Waveform relaxation methods for stochastic differential equations

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Abstract

An operator equation $X = \mathbb{T}X + G$ in a Banach space $\mathcal{E}$ of $\mathcal{F}_t$-adapted random elements describing an initial- or boundary value problem of a system of stochastic differential equations (SDEs) is considered. Our basic assumption is that the underlying system consists of weakly coupled subsystems. The proof of the convergence of corresponding waveform relaxation methods depends on the property that the spectral radius of an associated matrix is less than one. The entries of this matrix depend on the Lipschitz-constants of a decomposition of $\mathbb{T}$. In proving an existence result for the operator equation we show how the entries of the matrix depend on the right hand side of the stochastic differential equations. We derive conditions for the convergence under "classical" vector-valued Lipschitz-continuity of an appropriate splitting of the system of stochastic ODEs. A generalization of these key results under one-sided Lipschitz continuous and anticoercive drift coefficients of SDEs is also presented. Finally, we consider a system of SDEs with different time scales (singularly perturbed SDEs) as an illustrative example.

Key words: Waveform Relaxation Methods; Stochastic Differential Equations; Stochastic-Numerical Methods; Iteration Methods; Large Scale Systems.

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1 Introduction

The solution of complex and large scale systems plays a crucial role in recent scientific computations. In particular, large scale stochastic dynamical systems represent very complex systems incorporating the random appearances of physical processes in nature. The development of efficient numerical methods to study such large scale systems, which can be characterized as weakly coupled subsystems with quite different behavior, is an important challenge. Under some conditions, block-iterative methods are very efficient. One of these methods to solve large scale systems is given by the *waveform relaxation method*. This method was first proposed by Lelarasmee, Ruehli and Sangiovanni–Vincentelli [25] for the time-domain analysis of large scale integrated circuits. For the waveform algorithm concerning deterministic processes and related aspects, many research papers can be found, e.g. Bremer and Schneider [5], Bremer [6], Burrage [7], in't Hout [12], Jackiewicz and Kwapisz [16], Jansen et al. [17], Jansen and Vandewalle [18], Leimkuhler [23, 24], Miekkala and Nevanlinna [27, 28], Nevanlinna and Odeh [30], Sand and Burrage [34], Schneider [35, 36, 37], Ta'asan and Zhang [42], Zennaro [47], Zubik–Koval and Vandewalle [48], among many others.

To our knowledge, there is no application of the waveform relaxation methods to stochastic processes in the refereed literature. In what follows we present a theoretical foundation for the construction and convergence of waveform iterations applied to systems of ordinary stochastic differential equations (SDEs). The attention is restricted to Itô-interpreted SDEs (for original papers, see Itô [13, 14, 15], i.e. where occurring stochastic integration is interpreted in the sense of Itô [15]. For basic aspects on the theory of SDEs in the spirit of Itô [13], see e.g. Anulova et al. [1], Arnold [2, 3], Dynkin [9], Gard [11], Itô [13, 14, 15], Khas'minskij [19], Krylov [22], Mao [26], Protter [32] and Revuz and Yor [33]. We see our main contribution in deriving bounds for the Lipschitz-constants of the corresponding stochastic integral operator and in describing their dependence on the involved stochastic process.

The paper is organized as follows. In Section 2 we describe the key idea of waveform relaxation method. Section 3 presents a proof for the existence and uniqueness of an initial value problem for stochastic differential equations (SDEs) using Banach's fixed point principle for vector-valued Lipschitz continuous random operators in random product Banach spaces with appropriate norms. This key result can be used to derive conditions for the convergence of the waveform relaxation method in the case of Itô SDEs. Section 4 generalizes this idea to the case of one-sided Lipschitz-continuity of the drift part, restricted to anticoercive drift coefficients of SDEs. An illustrative example is given in Section 5. Section 6 closes this contribution with some conclusions, final remarks and interesting open problems.

2 The general idea of waveform relaxation methods

There are numerous initial- and boundary-value problems of differential equations which can be formulated as fixed point problems. Therefore, in the following we consider nonlinear equations of the type

$$x = \mathbb{T} x + g \tag{1}$$

where $\mathbb{T}$ maps the function space $\mathcal{U}$ into itself, and $g \in \mathcal{U}$. There are several techniques to find appropriate conditions on the operator $\mathbb{T}$ guaranteeing a unique solution $x^* \in \mathcal{U}$ of system (1) and resulting in an efficient algorithm to approximate x^*. In the case that (1) represents a network of weakly connected subsystems with quite different behavior, i.e. (1) carries the feature of a large scale system, the *waveform relaxation method* is an efficient approach to approximate x^*. Its *key steps* can be formulated as follows:

(i) *Decomposition step*: Find a suitable representation of $\mathcal{U}$ as a product of subspaces $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n$, i.e.

$$\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times ... \times \mathcal{U}_n\,, \tag{2}$$

and a corresponding splitting of $\mathbb{T}$ into $\mathbb{T}_1$, ... , $\mathbb{T}_n$ and g into $g_1, ..., g_n$ such that the fixed point problem (1) is equivalent to the system

$$\begin{aligned} x^{(1)} &= \mathbb{T}_1(x^{(1)}, ..., x^{(n)}) + g_1, \\ x^{(2)} &= \mathbb{T}_2(x^{(1)}, ..., x^{(n)}) + g_2, \\ \cdots\cdots\ \cdot\cdot\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x^{(n)} &= \mathbb{T}_n(x^{(1)}, ..., x^{(n)}) + g_n \end{aligned} \tag{3}$$

where $x^{(k)}, g_k \in \mathcal{U}_k$, and $\mathbb{T}_k$ maps the space $\mathcal{U}$ into the subspace $\mathcal{U}_k$ for $k = 1, 2, ..., n$.

(ii) *Solution step*: Solve the k-th subsystem

$$x^{(k)} = \mathbb{T}_k(x^{(1)}, ..., x^{(k-1)}, x^{(k)}, x^{(k+1)}, ..., x^{(n)}) + g_k \tag{4}$$

by an appropriate procedure. Here, $x^{(j)}, j = 1, 2, ..., n$ with $j \neq k$ are the inputs from other subsystems.

(iii) *Relaxation step*: Derive conditions such that the successive solution of the subsystems (4) leads to the unique solution of the large scale system.

The steps (ii) and (iii) can be combined to some "diagonalized" iteration scheme (see Schneider [37] for details). In the case of the Gauss–Jacobi procedure

$$x_i^{(1)} = \mathbb{T}_1(x_i^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_1,$$

$$
\begin{aligned}
&x_i^{(2)} = \mathbb{T}_2(x_{i-1}^{(1)}, x_i^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_2,\\
&.......\ ..\ ....\\
&x_i^{(n)} = \mathbb{T}_n(x_{i-1}^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_i^{(n)}) + g_n
\end{aligned}
\tag{5}
$$

we get the diagonalized iteration scheme

$$
\begin{aligned}
&x_i^{(1)} = \mathbb{T}_1(x_{i-1}^{(1)}, x_{i-1}^{(2)}, \ldots, x_{i-1}^{(n)}) + g_1,\\
&x_i^{(2)} = \mathbb{T}_2(x_{i-1}^{(1)}, x_{i-1}^{(2)}, \ldots, x_{i-1}^{(n)}) + g_2,\\
&.......\ ..\ ....\\
&x_i^{(n)} = \mathbb{T}_n(x_{i-1}^{(1)}, x_{i-1}^{(2)}, \ldots, x_{i-1}^{(n)}) + g_n
\end{aligned}
\tag{6}
$$

which represents a block Picard iteration. To prove the convergence of (6) we assume

(H$_1$) For $k = 1, ..., n$, $\mathcal{U}_k$ is a complete metric space with norm $||.||_k$.
(H$_2$) For $k = 1, \ldots, n$, $\mathbb{T}_k : \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_n \to \mathcal{U}_k$ is a globally Lipschitz continuous, nonlinear mapping, i.e.

$$
\begin{aligned}
&||\mathbb{T}_k(x^{(1)}, \ldots, x^{(n)}) - \mathbb{T}_k(\bar{x}^{(1)}, \ldots, \bar{x}^{(n)})||_k\\
&\leq\ l_{k1}||x^{(1)} - \bar{x}^{(1)}||_1 + \ldots + l_{kn}||x^{(n)} - \bar{x}^{(n)}||_n
\end{aligned}
\tag{7}
$$

for all $x^{(1)}, \bar{x}^{(1)} \in \mathcal{U}_1, \ldots, x^{(n)}, \bar{x}^{(n)} \in \mathcal{U}_n$.

Let $L := (l_{kj}), 1 \leq k, j \leq n$, be the matrix of Lipschitz constants l_{kj} of operators $\mathbb{T}_k$, $k = 1, 2, ..., n$.

Theorem 1 *We assume the hypotheses (H$_1$) and (H$_2$) to be satisfied. Under the additional assumption that the spectral radius $\varrho(L)$ of matrix L is lesser than one, the iteration scheme (6) converges in $\mathcal{U}$ with respect to an appropriate norm $|||.|||$ (for its definition, see (9) in the proof below).*

PROOF. Without loss of generality we may assume that all entries of L are strictly positive. Then, by a theorem of Perron (see [10]), the fact $\varrho(L) < 1$ implies that $\varrho(L)$ is an eigenvalue of L to which an eigenfunction e with strictly positive components $e_1, \ldots, e_n$ exists. From (6) and (7) we get for $k = 1, \ldots, n$

$$||x_i^{(k)} - \bar{x}_{i-1}^{(k)}||_k \leq l_{k1}||x_{i-1}^{(1)} - \bar{x}_{i-2}^{(1)}||_1 + \ldots + l_{kn}||x_{i-1}^{(n)} - \bar{x}_{i-2}^{(n)}||_n.$$

Hence, we have

$$
\begin{aligned}
&e_1||x_i^{(1)} - x_{i-1}^{(1)}||_1 + \ldots + e_n||x_i^{(n)} - x_{i-1}^{(n)}||_n \\
&\leq (e_1 l_{11} + e_2 l_{21} + \ldots + e_n l_{n1})||x_{i-1}^{(1)} - x_{i-2}^{(1)}||_1 + \ldots \\
&\quad + (e_1 l_{n1} + e_2 l_{n2} + \ldots + e_n l_{nn})||x_{i-1}^{(n)} - x_{i-2}^{(n)}||_n \\
&= \varrho(L)\,(e_1||x_{i-1}^{(1)} - x_{i-2}^{(1)}||_1 + \ldots + e_n||x_{i-1}^{(n)} - x_{i-2}^{(n)}||_n)\,.
\end{aligned} \tag{8}
$$

Now we introduce a norm $|||.|||$ in $\mathcal{U} := \mathcal{U}_1 \times \ldots \times \mathcal{U}_n$ by

$$
|||x||| := e_1||x^{(1)}||_1 + \ldots + e_n||x^{(n)}||_n. \tag{9}
$$

Using this norm we obtain from (8)

$$
|||x_i - x_{i-1}||| \leq \varrho(L)\,|||x_{i-1} - x_{i-2}|||.
$$

Thus, the iteration scheme (6) is convergent in $\mathcal{U}$ with respect to the norm (9), provided that $\varrho(L) < 1$. $\diamond$

Similar convergence results can be derived for modified schemes. The iterative methods to solve the subsystems can be applied in form of Gauss–Jacobi, Gauss–Seidel, successive overrelaxation (SOR) or Picard iterations in general, where the related spectral radii control the convergence of these algorithms in appropriate Banach spaces. For example, if we replace the Gauss–Jacobi procedure (6) by the Gauss–Seidel iteration

$$
\begin{aligned}
x_i^{(1)} &= \top\!\!\top_1(x_{i-1}^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, \ldots, x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_1, \\
x_i^{(2)} &= \top\!\!\top_2(x_i^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, \ldots, x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_i^{(n)} &= \top\!\!\top_n(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots, x_i^{(n-1)}, x_{i-1}^{(n)}) + g_n,
\end{aligned} \tag{10}
$$

then the corresponding matrix $\tilde{L} = (\tilde{l}_{k,j})$ of Lipschitz-constants can be determined from the estimates

$$
\begin{aligned}
\Delta x_i^{(1)} &\leq l_{11}\Delta x_{i-1}^{(1)} + l_{12}\Delta x_{i-1}^{(2)} + \cdots + l_{1n}\Delta x_{i-1}^{(n)}, \\
\Delta x_i^{(2)} &\leq l_{21}l_{11}\Delta x_i^{(1)} + (l_{21}l_{12} + l_{22})\Delta x_{i-1}^{(2)} + \cdots + (l_{21}l_{1n} + l_{2n})\Delta x_{i-1}^{(n)}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{11}
$$

where $\Delta x_i^{(k)} = ||x_i^{(k)} - x_{i-1}^{(k)}||_k$.

In the case $n = 3$, for the Gauss–Seidel iteration

$$
\begin{aligned}
x_i^{(1)} &= \Pi_1(x_{i-1}^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}) + g_1, \\
x_i^{(2)} &= \Pi_2(x_i^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}) + g_2, \\
x_i^{(3)} &= \Pi_n(x_i^{(1)}, x_i^{(2)}, x_{i-1}^{(3)}) + g_3
\end{aligned}
\tag{12}
$$

we obtain the matrix $\tilde{\mathbf{L}}$

$$
\tilde{\mathbf{L}} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21}l_{11} & l_{21}l_{12} + l_{22} & l_{21}l_{13} + l_{23} \\ l_{31}l_{11} + l_{32}l_{21}l_{11} & l_{31}l_{12} + l_{32}(l_{21}l_{12} + l_{22}) & l_{31}l_{13} + l_{33} + l_{32}(l_{21}l_{13} + l_{23}). \end{pmatrix}.
$$

Thus, $\varrho(\tilde{\mathbf{L}}) < 1$ implies the convergence of the iteration scheme (12). Consequently, Theorem 1 can be modified for this iteration scheme as well. General convergence theorems for iteration methods are also found in standard references, e.g. Zeidler [46].

Remark 2 *Theorem 1 is applicable to operators describing deterministic as well as stochastic processes. The main problem to be tackled in applying the waveform relaxation method to stochastic systems consists of estimating the influence of stochastic terms on the Lipschitz-constants. A first approach is presented in the next section.*

Remark 3 *It is worth noting that system (5) permits the application of multi-processor computers (parallel computing) – a fact which renders the waveform algorithm to be very attractive for numerical solving of large scale systems.*

3 Waveform relaxation methods for SDEs

3.1 Notation and main assumptions

Let $< .,. >_d$ denote the Euclidean scalar product defined by $< x, y >_d = \sum_{i=1}^d x_i y_i$ for vectors x, y in $\mathbb{R}^d$, $d \geq 1$ the current dimension, and $\|.\|_d$ the Euclidean vector norm in $\mathbb{R}^d$. Throughout this paper $\mathcal{B}^d$ represents the set of all Borel-measurable sets of $\mathbb{R}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space, and $\mathcal{T} = [0, T]$ a fixed finite time interval. Suppose that $(\mathcal{F}_t)_{t\in\mathcal{T}}$ performs a filtration such that $(\Omega, \mathcal{F}, \mathcal{F}_{t\in\mathcal{T}}, \mathbb{P})$ presents a complete stochastic basis. In the following we consider only $\mathcal{F}_t$-adapted stochastic processes $(X_t)_{t\in\mathcal{T}}$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_{t\in\mathcal{T}}, \mathbb{P})$, with finite p-th absolute moments for all times $t \in \mathcal{T}$, where $p \geq 1$. Recall that a stochastic process is called "cadlag (a.s.)" if and only if all trajectories are continuous from the right side, and left hand limits exist almost surely (with respect to probability measure $\mathbb{P}$). For more detailed information on stochastic calculus, see e.g. Anulova et al. [1].

Definition 4 *The space $\mathcal{E}_{p,d}$ is defined to be*

$$\mathcal{E}_{p,d} := \left\{ (X_t)_{0\le t\le T} : \begin{array}{c} X_t = X_t(\omega) \text{ is a cadlag (a.s.) stochastic process,} \\ X_t(\omega) : [0,T] \times (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}) \longrightarrow (\mathbb{R}^d, \mathcal{B}^d), \\ X_t \text{ is } \mathcal{F}_t\text{-adapted, } \mathbb{E}\ \sup_{0\le t\le T} \|X_t\|_d^p < +\infty \end{array} \right\} \tag{13}$$

and the space $\mathcal{E}^0_{p,d}$

$$\mathcal{E}^0_{p,d} := \left\{ (X_t)_{0\le t\le T} : \begin{array}{c} X_t = X_t(\omega) \text{ is a continuous (a.s.) stochastic process,} \\ X_t(\omega) : [0,T] \times (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}) \longrightarrow (\mathbb{R}^d, \mathcal{B}^d), \\ X_t \text{ is } \mathcal{F}_t\text{-adapted, } \mathbb{E}\ \max_{0\le t\le T} \|X_t\|_d^p < +\infty \end{array} \right\}. \tag{14}$$

Proposition 5 *The spaces $\mathcal{E}_{p,d}, \mathcal{E}^0_{p,d}$ are Banach spaces with respect to the norm*

$$\|X\|_{\mathcal{E}_{p,d}} = \left(\mathbb{E} \sup_{0\le t\le T} \|X_t\|_d^p \right)^{1/p} \tag{15}$$

for $X \in \mathcal{E}_{p,d}$ or $X \in \mathcal{E}^0_{p,d}$, respectively.

PROOF. The proofs of this assertion for $\mathcal{E}_{p,d}$ and $\mathcal{E}^0_{p,d}$ are similar, hence we restrict ourselves to the case of $\mathcal{E}^0_{p,d}$. The fact that $\mathcal{E}^0_{p,d}$ is a normed linear space follows from the linearity of $\mathbb{E}$-operation and properties of real vector norm $\|.\|_d$ in $\mathbb{R}^d$. It remains to show the completeness of $\mathcal{E}^0_{p,d}$. Let $(X^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in space $\mathcal{E}^0_{p,d}$. That is, we know that

$$\forall \varepsilon > 0\ \exists n_0(\varepsilon) \in \mathbb{N}\ \forall n, m \ge n_0(\varepsilon)\ :\ \|X^{(n)} - X^{(m)}\|_{\mathcal{E}^0_{p,d}} < \varepsilon\,.$$

Let $X^{(n)}$ converge to $\hat{X}$. Then, for all $n, m \ge n_0(\varepsilon)$, it follows that

$$\begin{aligned} \|\hat{X} - X^{(m)}\|^p_{\mathcal{E}^0_{p,d}} &= \mathbb{E} \sup_{0\le t\le T} \|\hat{X}_t - X^{(m)}_t\|_d^p \\ &\le \sup_{n\ge m} \left(\mathbb{E} \sup_{0\le t\le T} \|X^{(n)}_t - X^{(m)}_t\|_d^p \right) \le \varepsilon^p. \end{aligned}$$

Hence, by the Lemma of Fatou (see Bauer [5], p. 92), we get $\hat{X} - X^{(m)} \in \mathcal{E}^0_{p,d}$ for all $m \geq n_0(\varepsilon)$. Therefore

$$\hat{X} \;=\; \hat{X} \;-\; X^{(m)} \;+\; X^{(m)} \;\in\; \mathcal{E}^0_{p,d}\,.$$

Thus, the proof is completed. $\diamond$

Remark 6 *For $p = 2$, the function spaces $\mathcal{E}_{p,d}, \mathcal{E}^0_{p,d}$ form Hilbert spaces endowed with the naturally induced scalar product. For fixed parameters p, d, one finds the natural inclusion $\mathcal{E}^0_{p,d} \subset \mathcal{E}_{p,d}$.*

Our goal is to study the class of Itô-interpreted stochastic differential equations (SDEs) in conjunction with convergence of waveform relaxation methods. Let W_t^1, W_t^2, ..., W_t^m be m given independent, one-dimensional Wiener processes adapted to the filtration $\mathcal{F}_t$. Define $W_t^0 = t$ for all $t \in [0,T]$. In what follows we consider the initial value problem for the d-dimensional system of SDEs driven by the Wiener process $W_t = (W_t^1, W_t^2, ..., W_t^m)$

$$\begin{aligned} dX_t &= \sum_{j=0}^{m} f_j(t, X_t)\, dW_t^j \\ X_0 &= x_0(\omega) \text{ fixed and } \mathcal{F}_0\text{−measurable},\ 0 \leq t \leq T. \end{aligned} \tag{16}$$

The main emphasis of this paper is to derive conditions on the functions f_j in order to guarantee the convergence of waveform relaxation methods to the unique solution of (16) within the space $\mathcal{E}^0_{p,d}$. For this purpose, we take into account the following splitting of the d-dimensional system (16) into n interacting subsystems of dimension d_k

$$\begin{aligned} dX_t^{(1)} &= \sum_{j=0}^{m} f_{1,j}(t, X_t^{(1)}, X_t^{(2)}, ..., X_t^{(n)})\, dW_t^j, \\ dX_t^{(2)} &= \sum_{j=0}^{m} f_{2,j}(t, X_t^{(1)}, X_t^{(2)}, ..., X_t^{(n)})\, dW_t^j, \\ &\cdots .\ \cdots\cdots\cdots\cdots\cdots \\ dX_t^{(n)} &= \sum_{j=0}^{m} f_{n,j}(t, X_t^{(1)}, X_t^{(2)}, ..., X_t^{(n)})\, dW_t^j, \\ (X_0^{(1)}, X_0^{(2)}, ..., X_0^{(n)}) &= (x_0^{(1)}(\omega), x_0^{(2)}(\omega), ..., x_0^{(n)}(\omega)),\ 0 \leq t \leq T, \end{aligned} \tag{17}$$

where $f_j = (f_{1j}, f_{2j}, ..., f_{nj})^T, j = 0, 1, ..., m$, with $f_{kj} : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d_k}$, $d = \sum_{k=1}^{n} d_k$, $k = 1, 2, ..., n$. Concerning the functions f_j we assume

(A_0) $f_j, j = 0, 1, ..., m$ are Lebesgue-measurable.

(A_1) $f_{kj}(t,x), j = 0,1,...,m; k = 1,2,...,n$ are globally Lipschitz continuous in x , uniformly with respect to time $t \in [0,T]$, i.e. there are constants $L_{k,j}^{(i)} \in \mathbb{R}^1$ $(i = 1,2,...,n)$ such that

$$\forall (x^{(1)},...,x^{(n)}), (y^{(1)},...,y^{(n)}) \in \mathbb{R}^{d_1} \times ... \times \mathbb{R}^{d_n}$$

$$\| f_{k,j}(t, x^{(1)},...,x^{(n)}) - f_{k,j}(t, y^{(1)},...,y^{(n)}) \|_{d_k} \le \sum_{i=1}^{n} L_{i,j}^{(k)} \| x^{(i)} - y^{(i)} \|_{d_i}$$

for all times t with $0 \le t \le T$.

(A_2) For $k = 1,2,...,n$, and $j = 0,1,...,m$ it holds that

$$\sup_{0 \le t \le T} \inf_{y \in \mathbb{R}^d} \| f_{k,j}(t,y) \| \le K_B < +\infty \,.$$

3.2 An auxiliary result on linear boundedness

In most of the references on SDEs one finds the extra requirement of at most linear-polynomial boundedness of their coefficients. We shall show that this requirement can be replaced by the much weaker one of (A_2) as a consequence of requirement (A_1). For this purpose, the following auxiliary lemma is stated and proved.

Lemma 7 *Assume that the function $f_k : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d_k}$ satisfies the hypotheses*

(V_1) $f_k(t,x)$ is globally Lipschitz continuous in x, uniformly with respect to $t \in [0,T]$, i.e. it holds that

$$\exists L_k \in \mathbb{R}^1 \; \forall x,y \in \mathbb{R}^d \qquad \| f_k(t,x) - f_k(t,y) \|_{d_k} \le L_k \| x - y \|_d$$

for all times t with $0 \le t \le T$.

(V_2) $\sup_{0 \le t \le T} \inf_{y \in \mathbb{R}^d} \left(\| f_k(t,y) \| + L_k \| y \| \right) < +\infty$ for all indices k.

Then, there exists a nonnegative real constant

$$c_0 = c_0(f_k) = \sup_{0 \le t \le T} \inf_{y \in \mathbb{R}^d : \| f_k(t,y) \|_{d_k} < +\infty} \left\{ L_k \| y \|_d + \| f_k(t,y) \|_{d_k} \right\}$$

such that

$$\forall t \in [0,T] \; \forall x \in \mathbb{R}^d \; : \; \| f_k(t,x) \|_{d_k} \le c_0(f_k) + L_k \| x \|_d .$$

PROOF. Suppose that the function $f_k = f_k(t,x)$ satisfies hypotheses (V_1) and (V_2). Now, consider any value $(t,y) \in [0,T] \times \mathbb{R}^d$ with $\| f_k(t,y) \|_{d_k} < +\infty$. One finds the estimate

$$\|f_k(t,x)\|_{d_k} - \|f_k(t,y)\|_{d_k} \leq |\|f_k(t,x)\|_{d_k} - \|f_k(t,y)\|_{d_k}| \;\leq\; \|f_k(t,x) - f_k(t,y)\|_{d_k}$$
$$\leq L_k\|x-y\|_d \;\leq\; L_k\Big(\|x\|_d + \|y\|_d\Big)\,,$$

using the inverse triangular inequality and Lipschitz-continuity of function f. By algebraic rearrangements, and taking infimum and supremum on the right side, this implies

$$\forall t \in [0,T]\ \forall x \in \mathbb{R}^d$$
$$\|f_k(t,x)\|_{d_k} \leq \sup_{0\leq t\leq T}\ \inf_{y\in\mathbb{R}^d:\,\|f_k(t,y)\|_{d_k}<+\infty} \Big\{c_1\|y\|_d + \|f(t,y)\|_d\Big\} + c_1\|x\|_d$$

where

$$c_1 \;=\; c_1(f_k) \;=\; L_k\,.$$

Therefore, there exist nonnegative real constants $c_1 = c_1(f_k)$ and

$$c_0 \;=\; c_0(f_k) \;=\; \sup_{0\leq t\leq T}\ \inf_{y\in\mathbb{R}^d:\,\|f_k(t,y)\|_{d_k}<+\infty} \Big\{c_1(f_k)\|y\|_d + \|f_k(t,y)\|_{d_k}\Big\}$$

such that

$$\forall t \in [0,T]\ \forall x \in \mathbb{R}^d \;:\; \|f_k(t,x)\|_d \;\leq\; c_0(f_k) + c_1(f_k)\|x\|_d\,,$$

i.e. the linear-polynomial boundedness of globally Lipschitz continuous functions f_k.$\diamond$

Remark 8 *The hypotheses (V_1) and (V_2) guarantee the linear-polynomial boundedness of functions f_j and f_{kj} occurring as coefficients of considered SDEs. The result of Lemma 7 is trivial for functions f which are independent of time argument t (then, (V_1) implies (V_2)).*

3.3 On existence and uniqueness of the solution of (16)

In this subsection we present a constructive proof for the existence and uniqueness of the solution of the Cauchy problem (16) taking into account the splitting (17). The goal of this procedure is to extract conditions for the convergence of waveform relaxation methods applied to SDEs. For this purpose, we make use of the representation of the Banach space $\mathcal{E}^0_{p,d}$ as the product space

$$\mathcal{E}^0_{p,d} \;=\; \mathcal{E}^0_{p,d_1} \times \mathcal{E}^0_{p,d_2} \times \ldots \times \mathcal{E}^0_{p,d_n}$$

with $d = \sum_{k=1}^{n} d_k$. The spaces $\mathcal{E}^0_{p,d_k}$ are equipped with the norm $\|.\|_{\mathcal{E}^0_{p,d_k}}$, hence they are Banach spaces according to the Proposition 5. Later we shall introduce an appropriate norm in $\mathcal{E}^0_{p,d}$ which renders $\mathcal{E}^0_{p,d}$ to be a Banach space (This new norm is equivalent to the norm given by (15)).

We define the random operators $\Pi_k, k = 1, 2, ..., n$ by

$$[\Pi_k\left(X^{(1)}, X^{(2)}, ..., X^{(n)}\right)]_t \;=\; X_0^{(k)} + \sum_{j=0}^{m} \int_0^t f_{k,j}(s, X_s^{(1)}, X_s^{(2)}, ..., X_s^{(n)})\, dW_s^j \quad (18)$$

for all $X^{(k)} \in \mathcal{E}^0_{p,d_k}$, mapping $\mathcal{E}^0_{p,d}$ into $\mathcal{E}^0_{p,d_k}$. Then a solution of the initial value problem (17) is understood as a solution of the system of integral equations

$$[\Pi_k\left(X^{(1)}, X^{(2)}, ..., X^{(n)}\right)]_t \;=\; X_t^{(k)},\; k = 1, 2, ..., n. \quad (19)$$

Introducing the operator $\Pi = (\Pi_1, ..., \Pi_n)$, a solution of (19) corresponds to a fixed point of the operator Π. The proof of the following theorem relies on the contractivity of operator Π in the product Banach space $\mathcal{E}^0_{p,d}$.

Theorem 9 *Let $p \geq 1$. Assume that the given functions $f_{k,j}$ satisfy the conditions (A_0) - (A_2), and that $\mathbb{E}\,\|X_0^{(k)}\|_{d_k}^p < +\infty$ for all $k = 1, 2, ..., n; j = 0, 1, ..., m$.*
Then the initial value problem (17) has an unique, $\mathcal{F}_t$-adapted and continuous (a.s.) solution in the space $\mathcal{E}^0_{p,d}$.

PROOF. The proof is carried out in two main steps. First, we shall show that the decomposed operator Π is a mapping from the Banach space $\mathcal{E}^0_{p,d}$ into itself. Second, the operator Π forms a contraction in the Banach space $\mathcal{E}^0_{p,d}$ with respect to appropriately constructed norm. Then Banach's fixed point theorem provides us with the conclusion of Theorem 9.
Step 1: We prove that $\|\Pi_k(X)\|_{\mathcal{E}^0_{p,d_k}} < +\infty$ for $X \in \mathcal{E}^0_{p,d}$ whenever the functions $f_{k,j}(t,x)$ fulfill assumptions (A_0) - (A_2). Thanks to auxiliary Lemma 7, we know about the linear-polynomial boundedness of functions f_{kj} with corresponding constants $c_0(f_{kj})$ and $c_1(f_{kj})$, i.e.

$$\forall t \in [0,T]\; \forall x \in \mathbb{R}^d \;:\; \|f(t,x)\|_d \;\leq\; c_0(f) + c_1(f)\|x\|_d\,,$$

Using the latter fact, we can estimate the norm of images of operators Π_k. Remember $X_t = (X_t^{(1)}, ..., X_t^{(k)}, ..., X_t^{(n)})$. We obtain

$$\|[\Pi_k(X)]_t\|_{d_k}^p \;\leq\; \left(\|X_0^{(k)}\|_{d_k} + \|\int_0^t f_{k,0}(s,X_s)ds\|_{d_k} + \sum_{j=1}^{m} \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}\right)^p$$

$$\le (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + \|\int_0^t f_{k,0}(s,X_s)ds\|_{d_k}^p + \sum_{j=1}^m \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big)$$

$$\le (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + t^{p-1}\int_0^t \|f_{k,0}(s,X_s)\|_{d_k}^p ds + \sum_{j=1}^m \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big)$$

$$\le (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + 2^{p-1}(t^p c_0^p(f_{k,0}) + t^{p-1}c_1^p(f_{k,0})\int_0^t \|X_s\|_d^p ds)\Big)$$

$$+(m+2)^{p-1}\Big(\sum_{j=1}^m \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big)$$

$$\le (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + 2^{p-1}t^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\max_{0\le u\le t}\|X_u\|_d^p)\Big)$$

$$+(m+2)^{p-1}\Big(\sum_{j=1}^m \sup_{0\le u\le t}\|\int_0^u f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big)$$

$$\le (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + 2^{p-1}T^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\max_{0\le t\le T}\|X_t\|_d^p)$$

$$+(m+2)^{p-1}\Big(\sum_{j=1}^m \sup_{0\le t\le T}\|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big)$$

with appropriate constants $c_0(f_{k,0})$ and $c_1(f_{k,0})$ (see above). Using the Burkholder–Davis–Gundy inequality and basic properties of quadratic variation of Itô integrals with respect to Brownian motions W_s^j (see Revuz and Yor [33]), there are constants $c_{p,k,j}$ such that

$$\mathbb{E}\sup_{0\le t\le T}\|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p \le c_{p,k,j}\,\mathbb{E}\left(\int_0^T \|f_{k,j}(s,X_s)\|_{d_k}^2 d < W^j, W^j >_s\right)^{p/2}$$

$$= c_{p,k,j}\,\mathbb{E}\left(\int_0^T \|f_{k,j}(s,X_s)\|_{d_k}^2 ds\right)^{p/2}$$

where $< M, M >_s$ denotes the total quadratic variation of inscribed martingale M on $[0,s]$. In fact, applying the Burkholder inequality as stated in Protter [32, p. 174–175] to continuous time, local martingales (here represented by stochastic Itô integrals) and the constants $c_{p,k,j}$ can be chosen universally, e.g.

$$c_{p,k,j} \;\le\; \left(\Big(\frac{p}{p-1}\Big)^p\left(\frac{p(p-1)}{2}\right)\right)^{\frac{p}{2}}$$

for $p \ge 2$, see also Krylov [22, p. 160–163] for an alternative estimate with $p \in (0,+\infty)$. Note that a deterministic T naturally is a $\mathcal{F}_t$-stopping time and that here

$f_{k,j}(s, X_s)$ are bounded in the sense of norm $\|.\|_{\mathcal{E}_{p,d_k}}$, thus one has the right to apply the Burkholder–Davis–Gundy inequality. Using this fact, returning to the estimation of $\|[\top\!\!\top_k(X)]_t\|^p_{d_k}$, taking supremum and expectation $\mathbb{E}$, one receives

$$\begin{aligned}
\|\top\!\!\top_k(X)\|^p_{\mathcal{E}_{p,d_k}} &= \mathbb{E} \sup_{0\le t\le T} \|[\top\!\!\top_k(X)]_t\|^p_{d_k} \\
&\le (m+2)^{p-1}\Big(\mathbb{E}\,\|X_0^{(k)}\|^p_{d_k} + 2^{p-1}T^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\|X\|^p_{\mathcal{E}_{p,d}})\Big) \\
&\quad +(m+2)^{p-1}2^{p/2-1}\Big(\sum_{j=1}^m c_{p,k,j}\,\mathbb{E}\,\left(\int_0^T (c_0^2(f_{k,j}) + c_1^2(f_{k,j})\|X_s\|^2_d)ds\right)^{p/2}\Big) \\
&\le (m+2)^{p-1}\Big(\mathbb{E}\,\|X_0^{(k)}\|^p_{d_k} + 2^{p-1}T^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\|X\|^p_{\mathcal{E}^0_{p,d}})\Big) \\
&\quad +(m+2)^{p-1}2^{p/2-1}T^{p/2}\Big(\sum_{j=1}^m c_{p,k,j}(c_0^p(f_{k,j}) + c_1^p(f_{k,j})\|X\|^p_{\mathcal{E}^0_{p,d}})\Big) \\
&< +\infty\,,
\end{aligned}$$

with appropriate constants $c_0(f_{k,j})$ and $c_1(f_{k,j})$ (see above), since $X \in \mathcal{E}^0_{p,d}$. That is, the images of operators $\top\!\!\top_k$ cannot blow up (a.s.) at finite times $t \in [0,T]$. Therefore, and thanks to integral construction of operators $\top\!\!\top_k$, the non-blowing up (a.s.) images of operators $\top\!\!\top_k$ are continuous (a.s.) and $\mathcal{F}_t$-adapted stochastic processes $\top\!\!\top_k(X) \in \mathcal{E}^0_{p,d_k}$ whenever the domain element X to which the operator $\top\!\!\top_k$ is applied lies in the space $\mathcal{E}^0_{p,d}$, and the functions $f_{k,j}$ are globally Lipschitz continuous (A_1). As a consequence, the decomposed operator $\top\!\!\top = (\top\!\!\top_1, ..., \top\!\!\top_n)$ represents a mapping from the closed space $\mathcal{E}^0_{p,d}$ into itself.
Step 2: It remains to show the property of contractivity of the operator $\top\!\!\top$ with respect to an appropriate norm of the product space $\mathcal{E}^0_{p,d}$. Assume that $X_0^{(k)} = Y_0^{(k)}$ (a.s.), $k = 1, 2, ..., n$. Set

$$\Delta\top\!\!\top_k(t) \;:=\; [\top\!\!\top_k(X^{(1)}, ..., X^{(n)}) - \top\!\!\top_k(Y^{(1)}, ..., Y^{(n)})](t)$$

for all $t \in [0,T]$, and

$$\Delta f_{k,j}(s) \;:=\; f_{k,j}(s, X_s^{(1)}, ..., X_s^{(n)}) - f_{k,j}(s, Y_s^{(1)}, ..., Y_s^{(n)})$$

for all $s \in [0,T]$. For any fixed $(X^{(1)}, ..., X^{(n)}), (Y^{(1)}, ..., Y^{(n)}) \in \mathcal{E}^0_{p,d}$ one has

$$\begin{aligned}
\|\Delta\top\!\!\top_k(t)\|^p_{d_k} &\le \left(\|\int_0^t \Delta f_{k,0}(s)ds\|_{d_k} + \sum_{j=1}^m \|\int_0^t \Delta f_{k,j}(s)dW_s^j\|_{d_k}\right)^p \\
&\le (m+1)^{p-1}\left(\|\int_0^t \Delta f_{k,0}(s)ds\|^p_{d_k} + \sum_{j=1}^m \|\int_0^t \Delta f_{k,j}(s)dW_s^j\|^p_{d_k}\right)
\end{aligned}$$

$$\leq (m+1)^{p-1}\left(t^{p-1}\int_0^t \|\Delta f_{k,0}(s)\|_{d_k}^p ds + \sum_{j=1}^m \|\int_0^t \Delta f_{k,j}(s) dW_s^j\|_{d_k}^p \right)$$

using the triangle inequality and using the Hölder inequality several times. Recall that under global Lipschitz-continuity of $f_{k,j}$ it holds that

$$\|\Delta f_{k,j}(s)\|_{d_k}^p \ \leq \ n^{p-1}\sum_{i=1}^n (L_{i,j}^{(k)})^p \|X_s^{(i)} - Y_s^{(i)}\|_{d_i}^p$$

for $p \geq 1$. Therefore it follows that

$$\begin{aligned}
\|\Delta \mathbb{T}_k(t)\|_{d_k}^p &\leq (m+1)^{p-1} n^{p-1} t^p \sum_{i=1}^n (L_{i,0}^{(k)})^p \max_{0\leq s\leq t} \|X_s^{(i)} - Y_s^{(i)}\|_{d_i}^p \\
&\quad +(m+1)^{p-1}\sum_{j=1}^m \|\int_0^t \Delta f_{k,j}(s) dW_s^j\|_{d_k}^p \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^n (L_{i,0}^{(k)})^p \max_{0\leq t\leq T} \|X_t^{(i)} - Y_t^{(i)}\|_{d_i}^p \\
&\quad +(m+1)^{p-1}\sum_{j=1}^m \sup_{0\leq t\leq T} \|\int_0^t \Delta f_{k,j}(s) dW_s^j\|_{d_k}^p .
\end{aligned}$$

Now, by taking the operation of expectation $\mathbb{E}$ on both sides, this implies

$$\begin{aligned}
\|\Delta \mathbb{T}_k\|_{\mathcal{E}_{p,d_k}}^p &= \mathbb{E} \sup_{0\leq t\leq T} \|\Delta \mathbb{T}_k(t)\|_{d_k}^p \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^n (L_{i,0}^{(k)})^p \|X^{(i)} - Y^{(i)}\|_{\mathcal{E}_{p,d_i}^0}^p \\
&\quad +(m+1)^{p-1}\sum_{j=1}^m \mathbb{E} \sup_{0\leq t\leq T} \|\int_0^t \Delta f_{k,j}(s) dW_s^j\|_{d_k}^p .
\end{aligned}$$

The herein occurring terms $\int_0^t \Delta f_{k,j}(s) dW_s^j$ form continuous and $\mathcal{F}_t$-adapted martingales started at initial value 0 under the global Lipschitz-continuity (A_1) of functions $f_{k,j}$ and for $X^{(k)} \in \mathcal{E}_{p,d_k}^0$, where $k = 1,2,...,n; j = 1,2,...,m$. This can be shown in the same way as in step 1. Using the Burkholder–Davis–Gundy inequality and basic properties of quadratic variation of Itô integrals with respect to Brownian motions W_s^j (see Revuz and Yor [33, p. 153]), there are constants $C_{p,k,j}$ such that

$$\mathbb{E} \sup_{0\leq t\leq T} \|\int_0^t \Delta f_{k,j}(s) dW_s^j\|_{d_k}^p \leq C_{p,k,j}\, \mathbb{E} \left(\int_0^T \|\Delta f_{k,j}(s)\|_{d_k}^2 d < W^j, W^j >_s \right)^{p/2}$$

$$= C_{p,k,j}\, \mathbb{E} \left(\int_0^T \|\Delta f_{k,j}(s)\|^2_{d_k} ds \right)^{p/2}$$

where $< M, M >_s$ denotes the total quadratic variation of inscribed martingale M on $[0, s]$. As already stated, we can find an universal estimate of $C_{p,k,j}$ arising from the Burkholder inequality (see Protter [32, p. 174–175], as before), e.g. with

$$C_{p,k,j} \;\leq\; \left(\Big(\frac{p}{p-1}\Big)^p \left(\frac{p(p-1)}{2} \right) \right)^{\frac{p}{2}}$$

for $p \geq 2$, which still depends on p. Note that a deterministic T naturally is a $\mathcal{F}_t$-stopping time, and $\Delta f_{k,j}(s)$ are bounded in the sense of norm $\|.\|_{\mathcal{E}_{p,d_k}}$, thus one has the right to apply the Burkholder–Davis–Gundy inequality. Using the last observations and returning to the estimation of $\|\Delta \Pi_k\|^p_{\mathcal{E}_{p,d_k}}$, we have

$$\begin{aligned}
&\|\Delta \Pi_k\|^p_{\mathcal{E}_{p,d_k}} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^n (L^{(k)}_{i,0})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} \sum_{j=1}^m C_{p,k,j}\, \mathbb{E} \left(\int_0^T \|\Delta f_{k,j}(s)\|^2_{d_k} ds \right)^{p/2} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^n (L^{(k)}_{i,0})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} n^{p/2} \sum_{j=1}^m C_{p,k,j}\, \mathbb{E} \left(\sum_{i=1}^n (L^{(k)}_{i,j})^2 \int_0^T \|X^{(i)}_s - Y^{(i)}_s\|^2_{d_i} ds \right)^{p/2} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^n (L^{(k)}_{i,0})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} (nT)^{p/2} \sum_{j=1}^m C_{p,k,j}\, \mathbb{E} \left(\sum_{i=1}^n (L^{(k)}_{i,j})^2 \sup_{0 \leq t \leq T} \|X^{(i)}_t - Y^{(i)}_t\|^2_{d_i} \right)^{p/2} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^n (L^{(k)}_{i,0})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} n^{p-1} T^{p/2} \sum_{j=1}^m C_{p,k,j} \sum_{i=1}^n (L^{(k)}_{i,j})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}}
\end{aligned}$$

under Lipschitz-continuity of $f_{k,j}$. Hence, by taking the p-th root, we have

$$\|\Delta \Pi_k\|_{\mathcal{E}_{p,d_k}} \leq (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T} \sum_{i=1}^n \mathbf{k}_{i,k} \|X^{(i)} - Y^{(i)}\|_{\mathcal{E}^0_{p,d_i}}$$

where the coefficients $\mathbf{k}_{i,k}$ are given by

$$\mathbf{k}_{i,k} \;=\; \sqrt{T} L_{i,0}^{(k)} + \sum_{j=1}^{m} (C_{p,k,j})^{1/p} L_{i,j}^{(k)} \,.$$

Summarizing, we have the relation

$$\begin{pmatrix} \|\Delta \mathbb{T}_1\|_{\mathcal{E}^0_{p,d_1}} \\ \|\Delta \mathbb{T}_2\|_{\mathcal{E}^0_{p,d_2}} \\ \dots\dots\dots \\ \|\Delta \mathbb{T}_n\|_{\mathcal{E}^0_{p,d_n}} \end{pmatrix} \leq (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T}\, \mathbf{K} \begin{pmatrix} \|X^{(1)} - Y^{(1)}\|_{\mathcal{E}^0_{p,d_1}} \\ \|X^{(2)} - Y^{(2)}\|_{\mathcal{E}^0_{p,d_2}} \\ \dots\dots\dots \\ \|X^{(n)} - Y^{(n)}\|_{\mathcal{E}^0_{p,d_n}} \end{pmatrix} ,$$

for any $X^{(k)}, Y^{(k)} \in \mathcal{E}^0_{p,d_k}$ with $X_0^{(k)} = Y_0^{(k)}$ (a.s.), where the inequality sign $\leq$ is understood componentwise, and where $\mathbf{K}$ is the $n \times n$-matrix defined by $\mathbf{K} = (\mathbf{k}_{i,l})_{1 \leq i,l \leq n}$. Under the assumption that T is sufficiently small we can conclude that the spectral radius $\varrho(\mathbf{L})$ of the matrix $\mathbf{L} := (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T}\, \mathbf{K}$ is lesser than one. Thus, $\varrho(\mathbf{L})$ is an eigenvalue of $\mathbf{L}$ to which an eigenvector $(e_1, ..., e_n)$ with strictly positive components e_i corresponds. Now we introduce the norm

$$|||X|||_{\mathcal{E}^0_{p,d}} \;:=\; \left(\sum_{k=1}^{n} e_k \|X^{(k)}\|^p_{\mathcal{E}_{p,d_k}} \right)^{1/p} \tag{20}$$

in the Banach space $\mathcal{E}^0_{p,d}$. Then the vector-valued operator $\mathbb{T}$ mapping the closed set $\mathcal{E}^0$ into itself is strictly contractive with the contraction constant $\varrho(\mathbf{L})$. Consequently, the sequence generated by iterative application of operator $\mathbb{T}$ converges with respect to norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ to an unique element of $\mathcal{E}^0_{p,d}$ which is a solution of original system (17). Since the norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ is equivalent to the original norm $\|.\|_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$, we know that the solution of system (17) also lies in the original Banach space $\mathcal{E}^0_{p,d}$.

We have seen that the operator $\mathbb{T}$ is contractive in $\mathcal{E}^0_{p,d}$ for sufficiently small T. To get the result for any T we have to divide the interval $[0,T]$ in a finite number of sufficiently small subintervals and to repeat the stated proof-steps successively. Thus, the proof is completed. $\diamond$

Remark 10 *For $p = 2$, thanks to Doob's maximum inequality (see Revuz and Yor [33]), we can choose*

$$c_{2,k,j} \;=\; C_{2,k,j} \;=\; 4$$

in the estimation above. Following Protter [32, p. 174–175] we may apply the Burkholder inequality to continuous time, local martingales (here represented by stochastic Itô integrals), and the universal estimation

$$\max(c_{p,k,j}, C_{p,k,j}) \;\le\; \left(\left(\frac{p}{p-1}\right)^p \left(\frac{p(p-1)}{2}\right)\right)^{\frac{p}{2}} \tag{21}$$

is established for $p \geq 2$. Krylov [22] and Mao [26] have also proved some estimates for $p \in (0, +\infty)$.

Remark 11 *To get rid of dividing the interval $[0, T]$ in sufficiently small subintervals one may take weighted random norms on Banach spaces. One easily verifies that the appropriately weighted random norms are equivalent to the original norm (note that we make use of deterministic weights!).*

Remark 12 *In the case $m = 0$ (i.e. no stochastic terms) with $p = 1$, Theorem 9 yields a convergence criterion for the case of ordinary differential equations (here there is no dependence on the splitting parameter n).*

3.4 Convergence of waveform relaxation methods

The proof of Theorem 9 is based on general contraction principles and can be used to derive a sufficient condition for the convergence of the waveform relaxation method. If we consider the block Picard iteration as a special waveform relaxation technique for the fixed point problem (18), then we get the following sufficient condition for its convergence from the proof of Theorem 9.

Theorem 13 *Assume the hypotheses of Theorem 9 hold. Define $\mathbf{L} = (l_{ik})$ by*

$$l_{ik} := (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T} \Big(\sqrt{T} L_{i,0}^{(k)} + \sum_{j=1}^{m} (C_p)^{1/p} L_{i,j}^{(k)} \Big)$$

with corresponding universal constants C_p occurring at the right hand side of the Burkholder–Davis–Gundy inequality (or substituted by estimates as in (21)).
Then $\varrho(\mathbf{L}) < 0$ implies the convergence of the waveform relaxation algorithm based on the block Picard iterations (6) for the initial value problem (16) in the Banach space $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$ with norm $|||\cdot|||$ defined by (20), where $\mathcal{U}_k = \mathcal{E}^0_{p,d_k}$.
If we modify this algorithm with Gauss–Seidel iterations (10) applied to the initial value problem (16), then the condition $\varrho(\tilde{\mathbf{L}}) < 0$ implies its convergence with respect to corresponding norm $|||\cdot|||$.

PROOF. For the completion of the proof, it only remains to determine the matrix of Lipschitz-constants $\mathbf{L}$. These constants can be extracted from the last steps of the proof of previous Theorem 9 directly. Finally, one applies Theorem 1 to establish the claimed convergence with respect to the specifically constructed norm of $\mathcal{U}$. $\diamond$

4 The case of one-sided Lipschitz continuous and anticoercive drift

The conditions for convergence of waveform relaxation methods can be relaxed as follows. The global Lipschitz-continuity of drift coefficients of SDEs is replaced by local one, but, additionally, the one-sided Lipschitz-continuity and anticoercivity of the drift is required. We shall combine the idea of monotonicity of coefficients of SDEs, as indicated by Krylov [21, 22] for the analytical solution, and as used by Bremer [6] for the convergence of waveform relaxation methods for ODEs.

Definition 14 *A function $f_0 : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is said to be* (uniformly) one-sided Lipschitz continuous *if for the splitting $f_0 = (f_{1,0}, ..., f_{k,0}, ..., f_{n,0})^T$ there are constants $\hat{L}_{i,0}^{(k)} \in \mathbb{R}^1 (i, k = 1, 2, ..., n)$ such that*

$$(A_3) \qquad \forall x = (x^{(1)}, ..., x^{(n)}), y = (y^{(1)}, ..., y^{(n)}) \in \mathbb{R}^{d_1} \times ... \times \mathbb{R}^{d_n}$$

$$< f_{k,0}(t, x^{(1)}, ..., x^{(n)}) - f_{k,0}(t, y^{(1)}, ..., y^{(n)}), x^{(k)} - y^{(k)} >_{d_k} \leq \sum_{i=1}^{n} \hat{L}_{i,0}^{(k)} \|x^{(i)} - y^{(i)}\|_{d_i}^2$$

for all $t \in [0,T]$. A function $f : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is called (uniformly) anticoercive *if it satisfies*

$$(A_4) \qquad \exists c_a \in \mathbb{R}^1 \; \forall t \in [0,T] \; \forall x \in \mathbb{R}^d \; : < f(t,x), x >_d \leq \; c_a\Big(1 + \|x\|_d^2\Big) .$$

4.1 On existence and uniqueness of the solution of (16)

One encounters the following result. Assume measurebility (A_0) of all coefficients f_j.

Theorem 15 *Fix an exponent $p \geq 2$. Let the drift function $f_0 = f_0(t,x)$ be local and uniformly one-sided Lipschitz continuous (i.e. (A_3) holds), and the diffusion functions $f_{k,j} = f_{k,j}(t,x), j = 1, 2, ..., m; k = 1, 2, ..., n$ satisfy the conditions (A_1) of global Lipschitz-continuity and boundedness (A_2). Additionally, assume that f_0 possesses the property (A_4) of uniform anticoercivity, and $\mathbb{E} \, \|X_0^{(k)}\|_{d_k}^p < +\infty, k = 1, 2, ..., n$. Then the initial value problem (16) has an unique, $\mathcal{F}_t$-adapted and continuous (a.s.) solution in the space $\mathcal{E}_{p,d}^0$.*

PROOF. Again, the proof is carried out in two main steps. First, we shall show that the decomposed operator $\mathbb{T}$ is a mapping from the Banach space $\mathcal{E}^0_{p,d}$ into itself. Second, the operator $\mathbb{T}$ forms a contraction in the Banach space $\mathcal{E}_{p,d}$ with respect to an appropriately constructed norm. Then standard fixed point principles provide us with the conclusion of Theorem 15.

Step 1: Obviously, the existence of the unique solution of system (17) in any ball of $\mathbb{R}^d$ with finite radius $r > 0$ follows from the proof of Theorem 9 while assuming local Lipschitz-continuity of the components of f_0. That is that we can justify the unique solvability of the stopped system

$$dX^r_t \ = \ \chi_{\{\sup_{0\le s\le t}\|X^r_s\|_d < r\}}(t) \sum_{j=0}^{m} f_j(t, X^r_t)\, dW^j_t \tag{22}$$

in the space $\mathcal{E}^0_{p,d}$, where $\chi_{\{.\}}(t)$ represents the characteristic function of the subscribed set $\{.\}$ evaluated at time t. Here X^r_t denotes the solution of the system (22) truncating the system (16) such that the solutions X^r_t of (22) and X_t of (16) coincide up to the first exit time from the ball of radius r. It remains to show an aposterori estimate of the sequence $(X^r)_{r>0}$ of local and continuous (a.s.) solutions X^r of truncated system (22) such that its uniform limit uniquely exists in $\mathcal{E}^0_{p,d}$ as the radius r tends to infinity. Using the well-known Itô formula, the local Lipschitz-continuity and anticoercivity (A_4) of drift coefficient f_0 and the Lipschitz-continuity (A_1) of diffusion coefficients $f_{k,j}$ of the considered system of SDEs (17), one recognizes that the stopped solution processes X^r_t must satisfy

$$\|X^r_t\|^p_d \ = \ \|X^r_0\|^p_d + \sum_{j=0}^{m} \int_0^t \mathcal{L}^j\left(\|X^r_s\|^p_d\right) dW^j_s$$

with the operators $\mathcal{L}^j$ originating from the Itô formula. Thus, we have

$$\begin{aligned}
\mathcal{L}^0\left(\|x\|^p_d\right) &= p\, g(x) \|x\|^{p-2}_d\,, \\
g(x) &= < f_0(t,x), x >_d + \frac{1}{2}\sum_{j=1}^{m} \|f_j(t,x)\|^2_d + \frac{p-2}{2}\sum_{j=1}^{m} \frac{< f_j(t,x), x >^2_d}{\|x\|^2_d} \\
&\le < f_0(t,x), x >_d + \frac{p-1}{2}\sum_{j=1}^{m} \|f_j(t,x)\|^2_d\,, \\
\mathcal{L}^j\left(\|x\|^p_d\right) &= p < f_j(t,x), x >_d \|x\|^{p-2}_d \ \le \ p\, \|f_j(t,x)\|_d \|x\|^{p-1}_d
\end{aligned}$$

where $x \in \mathbb{R}^d$ and $j = 1, 2, ..., m$. For technical reasons, at first assume that we have $\mathbb{E}\, ||X^r_0||^{2p}_{\mathcal{E}_{p,d}} < +\infty$. Taking the supremum, taking into account the uniform anti-coercivity (A_4) of drift f_0 and the linear-polynomial boundedness of globally Lipschitz continuous diffusion functions $f_j (j = 1, 2, ..., m)$ under condition (A_2), and using the elementary inequality

$$(c_0 + c_1\|x\|^2)\|x\|^{p-2} \le c_0 + (c_0 + c_1)\|x\|^p$$

(a slightly more efficient estimate by application of the Hölder inequality would also be applicable here with $(c_0 + c_1\|x\|^2)\|x\|^{p-2} \le c_0\frac{2}{p} + (c_0\frac{p-2}{p} + c_1)\|x\|^p$) implies that

$$\begin{aligned}
\|X^r\|^p_{\mathcal{E}_{p,d}} \;\le\;& \mathbb{E}\,\|X^r_0\|^p_d + p\,\mathbb{E}\sup_{0\le t\le T}\int_0^t \Big(c_a(f_0)(1+\|X^r_s\|^2_d) + \\
&+\frac{p-1}{2}\sum_{j=1}^m (c_0(f_j) + c_1(f_j)\|X^r_s\|_d)^2\Big)\|X^r_s\|^{p-2}_d ds \\
&+\sum_{j=1}^m \mathbb{E}\sup_{0\le t\le T}\int_0^t \mathcal{L}^j(\|X^r_s\|^p_d)dW^j_s \\
\le\;& \mathbb{E}\,\|X^r_0\|^p_d + pT\Big(c_a(f_0) + (p-1)\sum_{j=1}^m c_0^2(f_j)\Big) \\
&+p\Big(c_a(f_0) + (p-1)\sum_{j=1}^m (c_0^2(f_j) + c_1^2(f_j))\Big)\int_0^T \mathbb{E}\,\|X^r_t\|^p_d dt \\
&+p2\sqrt{2}\sum_{j=1}^m \left(\mathbb{E}\int_0^T (c_0^2(f_j) + c_1^2(f_j)\|X^r_t\|^2_d)\|X^r_t\|^{2p-2}dt\right)^{1/2} \\
\le\;& \mathbb{E}\,\|X^r_0\|^p_d + pT\Big(c_a(f_0) + (p-1)\sum_{j=1}^m c_0^2(f_j)\Big) \\
&+p\Big(c_a(f_0) + (p-1)\sum_{j=1}^m (c_0^2(f_j) + c_1^2(f_j))\Big)\int_0^T \mathbb{E}\,\|X^r_t\|^p_d dt \\
&+p2\sqrt{2}\sum_{j=1}^m \left(\sqrt{T}c_0(f_j) + (c_0(f_j) + c_1(f_j))\Big(\int_0^T \mathbb{E}\,\|X^r_t\|^{2p}dt\Big)^{1/2}\right)
\end{aligned}$$

for all radii $r > 0$, where we have also applied Doob's maximum inequality to the occurring stochastic integrals (as in the proof above). Note that $c_a(f_0)$ represents the constant of anticoercivity (A_4) of drift function f_0 and $c_0(f_j), c_1(f_j)$ the constants of linear-polynomial growth of globally Lipschitz continuous diffusion functions f_j, respectively. Now, one can show that

$$\int_0^T \mathbb{E}\,\|X^r_t\|^p dt \;\le\; T\sup_{r>0}\sup_{0\le t\le T}\mathbb{E}\,\|X^r_t\|^p \;<\; +\infty$$

and

$$\int_0^T \mathbb{E}\,\|X_t^r\|^{2p} dt \;\leq\; T \sup_{r>0} \sup_{0\leq t\leq T} \mathbb{E}\,\|X_t^r\|^{2p} \;<\; +\infty$$

by applying Dynkin's formula (see Dynkin [9] or Khas'minskij [15]) to the functionals $\mathbb{E}\,\|X_t^r\|_d^p$ and $\mathbb{E}\,\|X_t^r\|_d^{2p}$, respectively, while $\sup_{r>0} \mathbb{E}\,\|X_0^r\|_d^{2p} < +\infty$. After that step and using Gronwall–Bellman inequality, one finds

$$\lim_{r\to+\infty} \|X^r\|_{\mathcal{E}_{p,d}} \;\leq\; \sup_{r>0} \|X^r\|_{\mathcal{E}_{p,d}} \;<\; +\infty\,.$$

Now, by use of standard localization procedures, one may relax the assumption $\mathbb{E}\,||X_0^r||^{2p} < +\infty$ to the weaker requirement $\mathbb{E}\,||X_0^r||^p < +\infty$.

Thus, from uniform anticoercivity (A_4) of functions f_j and $\mathbb{E}\,\|X_0^r\|_d^p < +\infty$, we know that uniform limit of continuous (a.s.) stochastic processes X^r as the radius r tends to infinity must exist with finite norm $\|.\|_{\mathcal{E}_{p,d}}$. Therefore, by the completeness of space $\mathcal{E}_{p,d}^0$, the limit process $\lim_{r\to+\infty} X^r$ which also solves the original system (16) must exist, be continuous (a.s.), be $\mathcal{F}_t$-adapted and have a finite norm $\|.\|_{\mathcal{E}_{p,d}}$. Consequently, the decomposed operator $\mathbb{T}$ is a mapping from Banach space $\mathcal{E}_{p,d}^0$ into itself.
Step 2: Contractivity of operator $\mathbb{T}$ on the space $\mathcal{E}_{p,d}^0$. Assume that $X_0^{(k)} = Y_0^{(k)}$ (a.s.). Take $\Delta X_s^{(k)} = X_s^{(k)} - Y_s^{(k)}$ for $k = 1, 2, ..., n$, and $\Delta X_s = X_s - Y_s$. Set

$$\Delta\mathbb{T}_k(t) \;:=\; [\mathbb{T}_k(X^{(1)}, ..., X^{(n)}) - \mathbb{T}_k(Y^{(1)}, ..., Y^{(n)})](t)$$

for all $t \in [0, T]$, and

$$\Delta f_{k,j}(s) \;:=\; f_{k,j}(s, X_s^{(1)}, ..., X_s^{(n)}) - f_{k,j}(s, Y_s^{(1)}, ..., Y_s^{(n)})$$

for all $s \in [0, T]$. Fix any $(X^{(1)}, ..., X^{(n)}), (Y^{(1)}, ..., Y^{(n)}) \in \mathcal{E}_{p,d}^0$, where $X^{(k)} \neq Y^{(k)}$ (a.s.). Define

$$g_k(x,y) := < f_{k,0}(t,x) - f_{k,0}(t,y), x^{(k)} - y^{(k)} >_{d_k} + \frac{1}{2}\sum_{j=1}^m \|f_{k,j}(t,x) - f_{k,j}(t,y)\|_{d_k}^2$$
$$+ \frac{p-2}{2}\sum_{j=1}^m \frac{< f_{k,j}(t,x) - f_{k,j}(t,y), x^{(k)} - y^{(k)} >_{d_k}^2}{\|x-y\|_{d_k}^2}$$

$$\leq < f_{k,0}(t,x) - f_{k,0}(t,y), x^{(k)} - y^{(k)} >_{d_k} + \frac{p-1}{2}\sum_{j=1}^m \|f_{k,j}(t,x) - f_{k,j}(t,y)\|_{d_k}^2$$

where $x = (x^{(1)}, ..., x^{(k)}, ..., x^{(n)})^T, y = (y^{(1)}, ..., y^{(k)}, ..., y^{(n)})^T \in \mathbb{R}^d$. In the following let $[.]_+$ denote the nonnegative part of the inscribed expression. Then one has

$$
\begin{aligned}
\|\Delta \amalg_k(t)\|_{d_k}^p \ &= \int_0^t \mathcal{L}^0\Big(\|\Delta X_s^{(k)}\|_{d_k}^p\Big) ds + \sum_{j=1}^m \int_0^t \mathcal{L}^j\Big(\|\Delta X_s^{(k)}\|_{d_k}^p\Big) dW_s^j \\
&= p\int_0^t g_k(X_s, Y_s)\|\Delta X_s^{(k)}\|_{d_k}^{p-2} ds \\
&\quad + p\sum_{j=1}^m \int_0^t < \Delta f_{k,j}(s), \Delta X_s^{(k)} >_{d_k} \|\Delta X_s^{(k)}\|_{d_k}^{p-2} dW_s^j \\
&\leq p\int_0^t \Big(\sum_{i=1}^n \hat{L}_{i,0}^{(k)}\|\Delta X_s^{(i)}\|_{d_i}^2 + \frac{p-1}{2}\sum_{j=1}^m (\sum_{l=1}^n L_{l,j}^{(k)}\|\Delta X_s^{(l)}\|)^2\Big)\|\Delta X_s^{(k)}\|_{d_k}^{p-2} ds \\
&\quad + p\sum_{j=1}^m |\int_0^t < \Delta f_{k,j}(s), \Delta X_s^{(k)} >_{d_k} \|\Delta X_s^{(k)}\|_{d_k}^{p-2} dW_s^j| \\
&\leq p\int_0^t \Big(\sum_{i=1}^n (\hat{L}_{i,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^m (L_{i,j}^{(k)})^2)\|\Delta X_s^{(i)}\|_{d_i}^2\Big)\|\Delta X_s^{(k)}\|_{d_k}^{p-2} ds \qquad (23) \\
&\quad + p\sum_{j=1}^m |\int_0^t < \Delta f_{k,j}(s), \Delta X_s^{(k)} >_{d_k} \|\Delta X_s^{(k)}\|_{d_k}^{p-2} dW_s^j| \\
&\leq p\int_0^t \Big(\sum_{l=1}^n [\hat{L}_{l,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^m (L_{l,j}^{(k)})^2]_+\Big)\sum_{i=1}^n \|\Delta X_s^{(i)}\|_{d_i}^p ds \\
&\quad + p\sum_{j=1}^m |\int_0^t < \Delta f_{k,j}(s), \Delta X_s^{(k)} >_{d_k} \|\Delta X_s^{(k)}\|_{d_k}^{p-2} dW_s^j| \\
&\leq pt\Big(\sum_{l=1}^n [\hat{L}_{l,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^m (L_{l,j}^{(k)})^2]_+\Big)\sum_{i=1}^n \sup_{0\leq s\leq t} \|\Delta X_s^{(i)}\|_{d_i}^p \\
&\quad + p\sum_{j=1}^m |\int_0^t < \Delta f_{k,j}(s), \Delta X_s^{(k)} >_{d_k} \|\Delta X_s^{(k)}\|_{d_k}^{p-2} dW_s^j| \\
&\leq pT\Big(\sum_{l=1}^n [\hat{L}_{l,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^m (L_{l,j}^{(k)})^2]_+\Big)\sum_{i=1}^n \sup_{0\leq t\leq T} \|\Delta X_t^{(i)}\|_{d_i}^p \\
&\quad + p\sum_{j=1}^m \sup_{0\leq t\leq T} |\int_0^t < \Delta f_{k,j}(s), \Delta X_s^{(k)} >_{d_k} \|\Delta X_s^{(k)}\|_{d_k}^{p-2} dW_s^j|
\end{aligned}
$$

using the Itô lemma applied to $\|\Delta X_s\|_{d_k}^p$, triangle inequality, the Hölder inequality, and the Lipschitz conditions (A_1) and (A_3), respectively. Note that the operators $\mathcal{L}^0$ and $\mathcal{L}^j$ are those operators arising at the application of Itô formula. Now, by taking the operation of expectation $\mathbb{E}$ on both sides, this implies

$$
\begin{aligned}
\|\Delta \Pi_k\|^p_{\mathcal{E}_{p,d_k}} &= \mathbb{E} \max_{0\le t\le T} \|\Delta \Pi_k(t)\|^p_{d_k} \\
&\le pT\Big(\sum_{l=1}^n [\hat{L}^{(k)}_{l,0} + n\frac{p-1}{2}\sum_{j=1}^m (L^{(k)}_{l,j})^2]_+\Big)\sum_{i=1}^n \|\Delta X^{(i)}\|^p_{\mathcal{E}_{p,d_i}} \\
&\quad + p\left(\sum_{j=1}^m \mathbb{E} \max_{0\le t\le T} |\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k} dW^j_s|\right) .
\end{aligned}
$$

The occurring terms $\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k} dW^j_s$ form continuous and $\mathcal{F}_t$-adapted martingales started at initial value 0 under the global Lipschitz-continuity (A_1) of diffusion functions $f_{k,j}$ and for $X^{(k)} \in \mathcal{E}_{p,d_k}$, where $k = 1, 2, ..., n; j = 1, 2, ..., m$. As in proof of Theorem 9, using the Burkholder–Davis–Gundy inequality and basic properties of quadratic variation of Itô integrals with respect to Brownian motions W^j_s, there are constants $\hat{C}_{p,k,j}$ such that

$$
\begin{aligned}
&\mathbb{E} \max_{0\le t\le T} |\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k} dW^j_s| \\
&\le \hat{C}_{p,k,j}\, \mathbb{E} \left(\int_0^T | < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} |^2 \Delta X^{(k)}_s\|^{2(p-2)}_{d_k} d < W^j, W^j >_s\right)^{1/2} \\
&= \hat{C}_{p,k,j}\, \mathbb{E} \left(\int_0^T | < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} |^2 \|\Delta X^{(k)}_s\|^{2(p-2)}_{d_k} ds\right)^{1/2} \\
&\le \hat{C}_{p,k,j}\, \mathbb{E} \left(\int_0^T \|\Delta f_{k,j}(s)\|^2 \|\Delta X^{(k)}_s\|^{2p-2}_{d_k} ds\right)^{1/2} \\
&\le \hat{C}_{p,k,j}\, \mathbb{E} \left(\int_0^T (\sum_{i=1}^n L^{(k)}_{i,j} \|\Delta X^{(i)}_s\|_{d_i})^2 \|\Delta X^{(k)}_s\|^{2p-2}_{d_k} ds\right)^{1/2} \\
&\le \hat{C}_{p,k,j}\sqrt{n}\, \mathbb{E} \left(\int_0^T \sum_{i=1}^n (L^{(k)}_{i,j})^2 \|\Delta X^{(i)}_s\|^2_{d_i} \|\Delta X^{(k)}_s\|^{2p-2}_{d_k} ds\right)^{1/2} \qquad (24) \\
&\le \hat{C}_{p,k,j}\sqrt{n(\sum_{i=1}^n (L^{(k)}_{i,j})^2)}\, \mathbb{E} \left(\int_0^T \sum_{i=1}^n \|\Delta X^{(i)}_s\|^{2p}_{d_i} ds\right)^{1/2} \\
&\le \hat{C}_{p,k,j}\sqrt{nT(\sum_{i=1}^n (L^{(k)}_{i,j})^2)}\, \mathbb{E} \left(\sum_{i=1}^n \max_{0\le t\le T} \|\Delta X^{(i)}_t\|^{2p}_{d_i}\right)^{1/2} \\
&\le \hat{C}_{p,k,j}\sqrt{nT(\sum_{i=1}^n (L^{(k)}_{i,j})^2)}\, \sum_{i=1}^n \mathbb{E} \max_{0\le t\le T} \|\Delta X^{(i)}_t\|^p_{d_i} \\
&= \hat{C}_{p,k,j}\sqrt{nT(\sum_{i=1}^n (L^{(k)}_{i,j})^2)}\, \sum_{i=1}^n \|\Delta X^{(i)}\|^p_{\mathcal{E}_{p,d_i}} .
\end{aligned}
$$

Using the last estimate and returning to the estimation of $\|\Delta \top\!\!\top_k\|^p_{\mathcal{E}_{p,d_k}}$, we have

$$\|\Delta \top\!\!\top_k\|^p_{\mathcal{E}_{p,d_k}} \leq pT\Big(\sum_{l=1}^{n}[\hat{L}^{(k)}_{l,0} + n\frac{p-1}{2}\sum_{j=1}^{m}(L^{(k)}_{l,j})^2]_+\Big)\sum_{i=1}^{n}\|\Delta X^{(i)}\|^p_{\mathcal{E}_{p,d_i}}$$
$$+p\left(\sum_{j=1}^{m}\hat{C}_{p,k,j}\sqrt{nT(\sum_{i=1}^{n}(L^{(k)}_{i,j})^2)\sum_{i=1}^{n}\|\Delta X^{(i)}\|^p_{\mathcal{E}_{p,d_i}}}\right)$$

under one-sided Lipschitz-continuity (A_3) of $f_{k,0}$. Hence, one finds

$$\|\Delta \top\!\!\top_k\|_{\mathcal{E}_{p,d_k}} \leq \sqrt[p]{p}\,\sqrt[2p]{T}\sum_{i=1}^{n}\hat{\mathbf{k}}_{i,k}\|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}$$

by taking the p-th root, where the coefficients $\hat{\mathbf{k}}_{i,k}$ are given by

$$\hat{\mathbf{k}}_{i,k} \;=\; \sqrt[2p]{T}\sum_{l=1}^{n}[\hat{L}^{(k)}_{l,0} + n\frac{p-1}{2}\sum_{j=1}^{m}(L^{(k)}_{l,j})^2]_+^{1/p} + \sum_{j=1}^{m}(\hat{C}_{p,k,j})^{1/p}\sqrt[2p]{n(\sum_{l=1}^{n}(L^{(k)}_{l,j})^2)}\,.$$

Summarizing the main result, we have shown the relation

$$\begin{pmatrix} \|\Delta \top\!\!\top_1\|_{\mathcal{E}_{p,d_1}} \\ \|\Delta \top\!\!\top_2\|_{\mathcal{E}_{p,d_2}} \\ \cdots\cdots \\ \|\Delta \top\!\!\top_n\|_{\mathcal{E}_{p,d_n}} \end{pmatrix} \leq \sqrt[p]{p}\,\sqrt[2p]{T}\,\hat{\mathbf{K}} \begin{pmatrix} \|X^{(1)} - Y^{(1)}\|_{\mathcal{E}_{p,d_1}} \\ \|X^{(2)} - Y^{(2)}\|_{\mathcal{E}_{p,d_2}} \\ \cdots\cdots \\ \|X^{(n)} - Y^{(n)}\|_{\mathcal{E}_{p,d_n}} \end{pmatrix},$$

for all $X^{(k)}, Y^{(k)} \in \mathcal{E}_{p,d_k}$ with $X^{(k)}_0 = Y^{(k)}_0$ (a.s.), where the inequality sign $\leq$ is understood componentwise, and where the $n \times n$-matrix $\hat{\mathbf{K}}$ is given by $\hat{\mathbf{K}} = (\hat{\mathbf{k}}_{i,l})_{1\leq i,l \leq n}$. Under the assumption that T is sufficiently small, we can conclude that the spectral radius $\varrho(\hat{\mathbf{L}})$ of the matrix $\hat{\mathbf{L}} := \sqrt[p]{p}\,\sqrt[2p]{T}\,\hat{\mathbf{K}}$ is less than one. Thus, $\varrho(\hat{\mathbf{L}})$ is an eigenvalue of $\hat{\mathbf{L}}$ to which an eigenvector with strictly positive components $(e_1, ..., e_n)$ corresponds. Now we introduce the norm

$$|||X|||_{\mathcal{E}^0_{p,d}} \;:=\; \left(\sum_{k=1}^{n} e_k\|X^{(k)}\|^p_{\mathcal{E}_{p,d_k}}\right)^{1/p} \tag{25}$$

in the Banach space $\mathcal{E}^0_{p,d}$. Then the vector-valued operator $\top\!\!\top$ mapping the closed set $\mathcal{E}^0_{p,d}$ into itself is strictly contractive with the contraction constant $\varrho(\hat{\mathbf{L}})$. Consequently, the sequence generated by iterative application of operator $\top\!\!\top$ converges with respect to norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ to an unique element of $\mathcal{E}^0_{p,d}$ which is a solution of the original system (17). Since the norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ is equivalent to the original norm $\|.\|_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$, we know that the solution of system (17) also lies in the original Banach space $\mathcal{E}^0_{p,d}$.

We have seen that the operator $\mathbb{T}$ is contractive in $\mathcal{E}^0_{p,d}$ for sufficiently small T. To get the result for any T we have to divide the interval $[0,T]$ in a finite number of sufficiently small subintervals and to repeat the stated proof-steps successively. Thus, the proof is completed. $\diamond$

4.2 Convergence of waveform relaxation methods

The contractivity of operator $\mathbb{T}$ can be used to establish a theorem on the convergence of waveform relaxation methods. Analogous to Theorem 13 we have

Theorem 16 *Assume the hypotheses of Theorem 15 are valid. Define* $\hat{\mathbf{L}} = (\hat{l}_{ik})$ *by*

$$\hat{l}_{ik} := \left(p\sqrt{T} \left[\sqrt{T} \sum_{l=1}^{n} [\hat{L}^{(k)}_{l,0} + n\frac{p-1}{2} \sum_{j=1}^{m} (L^{(k)}_{l,j})^2]_+ + \sum_{j=1}^{m} \hat{C}_p \sqrt{n(\sum_{l=1}^{n}(L^{(k)}_{l,j})^2)} \right] \right)^{1/p}$$

with corresponding universal constants C_p occurring at the right hand side of the Burkholder–Davis–Gundy inequality (or substituted by estimates as in (21)).
Then $\varrho(\hat{\mathbf{L}}) < 0$ implies the convergence of the waveform relaxation algorithm based on the block Picard iterations (6) for the initial value problem (16) in the Banach space $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$ with norm $|||\cdot|||$ defined by (25), where $\mathcal{U}_k = \mathcal{E}^0_{p,d_k}$.
If we modify this algorithm with Gauss–Seidel iterations (10) applied to the initial value problem (16), then the condition $\varrho(\tilde{\hat{\mathbf{L}}}) < 0$ implies its convergence with respect to corresponding norm $|||\cdot|||$.

The proof of Theorem 16 is omitted since the conclusion can be extracted immediately from the proof of Theorem 15.

4.3 Further remarks

One could think of slight improvements in the estimation of the coefficients $\hat{k}_{ik}$ from the proof of Theorem 15 and $\hat{l}_{ik}$ from the Theorem 16. For this purpose one returns to inequalities (23) and (24), respectively. Now one can make use of the elementary inequalities

$$\sum_{i=1}^{n} c_{ik}\, x_i\, x_k^{p-1} \;\leq\; \frac{1}{p} \sum_{\substack{i=1\\ i\neq k}}^{n} c_{ik}\, x_i^p \;+\; \left(\frac{p-1}{p} \sum_{\substack{i=1\\ i\neq k}}^{n} c_{ik} + c_{kk} \right) x_k^p$$

with $p \geq 1$ and

$$\sum_{i=1}^{n} c_{ik}\, x_i^2\, x_k^{p-2} \;\leq\; \frac{2}{p} \sum_{\substack{i=1 \\ i\neq k}}^{n} c_{ik}\, x_i^p \;+\; \left(\frac{p-2}{p} \sum_{\substack{i=1 \\ i\neq k}}^{n} c_{ik} + c_{kk} \right) x_k^p$$

with $p \geq 2$, where c_{ik}, x_i, x_k are nonnegative numbers. In passing note that these inequalities are obtained by the application of the well-known Young's inequality. Let $[.]_+$ denote the nonnegative part of the inscribed expression. So one would arrive at coefficients

$$\begin{aligned}
(\hat{\mathbf{k}}_{ik})^p = \sqrt{T}\,\Big[\Big(\frac{2}{p}\Big)^{1-\delta_{i,k}} &[L_{i,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^{m}(L_{i,j}^{(k)})^2]_+ + \\
+\delta_{i,k}\frac{p-2}{2}&\sum_{\substack{l=1 \\ l\neq k}}^{n}[L_{l,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^{m}(L_{l,j}^{(k)})^2]_+\Big] + \\
+\sqrt{n}\,\Big[(1-\delta_{i,k})&\sqrt{\frac{1}{p}}\sum_{j=1}^{m}\hat{C}_{p,k,j}L_{i,j}^{(k)} + \delta_{i,k}\sum_{j=1}^{m}\hat{C}_{p,k,j}\Big(\sqrt{\frac{p-1}{p}}\sum_{\substack{l=1 \\ l\neq k}}^{n}L_{l,j}^{(k)} + L_{k,j}^{(k)}\Big)\Big]
\end{aligned}$$

occurring at

$$\|\Delta \top\!\!\top_k\|_{\mathcal{E}_{p,d_k}} \leq (p\sqrt{T})^{1/p} \sum_{i=1}^{n} \hat{\mathbf{k}}_{i,k}\, \|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}\,,$$

where $\delta_{i,k}$ represents the Kronecker symbol. However, the evaluation of this result leads to more complex expressions for the spectral radius of the matrix $\hat{\mathbf{L}} = (\hat{l}_{ik})$ with

$$\hat{l}_{ik} \;=\; (p\sqrt{T})^{1/p}\, \hat{\mathbf{k}}_{i,k}$$

controlling the convergence of the waveform iterations for SDEs with one-sided Lipschitz continuous drift part. This is the reason why we preferred to use the more elementary estimates

$$\sum_{i=1}^{n} c_{ik}\, x_i^2\, x_k^{p-2} \;\leq\; \sum_{l=1}^{n} c_{lk} \cdot \sum_{i=1}^{n} x_i^p$$

with $p \geq 2$ after the inequality (23), and

$$\sum_{i=1}^{n} c_{ik}\, x_i\, x_k^{p-1} \;\leq\; \sum_{l=1}^{n} c_{lk} \cdot \sum_{i=1}^{n} x_i^p$$

with $p \geq 1$ after the inequality (24), where $c_{ik}, x_i, x_k \geq 0$.

The assertions of Theorems 15, 16 remain valid in case $1 \le p < 2$. In that case one needs slight modifications in some estimations of corresponding proof-steps.

The crucial point in all generalizations with locally Lipschitz continuous coefficients is to find an appropriate aposteriori estimation such that the limit process $\lim_{r\to+\infty} X^r$, where $X^r = (X^{r,(1)}, ..., X^{r,(n)})^T$ represents the solution of the corresponding truncated system (22), cannot blow up (a.s) at finite times. However, generically, the solutions do not lie in the original Banach space $\mathcal{E}_{p,d}$ anymore.

As a by-product, we have shown that any solution of system (16) also possesses the property

$$\sup_{0\le t\le T} \mathbb{E}\,\|X_t\|_d^{2p} \;<\; +\infty$$

under the assumptions of Theorem 15 and with initial condition $\mathbb{E}\,\|X_0\|^{2p} < +\infty$.

Similar assertions as in Theorem 15 can be formulated under the assumptions of nonlinear Lipschitz-type conditions, like the *Osgood–Bihari-type requirement.*

Definition 17 *A drift function* $f_0 : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *is said to be* (uniformly) one-sided OB-Lipschitz-continuous *if there exist Lebesgue-measurable, piecewise in z monotone functions* $w_i : [0,T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ $(i = 1,2,...,n)$ *with* $w_i(t,z) = 0$ *for* $z = 0$ *and* $t \in [0,T]$, *and for the splitting* $f_0(t,x) = (f_{1,0}(t,x), ..., f_{k,0}(t,x), ..., f_{n,0}(t,x))^T$ *there are constants* $\hat{L}^{(i)}_{k,0} \in \mathbb{R}^1 (i,k = 1,2,...,n)$ *such that for* $t \in [0,T]$

$$(A_5) \quad \forall x = (x^{(1)}, ..., x^{(k)}, ..., x^{(n)}), y = (y^{(1)}, ..., y^{(k)}, ..., y^{(n)}) \in \mathbb{R}^{d_1} \times ... \times \mathbb{R}^{d_n}$$

$$< f_{k,0}(t, x^{(1)}, ..., x^{(n)}) - f_{k,0}(t, y^{(1)}, ..., y^{(n)}), x^{(k)} - y^{(k)} >_{d_k}$$

$$\le \sum_{i=1}^{n} \hat{L}^{(k)}_{i,0} w_i(t, \|x^{(i)} - y^{(i)}\|^2_{d_i})\,.$$

A function $f : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *is called* uniformly w-anticoercive *if there is a Lebesgue-measurable, piecewise in z monotone function* $w : [0,T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *with* $w = w(t,z)$ *satisfying*

$$(A_6) \qquad \exists c_a \in \mathbb{R}^1\ \forall t \in [0,T]\ \forall x \in \mathbb{R}^d\ : < f(t,x), x >_d \le\ c_a + w(t, \|x\|^2_d)\,.$$

Then corresponding assertions of Theorem 15 can be found and proved under the additional assumptions of OB-Lipschitz-continuous and w-anticoercive drift functions a with respect to concave and Lebesgue-integrable functions w_i, w, and Lipschitz continuous diffusion parts b^j. However, the proof is somewhat delicate and is omitted (Note that the concavity is needed to control the stochastic terms by Jensen's inequality.).

5 An illustrative example with different time scales

There are a lot of real-life processes containing several time scales. For example, a rich class is given by biochemical processes. The presence of fast and slow variables can be expressed by *singularly perturbed differential equations* of the type

$$\begin{aligned} \frac{dx}{ds} &= f(x,y,s), \\ \varepsilon\,\frac{dy}{ds} &= g(x,y,s). \end{aligned} \tag{26}$$

By introducing the fast time $t = s/\varepsilon$ we get the system

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon\, f(x,y,\varepsilon t), \\ \frac{dy}{dt} &= g(x,y,\varepsilon t). \end{aligned} \tag{27}$$

In what follows, we suppose that system (27) is randomly perturbed in its first component by a stochastic term $\sqrt{\varepsilon}h(x,y,\varepsilon t)dW_t$ where $W = (W_t)_{t\in[0,T/\varepsilon]}$ is a standard Brownian motion. The system we obtain, which is to be understood in integral sense, is represented in the form

$$\begin{aligned} dX_t &= \varepsilon f(X_t, Y_t, \varepsilon t)\,dt + \sqrt{\varepsilon}h(X_t, Y_t, \varepsilon t)\,dW_t, \\ dY_t &= g(X_t, Y_t, \varepsilon t)\,dt\,. \end{aligned} \tag{28}$$

The singularly perturbed differential equations (28) with their naturally inherited splitting into slowly and fastly varying components form a suitable class for an application of waveform iteration techniques. The (stochastic) waveform iteration technique can be applied to approximate the solution of the initial value problem to (28) as follows. First, fix some initial guess $X_t^{(0)}$ for X_t, e.g. $X_t^{(0)} = X_0$. Second, we compute an approximation for the solution $Y = (Y_t)_{t\in[0,T/\varepsilon]}$ of the initial value problem of

$$dY_t^{(k)} \;=\; g(X_t^{(k-1)}, Y_t^{(k)}, \varepsilon t)\,dt$$

while freezing the first component, for example, pathwise by deterministic numerical methods. Afterwards, by plugging the obtained result $Y_t^{(k)}$ into the first equation one solves the system

$$dX_t^{(k)} \;=\; \varepsilon f(X_t^{(k)}, Y_t^{(k)}, \varepsilon t)\,dt \;+\; \sqrt{\varepsilon}h(X_t^{(k)}, Y_t^{(k)}, \varepsilon t)\,dW_t$$

by stochastic-numerical methods. This procedure will be repeated iteratively until a required accuracy has been reached.

To guarantee the convergence of the waveform algorithm applied to systems (28) one has to check the spectral radius criterion of corresponding matrix of Lipschitz-coefficients. Concerning the functions f, g and h, we assume that they are continuous and globally Lipschitz continuous in x and y uniformly with respect to t, i.e.

$$\begin{aligned} \|f(x,y,t) - f(\bar{x},\bar{y},t)\|_1 &\leq L^1_{1,0}\|x-\bar{x}\|_1 + L^1_{2,0}\|y-\bar{y}\|_2, \\ \|g(x,y,t) - g(\bar{x},\bar{y},t)\|_2 &\leq L^2_{1,0}\|x-\bar{x}\|_1 + L^2_{2,0}\|y-\bar{y}\|_2, \\ \|h(x,y,t) - h(\bar{x},\bar{y},t)\|_1 &\leq L^1_{1,1}\|x-\bar{x}\|_1 + L^1_{2,1}\|y-\bar{y}\|_2 \end{aligned} \tag{29}$$

for all $x, \bar{x} \in \mathbb{R}^{d_1}, y, \bar{y} \in \mathbb{R}^{d_2}, t \in [0,T]$, where $\|.\|_i$ represents the Euclidean norm in $\mathbb{R}^{d_i}$. Taking into account $L^2_{1,1} = L^2_{2,1} = 0$ we arrive at the 2×2 matrix $\mathbf{L} = (l_{i,k})$ of Lipschitz-constants

$$\mathbf{L} \;=\; 4^{(p-1)/p}\sqrt{T}\begin{pmatrix} (\varepsilon\sqrt{T}L^1_{1,0} + \sqrt{\varepsilon}\,C_p^{1/p}L^1_{1,1}) & (\varepsilon\sqrt{T}L^1_{2,0} + \sqrt{\varepsilon}\,C_p^{1/p}L^1_{2,1}) \\ \sqrt{T}L^2_{1,0} & \sqrt{T}L^2_{2,0} \end{pmatrix} \tag{30}$$

as found at the end of the proof of Theorem 9. Recall that the constant C_p arises as the constant on the right side of the well-known Burkholder–Davis–Gundy inequality and can be replaced by any of their majorants, e.g.

$$\tilde{C}_p = C_p{}^{\frac{1}{p}} \leq \sqrt{\left(\frac{p}{p-1}\right)^p\left(\frac{p(p-1)}{2}\right)}$$

where $p \geq 1$. Finally, the condition $\varrho(\mathbf{L}) < 1$ on the spectral radius $\varrho(\mathbf{L})$ controls the convergence of corresponding Picard iterations. Correspondingly, the condition $\varrho(\tilde{\mathbf{L}}) < 1$ on the spectral radius $\varrho(\tilde{\mathbf{L}})$ of matrix

$$\tilde{\mathbf{L}} \;=\; \begin{pmatrix} l_{11} & l_{12} \\ l_{21}l_{11} & l_{21}l_{12} + l_{22} \end{pmatrix}$$

guarantees the convergence of the waveform method based on Gauss–Seidel iteration (cf. matrix $\tilde{\mathbf{L}}$ belonging to (12)).

We omit more detailed numerical experiments here, and leave them to the future. In passing we note that the obtained random model is quite natural due to uncertainties of modeling and random environmental behavior which may result from the nature of random vibrations. The form of factor $\sqrt{\varepsilon}$ in the random terms can be justified by physical arguments (use Dissipation-Fluctuation Theorem of Theoretical Physics) and understood best with one-sided Lipschitz continuous and anticoercive drift terms.

6 Conclusions and remarks

This paper is an continuation of our works [35] - [41] concerning the approximation of the solution of initial value problems for systems of explicit differential equations. Here, we extended the standard idea of waveform iteration method to nonlinear stochastic differential equations (SDEs) driven by standard Wiener processes. It turns out that the Lipschitz-continuity of the coefficients of SDEs is crucial to establish the convergence of waveform relaxation methods. In particular, the Lipschitz-coefficients determine the length of integration intervals to which the waveform iterations are applied (windowing techniques).

Waveform iteration methods provide an alternative approach to approximating the solution of a system of stochastic differential equations. Compared with the traditional time-incremental methods as described in [20], [29] or [43], the waveform relaxation technique forms a global iteration scheme on a given time interval. Its efficiency depends on an appropriate decomposition of the large original system into weakly interacting subsystems. These methods are particularly designed to treat very large scale systems by parallel computations.

We have presented only some first theoretical foundations of waveform relaxation methods applied to systems of SDEs. Many directions of further investigation are possible. For example, waveform relaxation methods for some classes of stochastic partial differential equations or differential-algebraic systems where we expect more complicated expressions for the control of waveform iterations. An extended testing of computational efficiency, windowing techniques and practical implementation of waveform relaxation methods applied to SDEs should follow our considerations. We have not touched either the questions of numerical stability and contractivity of arising stochastic algorithms (for a recent monograph on basic aspects of stochastic-numerical stability theory, see [39]) nor their convergence along given functionals. Any extensions of deterministic numerical algorithms and approximations have to be done according to the main principles of approximation theory of stochastic processes on Banach spaces (e.g. see [41]). For a more recent survey on numerical analysis of (ordinary) stochastic differential equations, see [40].

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