t)) = |Du(t)|(\Omega) \tag{3.25}$$

$$[z(t), \nu] \in sign(-u(t)) \quad H^{N-1} - a.e. \ \text{on} \ \partial\Omega. \tag{3.26}$$

$$\begin{aligned} -\int_\Omega (w - u(t))u'(t) \leq \int_\Omega (z(t), Dw) - |Du(t)|(\Omega) \\ -\int_{\partial\Omega} [z(t), \nu] w - \int_{\partial\Omega} |u(t)| \, dH^{N-1} \end{aligned} \tag{3.27}$$

for every $w \in BV(\Omega) \cap L^2(\Omega)$. Le $q \geq 1$, and $\varphi(r) := |r|^{q-1} r$. Then, taking $w = u(t) - \varphi(u(t))$ as test function in (3.27), it yields

$$\int_\Omega \varphi(u(t))u'(t) \leq -\int_\Omega (z(t), D\varphi(u(t))) + \int_{\partial\Omega} [z(t), \nu]\varphi(u(t)) \, dH^{N-1}.$$

Now, by Proposition 2.8 of [7] and having in mind (3.25), we have

$$\int_\Omega (z(t), D\varphi(u(t))) = \int_\Omega \theta(z(t), D\varphi(u(t)), x)|D\varphi(u(t))| = |D\varphi(u(t))|(\Omega).$$

Moreover, by (3.26)

$$[z(t), \nu]\varphi(u(t)) = -|u(t)|^q \qquad H^{N-1} - a.e. \ \text{on} \ \partial\Omega.$$

Consequently, we get

$$\frac{1}{q+1}\frac{d}{dt}\int_\Omega |u(t)|^{q+1} + |D\varphi(u(t))|(\Omega) + \int_{\partial\Omega} |u(t)|^q \, dH^{N-1} \leq 0. \tag{3.28}$$

If we denote

$$v(t)(x) := \begin{cases} \varphi(u(t))(x) & \text{if } x \in \Omega \\ \\ 0 & \text{if } x \in I\!R^N \setminus \overline{\Omega}, \end{cases}$$

by Theorem 5.4.1 of [17], $v(t) \in BV(I\!R^N)$ and

$$|Dv(t)|(I\!R^N) = |D\varphi(u(t))|(\Omega) + \int_{\partial\Omega} |u(t)|^q \, dH^{N-1}.$$

Moreover, by Sobolev's inequality for BV functions (see Theorem 5.6.1 of [17]) we obtain that

$$\| |u(t)|^q \|_{L^{N/N-1}(\Omega)} = \|v(t)\|_{L^{N/N-1}(I\!R^N)} \leq C|Dv(t)|(I\!R^N).$$

Therefore, from (3.28), we obtain that

$$\frac{1}{q+1} \frac{d}{dt} \int_\Omega |u(t)|^{q+1} + \frac{1}{C} \| |u(t)|^q \|_{L^{N/N-1}(\Omega)} \leq 0.$$

Then, taking $q = N-1$, we get

$$\frac{d}{dt} \int_\Omega |u(t)|^N + \frac{N}{C} \left(\int_\Omega |u(t)|^N \right)^{\frac{N-1}{N}} \leq 0. \tag{3.29}$$

Hence

$$\frac{d}{dt} \left[\left(\int_\Omega |u(t)|^N \right)^{\frac{1}{N}} \right] + \frac{1}{C} \leq 0. \tag{3.30}$$

Then, given $0 \leq t \leq T^*(u_0)$, integrating (3.30) from t to $T^*(u_0)$ we obtain (3.23).

(ii) Since, $u(T^*(u_0)) = 0$, from Lemma 2, if $t \geq \tau > 0$, we get

$$\left| \frac{u(t,x)}{T^*(u_0) - t} \right| = \frac{|u(T^*(u_0), x) - u(t,x)|}{T^*(u_0) - t} = \frac{1}{T^*(u_0) - t} \left| \int_t^{T^*(u_0)} u'(s) \, ds \right|$$

$$\leq \frac{1}{T^*(u_0) - t} \int_t^{T^*(u_0)} \frac{2}{s} \|u_0\|_\infty \, ds \leq \frac{2}{\tau} \|u_0\|_\infty,$$

and (3.24) follows. □

Proof of Theorem 3: Since $u(t) \in BV(\Omega)$ for almost any $t > 0$, without loss of generality, we may assume that $u_0 \in BV(\Omega)$. We make a change of scale in time $t = \varphi(\tau)$ so that $\varphi(+\infty) = T^*(u_0)$. Let $\varphi(\tau) := T^*(u_0)(1 - e^{-\tau})$. Hence, if we define

$$v(\tau) := \frac{u(\varphi(\tau))}{T^*(u_0)} e^\tau,$$

we have

$$v'(\tau) = u'(\varphi(\tau)) + v(\tau).$$

Now, since the operator $\partial\Phi$ is positively homogeneous of zero degree, we have that

$$(v(\tau), -v'(\tau) + v(\tau)) \in \partial\Phi \quad \text{for almost all} \quad \tau > 0. \tag{3.31}$$

Therefore, $v(\tau)$ is a strong solution of the problem

$$v'(\tau) + \partial\Phi(v(\tau)) \ni v(\tau).$$

Let us see that there exists an increasing sequence $\tau_n \to +\infty$ and a function $v^* \in BV(\Omega)$, such that $\lim_{n\to\infty} v(\tau_n) = v^*$ in $L^p(\Omega)$, which implies the existence of an increasing sequence $t_n \to T^*(u_0)$ such that $\lim_{n\to\infty} w(t_n) = v^*$ in $L^p(\Omega)$.

First, observe that, using (3.24), we have

$$\|v(\tau)\|_\infty = \frac{e^\tau}{T^*(u_0)} \|u(\varphi(\tau))\|_\infty \leq \frac{2\|u_0\|_\infty}{\tau_0} \quad \text{for all } \tau \geq \tau_0 > 0. \tag{3.32}$$

On the other hand, by Lemma 3.3 of [12] (pag. 73), we have

$$\frac{d}{d\tau}\Phi(v(\tau)) = (-v'(\tau) + v(\tau), v'(\tau)) = -\int_\Omega v'(\tau)^2 + \int_\Omega v(\tau)v'(\tau),$$

i.e.,

$$\frac{d}{d\tau}\left(|Dv(\tau)|(\Omega) + \int_{\partial\Omega} |v(\tau)| - \frac{1}{2}\int_\Omega v(\tau)^2\right) = -\int_\Omega v'(\tau)^2 \leq 0. \tag{3.33}$$

Integrating from 0 to τ we obtain

$$\begin{aligned} &|Dv(\tau)|(\Omega) + \int_{\partial\Omega} |v(\tau)| - \frac{1}{2}\int_\Omega v(\tau)^2 \leq \\ &|Dv(0)|(\Omega) - \frac{1}{2}\int_\Omega v(0)^2 + \int_{\partial\Omega} |v(0)| \quad \forall\, \tau \geq 0. \end{aligned} \tag{3.34}$$

Estimates (3.32) and (3.34) prove that $\{v(\tau) \ : \ \tau \geq 0\}$ is bounded in $BV(\Omega)$. Hence, by the Compact Embedding Theorem for BV-functions (see for instance [2]) $\{v(\tau) \ : \ \tau \geq 0\}$ is relatively compact in $L^p(\Omega)$ for $1 \leq p < \frac{N}{N-1}$, and consequently, there exists $\tau_n \to \infty$ and $v^* \in L^p(\Omega) \cap BV(\Omega)$, such that $v(\tau_n) \to v^*$ in $L^p(\Omega)$. Moreover, by (3.32) we can assume that $v(\tau_n) \to v^*$ in $L^q(\Omega)$ for all $1 \leq q < \infty$. On the other hand, by (3.23), we have that

$$\|v(\tau)\|_N \geq C \qquad \forall\, \tau \geq 0.$$

Then, we get $v^* \neq 0$

Finally, let us prove that v^* is a solution of the stationary problem (S_D) which minimizes $\Phi(\cdot) - < \cdot, v^* >$ in $BV(\Omega) \cap L^2(\Omega)$. Let $(T(t))_{t\geq 0}$ be the semigroup in $L^1(\Omega)$ generated by $\mathcal{B} - I$. Then, we prove that $T(t)v^* = v^*$ for all $t \geq 0$. In fact, by (3.33), we have

$$\int_t^s \int_\Omega v'(\tau)^2 \leq |Dv(t)|(\Omega) + \int_{\partial\Omega} |v(t)| + \frac{1}{2}\int_\Omega v(s)^2 \leq M, \tag{3.35}$$

for all $0 < t \leq s$. Now,

$$\|v(t+\tau_n) - v(\tau_n)\|_2^2 = \int_\Omega \left| \int_{\tau_n}^{t+\tau_n} v'(s)\; ds \right|^2 \leq t \int_\Omega \int_{\tau_n}^{t+\tau_n} |v'(s)|^2 \; ds,$$

hence by (3.35), it follows that there exists $\epsilon_n \to 0$ such that

$$\|v(t+\tau_n) - v(\tau_n)\|_2^2 \leq t\epsilon_n \qquad \forall\, n \in I\!N. \tag{3.36}$$

Fix $t > 0$. Then, since $v(t) = T(t)(\frac{u_0}{T^*(u_0)})$, we have

$$\begin{aligned} \|T(t)v^* - v^*\|_2 &\leq \|T(t)v^* - v(t+\tau_n)\|_2 + \|v(t+\tau_n) - v(\tau_n)\|_2 + \|v(\tau_n) - v^*\|_2 \\ &\leq e^t\ \|v(\tau_n) - v^*\|_2 + \|v(t+\tau_n) - v(\tau_n)\|_2 + \|v(\tau_n) - v^*\|_2, \end{aligned}$$

and, having in mind (3.36), it follows that $T(t)v^* = v^*$. Thus $0 \in \partial\Phi(v^*) - v^*$, in other words, v^* minimizes $\Phi(\cdot) - < \cdot, v^* >$ in $BV(\Omega) \cap L^2(\Omega)$. □

4 Solutions of S_D in the radial case

In Theorem 3 we have shown that the asymptotic profile of the solutions of problem (P_D) are solutions of problem (S_D). In this section we are going to study this class of solutions of problem (S_D) in the radial case.

In order to state our next result it is convinient to recall that any set of finite perimeter X in $I\!R^N$ can be decomposed into connected components (in the BV sense, [1]) X_i, $i \in I$, I being at most countable, in such a way that $|X| = \sum_{i \in I} |X_i|$ and $H^{N-1}(\partial^M X) = \sum_{i \in I} H^{N-1}(\partial^M X_i)$ where $\partial^M X$, $\partial^M X_i$ denote the measure theoretic boundaries of X and X_i, repectively ([1]).

Proposition 1 *Let v be a solution of problem (S_D) which is a minimizer of $\Phi(\cdot) - < \cdot, v >$ in $BV(\Omega) \cap L^2(\Omega)$.*

(i) Assume that $v \geq 0$ has its support contained in a ball $B \subset\subset \Omega$. Then, for almost all $k \geq 0$, the BV connected components of $[v \geq k] := \{x \in \Omega : v(x) \geq k\}$ are convex.

(ii) Assume $\Omega = B(0,R)$, $R > 0$ and let $v \geq 0$ be a radially symmetric function in $B(0,R)$. Then, for almost all $k \in I\!R$, the BV connected components of $[v \geq k]$ are convex and consequently, $v(x) = g(\|x\|)$ where g is a decreasing function of $r > 0$.

Proof: (i) Let k be such that $[v \geq k]$ is a set of finite perimeter in Ω, hence in $I\!R^N$. Let $X_i(k)$, $i \in I$, be the BV-components of $[v \geq k]$ ([1]). Let $co(X_i(k))$ be the convex enveloppe of $X_i(k)$, $i \in I$. Let $A(k) = \cup_{i \in I} co(X_i(k))$. Now, observe that if $k \geq k'$ are such that $[v \geq k]$, $[v \geq k']$ are sets of finite perimeter in $I\!R^N$, then $A(k) \subseteq A(k')$ (modulo a null set). Indeed, since $k \geq k'$, we have that $X_i(k) \subseteq X_i(k')$ (modulo a null set), and, hence, also $co(X_i(k)) \subseteq co(X_i(k'))$. Thus, $A(k) \subseteq A(k')$. Let w be the L^∞ function such that $[w \geq k] = A(k)$ a.e. for almost all $k \in I\!R$ ([1]). Since $[v \geq k] \subseteq A(k)$ for almost all $k \in I\!R$, we have that $v \leq w$. Now, since $H^{N-1}(\partial^M co(X_i(k))) \leq H^{N-1}(\partial^M X_i(k))$, using the coarea formula (see, [2] or [17]), we have that

$$\int_\Omega |Dw| \leq \int_\Omega |Dv|$$

Hence, $w \in BV(\Omega)$. Now, if for a non null set K of $k \in I\!R$, $X_i(k)$ is not convex, we have that $H^{N-1}(\partial^M co(X_i(k))) < H^{N-1}(\partial^M X_i(k))$, the n

$$\int_\Omega |Dw| < \int_\Omega |Dv|$$

Finally, observe that $v = w = 0$ in $\partial\Omega$. Therefore

$$\int_\Omega |Dw| + \int_{\partial\Omega} |w| - \int_\Omega wv < \int_\Omega |Dv| + \int_{\partial\Omega} |v| - \int_\Omega v^2,$$

and v cannot be a minimizer of $\Phi(\cdot) - < \cdot, v >$ in $BV(\Omega) \cap L^2(\Omega)$.

(ii) In this case the proof is similar to the one of (i), we only need to observe that since v is radial we do not need that v has its support strictly contained in $B(0,R)$ to conclude the function w satisfy $w = v$ on $\partial\Omega$. Finally, since almost all upper level sets of v have convex BV-connected components and v is radially symmetric this implies that, for almost all $k \in I\!R$, $[v \geq k]$ is a ball centered at 0. This implies that $v(x) = g(\|x\|)$ where g is a decreasing function of $r > 0$. □

By Proposition 1, we know that if $\Omega = B(0,R)$, $R > 0$, the radial solutions v of (S_D) are of the form $v(x) = g(\|x\|)$ for some decreasing function $g(r)$. By modifying, if necessary, v in a set of measure zero, we may assume that g is upper semicontinuous in $[0,R]$. Consequently, the set $[v \geq k] = \{x \in B(0,R) : \|x\| \leq f(k)\}$, where f is the decreasing function $f(k) := \sup\{r \in [0,R] \; : \; g(r) \geq k\}$, $k \in [g(R), g(0)]$. Moreover, since

$$Per([v \geq k]) = Per(\{x \in B(0,R) : \|x\| \leq f(k)\}) = 2\pi f(k)$$

$f(k)$ can be identified as

$$f(k) = \frac{1}{2\pi} Per([v \geq k]).$$

Let us prove that

$$Per([v \geq k]) = \int_{[v \geq k]} v(x)\, dx \qquad \forall\, k \in]g(R), g(0)]. \tag{4.1}$$

Indeed, since v is a solution of (S_D) there exists $z \in X(\Omega)$ satisfying: $v = -div(z)$ in $\mathcal{D}'(\Omega)$, $\int_\Omega (z, Dv) = \|Dv\|(\Omega)$ and $[z, \nu] \in sign(-v)$. Hence, if $k > g(R)$, using Green's formula we have

$$\int_{[v \geq k]} v\, dx = \int_\Omega v \chi_{[v \geq k]}\, dx = -\int_\Omega div(z) \chi_{[v \geq k]}\, dx =$$

$$\int_\Omega (z, D\chi_{[v \geq k]}) - \int_{\partial\Omega} [z, \nu] \chi_{[v \geq k]}\, dH^{N-1} = \int_\Omega (z, D\chi_{[v \geq k]}).$$

Now, by the coarea formula, we have

$$\int_0^\infty |D\chi_{[v \geq t]}|\, dt = |Dv|(\Omega) = \int_\Omega (z, Dv) = -\int_\Omega div(z)\, v\, dx - \int_{\partial\Omega} v\, dH^{N-1}$$

$$= \int_0^\infty \left(\int_\Omega -div(z)\, \chi_{[v \geq t]}\, dx + \int_{\partial\Omega} [z, \nu] \chi_{[v \geq t]}\, dH^{N-1} \right)$$

$$= \int_0^\infty \int_\Omega (z, D\chi_{[v \geq t]}) \leq \int_0^\infty |D\chi_{[v \geq t]}|\, dt.$$

It follows that

$$\int_\Omega (z, D\chi_{[v \geq t]}) = |D\chi_{[v \geq t]}|(\Omega),$$

and, consequently, (4.1) holds.

On the other hand, since $0 \leq v(x) = g(\|x\|)$ and g is decreasing, we have that

$$\int_{[v \geq k]} v(x)\, dx = \int_{[v \geq k]} \int_0^{+\infty} \chi_{[v \geq t]}(x)\, dt\, dx$$

$$= \int_{[v \geq k]} (k + \int_k^{g(0)} \chi_{[v \geq t]}(x)\, dt)\, dx = k|[v \geq k]| + \int_k^{g(0)} |[v \geq t]|\, dt$$

Then, a.e. in $k \in [g(R), g(0)]$, we have that

$$\frac{d}{dk} Per([v \geq k]) = k \frac{d}{dk} |[v \geq k]|,$$

which, written in terms of $f(k)$ is

$$\frac{d}{dk} 2\pi f(k) = k \frac{d}{dk} \pi f(k)^2,$$

i.e.,

$$\frac{d}{dk} f(k) = k f(k) \frac{d}{dk} f(k).$$

Then, we have that either $f(k) = \frac{1}{k}$ or $f'(k) = 0$ for almost all $k \in [g(R), g(0)]$. Since f is a (pseudo)inverse of g, in terms of g this gives that either $g(r) = \frac{1}{r}$ or $g'(r) = 0$, a.e. in $r \in (0, R)$. Summarizing, we have proved the following result.

Corollary 3 *Let $\Omega = B(0,R)$, $R > 0$, and $u_0 \geq 0$ be a radial function in $B(0,R)$. If v^* is the asymptotic profile of the solution of (P_D) with initial datum u_0, then there exists a decreasing function $g : [0,R] \to [0, \|u_0\|_\infty]$ satisfying $g(r) = \frac{1}{r}$ or $g'(r) = 0$, a.e. in $r \in (0,R)$, such that $v^*(x) = g(\|x\|)$.*

Proof: The result follows as a consequence of the above computations having in mind that, since u_0 is a radially symmetric function, we have that v^* is also a radially symmetric function. $\square$

We finish this section by giving some explicit solutions of (S_D) in the radial case and by showing a procedure to construct many other explicit radial solutions wich could be called as *towers* (Corollary 3).

Proposition 2 *The following functions are solutions of (S_D) in $B(0,R)$:*

$$u_1(x) = \frac{N-1}{\|x\|},$$

$$u_2(x) = \frac{Per(B(p,r))}{|B(p,r)|}\chi_{B(p,r)}(x), \quad where \quad B(p,r) \subseteq B(0,R),$$

and

$$u_3(x) = \begin{cases} \dfrac{N}{r} & if \quad x \in B(0,r) \subseteq B(0,R) \\[2ex] \dfrac{N-1}{\|x\|} & if \quad x \in B(0,R) \setminus B(0,r). \end{cases}$$

Proof: Working as in the proof of Lemma 1 it is easy to see that u_1, u_2 and u_3 are solutions of (S_D) in $B(0,R)$ whose associated vector fileds are

$$z_1(x) = -\frac{x}{\|x\|},$$

$$z_2(x) = \begin{cases} \dfrac{-x}{r} & \text{if} \quad x \in B(0,r) \\[2ex] -r^{N-1}\dfrac{x}{\|x\|^N} & \text{if} \quad x \in B(0,R) \setminus B(0,r), \end{cases}$$

and

$$z_3(x) = \begin{cases} \dfrac{-x}{r} & \text{if} \quad x \in B(0,r) \\[2ex] -\dfrac{x}{\|x\|} & \text{if} \quad x \in B(0,R) \setminus B(0,r), \end{cases}$$

respectively. $\square$

Proposition 3 *Let $R_1 < R_2 \leq R$, $B_1 = B(0,R_1)$, $B_2 = B(0,R_2)$. Then*

$$u(x) = \frac{Per(B_1)}{|B_1|}\chi_{B_1}(x) + \frac{Per(B_2) - Per(B_1)}{|B_2| - |B_1|}\chi_{B_2 \setminus B_1}(x)$$

is a solution of (S_D) in $B(0,R)$.

Proof: Let $\Omega = B_2 \setminus \overline{B}_1$, $\Gamma_1 = \partial B_1$, $\Gamma_2 = \partial B_2$. Define $z(x) = -\frac{x}{R_1}$ in B_1. Then

$$-div(z) = \frac{Per(B_1)}{|B_1|} \quad \text{in } B_1.$$

Since the vector field z outside B_2 is given by $z(x) = -R\frac{x}{\|x\|^2}$ we only need to construct a vector field z in Ω such that $\|z\| \leq 1$, $-div(z) = \frac{Per(B_2)-Per(B_1)}{|B_2|-|B_1|}$ in Ω and such that $z(x) \cdot \frac{x}{\|x\|} = -1$ on $\Gamma_1 \cup \Gamma_2$. For that let us consider the following *capillarity problem*

$$-div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = k_0 \cos\gamma \quad \text{in } \Omega \tag{4.2}$$

with boundary conditions

$$-\frac{Du}{\sqrt{1+|Du|^2}}\cdot\nu = \begin{cases} +cos\,\gamma & \text{on} \quad \Gamma_2 \\ -cos\,\gamma & \text{on} \quad \Gamma_1, \end{cases}$$

where ν denotes the outer unit normal to Ω, $\gamma \in (0, \frac{\pi}{2})$ and $k_0 = \frac{Per(B_2)-Per(B_1)}{|B_2|-|B_1|}$. Observe that, integrating by parts and using the boundary conditions, k_0 is fixed as the above value. Then, after proving that (4.2) has a solution, we define $z_\gamma(x) = \frac{Du}{\sqrt{1+|Du|^2}}$. Observe that $\|z_\gamma\| \le 1$, and $z_\gamma(x)\cdot\frac{x}{\|x\|} = -cos\,\gamma$ on $\Gamma_1 \cup \Gamma_2$. Letting $\gamma \to 0+$ we obtain the desired vector field z. The existence of solutions of (4.2) follows from a result of E. Giusti ([18], Sect. 4). A sufficient condition for the existence of a solution of (4.2) is that there exists $\alpha_0 > 0$ such that the following inequality

$$\begin{gathered} cos\,\gamma\,|H^{N-1}(\partial^M E\cap\Gamma_1) - H^{N-1}(\partial^M E\cap\Gamma_2) + k_0|E|| \le q \\ H^{N-1}(\partial^M E\cap\Omega) - \alpha_0\, min(|E|, |\Omega\setminus E|) \end{gathered} \tag{4.3}$$

holds for any rectifiable subset contained in Ω. This will be a consequence of the following two Lemmas.

Lemma 4 *The only minimizers of*

$$Per(E) - k_0|E| \tag{4.4}$$

in the class of rectifiable sets E such that $B_1 \subseteq E \subseteq B_2$ are $E = B_1$ or $E = B_2$.

Proof: Let E_0 be a minimizer of (4.4). Let $r > 0$ be such that $|E_0| = |B(0,r)|$. Since $B_1 \subseteq E_0 \subseteq B_2$ we also have that $B_1 \subseteq B(0,r) \subseteq B_2$. If $|E_0 \Delta B(0,r)| > 0$, by the isoperimetric inequality, we know that $Per(E_0) > Per(B(0,r))$. This implies that

$$Per(B(0,r)) - k_0|B(0,r)| < Per(E_0) - k_0|E_0|$$

contradicting the fact that E_0 is a minimizer of (4.4). Hence $E_0 = B(0,r)$. Now the minima of (4.4) on the family of balls $B(0,t)$, $t \in [R_1, R_2]$, is attained when $t = R_1$ or $t = R_2$.

Let $\mathcal{G}$ be the family of rectifiable subsets of Ω.

Lemma 5 *For any $E \in \mathcal{G}$ we have*

$$H^{N-1}(\partial^M E\cap\Gamma_1) \le H^{N-1}(\partial^M E\cap\Gamma_2) + H^{N-1}(\partial^M E\cap\Omega) - k_0|E|, \tag{4.5}$$

and

$$H^{N-1}(\partial^M E\cap\Gamma_2) \le H^{N-1}(\partial^M E\cap\Gamma_1) + H^{N-1}(\partial^M E\cap\Omega) + k_0|E|. \tag{4.6}$$

Proof: For any $E \in \mathcal{G}$, we define

$$G(E) = H^{N-1}(\partial^M E\cap\Gamma_2) - H^{N-1}(\partial^M E\cap\Gamma_1) + H^{N-1}(\partial^M E\cap\Omega) - k_0|E|.$$

Let E_0 be a minimizer of $G(E)$ on the family $\mathcal{G}$. Observe that $G(\Omega) = 0$, hence

$$\begin{gathered} H^{N-1}(\partial^M E_0\cap\Gamma_2) - H^{N-1}(\partial^M E_0\cap\Gamma_1) + H^{N-1}(\partial^M E_0\cap\Omega) \\ -k_0|E_0| = G(E_0) \le G(\Omega) = 0. \end{gathered} \tag{4.7}$$

On the other hand, by the above Lemma, we have that

$$Per(B_1) - k_0|B_1| \le Per(E_0\cup B_1) - k_0|E_0\cup B_1|. \tag{4.8}$$

Adding both inequalities, we obtain

$$H^{N-1}(\partial^M E_0 \cap \Gamma_2) + Per(B_1) - H^{N-1}(\partial^M E_0 \cap \Gamma_1) + H^{N-1}(\partial^M E_0 \cap \Omega)$$
$$\leq Per(E_0 \cup B_1)$$

which is indeed an equality. Hence, (4.7) and (4.8) must be equalities. In particular,

$$Per(B_1) - k_0|B_1| = Per(E_0 \cup B_1) - k_0|E_0 \cup B_1|,$$

i.e., $E_0 \cup B_1$ is a minimizer of (4.4). By the above Lemma, $E_0 \cup B_1 = B_1$ or $E_0 \cup B_1 = B_2$, i.e., $E_0 = \emptyset$ or $E_0 = \Omega$. In any case, $G(E_0) = 0$. It follows that

$$0 \leq G(E)$$

for any $E \in \mathcal{G}$, which gives (4.5). To prov e (4.6), it suffices to take $\Omega \setminus E$ in place of E in (4.5). □

We can now prove (4.3) and this will complete the proof of Proposition 3. Both inequalities (4.5), (4.6) give that

$$|H^{N-1}(\partial^M E \cap \Gamma_1) - H^{N-1}(\partial^M E \cap \Gamma_2) + k_0|E|| \leq H^{N-1}(\partial^M E \cap \Omega). \tag{4.9}$$

Now, observe that

$$H^{N-1}(\partial^M E \cap \Omega) \leq \frac{H^{N-1}(\partial^M E \cap \Omega)}{cos\,\gamma} - \frac{\alpha_0}{cos\,\gamma} min(|E|, |\Omega \setminus E|), \tag{4.10}$$

for some constant $\alpha_0 > 0$. Indeed, by the relative isoperimetric inequality ([17], [26]) we have that

$$H^{N-1}(\partial^M E \cap \Omega) \geq C min(|E|, |\Omega \setminus E|)^{\frac{N-1}{N}},$$

for some constant $C > 0$, and, since

$$min(|E|, |\Omega \setminus E|)^{\frac{N-1}{N}} \geq \frac{min(|E|, |\Omega \setminus E|)}{|\Omega|^{\frac{1}{N}}},$$

we obtain

$$H^{N-1}(\partial^M E \cap \Omega) \geq \frac{C}{|\Omega|^{\frac{1}{N}}} min(|E|, |\Omega \setminus E|).$$

This implies (4.10). Observe that (4.9) and (4.10) prove (4.3).

5 The Neumann problem

In [3], it was shown that the weak solutions of problem (P_N) stabilize as $t \to \infty$ by converging in the L^1-norm to the average of the initial datum. In this section we are going to prove, by energy methods, like in [5] (see also the monograph [6]), that in the two dimensional case, in fact, this asymptotic state is reached in finite time.

Theorem 5 *Suppose $N = 2$. Let $u_0 \in L^2(\Omega)$ and $u(t,x)$ the unique weak solution of problem (P_N). Then there exists a finite time T_0 such that*

$$u(t) = \overline{u_0} := \frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx \qquad \forall\, t \geq T_0.$$

Proof: Since u is a weak solution of problem (P_N), there exists $z \in L^\infty(Q)$ with $\|z\|_\infty \leq 1$, $u_t = div(z)$ in $\mathcal{D}'(Q)$ such that

$$\int_\Omega (u(t) - w)u_t(t) = \int_\Omega (z(t), Dw) - |Du(t)|(\Omega) \tag{5.1}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$. Hence, taking $w = \overline{u_0}$ as test function in (5.1), it yields

$$\int_\Omega (u(t) - \overline{u_0})u_t(t) = -|Du(t)|(\Omega).$$

Now, by Poincaré inequality for BV functions (see [17] or [26]) and having in mind that we have conservation of mass, we obtain

$$\|u(t) - \overline{u_0}\|_2 \leq C|Du(t)|(\Omega).$$

Thus, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - \overline{u_0})^2 + \frac{1}{C}\|u(t) - \overline{u_0}\|_2 \leq 0. \tag{5.2}$$

Therefore, the function

$$y(t) := \int_\Omega (u(t) - \overline{u_0})^2$$

satisfies the inequality

$$y'(t) + My(t)^{1/2} \leq 0,$$

from where it follows that there exists $T_0 > 0$ such that $y(t) = 0$ for all $t \geq T_0$. □

By Theorem 5, given $u_0 \in L^2(\Omega)$, if $u(t,x)$ is the unique weak solution of problem (P_N), then

$$T^*(u_0) := \inf\{t > 0 \ : \ u(t) = \overline{u_0}\} < \infty.$$

The study the behaviour of $u(t)$ near $T^*(u_0)$ can be carryed out as in the case of the Dirichlet problem. As in that case, before proving the result, lower and upper bounds on the rate of decay of $\|u(t) - \overline{u_0}\|_2$ are established.

Lemma 6 *Suppose $N = 2$. Let $u_0 \in L^\infty(\Omega)$ and let $u(t,x)$ be the unique solution of problem (P_N). Then, we have:*

(i) There exists a constant C_1 independent of the initial data, such that

$$C_1(T^*(u_0) - t) \leq \|u(t) - \overline{u_0}\|_2 \qquad \text{for} \quad 0 \leq t \leq T^*(u_0). \tag{5.3}$$

(ii) Given $0 < \tau < T^*(u_0)$, we have

$$\|u(t) - \overline{u_0}\|_\infty \leq \frac{2\|u_0\|_\infty}{\tau}(T^*(u_0) - t) \qquad \text{for} \quad \tau \leq t \leq T^*(u_0). \tag{5.4}$$

Proof: (i): Working as in the proof of Theorem 5, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - \overline{u_0})^2 + C_1\|u(t) - \overline{u_0}\|_2 \leq 0.$$

Hence

$$\frac{d}{dt}\left[\left(\int_\Omega |u(t) - \overline{u_0}|^2\right)^{\frac{1}{2}}\right] + C_1 \leq 0. \tag{5.5}$$

Then, given $0 \leq t \leq T^*(u_0)$, integrating (5.5) from t to $T^*(u_0)$ we obtain (5.3).

The proof of (ii) is again a consequence of the regularizing effect due to the homogeneity of the operator $\mathcal{A}$ (defined in Section 2) which implies the estimate

$$|u'(t)| \leq \frac{2}{t}|u_0| \quad \text{for almost all} \quad t > 0. \tag{5.6}$$

□

As in the case of the Dirichlet problem we prove the following result.

Theorem 6 *Suppose $N = 2$. Let $u_0 \in L^\infty(\Omega)$ and let $u(t,x)$ be the unique weak solution of problem (P_N). Let*

$$w(t,x) := \begin{cases} \dfrac{u(t,x) - \overline{u_0}}{T^*(u_0) - t} & \text{if } \ 0 \le t < T^*(u_0), \\ \\ 0 & \text{if } \ t \ge T^*(u_0). \end{cases}$$

Then, there exists an increasing sequence $t_n \to T^*(u_0)$, and a solution $v^* \neq 0$ of the stationary problem

$$S_N \begin{cases} -div\left(\dfrac{Dv}{|Dv|}\right) = v & in \quad \Omega \\ \\ \dfrac{\partial v}{\partial \eta} = 0 & on \quad \partial\Omega \end{cases}$$

such that

$$\lim_{n\to\infty} w(t_n) = v^* \quad \text{in} \quad L^p(\Omega),$$

for all $1 \le p < \infty$. Moreover v^* is a minimizer of $\Psi(\cdot) - < \cdot, v^* >$ in $BV(\Omega) \cap L^2(\Omega)$.

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