# THE 3D QUASIGEOSTROPHIC EQUATION UNDER RANDOM PERTURBATION

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ABSTRACT. The three-dimensional baroclinic quasigeostrophic flow model has been widely used to study basic mechanisms in oceanic flows and climate dynamics. In this paper, we consider this flow model under random wind forcing and time-periodic fluctuations on fluid boundary (the interface between the oceans and the atmosphere). The time-periodic fluctuations are due to periodic rotation of the earth and thus periodic exposure of the earth to the solar radiation. After establishing the well-posedness of the baroclinic quasigeostrophic flow model in the state space, we demonstrate the existence of the random attractors, again in the state space. We also discuss the relevance of our result to climate modeling.

## 1. INTRODUCTION

In the following we will study the quasigeostrophic equation (QGE) for the ocean in a three dimensional domain. It has been formally derived as an approximation of the rotating three-dimensional primitive equations [15]. Bourgeois and Beale [3], Embid and Majda [9], and Desjardins and Grenier [5] have recently shown that the three-dimensional quasigeostrophic equation is a valid approximation of the primitive equations in the limit of zero Rossby number. Holm [12] has established the Hamiltonian formulation for the inviscid quasigeostrophic equation. In particular, we will consider the following version [25, 7] of the equation in terms of the stream function $\psi(x, y, z, t)$:

$$\tilde{\Delta}\psi_t + J(\psi, \tilde{\Delta}\psi) + \beta\psi_x = \nu\tilde{\Delta}\tilde{\Delta}\psi$$

$$\tilde{\Delta} = \psi_{xx} + \psi_{yy} + \left(\frac{f_0^2}{N(z)}\psi_z\right)_z$$

Here $x$, $y$, $z$ are Cartesian coordinates in zonal (east), meridional (north), vertical directions, respectively; $f_0 + \beta y$ (with $f_0, \beta$ constants) is the Coriolis parameter; $N(z) > 0$, $N(z)_z \neq 0$ is the Brunt-Vaisala stratification frequency, and $\nu > 0$ is viscosity. Moreover, $J(f, g) = f_x g_y - f_y g_x$ is the Jacobi operator and potential vorticity is defined as $\tilde{\Delta}\psi + f_0 + \beta y$. Note that $\tilde{\Delta}\psi$ can be regarded as a modified Laplacian operator where the coefficient in the vertical $z$ direction is adjusted due to the density stratification, and the coefficients in $x, y$ directions are constants due to the horizontal density homogenity in the 3D quasigeostrophic flow model formulation. Bennett and Kloeden [6] have also used a similar modified Laplacian

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viscous term in a more complicated 3D quasigeostrophic flow model involving thermodynamics as well as hydrodynamics.
Our aim is to study the potential verticity evolution in an ocean under the influence of the atmosphere. Following an idea of Hasselmann [11] one can divide the geophysical or meteorological flow into two parts. These two parts are the slowly changing climate part and rapidly changing weather part. The weather part can be modeled by a stochastic process such as white noise, see Hasselmann [11], Arnold [2] and Saltzman [20]. Oceanic flows are affected (on the ocean surface) by these short time influences due to weather variations which are usually called wind forcing. Moreover, oceanic flows are also affected by climatic variations due to periodic rotation of the earth and thus periodic exposure of the earth to the solar radiation; see [16], Chapter 6 and [13], Chapter 11.
Since the exchange between the atmosphere and an ocean takes place at the surface of the ocean, we will consider the above quasigeostrophic partial differential equation with white noise Neumann boundary condition on the top surface of the ocean [15]. We will also add some time-periodic boundary condition on the ocean surface due to periodic rotation of the earth and the solar radiation. Since there is no influence of the weather at the bottom of the ocean we will have the homogeneous boundary condition there. The boundary conditions in horizontal directions are assumed to be periodic for mathematical convenience as in other recent works [9, 3]. However, since we have a forth order differential operator at the right hand side of the quasigeostrophic equation, we need a second group of boundary conditions, say, for $\tilde{\Delta}\psi$, and these conditions are the ones used in [25], [7] to be specified below. For simplicity we will suppose that this second group of boundary conditions are deterministic. But generalizations are possible.

Our aim is to find structures in the dynamics of the QGE. As we have time dependent random and time-periodic boundary conditions, we will obtain a nonautonomous dynamical system with random influences. We will show how to find attractors for such a dynamical system. The existence and interpretation of climatic attractors have been controversial and have caused a lot of debate [14]. A low dimensional climatic attractor was regarded as an indication that the main feature of long-time climatic evolution may be viewed as the manifestation of a *deterministic* dynamics. Our result is about *random* attractors, and thus the long time regimes that such attractors may represent still carry the stochastic information of the geophysical flow system.
The QGE can be transformed into an evolution equation with standard boundary conditions. For this transformation we need an Ornstein-Uhlenbeck process fulfilling our dynamical random or time-periodic boundary conditions. This transformation will be introduced in section 2.
In the third section we investigate the coefficients of the transformed evolution equation, and further obtain a global existence and uniqueness result and some regularity result.
In the forth section, we study the random dynamics of the transformed QGE. Based on the uniqueness result above, the transformed evolution equation generates a nonautonomous dynamical system. In addition, if we restrict this system to discrete time step of the period of the periodic rotation of the earth, we obtain a random dynamical system. This random dynamical system has a random attractor. This result can be extended to the dynamical system on the real time axis.

The fifth section contains the proofs. Finally, we summarize our results in the sixth section.

## 2. The 3D quasigeostropic equation

Let $O=(0,2\pi)^3$ be the cube which is a model for a piece of the ocean. For $x,\,y,\,z\in(0,2\pi)$ and smooth functions $u(x,y,z)$, $v(x,y,z)$ we define the Jacobi operator

$$J(u,v)=u_xv_y-u_yv_x.$$

In addition, the differential operator $\tilde\Delta$ is defined by

$$\psi_{xx}+\psi_{yy}+\left(\frac{f_0^2}{N(z)}\psi_z\right)_z$$

for $f_0\in\mathbb{R}$ and $N(z)$ is defined to be a positive $C^1$-smooth function $N(z)>0$, $N(z)_z\neq 0$ on $[0,2\pi]$. Let $\nu$, $\beta$ be positive constants. In the following we investigate the 3D QGE flow model [15, 19, 25]:

$$\tilde\Delta\psi_t+J(\psi,\tilde\Delta\psi)+\beta\psi_x=\nu\tilde\Delta\tilde\Delta\psi. \tag{1}$$

We impose the following boundary conditions for this equation. Let $O_{\cdot,\cdot,0}$, $O_{\cdot,\cdot,2\pi},\cdots$ be the faces of the cube $O$ then we have periodic boundary conditions in $x$, $y$ direction

$$\tilde\psi|_{O_{0,\cdot,\cdot}}=\tilde\psi|_{O_{2\pi,\cdot,\cdot}}\qquad \tilde\psi_x|_{O_{0,\cdot,\cdot}}=\tilde\psi_z|_{O_{2\pi,\cdot,\cdot}}$$

and similar for the faces $O_{\cdot,0,\cdot}$, $O_{\cdot,2\pi,\cdot}$ with derivative in $y$ direction. With respect to the $z$ direction $\psi$ has to fulfill Neumann boundary conditions; see [15], p.367. Namely, for $(x,y)\in O_{\cdot,\cdot,0}$ we impose that

$$\frac{\partial\psi}{\partial n}=\psi_z=0$$

and

$$\frac{\partial\psi}{\partial n}=\psi_z=\dot W\quad\text{for }(x,y)\in O_{\cdot,\cdot,2\pi}.$$

where $\dot W$ is a temporal white noise and $n$ denotes the outer normal.
Moreover, $\tilde\Delta\psi$ is supposed to be periodic in $x,y$ directions:

$$\tilde\Delta\psi|_{O_{0,\cdot,\cdot}}=\tilde\Delta\psi|_{O_{2\pi,\cdot,\cdot}}\qquad \tilde\Delta\psi_x|_{O_{0,\cdot,\cdot}}=\tilde\Delta\psi_z|_{O_{2\pi,\cdot,\cdot}}$$

and similar for the faces $O_{\cdot,0,\cdot}$, $O_{\cdot,2\pi,\cdot}$ with derivative in $y$ direction. For the other faces on the top and the bottom of the ocean, we impose homogeneous Neumann boundary conditions [25, 7, 6]

$$\tilde\Delta\psi_z|_{O_{\cdot,\cdot,0}}=0,\qquad \tilde\Delta\psi_z|_{O_{\cdot,\cdot,2\pi}}=0$$

In addition, we impose that

$$\int_O\psi dO=0,$$

and

$$\int_O\tilde\Delta\psi dO=0.$$

We also assume an appropriate initial condition

$$\psi(x,y,z,0)=\psi_0(x,y,z).$$

Later on we will see that we can find an Ornstein-Uhlenbeck process $\eta$ fulfilling the linear differential equation

$$\eta_t = \nu\tilde{\Delta}\eta \tag{2}$$

where $\eta$ fulfills the same boundary conditions as $\psi$ in some sense.
We now transform formally the above equation to another parabolic differential equation with random coefficients but with standard boundary conditions. To do this we have to introduce the solution operator $G$ of the following elliptic problem:

$$\tilde{\Delta}\xi = f \tag{3}$$

on $O$ with periodic boundary conditions in $x, y$ directions and *homogeneous* Neumann boundary conditions in $z$ direction. For existence and regularity properties of this operator see below. $G$ can be considered as a linear continuous operator. Here $f$ is an element in a particular Hilbert space.
For the difference of $\psi$ and $\eta$ we obtain for (1)

$$\begin{aligned}(\tilde{\Delta}\psi - \tilde{\Delta}\eta)_t \quad &+ \quad J(\psi-\eta, \tilde{\Delta}\psi - \tilde{\Delta}\eta) + J(\eta, \tilde{\Delta}\psi - \tilde{\Delta}\eta) + J(\psi-\eta, \tilde{\Delta}\eta) + J(\eta, \tilde{\Delta}\eta) \\ &+ \quad \beta(\psi-\eta)_x + \beta\eta_x = \nu\tilde{\Delta}\tilde{\Delta}(\psi-\eta)\end{aligned}$$

where $\eta$ is assumed to be sufficiently regular. The difference $\psi - \eta$ satisfies the same boundary condition as $\xi$ in (3). Hence by the formal substitution $v = \tilde{\Delta}\psi - \tilde{\Delta}\eta$ we can write

$$\begin{aligned}&v_t + J(G(v), v) + J(\eta, v) + J(G(v), \tilde{\Delta}\eta) + J(\eta, \tilde{\Delta}\eta) \\ &\quad + \beta G(v)_x + \beta\eta_x = \nu\tilde{\Delta}v.\end{aligned}$$

The problem (2) satisfies only standard boundary conditions with respect to $\eta$ but not with respect to $\tilde{\Delta}\eta$. To obtain an equation with standard boundary conditions we need the transformation $u = v + \tilde{\Delta}\eta$ assuming formally that $\eta$ is sufficiently regular. Then $u$ fulfills periodic boundary conditions in $x, y$ directions, and satisfies the following random evolution:

$$u_t + J(G(u), u) + J(\eta - G(\tilde{\Delta}\eta), u) + \beta G(u)_x - \beta(G(\tilde{\Delta}\eta))_x - \eta_x) = \nu\tilde{\Delta}u. \tag{4}$$

This equation will be treated in the rest of the paper.

**Remark 2.1.** To obtain a solution of the original equation all reverse transformations make sense if $\eta$ is contained in a Sobolev space of second order. Hence we can find a solution of this equation contained in this Sobolev space. Thus, in the following we will formulate our results for this transformed random evolution equation (4).

## 3. Existence and uniqueness of the solution of the QGE

In this section we consider the well-posedness of the transformed QGE (4). We intend to consider equation (4) as a evolution equation on a rigged space $V \subset H \subset V'$. Since this equation is given with respect to periodic boundary conditions in horizontal direction the space $H$ is defined to be

$$\dot{L}_2(O) = \{u \in L_2(O), \int_O u dO = 0\}.$$

The inner product on $H$ is denoted by $(\cdot, \cdot)$. Let $H^k$, $k \in \mathbb{N}$ be the usual Sobolev space consisting of functions with square integrable derivative up to second order.

If $k$ is not an integer these spaces are the usual Slobodeckij spaces. For $V \subset H^1 \cap H$ we choose the space of functions such that

$$u|_{Q_{0,\cdot,\cdot}} = u|_{Q_{2\pi,\cdot,\cdot}}, \qquad u|_{Q_{\cdot,0,\cdot}} = u|_{Q_{\cdot,2\pi,\cdot}}.$$

Note that for functions in $H^1$ the trace on the boundary is well defined. This set will be equipped with the usual $H^1$ inner product denoted by $(\cdot,\cdot)_V$. Let $A$ be a linear bounded operator

$$A : V \to V'$$

which is the usual operator stemming from the bilinear form $a(u,v)$ defined by $-\tilde{\Delta}$ with periodic boundary conditions in $x-y$ direction and homogeneous Neumann boundary condition in $z$ direction. Note that $-\tilde{\Delta}$ is symmetric with respect to these boundary conditions. We have that

$$\langle Au, u\rangle = \|u\|_V^2.$$

$\langle\cdot,\cdot\rangle$ denotes the dual pairing between $V'$ and $V$. The operator $A$ is an isomorphism from $V$ to $V'$. Since $V = D(A^{\frac{1}{2}})$ is compactly embedded in $H$ we can define $G(f) = A^{-1}f$ which is a continuous operator for instance from $H$ to $D(A)$ or more generally from $D(A^s)$ to $D(A^{s+1})$, $s \in \mathbb{R}$. For the definition of the spaces $D(A^s)$ and their norms $\|\cdot\|_{D(A^s)}$ see Temam [24] Section II.2. In particular, for $f \in H$ the function $G(f)$ is periodic in $x, y$ directions.
On $H$ the operator $A$ has the spectrum $0 < \lambda_1 \le \lambda_2 \le \cdots$. The associated eigenfunctions are denoted by $e_1, e_2, \cdots$.
The embedding constant between $V$ and $H$ is given by $\lambda_1$:

$$\lambda_1\|u\|_H^2 \le \|u\|_V^2.$$

We now investigate the properties of the operator $J$ which defines the nonlinearity of (4).

**Lemma 3.1.** *Suppose that $m_1$, $m_2$, $m_3$ are three nonnegative numbers less than $\frac{3}{2}$ with*

$$\sum_{i=1}^{3} m_i \ge \frac{3}{2}.$$

*Then there exists a constant $c_1 > 0$ such that for $u \in H^{m_1+1}$, $v \in H^{m_2+1}$ and $w \in H^{m_3}$*

$$|\langle J(u,v), w\rangle| \le c_1\|u\|_{H^{m_1+1}}\|v\|_{H^{m_2+1}}\|w\|_{H^{m_3}}.$$

*Proof.* On account of the embedding theorem $H^m \subset L_q(\mathcal{O})$ where $\frac{1}{q} = \frac{1}{2} - \frac{m}{2}$ for $q > 1$, $m \ge 0$, $m \ne \frac{3}{2}$, and the Hölder inequality, we get

$$|\langle J(u,v), w\rangle| \le \|\nabla u\|_{L_{q_1}}\|\nabla v\|_{L_{q_2}}\|u\|_{L_{q_3}}, \qquad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \le 1.$$

The embedding theorem gives the conclusion. For the idea of the proof see Temam [23], Section 2.3. □

**Remark 3.2.** Suppose two of the $m_i$'s in the last lemma, say $m_2$, $m_3$, have the value zero. Then if $m_1$ has is chosen bigger than $\frac{3}{2}$ the conclusion of the last lemma remains true. This follows if we apply at first the Sobolev lemma and then the Cauchy- Schwarz inequality.

From Lemma 3.1 we can derive some a priori estimates for the nonlinearity of (4).

**Corollary 3.3.** *Suppose that* $u \in D(A^\alpha)$, $\alpha > \frac{3}{4}$ *and* $v,\, w \in V$. *Then*

$$|\langle J(u,v),w\rangle| \le c_1\|u\|_{D(A^\alpha)}\|v\|_V\|w\|_V,$$

*and hence for* $u \in H$, $v,\, w \in V$

$$|\langle J(G(u),v),w\rangle| \le c_1\|u\|_H\|v\|_V\|w\|_V.$$

*Suppose now that* $u \in D(A^\alpha)$, $\alpha > \frac{5}{4}$ *and* $v \in V$, $w \in H$. *Then*

$$|\langle J(u,v),w\rangle| \le c_1\|u\|_{D(A^\alpha)}\|v\|_V\|w\|_H,$$

*and hence for* $u \in D(A^\alpha)$, $\alpha > \frac{1}{4}$ *and* $v \in V$, $w \in H$

$$|\langle J(G(u),v),w\rangle| \le c_1\|u\|_{D(A^\alpha)}\|v\|_V\|w\|_H.$$

*Furthermore, if* $u \in H^2$ *and* $v,\, w \in V$ *we have the inequality*

$$|\langle J(u,v),w\rangle| \le c_1\|u\|_{H^2}\|v\|_V\|w\|_V.$$

Based on this Corollary, we also get

**Lemma 3.4.** *For any* $\mu > 0$ *there exists a constant* $c_{2,\mu}$ *such that for* $v \in H$ *and* $u \in H^2$

$$|\langle J(u),v),Av)\rangle| \le c_{2,\mu}\|u\|_{H^2}^2\|v\|_V^2 + \mu\|Av\|^2.$$

Later on we can use this lemma to drive additional regularity properties of (4). Also on account of Lemma 3.1 we obtain the following algebraic properties of $J$.

**Lemma 3.5.** *If* $v,\, w \in V$ *and* $u \in H^2$ *and periodic in* $x,\, y$ *directions, then we have*

$$\langle J(u,v),w\rangle = -\langle J(u,w),v\rangle.$$

*Hence we have* $\langle J(u,v),v\rangle = 0$.

*Proof.* We choose $u$ from a set of sufficiently smooth functions which is dense in $D(A^\alpha)$. Integration by parts formulae yields

$$\begin{aligned}
&\int_0^{2\pi}\int_{O_{\cdot,\cdot,z}} u_x v_y w\,dxdydz - \int_0^{2\pi}\int_{O_{\cdot,\cdot,z}} u_y v_x w\,dxdydz \\
&\quad = -\int_0^{2\pi}\int_{O_{\cdot,\cdot,z}} u_{xy} v w\,dxdydz + \int_0^{2\pi}\int_{O_{\cdot,\cdot,z}} u_{yx} v w\,dxdydz \\
&\quad - \int_0^{2\pi}\int_{O_{\cdot,\cdot,z}} u_x v w_y\,dxdydz + \int_0^{2\pi}\int_{O_{\cdot,\cdot,z}} u_y v w_x\,dxdydz \\
&\quad + \int_0^{2\pi}\int_0^{2\pi} u_x v w\Big|_{y=0}^{y=2\pi} dxdz - \int_0^{2\pi}\int_0^{2\pi} u_y v w\Big|_{x=0}^{x=2\pi} dydz.
\end{aligned}$$

Note that the two boundary terms are zero. Indeed, since $u(x,0,z) = u(x,2\pi,z)$, we know that $u_x(x,0,z) = u_x(x,2\pi,z)$, and since $v$, $w$ are $2\pi$–periodic with respect to $x$. By the smoothness assumption we can suppose that the derivatives on the boundary are well defined. Thus the first boundary term is zero. Similarly, the second boundary term is also zero. The second claim is due to antisymmetric property of the first claim of this corollary. □

The following lemma will be used to obtain the continuity of the solution operator of (4).

**Lemma 3.6.** *There exists a constant $c_3 > 0$ such that for $u_1$, $u_2 \in V$*

$$|\langle J(G(u_1), u_1) - J(G(u_2), u_2), u_1 - u_2\rangle| \leq c_3\|u_1 - u_2\|_V\|u_1 - u_2\|_H\|u_1\|_V.$$

*Proof.* By Lemma 3.5 the left hand side above expression is equal to

$$\begin{aligned}|\langle J(G(u_1), u_1), u_2\rangle + \langle J(G(u_2), u_2), u_1\rangle| \\ &= |\langle J(G(u_1), u_1), u_2 - u_1\rangle - \langle J(G(u_2), u_1), u_2 - u_1\rangle| \\ &= |\langle J(G(u_1) - G(u_2), u_1), u_1 - u_2\rangle|.\end{aligned}$$

Corollary 3.3 gives the conclusion. □

The equation (4) can be considered as an evolution equation on the rigged space $V \subset H \subset V'$ introduced at the beginning of this section.
We now explain the properties of coefficients which are contained in (4). Due to Corollary 3.3 we have a bilinear continuous operator

$$B(\cdot,\cdot) \to V \times V \to H, \quad B(u, v) := J(G(u), v).$$

In addition, we have a time dependent linear operator

$$C(t, u) := J(\eta(t) - G(\tilde{\Delta}\eta(t)), u), \quad C(t) : V \to V'.$$

**Lemma 3.7.** *Suppose that $\eta(\cdot) \in L_{2,loc}(0, \infty; H^2)$. and $\eta$ is periodic in x,y directions. Then we have for almost any $t \geq 0$:*

$$\|C(t, \cdot)\|_{V,V'} \leq c_4\|\eta(t)\|_{H^2}, \qquad \langle C(t, u), u\rangle = 0.$$

Indeed, because $\eta(t) - G(\Delta\eta(t)) \in H^2$ for any $t \geq 0$ we obtain the first conclusion by Corollary 3.3. The second Conclusion follows from Lemma 3.5.

We now investigate the last linear operator appearing in (4) which is defined by

$$D(u) = \beta G(u)_x : V \to H.$$

We obtain straightforwardly $\|D(\cdot)\|_{V,H} \leq c_5$.

**Lemma 3.8.** *For $u \in V$ we have*

$$(D(u), u) = 0$$

*Proof.* Denoting $G(u) \in D(A)$ by $\xi$ we have

$$\begin{aligned}\beta^{-1}(D(u), u) =&(\xi_x, \tilde{\Delta}\xi) = \frac{1}{2}\bigg(\int_O (\xi_x^2)_x dxdydz - (\xi_y^2)_x dxdydz \\ &- \int_0^{2\pi}\int_0^{2\pi}\frac{f_0^2}{N(z)}\int_0^{2\pi}(\xi_z^2)_x dxdydz\bigg)\end{aligned}$$

via the integration by parts. The second term under the integral generates the boundary term

$$\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\xi_x\xi_y\Big|_{y=0}^{y=2\pi}dxdz.$$

Indeed, for sufficiently smooth $\xi$ from a dense set in $V$ we have $\xi(x, 0, z) = \xi(x, 2\pi, z)$ and thus $\xi_x(x, 0, z) = \xi_x(x, 2\pi, z)$. It follows from the periodicity in $y$ we have $\xi_y(x, 0, z) = \xi_y(x, 2\pi, z)$ such that this boundary term is zero.
For the last term the following boundary term appears

$$\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\frac{f_0^2}{N(z)}\xi_x\xi_z\Big|_{z=0}^{z=2\pi}dxdy$$

which is zero by the homogeneous Neumann boundary conditions. This is fulfilled for $G(u) = \xi$. Integration with respect to $x$ and using the periodicity as in the proof of Lemma 3.1, we get the conclusion. □

We can formulate the evolution equation (4) on $V \subset H \subset V'$

$$u_t + \nu Au + B(u,u) + C(t,u) + D(u) = f(t), \quad u(0) = u_0 \in H \tag{5}$$

for $f(t) = \beta(G(\Delta\eta(t))_x - \eta(t)_x)$ which is contained in $L_{2,loc}(0,\infty,H)$.
Apart from the linear operators $C(t)$ and $D$ equation (5) has the form of equations of 2D Navier -Stokes type; see Temam [22], Chapter 3, for which we have existence and uniqueness. Here *2D* means that the conclusion of Lemma 3.6 is fulfilled which is responsible for a uniqueness theorem. By the regularity properties of the operators $B$, $C(t)$ and $D$ the same method as for the 2D Navier-Stokes equations ensures existence and uniqueness for (5). Thus we get the following main result in this section, about the well-posedness for 3D quasigeostrophic flows under random wind forcing on ocean surface.

**Theorem 3.9.** *(Well-posedness) Suppose that $\eta \in L_{2,loc}(0,\infty;H^2)$. Then the 3D quasigeostrophic equation (5) or (4) has a unique solution $u(t) \in L_{2,loc}(0,\infty;V) \cap C([0,\infty];H)$ for any $T \geq 0$. In addition, for any $t > 0$ the solution $u(t) \in V$.*

**Remark 3.10.** i) On account of Lemma 3.6 we can prove that $u(t)$ depends continuously on the initial conditions $x \in H$. This will be used later on.
ii) Due to Lemma 3.4 for $t > 0$ and a bounded set of initial conditions in $H$ the image with respect to the solution operator $u_0 \to u(t)$ is bounded in $V$. Hence the solution operator is compact for $t > 0$.

## 4. The dynamical system of 3D quasigeostrophic flows

In this section we study the dynamical behavior of QGE (5). In the following we are going to describe the background perturbations defined on the boundary of $O$ which will influence the dynamical system generated by (5). We will have two different influences. The first perturbation is a white noise which models the weather or the small scale impact of the atmospheric motion through wind forcing on the surface. The other one is a periodic motion which serves as a model for the impact due to periodic rotation of the earth and thus periodic exposure of the earth to the solar radiation; see [16], Chapter 6 and [13], Chapter 11.
In the first part of this section we are going to explain a dynamical model of the boundary conditions.
We consider the elliptic differential equation

$$\tilde{\Delta}u = 0, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \mathcal{O}_{\cdot,\cdot,0} \quad \frac{\partial u}{\partial n} = f \in H^{\frac{1}{2}}(O_{\cdot,\cdot,2\pi}) \cap \dot{L}_2(O_{\cdot,\cdot,2\pi}).$$

On the other faces we have periodic boundary conditions. This equation has a unique solution; see Egorov and Shubin [8] Page 130f. The solution operator is a linear continuous operator with the image in $H^2$. We denote this operator by $\tilde{G}(f)$. $H^{\frac{1}{2}}$ denotes a usual boundary space.

At first we consider the random part of the boundary conditions. Let $W$ be a continuous temporal Wiener process with values in a Hilbert space $U \subset \dot{L}_2(O_{\cdot,\cdot,2\pi})$ of sufficiently regular functions. For instance, we can take $U = H^{\frac{1}{2}}(O_{\cdot,\cdot,2\pi}) \cap \dot{L}_2(O_{\cdot,\cdot,2\pi})$ This Wiener process is defined for positive and negative times; see Arnold [1] Page

547. The covariance operator of $W$ is denoted by $Q$, which is a positive symmetric linear operator on $U$. The dynamics of $W$ is given by the *metric dynamical system* consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a flow $\theta$, $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ where $\mathbb{P}$ is the Wiener measure with covariance $Q$ and $\theta = (\theta_t)_{t\in\mathbb{R}}$ is the flow of the Wiener shift:

$$W(\cdot, \theta_t\omega) = W(\cdot + t, \omega) - W(t, \omega) \quad \text{for } t \in \mathbb{R}.$$

The mapping $\theta$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$–measurable and fulfills the property

$$\theta_{t+\tau} = \theta_t \circ \theta_\tau, \qquad t,\, \tau \in \mathbb{R}. \tag{6}$$

For instance, we can choose $\Omega$ to be the set of continuous functions $C_0(\mathbb{R}, U)$ which are zero at zero and for $\mathcal{F}$ we choose the Borel-$\sigma$-algebra of $C_0(\mathbb{R}, U)$. Note that $\mathbb{P}$ is ergodic with respect to $\theta$.

We now show the existence of a solution of (2) satisfying particular properties. At first we show that the following problem has a *stationary* solution:

$$\eta_{1\,t} = \tilde{\Delta}\eta_1, \quad \frac{\partial \eta_1}{\partial n} = 0, \quad (x,y) \in O_{\cdot,\cdot,0}, \quad \frac{\partial \eta_1}{\partial n} = \dot{W}(t), \quad (x,y) \in O_{\cdot,\cdot,2\pi} \tag{7}$$

under periodic boundary conditions with respect to the other faces of the cube $\mathcal{O}$. This solution will serve as a process which compensates the nonhomogeneous boundary conditions in (4). A similar problem has been considered in Da Prato and Zabczyk [18] Chapter 13 or [17].

In contrast to (7) we consider boundary conditions which are defined to be *homogeneous* Neumann boundary on $O_{\cdot,\cdot,2\pi}$. The solution of this differential equation generates a semigroup $(S(t))_{t\ge 0}$. This semigroup has a strongly continuous generator, which will be denoted also by $-A$ where $A$ is equivalent to the operator introduced in the last section.

The following lemma allows us to define a stationary Ornstein-Uhlenbeck process which fulfills particular regularity assumptions.

**Lemma 4.1.** *Let $W$ be the Wiener process introduced above such that*

$$\int_0^t \|AS(\tau)\tilde{G}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(U,V)} d\tau < \infty.$$

*In this formula $\mathcal{L}_2$ denotes the space of Hilbert-Schmidt operators. Then there exists a continuous stationary solution written as*

$$t \to \eta_1(\theta_t\omega),$$

*where $\eta_1$ is a random variable with values in $\in V$.*

*Proof.* As in DaPrato and Zabczyk [18] Chapter 13,

$$t \to A\int_0^t S(t-\tau)d\tilde{G}(W(\tau,\omega)) =: \eta_1(t,\omega), \quad t \ge 0$$

is a continuous solution in $V$ of (7) with initial condition zero. Since $\tilde{G}$ is a linear bounded operator $\tilde{G}(W)$ is a Wiener process. We have to show that the random variable $\eta_1$ which determines the stationary solution is given by

$$\eta_1(\omega) = (L_2)\lim_{t\to\infty} \eta_1(t, \theta_{-t}\omega) = A\int_{-\infty}^0 S(-\tau)d\tilde{G}(W(\tau,\omega)).$$

By a simple integral transformation we can conclude that

$$\eta_1(t, \theta_{-t}\omega) = A\int_{-t}^0 S(-\tau)d\tilde{G}(W(\tau,\omega)).$$

The Fourier method with respect to the base $(e_j)$ applied to expand $S(t)$ yields $\|S(t)\|^2_{H,H} \le e^{-2\lambda_1 t}$. We obtain for $0 < t_1 < t_2$:

$$\begin{aligned}
\mathbb{E}\|\eta(t_2,\theta_{-t_2}\omega) - \eta(t_1,\theta_{-t_1}\omega)\|_V^2 &= \mathbb{E}\left\| A\int_{-t_2+t_1}^0 S(t_1)S(-\tau)d\tilde{G}(W(\tau,\omega))\right\|_V^2 \\
&\le e^{-2\lambda_1 t_1}\int_0^{t_2-t_1} \|AS(\tau)\tilde{G}Q^{\frac12}\|^2_{\mathcal{L}_2(U,V)}d\tau \\
&\le e^{-2\lambda_1 t_1}\int_0^{\infty} \|AS(\tau)\tilde{G}Q^{\frac12}\|^2_{\mathcal{L}_2(U,V)}d\tau \\
&\le e^{-2\lambda_1 t}\int_0^{\infty} e^{-2\lambda_1\tau}\|A\tilde{G}Q^{\frac12}\|^2_{\mathcal{L}_2(U,V)}d\tau \le Ce^{-2\lambda_1 t_1}
\end{aligned}$$

which ensures the above convergence. Consequently, we have

$$\eta_1(\omega) = A\int_{-\infty}^0 S(-\tau)d\tilde{G}(W(\tau,\omega)).$$

To see that $t \to \eta_1(\theta_t\omega)$ solves (7) we consider the equation (7) with initial condition $\eta_1(\omega)$:

$$S(t)A\int_{-\infty}^0 S(-\tau)d\tilde{G}(W(\tau,\omega)) + A\int_0^t S(t-\tau)d\tilde{G}(W(\tau,\omega)) = \eta_1(\theta_t\omega).$$

Indeed, we can exchange $S(t)$ and $A$.

We now introduce the term *tempered random variable*. A random variable $\xi$ is called *tempered* if this random variable has a subexponential growth:

$$\limsup_{t\to\infty, t\in\mathbb{T}} \frac{\log^+(\xi(\theta_t\omega))}{|t|} = 0.$$

In the case of an ergodic measure $\mathbb{P}$, it is well known that there is only one alternative, i.e., the lim sup in the last formula is $+\infty$; see Arnold [1] Page 165, Proposition 4.1.3. ☐

**Lemma 4.2.** *There exists a $(\theta_t)_{t\in\mathbb{R}}$-invariant set $\Omega_1$ of full $\mathbb{P}$-measure such that for any $\omega \in \Omega_1$ , $\eta(\omega)$ is well defined and*

$$\lim_{t\to\pm\infty} \frac{\log^+\|\eta_1(\theta_t\omega)\|_H}{t} = 0$$

*which means that the $H$ norm of $\eta_1$ is tempered.*

*Proof.* The proof is based on the following observation: Similar to Da Prato and Zabczyk [18] Proposition 13.2.4 and [17] Theorem 5.9 and Remark 5.11. We have

$$\mathbb{E}\sup_{s\in[0,1]}\left\|A\int_0^s S(\tau)d\tilde{G}(W(\tau,\omega))\right\|_H < \infty$$

which allows us to show that

$$\mathbb{E}\sup_{s\in[0,1]}\|\eta_1(\theta_s\omega)\|_H < \infty.$$

It is important to note here that the assumption of Lemma 4.1 is formulated with respect to the $V$ norm but here we consider the $H$ norm. Hence by the Birkhoff ergodic theorem, we have a $(\theta_t)_{t\in\mathbb{Z}}$–invariant set $\Omega_1 \subset \Omega$ such that

$$\lim_{n\to\pm\infty} \frac{\sup_{s\in[0,1]} \|\eta_1(\theta_{s+n}\omega)\|_H}{n} = 0.$$

On the other hand for $t > 0$:

$$\|\eta_1(\theta_t\omega)\|_H \le \sup_{s\in[0,1]} \|\eta_1(\theta_{s+[t]}\omega)\|_H$$

and similarly for $t < 0$, which is sufficient for the $(\theta_t)_{t\in\mathbb{R}}$–invariance of $\Omega_1$. □

To satisfy the assumptions of Theorem 3.9 we have to show that $\eta_1 \in H^2$.

**Lemma 4.3.** *The mapping* $t \to \eta_1(\theta_t\omega) \subset L_{2,loc}(0,\infty;H^2)$.

*Proof.* We use the idea of Da Prato and Zabczyk [18] Page 243f. for $\eta_1(\theta_t\omega) = v(t) + \tilde{G}(W(t))$. The process $v$ fulfills the equation

$$dv + Av\,dt = -d\tilde{G}(W(t)), \qquad v(0) = \eta_1(\omega) \in V \tag{8}$$

with homogeneous Neumann boundary conditions in $z$-direction and periodic boundary conditions in $x$, $y$ directions. Hence $v_z$ satisfies

$$dv_z + Av_z\,dt = -d\tilde{G}(W)_z + \left(\frac{f_0^2}{N(z)}\right)_z v_z, \quad v_z(0) = \eta_1(\omega)_z \in H. \tag{9}$$

Since (8) can be formulated with respect to homogeneous Neumann boundary conditions in $z$-direction and periodic boundary conditions on the other faces, (9) can be considered with *homogeneous* Dirichlet conditions on $\mathcal{O}_{\cdot,\cdot,0}$, $\mathcal{O}_{\cdot,\cdot,2\pi}$. Hence this equation can be interpreted as an evolution equation $\tilde{V} \subset H \subset \tilde{V}'$, $\tilde{V} \subset H^1$. Since we know that there exists a solution of $v(\cdot) \in L_{2,loc}(0,\infty;V)$ the second part on the right hand side can be handled as a known element in $L_{2,loc}(0,\infty;H)$. Standard arguments and the *energy inequality* for parabolic differential equations show that $D_z v \in L_{2,loc}(0,\infty;\tilde{V})$. Similarly we get $D_x v$, $D_y v \in L_{2,loc}(0,\infty;V)$ without changing the boundary conditions. Consequently, since $v \in V \subset H^1$ and $\tilde{G}(W(t)) \in H^2$ we have that $\eta_1(\theta.\omega) \in L_{2,loc}(0,\infty;H^2)$ almost surely and by the stationarity we have $\eta(\omega) \in H^2$ almost surely. □

In the following we are going to describe the dynamics of our Wiener process by $(\Omega_1, \mathcal{F}, \mathbb{P}, \theta^1)$ where $\mathcal{F}$, $\mathbb{P}$ are restrictions of the Borel-$\sigma$-algebra, Wiener measure with respect to $\Omega_1$. We will suppose that the noise is driven by a covariance $Q$ such that the above assumptions are fulfilled and that $\eta_1$ is sufficiently regular such that the trace operator with respect to the boundary can be applied.

In a similar manner we can consider (7) with time-periodic boundary condition representing the impact of the earth's rotation on the fluid :

$$\eta_{2t} = \tilde{\Delta}\eta_2\ \frac{\partial\eta_2}{\partial n} = 0, \text{on } O_{\cdot,\cdot,0}, \quad \frac{\partial\eta_1}{\partial n} = u_0\sin(2\pi t) \text{ on } O_{\cdot,\cdot,2\pi} \tag{10}$$

for $u_0 \in H^{\frac{1}{2}} \cap \dot{L}_2(\mathcal{O}_{\cdot,\cdot,2\pi})$ and (spatial) periodic boundary conditions on the other faces. We obtain without proof the following :

**Lemma 4.4.** *Suppose that* $u_0 \in H^{\frac{1}{2}}(O_{\cdot,\cdot,2\pi}) \cap \dot{L}_2(O_{\cdot,\cdot,2\pi})$. *Then there exists a continuous periodic solution*

$$t \to \eta_2(t) \in H^2$$

*which satisfies (10). In particular,* $\eta_2(t)$ *is also periodic in* $x$, $y$ *directions.*

The proof is based on the properties of the operator $\tilde{G}$.

Let $\theta^2 = (\theta^2)_{t\in\mathbb{R}}$ be the shift operator $\theta_t^2 f(\cdot) = f(t+\cdot)$ for appropriate functions $f$. We consider the hull of $u_0 \sin t$ with respect to $\theta^2$:

$$\Omega_2 = \bigcup_{t\in\mathbb{R}} \theta_t^2(u_0 \sin(2\pi\cdot)) = \bigcup_{t\in\mathbb{R}} u_0 \sin(2\pi(t+\cdot)) = \bigcup_{t\in[0,2\pi)} u_0 \sin(2\pi(t+\cdot)).$$

Summarizing, we have found a process $\eta = \eta_1 + \eta_2$ which will serve as a model for the perturbation on the ocean surface.

After these preparations we can introduce a nonautonomous/random dynamical system. Let $\theta$ be a flow on a set $\Omega = \Omega_1 \times \Omega_2$ (such that (6) is fulfilled).

**Definition 4.5.** *A nonautonomous dynamical system $\phi$ on a phase space $H$ with respect to a flow $\theta$ is a mapping*

$$\phi: \mathbb{T}^+ \times \Omega \times H \to H$$

*fulfilling the cocycle property*

$$\begin{aligned} &\phi(t+\tau,\omega,\cdot) = \phi(t,\theta_\tau\omega,\phi(\tau,\omega,\cdot)) \quad \text{for } t,\, \tau \in \mathbb{T}^+ \\ &\phi(0,\omega,\cdot) = \mathrm{id}_H \end{aligned}$$

*for $\omega \in \Omega$ and $t$, $\tau \in \mathbb{T}^+$. Suppose that the flow $\theta$ is carried by a metric dynamical system and $\phi$ is supposed to be measurable then $\phi$ forms a random dynamical system.*

We now consider the flow $\theta = (\theta^1, \theta^2)$ on $\Omega = \Omega^1 \times \Omega^2$. By the global forward existence and uniqueness of the solution of (5) for $\eta(\theta_t\omega) = \eta_1(\theta_t^1\omega_1) + \eta_2(\theta_t^2\omega_2)$, $\omega = (\omega_1,\omega_2) \in \Omega$, the solution operator of (5), which maps an initial condition $u_0 \in H$ and a sample point $\omega$ to the solution at time $t$, has the cocycle property for $H = \dot{L}_2(O)$. We will denote this solution operator by $\phi(t,\omega,x)$. Note that by the periodicity of $\eta_2$ the restriction of $\phi$ on $\mathbb{Z}$ is a *random dynamical system* for any $\omega_2 \in \Omega_2$. Indeed, $\theta_i\omega = (\theta_i^1\omega_1, \omega_2)$ which can be identified with $\theta^1$ and which leaves $\mathbb{P}$ invariant ($\mathbb{P}$ is ergodic with respect to $(\theta_i^1)_{i\in\mathbb{Z}}$).

**Remark 4.6.** Another opportunity to get an example for a *complete* random dynamical system would be to equip $\Omega_2$ with an ergodic measure. But in contrast to the fact that the daily or yearly rotation of the earth is well determined such a random ansatz would express that the beginning of these periods is rather random.

The main result of this article is the existence of an attractor for the nonautonomous dynamical system. This attractor will attract random sets in probability. Before we give the main theorem, we make some basic remarks on *random sets*.
Suppose that $H$ is a Polish space. A set function $\omega \to D(\omega)$ with closed and nonempty images is called a closed random set over $(\Omega_1, \mathcal{F}, \mathbb{P})$ if and only if there exists a countable number of random variables

$$\xi_i : (\Omega_1, \mathcal{F}, \mathbb{P}) \to H, \quad i \in \mathbb{N}$$

such that

$$D(\omega) = \overline{\bigcup_{i\in\mathbb{N}} \xi_i(\omega)}$$

see Castaing and Valadier [4] Chapter 3.

A random set $D$ is called tempered if the random variable $\mathrm{dist}_H(D(\omega),\{0\})$ is tempered where

$$\mathrm{dist}_H(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_H.$$

We now define the term *random attractor.*

**Definition 4.7.** *Let $\phi$ be a random dynamical system over the metric dynamical system $\theta$. A tempered random set $(A(\omega))_{\omega\in\Omega}$ with compact and nonempty images $A(\omega)$ is called random attractor if*

$$\phi(t,\omega,A(\omega))=A(\theta_t\omega),\qquad t\in\mathbb{T}^+ \tag{11}$$

*and for any random tempered closed random set $D(\omega)$:*

$$(\mathbb{P})\lim_{t\to\infty,t\in\mathbb{T}^+}\mathrm{dist}_H(\overline{\phi(t,\omega,D(\omega))},A(\theta_t\omega))=0. \tag{12}$$

In the next section we will show that the dynamical system generated by (5) restricted to $\mathbb{T}=\mathbb{Z}$ has such a random attractor. However, because of time-periodic perturbation, we do not have a random dynamical system. Therefore we have to modify this conclusion for $\mathbb{T}=\mathbb{Z}$ a little bit, and thus we obtain the following main result in this section.

**Theorem 4.8.** *(Random attractors) There exists a random attractor $A=(A(\omega))_{\omega\in\Omega}$ for the 3D quasigeostrophic equation (5) under random plus time-periodic forcing on the ocean surface. Moreover, the mapping*

$$t\to A(\theta_t\omega)$$

*has a subexponentially growth.*

Note that the dynamical system generated by (5) defines a nonautonomous dynamical system and not a random dynamical system in general, due to the time-periodic boundary condition.
In the next section, we prove this main theorem 4.8.

## 5. Proof of the main theorem

The proof of the main theorem 4.8 in the last section is based on checking following conditions. We will consider the set of families of sets $D$ such that $D(\omega)$ is closed and nonempty and $t\to D(\theta_t(\omega_1,\omega_2))$ has a subexponintially growth. In particular, if we fix an $\bar{\omega}_2$. The restriction to $\mathbb{Z}$ is supposed to be tempered. The set of these set families is denoted by $\mathcal{D}$.

**Theorem 5.1.** *Suppose that $\theta^1$ be a metric dynamical system and suppose that for $\omega_1\in\Omega_1$, $t\in\mathbb{Z}^+$ the mappings $\phi(t,\omega_1,\cdot)$ are continuous and that there exists a set function $B\in\mathcal{D}$ having compact images which is absorbing:*

$$\phi(t,\theta^1_{-t}\omega_1,D(\theta^1_{-t}\omega_1))\subset B(\omega_1) \tag{13}$$

*for $\mathbb{Z}^+\ni t\geq t_0(D,\omega)$. Then there exists a unique random attractor $A$ in $\mathcal{D}$.*

*Proof.* The proof of this theorem can be found in Flandoli and Schmalfuß [10].
Now let us use this result to prove our main theorem 4.8.
*Proof* of Theorem 4.8:

Calculating the inner product in $H$ we obtain by Lemma 3.5, 3.7, 3.8.

(14)
$$\begin{aligned}\|u(t)\|_H^2 + 2\nu\int_0^t \|u(\tau)\|_V^2 d\tau &= \|u_0\|_H^2 + 2\beta\int_0^t \langle G(\tilde\Delta\eta)_x - \eta_x, u\rangle d\tau\\ &\le \|u_0\|_H^2 + \frac{\beta^2}{\nu}\int_0^t \|G(\tilde\Delta\eta(\theta_\tau\omega))_x - \eta(\theta_\tau\omega)_x\|_{V'}^2 d\tau + \nu\int_0^t \|u(\tau)\|_V^2 d\tau.\end{aligned}$$

Parallel to this inequality we consider the equation

$$\begin{aligned}&\xi(t) + \nu\lambda_1\int_0^t \xi(\tau)d\tau\\ &\quad = \|u_0\|_H^2 + \frac{\beta^2}{\nu}\int_0^t \|G(\tilde\Delta\eta(\theta_\tau\omega))_x - \eta(\theta_\tau\omega)_x\|_{V'}^2 d\tau.\end{aligned} \tag{15}$$

The solution of this equation $\xi(t,\omega,\|u_0\|_H^2)$ is a bound for $\|u(t)\|_H^2$. In addition, this equation has a unique forward and backward exponentially fast attracting solution

$$(\omega,t)\to\xi^*(\theta_t\omega),\quad \xi^*(\omega) = \frac{\beta^2}{\nu}\int_{-\infty}^0 e^{\nu\lambda_1\tau}\|G(\tilde\Delta\eta(\theta_\tau\omega))_x - \eta(\theta_\tau\omega)_x\|_{V'}^2 d\tau.$$

The mapping $t\to\xi^*(\theta_\omega)$ is subexponentially growing. We have

$$\begin{aligned}\frac{\beta^2}{2\nu}\xi^*(\omega) \le &\int_{-\infty}^0 e^{\nu\lambda_1\tau}\|G(\tilde\Delta\eta_1(\theta_\tau\omega))_x - \eta_1(\theta_\tau\omega)_x\|_{V'}^2 d\tau\\ &+ \int_{-\infty}^0 e^{\nu\lambda_1\tau}\|G(\tilde\Delta\eta_2(\theta_\tau\omega))_x - \eta_2(\theta_\tau\omega)_x\|_{V'}^2 d\tau.\end{aligned}$$

Indeed, the second term on the right hand side is bounded and the first is tempered, see Lemma 4.2 and ([21]). Since $\|\eta_1\|_H$ is tempered so is $\|G(\tilde\Delta\eta_1)_x\|_{V'}$
To see the backward convergence uniform with respect to $D$ we must prove that

$$\lim_{t\to\infty}\sup_{u_0\in D(\theta_{-t}\omega)} |\xi(t,\theta_{-t}\omega,\|u_0\|_H^2) - \xi^*(\omega)| = 0.$$

Because of

$$\xi^*(\omega) = \xi(t,\theta_{-t}\omega,\xi^*(\theta_{-t}\omega)))$$

we have

$$\sup_{u_0\in D(\theta_{-t}\omega)} |\xi(t,\theta_{-t}\omega,\|u_0\|_H^2) - \xi^*(\omega)| \le \sup_{u_0\in D(\theta_{-t}\omega)} |\|u_0\|_H^2 - \xi^*(\theta_{-t}\omega)|e^{-\lambda_1 t}.$$

Indeed, $\xi$ and $\xi^*$ are solutions of one dimensional affine differential equations. Since the first factor on the right hand side is only subexponentially growing the convergence conclusion follows for $t\to\infty$. Hence the ball with center zero and radius $2\xi^*(\omega)$ forms an absorbing set (in the sense of (13)) $\tilde B(\omega)$. This is also true if we restrict this dynamical system to the time set $\mathbb{Z}$. The forward convergence follows straightforwardly by the solution formula for an affine equation

$$|\sup_{u_0\in D(\omega)}\psi(t,\omega,\|u_0\|^2) - \xi^*(\theta_t\omega)| \le e^{-\lambda_1 t}\sup_{u_0\in D(\omega)} |\|u_0\|^2 - \xi^*(\omega)|$$

Plugging in $2\xi^*(\omega)$ into (15) for $\|u_0\|_H^2$ it is easily seen that $\tilde B$ is forward invariant:

$$\phi(t,\omega,\tilde B(\omega)) \subset \tilde B(\theta_t\omega).$$

On the other hand we can check that that

$$B(\omega) := \overline{\phi(1,\theta_{-1}\omega,\tilde B(\theta_{-1}\omega))} \subset \tilde B(\omega).$$

Note that $B$ is compact because $\phi$ is regularizing which follows by Remark 3.10 ii).

The continuity of $\phi(t,\omega,\cdot)$ follows because we have the estimate from Lemma 3.6. However, we need this technique later on once more such that we are going to explain this technique: We have by Lemma 3.6 and the properties of $C(t)$ and $D$:

$$\begin{aligned}
&|\langle B(u_1,u_1)-B(u_2,u_2),u_1-u_2\rangle| \le c_3\|u_1-u_2\|_H\|u_1-u_2\|_V\|u_1\|_V\\
&\langle C(t,u_1)-C(t,u_2),u_1-u_2\rangle = 0\\
&\langle D(u_1)-D(u_2),u_1-u_2\rangle = 0
\end{aligned}$$

Hence there exists a constant $c_6$ such hat for two solutions $u_1$, $u_2$ with initial conditions $u_{10}$, $u_{20}$

$$\frac{d}{dt}\|u_1(t)-u_2(t)\|_H^2 \le c_6(\|u\|_V^2)\|u_1(t)-u_2(t)\|_H^2.$$

It follows by the Gronwall lemma

$$\|u_1(t)-u_2(t)\|_H^2 \le \|u_{10}-u_{20}\|_H^2 e^{c_6\int_0^t \|u_1(\tau)\|_V^2 d\tau} \tag{16}$$

which gives the continuity of $\phi(t,\omega,\cdot)$.

We have checked all assumptions of the above theorem such that the discrete random dynamical system has a random attractor $A(\omega_1,\bar\omega_2)$. We now extend the definition of $A$ to $\Omega_2$. We set:

$$A(\theta_t(\omega,\bar\omega_2)) = \phi(t,\omega,A(\omega_1,\bar\omega_2)) \qquad \text{for } t\in\mathbb{R}^+$$

which is invariant in the sense of (11). Note that by the cocycle property this definition is correct. We show convergence (12):

$$(\mathbb{P}) \lim_{t\to\infty,t\in\mathbb{R}^+} \operatorname{dist}_H(\overline{\phi(t,(\omega_1,\bar\omega_2),D((\omega_1,\bar\omega_2)))},A(\theta_t(\omega_1,\bar\omega_2))) = 0.$$

for $D\in\mathcal{D}$ and any $\bar\omega_2\in\Omega_2$. Since $B(\omega)$ is an forward absorbing and forward invariant set it remains to check the convergence conclusion for $D=B$. On account of (14) we can notice for fixed $\bar\omega_2$ and $u_1=\phi(t,(\omega_1,\bar\omega_2),y)$

$$\begin{aligned}
\sup_{y\in B((\omega_1,\bar\omega_2))} &\nu\int_0^1 \|\phi(\tau,(\omega_1,\bar\omega_2),y)\|_V^2 d\tau\\
&\le 2\xi^*(\omega_1,\bar\omega_2) + \frac{\beta^2}{\nu}\int_0^1 \|G(\tilde\Delta\eta(\theta_\tau(\omega_1,\bar\omega_2)))_x - \eta(\theta_\tau(\omega_1,\bar\omega_2))_x\|_{V'}^2 d\tau.
\end{aligned}$$

Hence for an appropriate constant $c_7>0$ the expression

$$\sup_{x\in B((\omega_1,\bar\omega_2))} e^{c_7\int_0^1 \|G(\tilde\Delta\eta(\theta_\tau(\omega_1,\bar\omega_2))_x-\eta(\theta_\tau(\omega_1,\bar\omega_2))_x\|_{V'}^2 d\tau} =: Y(\omega_1,\bar\omega_2).$$

The mapping

$$(\omega,n)\to Y((\theta_n\omega_1,\bar\omega_2)$$

defines a stationary process. We obtain by (16):

$$\begin{aligned}
\sup_{\tau\in[0,1]} &\operatorname{dist}_H(\phi(\tau+n,\omega,B((\omega_1,\bar\omega_2)),A(\theta_{\tau+n}(\omega_1,\bar\omega_2)))\\
&\le \operatorname{dist}_H(\phi(n,(\omega_1,\bar\omega_2),B((\omega_1,\bar\omega_2)),A((\theta_n^1\omega_1,\bar\omega_2)))Y((\theta_n\omega_1,\bar\omega_2))^{\frac12}.
\end{aligned}$$

Since the first factor of the right hand side tends to zero in probability the product of the left hand side also tends to zero in probability which gives the convergence conclusion (12).
To obtain the subexponential growth of

$$t \to A(\theta_t\omega)$$

we need that $Y$ is tempered. Due to Arnold [1] Proposition 4.1.3 it is sufficient to show that

$$\mathbb{E}\sup_{s\in[0,1]}\int_0^1 \|G(\tilde{\Delta}\eta(\theta_{\tau+s}\omega))_x - \eta(\theta_{\tau+s}\omega)_x\|_{V'}^2 d\tau$$

The expression under the expectation can be estimated by

$$\int_0^2 \|G(\tilde{\Delta}\eta(\theta_\tau\omega))_x - \eta(\theta_\tau\omega)_x\|_{V'}^2 d\tau$$

which has a finite expectation. Consequently,

$$\sup_{x\in A(\theta_t\omega)}\|x\|_H \le Y(\omega)^{\frac{1}{2}}\mathrm{diam}(A(\omega)) + \sup_{x\in A(\omega)}\|x\|_H, \quad t\in[0,1]$$

for $\omega = (\omega_1, \bar{\omega}_2)$ by the triangle inequality. We have used that the product of two tempered random variables is tempered which gives the general convergence conclusion of the main theorem 4.8. □

## 6. Summary

We have studied the 3D baroclinic quasigeostrophic flow model under random wind forcing and time-periodic fluctuations on fluid boundary, i.e., on the interface between the oceans and the atmosphere. The time-periodic fluctuations are due to periodic rotation of the earth and thus periodic exposure of the earth to the solar radiation. We have established the well-posedness of the baroclinic quasigeostrophic flow model in the state space (Theorem 3.9), and we have demonstrated the existence of the random attractors (Theorem 4.8), again in the state space. We have also discussed the relevance of our results to climate modeling.

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