

# LOCAL DISCONTINUOUS GALERKIN METHODS FOR THE STOKES SYSTEM

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**Abstract.** In this paper, we introduce and analyze local discontinuous Galerkin methods for the Stokes system. For arbitrary meshes with hanging nodes and elements of various shapes we derive a priori estimates for the  $L^2$ -norm of the errors in the velocities and the pressure. We show that *optimal* order estimates are obtained when polynomials of degree  $k$  are used for each component of the velocity and polynomials of degree  $k - 1$  for the pressure, for any  $k \geq 1$ . We also consider the case in which *all* the unknowns are approximated with polynomials of degree  $k$  and show that, although the orders of convergence remain the same, the method is more efficient. Numerical experiments verifying these facts are displayed.

**Key words.** Finite elements, discontinuous Galerkin methods, Stokes system

**AMS subject classifications.** 65N30

**1. Introduction.** In this paper, we introduce and analyze local discontinuous Galerkin (LDG) methods for the Stokes system

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}_{\mathcal{D}} && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^d$  and the Dirichlet datum satisfies the usual compatibility condition  $\int_{\Omega} \mathbf{g}_{\mathcal{D}} \cdot \mathbf{n} \, ds = 0$ , with  $\mathbf{n}$  denoting the outward unit normal to  $\partial\Omega$ . We thus continue the study of LDG methods as applied to diffusion-dominated problems started by Castillo, Cockburn, Perugia and Schötzau [8], who carried out the analysis of general LDG methods for the Laplacian on general triangulations, and by Cockburn, Kanschat, Perugia and Schötzau [13], who obtained superconvergence results for Cartesian grids and a special LDG method. Our long-term goal is to study LDG methods for the *incompressible* Navier-Stokes equations; the analysis of the Stokes system is thus a necessary intermediate step.

There are mainly two motivations for using LDG methods for the Navier-Stokes equations. The first one is that these methods can easily handle meshes with hanging

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nodes, elements of general shapes, and local spaces of different types; this makes them ideally suited for *hp*-adaptivity. The second one, of no less importance, is that with their carefully devised *numerical fluxes* inherited from the corresponding DG discretizations of non-linear hyperbolic conservation laws, see the work by Cockburn and Shu [12, 15, 16, 17, 19], the LDG methods weakly enforce the conservation laws element-by-element and in a *conservative* way. This last property is highly appreciated by the practitioners of computational fluid dynamics, especially in situations where there are shocks, steep gradients or boundary layers. In fact, it was for the convection-dominated *compressible* Navier-Stokes equations that the DG discretization techniques were applied for the first time by Bassi and Rebay in [4] with excellent results; the LDG method was then introduced by Cockburn and Shu in [18] as an extension of Bassi and Rebay's method to general convection-diffusion problems. To give the reader a flavor of the LDG methods proposed in this paper, we briefly compare them with other methods.

• **Interior penalty methods.** In the framework of the Stokes system, the main difficulty to obtain numerical approximations is the enforcement of the incompressibility condition on the velocity. For continuous approximations of the velocity, it is well known that a pointwise enforcement could yield an over-constrained velocity and the only divergence-free function might turn out to be identically zero; this is the so-called *locking* phenomenon. However, in 1990, Baker, Jureidini and Karakashian [2] showed how to enforce the incompressibility condition pointwise *inside* each element and still obtain optimal error estimates. They achieved this by using interior penalty (IP) methods, that is, methods that take the velocity approximation to be *discontinuous* and penalize the size of its discontinuity jumps across the element boundaries; see also the recent extension of this method to the incompressible Navier-Stokes equations by Karakashian and Katsaounis [28]. Arnold, Brezzi, Cockburn and Marini [1] briefly review IP methods for purely elliptic problems and then relate and compare them to the LDG and other DG methods. A similar comparison can easily be developed for the Stokes system, but here we restrict ourselves to pointing out that, like the IP method of Baker, Jureidini and Karakashian, the LDG methods use a discontinuous approximate velocity whose discontinuity jumps across the element boundaries are also penalized. However, unlike the IP method of Baker, Jureidini and Karakashian, the LDG methods use discontinuous pressure approximations and (at least in this paper) do not try to impose the incompressibility condition pointwise inside the elements; instead, like in standard mixed methods, this condition is imposed weakly.

• **Standard mixed methods.** In his review of standard mixed methods for the Navier-Stokes equations, Fortin [21] points out that the use of discontinuous approximations for the pressure ensures a better conservation of mass in comparison with the use of continuous approximations and refers to the work of Fortin, Pelletier and Camarero [30] for situations that illustrate this point. This is a property that these methods have in common with the LDG methods, not only because of the use of discontinuous approximations of the pressure, but also because the LDG methods ensure mass conservation. Indeed, to obtain the LDG methods, we first rewrite the Stokes system as the following collection of conservation laws

$$\underline{\sigma} = \nabla \mathbf{u} \quad \text{in } \Omega, \quad (1.2)$$

$$-\nabla \cdot \underline{\sigma} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1.3)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.4)$$

$$\mathbf{u} = \mathbf{g}_D \quad \text{on } \partial\Omega, \quad (1.5)$$

and then discretize them by using the DG technique, that is, element-by-element and in a conservative way; this is what ensures mass conservation. Note that to achieve this, we introduced the stress tensor  $\underline{\sigma}$ . This could be considered a disadvantage of the LDG methods with respect to the classical mixed methods, but this is not so because  $\underline{\sigma}$  can be eliminated independently and in parallel on each grid cell, as we shall see.

Let us briefly digress to point out that the issue of the possible advantages of methods that, like the LDG methods, enforce the conservation laws locally and in a conservative way over finite element methods which cannot do that, and are typically based on continuous approximations, is the subject of an ongoing discussion which is far from being exhausted. Although it has been firmly established that this property is certainly desirable for convection-dominated problems, its possible advantages in other situations still remain to be thoroughly explored. About this very point, see the review of DG methods by Cockburn, Karniadakis and Shu [14] and the paper by Hughes, Engel, Mazzei, and Larson [25] where a comparison of discontinuous and continuous Galerkin methods is carried out.

• **Stabilized mixed methods.** Finally, let us emphasize that for the LDG methods, the approximation spaces for the velocity and the pressure can be chosen almost arbitrarily; only a mild local condition has to be satisfied. This is so because the LDG methods can be considered to be *stabilized* mixed methods; for a review of stabilized mixed methods, see the article by Franca, Hughes and Stenberg [22]. They are thus related to the Galerkin least squares (GLS) mixed methods introduced in 1986/1987 by Hughes, Franca and Balestra [27, 26] who used the jumps of the pressures across boundary elements and residuals inside the elements to render them stable. However, unlike these methods, LDG methods use discontinuous approximations to the velocity and employs stabilization terms which involve jumps across the element boundaries only. Variations of the LDG methods we study here could be easily constructed which are closely related to the ‘locally’ stabilized methods introduced and numerically studied in 1989 by Silvester and Kechkar [31] and then analyzed in 1992 by Kechkar and Silvester [29]; however, this subject will not be considered in this paper. Finally, we must also point out that in the GLS methods, one has, in particular for velocities which are piecewise quadratic or of higher degree, and also for curvilinear mapped elements, to evaluate the GLS stabilization terms which are quite costly due to the appearance of, e.g., the Laplacian in the bilinear forms. The LDG methods achieve, as we prove here, the same stabilization effect but, as a rule, do this without recourse to domain integrals of second-order derivatives of finite element functions. Rather, only edge/face integrals of jumps are evaluated.

Now, let us briefly describe our results. We show that if we use polynomials of degree  $k$  to approximate the pressure  $p$ , the stresses  $\underline{\sigma}$ , and the velocity  $\mathbf{u}$ , the order of convergence of  $k$  is obtained for the  $L^2$ -norm of  $p$  and  $\underline{\sigma}$ , and of  $k + 1$  for the  $L^2$ -norm of the velocity. These orders of convergence are *sharp*, as they are observed in our numerical experiments. We also explore the situation in which polynomials of degree  $k - 1$  are used to approximate the pressure  $p$  and the stress tensor  $\underline{\sigma}$ . In this case, we prove that the above mentioned orders of convergence remain *invariant*; in other words, in this case the error estimates are *optimal*. Our numerical experiments confirm this fact; moreover, they also show that this choice of approximating spaces gives rise to a method which is *less efficient* than the one obtained by using same approximation spaces for all the variables. In Table 1.1, we summarize our theoretical results and compare them with the orders of convergence obtained for the IP method of Baker, Jureidini and Karakashian [2] and the stabilized mixed methods of Hughes

and Franca [26] (see also Franca and Stenberg [23] for a unified error analysis). Note that when the approximations are continuous, the jumps across elements are zero and the corresponding penalization term vanishes; we indicate this by writing ‘none’.

TABLE 1.1  
Theoretical orders of convergence for  $k \geq 1$ .

method	penalization of the jumps of velocity and pressure		$\ \mathbf{u} - \mathbf{u}_N\ _0$	$\ p - p_N\ _0$
LDG	$\mathcal{O}(h^{-1})$	$\mathcal{O}(h)$	$k + 1$	$k$
IP [2]	$\mathcal{O}(h^{-1})$	none	$k + 1$	$k$
Stabilized mixed [26, 23]	none	$\mathcal{O}(h)$	$k + 1$	$k$

Finally, let us point out that the technique we use in our analysis is an extension of that used in [8] for the Laplacian. One of the contributions in this paper is that we make the technique work for local spaces that might be different for different unknowns. In fact, in *all* previous error analyses of LDG methods involving second-order operators, see [18, 7, 9, 11, 8, 13, 20], the local spaces for both the auxiliary stresses and the main unknowns have been taken to be identical. The second contribution is that we show how to obtain the inf-sup condition, which is non-standard given the discontinuous nature of our elements, in order to obtain error estimates for the pressure. Note that, unlike the analysis technique used by Hughes, Franca and Balestra [27, 26] who obtained error estimates of the pressure in certain mesh-dependent norms, we obtain an error of the pressure in the  $L^2$ -norm by using an inf-sup condition; in this respect, our technique is closer to that employed in 1991 by Franca and Stenberg [23].

The paper is organized as follows. In section 2, we introduce the method, show that it determines a unique approximate solution, and then state and discuss our main results. Finally, a brief overview of its proof is given which is then completed in full detail in section 3. Section 4 is devoted to numerical experiments devised to verify our theoretical results and to compare the effect that the use of different spaces has on the quality of the LDG approximate solution. We end in section 5 by describing extensions of our analysis and giving some concluding remarks.

**2. The main results.** In this section, we formulate the LDG method and show that it possesses a well-defined solution. We then state and discuss our main results and, finally, we present an abstract framework upon which our error analysis is based. We assume throughout this section, in order to avoid unnecessary technicalities, that the exact solution  $(\mathbf{u}, p)$  of (1.1) belongs at least to  $H^2(\Omega)^d \times H^1(\Omega)$ .

**2.1. Definition of the LDG method.** To define the LDG method, we consider the system of first-order conservation laws (1.2)–(1.5). We use the standard notation  $(\nabla \mathbf{v})_{ij} = \partial_j v_i$  and  $(\nabla \cdot \underline{\sigma})_i = \sum_{j=1}^d \partial_j \sigma_{ij}$ . We also denote by  $\mathbf{v} \otimes \mathbf{n}$  the matrix whose  $ij$ -th component is  $v_i n_j$  and write

$$\underline{\sigma} : \underline{\tau} := \sum_{i,j=1}^d \sigma_{ij} \tau_{ij}, \quad \mathbf{v} \cdot \underline{\sigma} \cdot \mathbf{n} := \sum_{i,j=1}^d v_i \sigma_{ij} n_j = \underline{\sigma} : (\mathbf{v} \otimes \mathbf{n}).$$

Multiplying equations (1.2), (1.3) and (1.4) by arbitrary, smooth test functions  $\underline{\tau}$ ,  $\mathbf{v}$ , and  $q$ , respectively, and integrating by parts over an arbitrary subset  $K$  of the domain  $\Omega$ , we obtain

$$\int_K \underline{\sigma} : \underline{\tau} \, d\mathbf{x} = - \int_K \mathbf{u} \cdot \nabla \cdot \underline{\tau} \, d\mathbf{x} + \int_{\partial K} \mathbf{u} \cdot \underline{\tau} \cdot \mathbf{n}_K \, ds, \quad (2.1)$$

$$\begin{aligned} \int_K \underline{\sigma} : \nabla \mathbf{v} \, d\mathbf{x} - \int_{\partial K} \underline{\sigma} : (\mathbf{v} \otimes \mathbf{n}_K) \, ds - \int_K p \nabla \cdot \mathbf{v} \, d\mathbf{x} + \int_{\partial K} p \mathbf{v} \cdot \mathbf{n}_K \, ds \\ = \int_K \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \end{aligned} \quad (2.2)$$

$$- \int_K \mathbf{u} \cdot \nabla q \, d\mathbf{x} + \int_{\partial K} \mathbf{u} \cdot \mathbf{n}_K q \, ds = 0, \quad (2.3)$$

where  $\mathbf{n}_K$  is the outward unit normal to  $\partial K$ . This is the weak form of the Stokes system that we shall use to define the LDG method. We enforce the above equations on each element  $K$  of a general triangulation  $\mathcal{T}$  of  $\Omega$  which can have hanging nodes and elements of various shapes. Thus, since the above equations are well defined for any functions  $(\underline{\sigma}, \mathbf{u}, p)$  and  $(\underline{\tau}, \mathbf{v}, q)$  in  $\underline{\Sigma} \times \mathbf{V} \times Q$  where

$$\begin{aligned} \underline{\Sigma} &:= \{ \underline{\sigma} \in L^2(\Omega)^{d^2} : \sigma_{ij}|_K \in H^1(K), \forall K \in \mathcal{T}, 1 \leq i, j \leq d \}, \\ \mathbf{V} &:= \{ \mathbf{v} \in L^2(\Omega)^d : v_i|_K \in H^1(K), \forall K \in \mathcal{T}, 1 \leq i \leq d \}, \\ Q &:= \{ q \in L^2(\Omega) : \int_{\Omega} q \, d\mathbf{x} = 0, q|_K \in H^1(K), \forall K \in \mathcal{T} \}, \end{aligned}$$

we seek to approximate the exact solution  $(\underline{\sigma}, \mathbf{u}, p)$  with functions  $(\underline{\sigma}_N, \mathbf{u}_N, p_N)$  in the finite element space  $\underline{\Sigma}_N \times \mathbf{V}_N \times Q_N \subset \underline{\Sigma} \times \mathbf{V} \times Q$ , where

$$\begin{aligned} \underline{\Sigma}_N &:= \{ \underline{\sigma} \in L^2(\Omega)^{d^2} : \sigma_{ij}|_K \in \mathcal{S}(K), \forall K \in \mathcal{T}, 1 \leq i, j \leq d \}, \\ \mathbf{V}_N &:= \{ \mathbf{v} \in L^2(\Omega)^d : v_i|_K \in \mathcal{V}(K), \forall K \in \mathcal{T}, 1 \leq i \leq d \}, \\ Q_N &:= \{ q \in L^2(\Omega) : \int_{\Omega} q \, d\mathbf{x} = 0, q|_K \in \mathcal{Q}(K), \forall K \in \mathcal{T} \}, \end{aligned}$$

and the *local* finite element spaces  $\mathcal{S}(K)$ ,  $\mathcal{V}(K)$  and  $\mathcal{Q}(K)$  typically consist of polynomials.

The approximate solution  $(\underline{\sigma}_N, \mathbf{u}_N, p_N)$  is now defined by imposing that for all  $K \in \mathcal{T}$ , for all  $(\underline{\tau}, \mathbf{v}, q) \in \mathcal{S}(K)^{d^2} \times \mathcal{V}(K)^d \times \mathcal{Q}(K)$ ,

$$\int_K \underline{\sigma}_N : \underline{\tau} \, d\mathbf{x} = - \int_K \mathbf{u}_N \cdot \nabla \cdot \underline{\tau} \, d\mathbf{x} + \int_{\partial K} \hat{\mathbf{u}}_{N,\sigma} \cdot \underline{\tau} \cdot \mathbf{n}_K \, ds, \quad (2.4)$$

$$\begin{aligned} \int_K \underline{\sigma}_N : \nabla \mathbf{v} \, d\mathbf{x} - \int_{\partial K} \hat{\underline{\sigma}}_N : (\mathbf{v} \otimes \mathbf{n}_K) \, ds - \int_K p_N \nabla \cdot \mathbf{v} \, d\mathbf{x} + \int_{\partial K} \hat{p}_N \mathbf{v} \cdot \mathbf{n}_K \, ds \\ = \int_K \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \end{aligned} \quad (2.5)$$

$$- \int_K \mathbf{u}_N \cdot \nabla q \, d\mathbf{x} + \int_{\partial K} \hat{\mathbf{u}}_{N,p} \cdot \mathbf{n}_K q \, ds = 0. \quad (2.6)$$

Here,  $\hat{\mathbf{u}}_{N,\sigma}$ ,  $\hat{\underline{\sigma}}_N$ ,  $\hat{p}_N$  and  $\hat{\mathbf{u}}_{N,p}$  are the so-called *numerical fluxes*, which are discrete approximations to traces on the boundary of the elements. Note how the numerical fluxes  $\hat{\mathbf{u}}_{N,\sigma}$  and  $\hat{\mathbf{u}}_{N,p}$  arise naturally from the weak formulation; although both are

approximations to the trace of the velocity  $\mathbf{u}$ , they are defined in very different ways since they are associated to different conservation laws.

To define these numerical fluxes, we need to introduce some notation associated with traces. Let  $K^+$  and  $K^-$  be two adjacent elements of  $\mathcal{T}$ ; let  $\mathbf{x}$  be an arbitrary point of the set  $e = \partial K^+ \cap \partial K^-$ , which is assumed to have a non-zero  $(d-1)$ -dimensional measure, and let  $\mathbf{n}^+$  and  $\mathbf{n}^-$  be the corresponding outward unit normals at that point. Let  $(\underline{\sigma}, \mathbf{u}, p)$  be a function smooth inside each element  $K^\pm$  and let us denote by  $(\underline{\sigma}^\pm, \mathbf{u}^\pm, p^\pm)$  the traces of  $(\underline{\sigma}, \mathbf{u}, p)$  on  $e$  from the interior of  $K^\pm$ . Then, we define the mean values  $\{\!\{ \cdot \}\!\}$  and jumps  $[\![ \cdot ]\!]$  at  $\mathbf{x} \in e$  as

$$\begin{aligned} \{\!\{ p \}\!\} &:= (p^+ + p^-)/2, & \{\!\{ \mathbf{u} \}\!\} &:= (\mathbf{u}^+ + \mathbf{u}^-)/2, & \{\!\{ \underline{\sigma} \}\!\} &:= (\underline{\sigma}^+ + \underline{\sigma}^-)/2, \\ [p] &:= p^+ \mathbf{n}^+ + p^- \mathbf{n}^-, & [\mathbf{u}] &:= \mathbf{u}^+ \cdot \mathbf{n}^+ + \mathbf{u}^- \cdot \mathbf{n}^-, & [\underline{\sigma}] &:= \underline{\sigma}^+ \cdot \mathbf{n}^+ + \underline{\sigma}^- \cdot \mathbf{n}^-. \end{aligned}$$

Note that the jumps  $[p]$  and  $[\underline{\sigma}]$  are both vectors whereas the jump  $[\mathbf{u}]$  is a scalar. We also need to define a jump of the velocity  $\mathbf{u}$  which is a matrix, namely,

$$[\mathbf{u}] := \mathbf{u}^+ \otimes \mathbf{n}^+ + \mathbf{u}^- \otimes \mathbf{n}^-.$$

In components, we have  $[\mathbf{u}]^2 = \sum_{i=1}^d (u_i^+ - u_i^-)^2$  and  $(\mathbf{u}^\pm \otimes \mathbf{n}^\pm)^2 = \sum_{i=1}^d (u_i^\pm)^2$ . Also, we remark that, since  $[\mathbf{u}] = \sum_{i=1}^d (u_i^+ - u_i^-) \mathbf{n}_i^+$ , we have  $[\mathbf{u}]^2 \leq [\underline{\mathbf{u}}]^2$ , that is, the norm of the scalar-valued jump of the velocity can be controlled by the norm of the matrix-valued jump.

We are now ready to introduce the numerical fluxes. We begin by defining the numerical fluxes  $\hat{\underline{\sigma}}$  and  $\hat{\mathbf{u}}_\sigma$  associated with the Laplacian. We pick a *direct* extension of the choice of numerical fluxes for the Laplace operator considered in [8] and [13]. That is, on a face  $e$  inside the domain  $\Omega$ , we take

$$\begin{bmatrix} \hat{\underline{\sigma}} \\ \hat{\mathbf{u}}_\sigma \end{bmatrix} := \begin{bmatrix} \{\!\{ \underline{\sigma} \}\!\} \\ \{\!\{ \mathbf{u} \}\!\} \end{bmatrix} - \begin{bmatrix} C_{11} [\underline{\mathbf{u}}] + [\underline{\sigma}] \otimes \mathbf{C}_{12} \\ -[\underline{\mathbf{u}}] \cdot \mathbf{C}_{12} \end{bmatrix}, \quad (2.7)$$

and if  $e$  lies on the boundary, we take

$$\begin{bmatrix} \hat{\underline{\sigma}} \\ \hat{\mathbf{u}}_\sigma \end{bmatrix} := \begin{bmatrix} \underline{\sigma}^+ - C_{11} (\mathbf{u}^+ - \mathbf{g}_D) \otimes \mathbf{n}^+ \\ \mathbf{g}_D \end{bmatrix}. \quad (2.8)$$

Note that since the numerical flux  $\hat{\mathbf{u}}_\sigma$  is independent of the variable  $\underline{\sigma}$ , it is possible to use the equation (2.4) to solve  $\underline{\sigma}_N$  in terms of  $\mathbf{u}_N$  only, element by element. This local solvability, which allows us to *eliminate* the stresses  $\underline{\sigma}_N$  from the equations, gives the name to the LDG method (see [8, 18] for more details).

The numerical fluxes associated with the incompressibility constraint,  $\hat{\mathbf{u}}_p$  and  $\hat{p}$ , are defined by using an analogous recipe. If the face  $e$  is on the interior of  $\Omega$ , we take

$$\begin{bmatrix} \hat{\mathbf{u}}_p \\ \hat{p} \end{bmatrix} := \begin{bmatrix} \{\!\{ \mathbf{u} \}\!\} \\ \{\!\{ p \}\!\} \end{bmatrix} + \begin{bmatrix} D_{11} [p] + D_{12} [\mathbf{u}] \\ -D_{12} \cdot [p] \end{bmatrix}, \quad (2.9)$$

and if  $e$  lies on the boundary, we take

$$\begin{bmatrix} \hat{\mathbf{u}}_p \\ \hat{p} \end{bmatrix} := \begin{bmatrix} \mathbf{g}_D \\ p^+ \end{bmatrix}. \quad (2.10)$$

The parameters  $C_{11}$ ,  $\mathbf{C}_{12}$ , and  $D_{11}$ ,  $\mathbf{D}_{12}$  depend on  $\mathbf{x} \in e$ . This completes the definition of the LDG method for the Stokes system (1.1).

We would like to stress the following points about this method:

- The numerical fluxes are consistent in the sense that equations (2.4)–(2.6) coincide with (2.1)–(2.3) for the exact solution  $(\underline{\sigma}, \mathbf{u}, p)$ . Note also that the boundary condition is taken into consideration *only* through the numerical fluxes  $\hat{\mathbf{u}}_\sigma$  and  $\hat{\mathbf{u}}_p$  on the boundary.
- The purpose of the coefficients  $C_{11}$  and  $D_{11}$  is to enhance the stability of the method. They are thus referred to as the stabilization coefficients. As we shall see, they can also affect the accuracy of the method. The parameters  $C_{12}$  and  $D_{12}$  can be chosen as to reduce the sparsity of the matrices and, in special cases, to enhance the accuracy of the method; see the case of the Laplacian treated in [13]. In this paper, we simply assume that they are of order one.
- Note that if we rewrite the conservation law (1.3) as

$$-\nabla \cdot (\underline{\sigma} - p \underline{I}) = \mathbf{f} \quad \text{in } \Omega,$$

where  $\underline{I}$  is the identity tensor, we see that we need to define a single numerical flux for  $(\underline{\sigma} - p \underline{I})$  which, in fact, has been taken to be  $\hat{\underline{\sigma}} - \hat{p} \underline{I}$ . We could have taken the following more general ansatz for the numerical flux for the pressure:  $\hat{p} = \{\{p\}\} - D_{12} \cdot \llbracket p \rrbracket + D_{22} \llbracket \mathbf{u} \rrbracket$ , but this would result in

$$\hat{\underline{\sigma}} - \hat{p} \underline{I} = \{\{\underline{\sigma}\}\} - \{\{p\}\} \underline{I} + D_{12} \cdot \llbracket p \rrbracket \underline{I} - (C_{11} \llbracket \mathbf{u} \rrbracket + D_{22} \llbracket \mathbf{u} \rrbracket \underline{I}).$$

Since, as we shall see, the role of the term  $C_{11} \llbracket \mathbf{u} \rrbracket$  is to control *all* the discontinuity jumps of the velocity  $\mathbf{u}$  whereas the term  $D_{22} \llbracket \mathbf{u} \rrbracket \underline{I}$  can only induce a control on the jumps of the *normal* component of the velocity, it is clear that we can always take  $D_{22} \equiv 0$ .

**2.2. The mixed setting.** The study of the LDG method is greatly facilitated if we recast its formulation in a classical mixed finite element setting. To do that, we denote by  $\mathcal{E}_i$  the *union* of all interior faces of the triangulation  $\mathcal{T}$  and by  $\mathcal{E}_D$  the union of faces lying on  $\partial\Omega$ . By summing equations (2.4), (2.5) and (2.6) over all elements and after simple algebraic manipulations, the LDG method can be reformulated more compactly as follows. Find  $(\underline{\sigma}_N, \mathbf{u}_N, p_N) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$  such that

$$\begin{aligned} a(\underline{\sigma}_N, \underline{\tau}) + b(\mathbf{u}_N, \underline{\tau}) &= f(\underline{\tau}), \\ -b(\mathbf{v}, \underline{\sigma}_N) + c(\mathbf{u}_N, \mathbf{v}) + d(\mathbf{v}, p_N) &= g(\mathbf{v}), \\ -d(\mathbf{u}_N, q) + e(p_N, q) &= h(q) \end{aligned} \quad (2.11)$$

for all  $(\underline{\tau}, \mathbf{v}, q) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$ .

Here,

$$\begin{aligned} a(\underline{\sigma}, \underline{\tau}) &:= \int_{\Omega} \underline{\sigma} : \underline{\tau} \, dx, \\ b(\mathbf{u}, \underline{\tau}) &:= \sum_{K \in \mathcal{T}} \int_K \mathbf{u} \cdot \nabla \cdot \underline{\tau} \, dx - \int_{\mathcal{E}_i} (\{\{\mathbf{u}\}\} + \llbracket \mathbf{u} \rrbracket \cdot \mathbf{C}_{12}) \cdot \llbracket \underline{\tau} \rrbracket \, ds, \\ c(\mathbf{u}, \mathbf{v}) &:= \int_{\mathcal{E}_i} C_{11} \llbracket \mathbf{u} \rrbracket : \llbracket \mathbf{v} \rrbracket \, ds + \int_{\mathcal{E}_D} C_{11} (\mathbf{u} \otimes \mathbf{n}) : (\mathbf{v} \otimes \mathbf{n}) \, ds, \\ d(\mathbf{v}, p) &:= - \sum_{K \in \mathcal{T}} \int_K p \nabla \cdot \mathbf{v} \, dx + \int_{\mathcal{E}_i} (\{\{p\}\} - D_{12} \cdot \llbracket p \rrbracket) \llbracket \mathbf{v} \rrbracket \, ds + \int_{\mathcal{E}_D} p \mathbf{v} \cdot \mathbf{n} \, ds, \\ e(p, q) &:= \int_{\mathcal{E}_i} D_{11} \llbracket p \rrbracket \cdot \llbracket q \rrbracket \, ds, \end{aligned}$$

and

$$\begin{aligned} f(\underline{\tau}) &:= \int_{\mathcal{E}_{\mathcal{D}}} \mathbf{g}_{\mathcal{D}} \cdot \underline{\tau} \cdot \mathbf{n} \, ds, \\ g(\mathbf{v}) &:= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\mathcal{E}_{\mathcal{D}}} C_{11}(\mathbf{g}_{\mathcal{D}} \otimes \mathbf{n}) : (\mathbf{v} \otimes \mathbf{n}) \, ds, \\ h(q) &:= - \int_{\mathcal{E}_{\mathcal{D}}} \mathbf{g}_{\mathcal{D}} \cdot \mathbf{n} \, q \, ds. \end{aligned}$$

Note that, by integration by parts, the forms  $b$  and  $d$  can also be expressed as

$$\begin{aligned} b(\mathbf{u}, \underline{\tau}) &= - \sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u} : \underline{\tau} \, d\mathbf{x} + \int_{\mathcal{E}_i} (\llbracket \underline{\tau} \rrbracket - \llbracket \underline{\tau} \rrbracket \otimes \mathbf{C}_{12}) : \llbracket \mathbf{u} \rrbracket \, ds + \int_{\mathcal{E}_{\mathcal{D}}} \underline{\tau} : (\mathbf{u} \otimes \mathbf{n}) \, ds, \\ d(\mathbf{v}, p) &= \sum_{K \in \mathcal{T}} \int_K \mathbf{v} \cdot \nabla p \, d\mathbf{x} - \int_{\mathcal{E}_i} (\llbracket \mathbf{v} \rrbracket + \mathbf{D}_{12}[\mathbf{v}]) \cdot \llbracket p \rrbracket \, ds. \end{aligned}$$

Finally, in order to analyze the method, we write the mixed system (2.11) in the following equivalent form:

Find  $(\underline{\sigma}_N, \mathbf{u}_N, p_N) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$  such that

$$\mathcal{A}(\underline{\sigma}_N, \mathbf{u}_N, p_N; \underline{\tau}, \mathbf{v}, q) = \mathcal{F}(\underline{\tau}, \mathbf{v}, q) \quad (2.12)$$

for all  $(\underline{\tau}, \mathbf{v}, q) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$ , by setting

$$\begin{aligned} \mathcal{A}(\underline{\sigma}, \mathbf{u}, p; \underline{\tau}, \mathbf{v}, q) &:= a(\underline{\sigma}, \underline{\tau}) + b(\mathbf{u}, \underline{\tau}) - b(\mathbf{v}, \underline{\sigma}) + c(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, p) - d(\mathbf{u}, q) + e(p, q), \\ \mathcal{F}(\underline{\tau}, \mathbf{v}, q) &:= f(\underline{\tau}) + g(\mathbf{v}) + h(q). \end{aligned}$$

**2.3. Existence and uniqueness of LDG solutions.** Next, we show that the LDG method defines a unique approximate solution provided that for each element  $K \in \mathcal{T}$  the following mild conditions on the local spaces hold:

$$u \in \mathcal{V}(K) : \int_K \nabla u \cdot \mathbf{v} \, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in \mathcal{S}^d(K) \quad \text{implies} \quad \nabla u \equiv \mathbf{0} \text{ on } K, \quad (2.13)$$

$$q \in \mathcal{Q}(K) : \int_K \mathbf{v} \cdot \nabla q \, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in \mathcal{V}^d(K) \quad \text{implies} \quad \nabla q \equiv \mathbf{0} \text{ on } K. \quad (2.14)$$

See [8] for simple examples of local spaces not satisfying the above conditions.

**PROPOSITION 2.1** (Well-posedness of the LDG method). *Consider the LDG method defined by the weak formulation (2.4)–(2.6) and by the numerical fluxes given by (2.7)–(2.10). Suppose that the coefficients  $C_{11}$  are  $D_{11}$  positive. Finally, assume that the conditions (2.13) and (2.14) on the local spaces are satisfied. Then the LDG method defines a unique approximate solution  $(\underline{\sigma}_N, \mathbf{u}_N, p_N) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$ .*

*Proof.* It is enough to show that the only possible solution to the system (2.11) with  $\mathbf{f} = \mathbf{0}$  and  $\mathbf{g}_{\mathcal{D}} = \mathbf{0}$  is  $(\underline{\sigma}_N, \mathbf{u}_N, p_N) = (\underline{\mathbf{0}}, \mathbf{0}, 0)$ . Indeed, taking  $\underline{\tau} = \underline{\sigma}_N$ ,  $\mathbf{v} = \mathbf{u}_N$ ,  $q = p_N$  in (2.11) and adding the three equations yields

$$a(\underline{\sigma}_N, \underline{\sigma}_N) + c(\mathbf{u}_N, \mathbf{u}_N) + e(p_N, p_N) = 0,$$

which implies  $\underline{\sigma}_N = \underline{\mathbf{0}}$ ,  $\llbracket \mathbf{u}_N \rrbracket = \underline{\mathbf{0}}$  on  $\mathcal{E}_i$ ,  $\mathbf{u}_N = \mathbf{0}$  on  $\mathcal{E}_{\mathcal{D}}$ , and  $\llbracket p_N \rrbracket = 0$  on  $\mathcal{E}_i$  since the coefficients  $C_{11}$  and  $D_{11}$  are positive. Consequently, the first equation in (2.11) reads

$$\sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u}_N : \underline{\tau} \, d\mathbf{x} = 0, \quad \forall \underline{\tau} \in \underline{\Sigma}_N.$$



Assumption (2.13) implies that  $\nabla \mathbf{u}_N = \mathbf{0}$  on every  $K \in \mathcal{T}$ , and, since  $\llbracket \mathbf{u}_N \rrbracket = \mathbf{0}$  on  $\mathcal{E}_i$  and  $\mathbf{u}_N = \mathbf{0}$  on  $\mathcal{E}_D$ , we must have  $\mathbf{u}_N = \mathbf{0}$ .

Taking  $\underline{\sigma}_N = \mathbf{0}$  and  $\mathbf{u}_N = \mathbf{0}$ , the second equation in (2.11) becomes

$$\sum_{K \in \mathcal{T}} \int_K \mathbf{v} \cdot \nabla p_N \, d\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathbf{V}_N.$$

Analogously, we conclude from assumption (2.14) that  $\nabla p_N = \mathbf{0}$  on every  $K \in \mathcal{T}$ , and, since  $\llbracket p_N \rrbracket = 0$ , that  $p_N$  is a constant. Since we also require that  $\int_{\Omega} p_N \, d\mathbf{x} = 0$ , we conclude that  $p_N = 0$ .  $\square$

**2.4. A priori estimates.** In this section we state and discuss our a priori error bounds for the LDG method. We assume that every element  $K$  of the triangulation  $\mathcal{T}$  is *affinely equivalent*, see [10, Section 2.3], to one of several reference elements in an arbitrary but fixed set; this allows us to use elements of various shapes with possibly curved boundaries. For each  $K \in \mathcal{T}$ , we denote by  $h_K$  the diameter of  $K$  and by  $\rho_K$  the diameter of the biggest ball included in  $K$ ; we set, as usual,  $h := \max_{K \in \mathcal{T}} h_K$ . The triangulations we consider can have hanging nodes but have to be *regular*, that is, there exists a positive constant  $\sigma_1$  such that

$$\frac{h_K}{\rho_K} \leq \sigma_1, \quad \forall K \in \mathcal{T}; \quad (2.15)$$

see [10, Section 3.1]. Moreover, we let the maximum number of neighbors of a given element  $K$  be arbitrary but fixed. To formally state this property, we need to introduce the set  $\langle K, K' \rangle$  defined as

$$\langle K, K' \rangle = \begin{cases} \emptyset & \text{if } \text{meas}_{(d-1)}(\partial K \cap \partial K') = 0, \\ \text{interior of } \partial K \cap \partial K' & \text{otherwise.} \end{cases}$$

Thus, we assume that there exists a positive constant  $\sigma_2 < 1$  such that, for each element  $K \in \mathcal{T}$ ,

$$\sigma_2 \leq \frac{h_{K'}}{h_K} \leq \sigma_2^{-1} \quad \forall K' : \langle K, K' \rangle \neq \emptyset. \quad (2.16)$$

These three hypotheses allow for quite general triangulations and are not restrictive in practice.

We assume that the local finite element spaces satisfy the following *inclusions*, for  $i = 1, \dots, d$ ,

$$\partial_i \mathcal{V}(K) \subseteq \mathcal{S}(K), \quad \partial_i \mathcal{S}(K) \subseteq \mathcal{V}(K), \quad \partial_i \mathcal{V}(K) \subseteq \mathcal{Q}(K), \quad \partial_i \mathcal{Q}(K) \subseteq \mathcal{V}(K). \quad (2.17)$$

Note that (2.17) also implies the assumptions (2.13) and (2.14) on the local spaces.

We denote by  $P^\kappa(K)$  the set of all polynomials of degree at most  $\kappa$  on  $K$ , and by  $Q^\kappa(K)$  the polynomials of degree at most  $\kappa$  in each variable. Then, in order to guarantee certain approximation properties of the local spaces, we assume that they contain at least the following polynomial spaces

$$P^k(K) \subseteq \mathcal{V}(K), \quad P^l(K) \subseteq \mathcal{S}(K), \quad P^m(K) \subseteq \mathcal{Q}(K), \quad (2.18)$$

with approximation orders  $k \geq 1$  and  $l, m \geq 0$ . Since  $\partial_i P^k(K) \subset P^{k-1}(K)$  and  $\partial_i Q^k(K) \subset Q^k(K)$ , conditions (2.17) and (2.18) are satisfied, for example, by

$$\mathcal{V}(K) = P^k(K), \quad \mathcal{S}(K) = P^l(K), \quad \mathcal{Q}(K) = P^m(K), \quad (2.19)$$

with  $k \geq 1$ ,  $l = k$  or  $l = k - 1$ , and  $m = k$  or  $m = k - 1$ , or by

$$\mathcal{V}(K) = Q^k(K), \quad \mathcal{S}(K) = Q^k(K), \quad \mathcal{Q}(K) = Q^k(K), \quad k \geq 1. \quad (2.20)$$

Next, we introduce a seminorm that appears in a natural way in the analysis of LDG methods. We denote by  $H^s(D)$ ,  $D$  being a domain in  $\mathbb{R}^d$ , the Sobolev spaces of integer orders, and by  $\|\cdot\|_{s,D}$  and  $|\cdot|_{s,D}$  the usual norms and seminorms in  $H^s(D)$ ,  $H^s(D)^d$  and  $H^s(D)^{d^2}$ ; we omit the dependence on the domain in the norms whenever  $D = \Omega$ . We define

$$|(\underline{\sigma}, \mathbf{u}, p)|_{\mathcal{A}}^2 := \|\underline{\sigma}\|_0^2 + \Theta^2(\mathbf{u}, p),$$

where

$$\Theta^2(\mathbf{u}, p) = \int_{\mathcal{E}_i} \left( C_{11} \llbracket \mathbf{u} \rrbracket^2 + D_{11} \llbracket p \rrbracket^2 \right) ds + \int_{\mathcal{E}_D} C_{11} (\mathbf{u} \otimes \mathbf{n})^2 ds.$$

We assume that the stabilization coefficients  $C_{11}$  and  $D_{11}$  defining the numerical fluxes in (2.7) and (2.9) are given by

$$C_{11}(\mathbf{x}) = \begin{cases} c_{11} \min\{h_{K^+}^\gamma, h_{K^-}^\gamma\} & \text{if } \mathbf{x} \in \langle K^+, K^- \rangle, \\ c_{11} h_{K^+}^\gamma & \text{if } \mathbf{x} \in \partial K^+ \cap \partial \Omega, \end{cases} \quad (2.21)$$

$$D_{11}(\mathbf{x}) = d_{11} \min\{h_{K^+}^\delta, h_{K^-}^\delta\} \quad \mathbf{x} \in \langle K^+, K^- \rangle, \quad (2.22)$$

with  $c_{11}, d_{11} > 0$ ,  $-1 \leq \gamma, \delta \leq 1$  independent of the meshsize and  $|\mathbf{C}_{12}|$  as well as  $|\mathbf{D}_{12}|$  of order one.

We are now ready to state our a priori error estimates for the LDG method. The first result is concerned with the error in the seminorm  $|\cdot|_{\mathcal{A}}$  and the  $L^2$ -error in the pressure.

**THEOREM 2.2.** *Let  $(\underline{\sigma}, \mathbf{u}, p)$  be the solution of (1.2)–(1.5) and let  $(\underline{\sigma}_N, \mathbf{u}_N, p_N)$  be the approximate solution given by the LDG method (2.4)–(2.6) with numerical fluxes (2.7)–(2.10). Assume the hypotheses (2.15), (2.16) on the triangulations, the hypotheses (2.17), (2.18) on the local spaces, with approximation orders  $k \geq 1$  and  $l, m \geq 0$ , and the hypotheses (2.21), (2.22) on the form of the stabilization parameters. For  $\underline{\sigma} \in H^{l+1}(\Omega)^{d^2}$ ,  $\mathbf{u} \in H^{k+1}(\Omega)^d$  and  $p \in H^{m+1}(\Omega)$ , we have that the errors  $\underline{e}_\sigma = \underline{\sigma} - \underline{\sigma}_N$ ,  $\mathbf{e}_u = \mathbf{u} - \mathbf{u}_N$  and  $e_p = p - p_N$  satisfy*

$$|(\underline{e}_\sigma, \mathbf{e}_u, e_p)|_{\mathcal{A}} \leq C \left[ h^{l+\frac{1-\gamma}{2}} \|\underline{\sigma}\|_{l+1} + h^k \|\mathbf{u}\|_{k+1} + h^{m+\frac{1+\min(-\gamma, \delta)}{2}} \|p\|_{m+1} \right],$$

as well as

$$\|e_p\|_0 \leq C \left[ h^{l+\frac{1-\gamma}{2}} \|\underline{\sigma}\|_{l+1} + h^k \|\mathbf{u}\|_{k+1} + h^{m+\frac{1+\min(-\gamma, \delta)}{2}} \|p\|_{m+1} \right],$$

where the constants  $C$  solely depend on  $\Omega$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $c_{11}$ ,  $d_{11}$ ,  $d$ , and the dimensions of the local spaces, but are independent of the meshsize  $h$ .

To prove a priori bounds for the  $L^2$ -error in  $\mathbf{u}$ , we assume elliptic regularity, that is, we assume that the solution  $(\mathbf{z}, q)$  of the homogeneous Stokes problem

$$-\Delta \mathbf{z} + \nabla q = \boldsymbol{\lambda} \quad \text{in } \Omega, \quad (2.23)$$

$$\nabla \cdot \mathbf{z} = 0 \quad \text{in } \Omega, \quad (2.24)$$

$$\mathbf{z} = \mathbf{0} \quad \text{on } \partial \Omega, \quad (2.25)$$

with right-hand side  $\boldsymbol{\lambda} \in L^2(\Omega)^d$  satisfies the estimate

$$\|\boldsymbol{z}\|_2 + \|\boldsymbol{q}\|_1 \leq C \|\boldsymbol{\lambda}\|_0 \quad (2.26)$$

for a constant  $C > 0$  just depending on  $\Omega$ . For the inequality (2.26) to hold, certain restrictions on  $\Omega$  are necessary; see, for example, Proposition 2.3 in Témam [32].

**THEOREM 2.3.** *Under the same assumptions as in Theorem 2.2 and the elliptic regularity assumption (2.26), we have that*

$$\|\boldsymbol{e}_u\|_0 \leq C \left[ h^{l+\min(-\gamma,\delta)+1} \|\underline{\boldsymbol{\sigma}}\|_{l+1} + h^{k+\frac{1+\min(-\gamma,\delta)}{2}} \|\boldsymbol{u}\|_{k+1} + h^{m+\min(-\gamma,\delta)+1} \|\boldsymbol{p}\|_{m+1} \right]$$

with a constant  $C$  that solely depends on  $\Omega$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $c_{11}$ ,  $d_{11}$ ,  $d$ , the dimensions of the local spaces and the constant in (2.26), but that is independent of the meshsize  $h$ .

Let us briefly discuss the results of Theorems 2.2 and 2.3:

- We begin by noting that the convergence orders are limited by the exponent  $\min(-\gamma, \delta)$  and we conclude that, in order to maximize the orders of convergence of the LDG approximation,  $\gamma$  in (2.21) should be chosen to be negative and  $\delta$  in (2.22) to be positive. In fact, the quantity  $\min(-\gamma, \delta)$  achieves its maximum, 1, for  $\gamma = -1$ ,  $\delta = 1$ , i.e., for  $C_{11} = \mathcal{O}(1/h)$  and  $D_{11} = \mathcal{O}(h)$ .
- When  $P^k$ - or  $Q^k$ -elements with  $k \geq 1$  are used for all field variables, i.e, the local spaces are chosen as in (2.19) with  $l = k$  and  $m = k$  or as in (2.20), we obtain for smooth solutions  $\underline{\boldsymbol{\sigma}} \in H^{k+1}(\Omega)^{d^2}$ ,  $\boldsymbol{u} \in H^{k+1}(\Omega)$ ,  $\boldsymbol{p} \in H^{k+1}(\Omega)$ , the error bounds

$$|(\underline{\boldsymbol{e}}_\sigma, \boldsymbol{e}_u, \boldsymbol{e}_p)|_{\mathcal{A}} + \|\boldsymbol{e}_p\|_0 \leq Ch^k, \quad \|\boldsymbol{e}_u\|_0 \leq Ch^{k+\frac{1+\min(-\gamma,\delta)}{2}},$$

for  $\gamma, \delta \in [-1, 1]$ . These estimates are summarized in Table 2.1 for some choices of  $C_{11}$  and  $D_{11}$ . Although these rates are sharp in the sense that they are actually observed in the numerical experiments of section 4, they are not optimal in terms of the approximation properties of the FE spaces.

TABLE 2.1  
Orders of convergence for smooth solutions and  $P^k$ - or  $Q^k$ -elements.

$C_{11}$	$D_{11}$	$ (\underline{\boldsymbol{e}}_\sigma, \boldsymbol{e}_u, \boldsymbol{e}_p) _{\mathcal{A}}$	$\ \boldsymbol{e}_p\ _0$	$\ \boldsymbol{e}_u\ _0$
$\mathcal{O}(1), \mathcal{O}(1/h)$	$\mathcal{O}(1)$	$k$	$k$	$k + 1/2$
$\mathcal{O}(1)$	$\mathcal{O}(h)$	$k$	$k$	$k + 1/2$
$\mathcal{O}(1/h)$	$\mathcal{O}(h)$	$k$	$k$	$k + 1$

- If the  $P$ -elements used for  $\underline{\boldsymbol{\sigma}}$  and  $\boldsymbol{p}$  are of one order lower than the ones used for the velocities  $\boldsymbol{u}$ , i.e., if we consider  $P$ -elements as in (2.19) with  $l = m = k - 1$  and  $k \geq 1$ , then an *optimal* order error estimate is obtained for  $\gamma = -1$ ,  $\delta = 1$ , i.e., for  $C_{11} = \mathcal{O}(1/h)$ ,  $D_{11} = \mathcal{O}(h)$ . In this case we have for  $\underline{\boldsymbol{\sigma}} \in H^k(\Omega)^{d^2}$ ,  $\boldsymbol{u} \in H^{k+1}(\Omega)^d$  and  $\boldsymbol{p} \in H^k(\Omega)$  the estimate

$$|(\underline{\boldsymbol{e}}_\sigma, \boldsymbol{e}_u, \boldsymbol{e}_p)|_{\mathcal{A}} + \|\boldsymbol{e}_p\|_0 \leq Ch^k, \quad \|\boldsymbol{e}_u\|_0 \leq Ch^{k+1},$$

which is optimal in terms of the approximation properties and of the regularity requirements of the exact solution.

**2.5. The setting for the error analysis.** The purpose of this section is to display as clearly as possible the main ingredients of the proof of our a priori results in section 2.4. To do so, we base our analysis on a similar abstract setting as the one introduced in [8] for the Laplacian.

We split the error  $(\underline{e}_\sigma, \mathbf{e}_u, e_p) = (\underline{\sigma} - \underline{\sigma}_N, \mathbf{u} - \mathbf{u}_N, p - p_N)$  into the following sum:

$$(\underline{e}_\sigma, \mathbf{e}_u, e_p) = (\underline{\sigma} - \underline{\Pi}\underline{\sigma}, \mathbf{u} - \underline{\Pi}\mathbf{u}, p - \underline{\Pi}p) + (\underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p),$$

where  $\underline{\Pi} : \underline{\Sigma} \rightarrow \underline{\Sigma}_N$ ,  $\underline{\Pi} : \mathbf{V} \rightarrow \mathbf{V}_N$  and  $\underline{\Pi} : Q \rightarrow Q_N$  are fixed projections onto the corresponding finite element spaces.

**The basic ingredients.** The basic ingredients of our error analysis are two. The first one is, as it is classical in finite element error analysis, the so-called Galerkin orthogonality property, namely,

$$\mathcal{A}(\underline{e}_\sigma, \mathbf{e}_u, e_p; \underline{\mathcal{T}}, \mathbf{v}, q) = 0 \quad \forall (\underline{\mathcal{T}}, \mathbf{v}, q) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N. \quad (2.27)$$

This property is a straightforward consequence of the consistency of the numerical fluxes and is valid since  $(\mathbf{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$ .

The second ingredient is a couple of inequalities that reflect the approximation properties of the projections  $\underline{\Pi}$ ,  $\underline{\Pi}$  and  $\underline{\Pi}$ , namely, we assume that there exist error bounds  $K_{\mathcal{A}}$  and  $K_{\mathcal{B}}$  such that

$$|\mathcal{A}(\underline{\sigma} - \underline{\Pi}\underline{\sigma}, \mathbf{u} - \underline{\Pi}\mathbf{u}, p - \underline{\Pi}p; \underline{\mathcal{T}} - \underline{\Pi}\underline{\mathcal{T}}, \mathbf{v} - \underline{\Pi}\mathbf{v}, q - \underline{\Pi}q)| \leq CK_{\mathcal{A}}(\underline{\sigma}, \mathbf{u}, p; \underline{\mathcal{T}}, \mathbf{v}, q) \quad (2.28)$$

for any  $(\underline{\sigma}, \mathbf{u}, p), (\underline{\mathcal{T}}, \mathbf{v}, q) \in \underline{\Sigma} \times \mathbf{V} \times Q$ , and

$$|\mathcal{A}(\underline{\sigma} - \underline{\Pi}\underline{\sigma}, \pm(\mathbf{u} - \underline{\Pi}\mathbf{u}), p - \underline{\Pi}p; \underline{\mathcal{T}}, \pm\mathbf{v}, q)| \leq C |(\underline{\mathcal{T}}, \mathbf{v}, q)|_{\mathcal{A}} K_{\mathcal{B}}(\underline{\sigma}, \mathbf{u}, p) \quad (2.29)$$

for any  $(\underline{\mathcal{T}}, \mathbf{v}, q) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$  and  $(\underline{\sigma}, \mathbf{u}, p) \in \underline{\Sigma} \times \mathbf{V} \times Q$  and with constants  $C$  which are independent of the meshsize (specific forms for  $K_{\mathcal{A}}$  and  $K_{\mathcal{B}}$  shall be provided below). As we show next, all the error estimates we are interested in can be obtained in terms of  $K_{\mathcal{A}}$  and  $K_{\mathcal{B}}$ .

**Error in the  $\mathcal{A}$ -seminorm.** The error in  $|\cdot|_{\mathcal{A}}$  can be estimated as follows.

LEMMA 2.4. *We have*

$$|(\underline{e}_\sigma, \mathbf{e}_u, e_p)|_{\mathcal{A}} \leq CK_{\mathcal{A}}^{1/2}(\underline{\sigma}, \mathbf{u}, p; \underline{\sigma}, \mathbf{u}, p) + CK_{\mathcal{B}}(\underline{\sigma}, \mathbf{u}, p),$$

with  $C$  independent of the meshsize.

*Proof.* This is a straightforward extension of [8, Lemma 2.3]. We present the proof for the sake of completeness. Since  $|\cdot, \cdot, \cdot|_{\mathcal{A}}$  is a seminorm, we have

$$|(\underline{e}_\sigma, \mathbf{e}_u, e_p)|_{\mathcal{A}} \leq |(\underline{\sigma} - \underline{\Pi}\underline{\sigma}, \mathbf{u} - \underline{\Pi}\mathbf{u}, p - \underline{\Pi}p)|_{\mathcal{A}} + |(\underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p)|_{\mathcal{A}}.$$

By the definition of  $\mathcal{A}$  in (2.12), by Galerkin orthogonality (2.27), and by assumption (2.29),

$$\begin{aligned} |(\underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p)|_{\mathcal{A}}^2 &= \mathcal{A}(\underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p; \underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p) \\ &= \mathcal{A}(\underline{\Pi}\underline{\sigma} - \underline{\sigma}, \underline{\Pi}\mathbf{u} - \mathbf{u}, \underline{\Pi}p - p; \underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p) \\ &\leq C |(\underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p)|_{\mathcal{A}} K_{\mathcal{B}}(\underline{\sigma}, \mathbf{u}, p), \end{aligned}$$

we have that

$$|(\underline{\Pi}\underline{e}_\sigma, \underline{\Pi}\mathbf{e}_u, \underline{\Pi}e_p)|_{\mathcal{A}} \leq CK_{\mathcal{B}}(\underline{\sigma}, \mathbf{u}, p), \quad (2.30)$$

and so,

$$|(\underline{e}_\sigma, \mathbf{e}_u, e_p)|_{\mathcal{A}} \leq |(\underline{\sigma} - \underline{\Pi}\sigma, \mathbf{u} - \mathbf{\Pi}\mathbf{u}, p - \Pi p)|_{\mathcal{A}} + CK_B(\underline{\sigma}, \mathbf{u}, p).$$

The estimate now follows from a simple application of assumption (2.28). This completes the proof.  $\square$

**Error in the pressure.** To obtain an error estimate in the pressure, we shall prove a stability result which allows us to measure the error of the pressure in the  $L^2$ -norm. It can be viewed as a discrete counter-part of the standard continuous inf-sup condition for the Stokes problem (see e.g., [6, 24]), adapted to the discontinuous spaces considered here. Its proof is obtained by following the techniques used by Franca and Stenberg [23] in subsection 3.4 below.

**PROPOSITION 2.5.** *Assume that  $C_{11}$  and  $D_{11}$  are of the form (2.21) and (2.22), respectively. Then there exist positive constants  $\kappa_1$  and  $\kappa_2$  independent of the meshsize such that for all  $(\underline{\mathcal{T}}, \mathbf{v}, q) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$  there is a  $\mathbf{w} \in \mathbf{V}_N$  with*

$$\mathcal{A}(\underline{\mathcal{T}}, \mathbf{v}, q; \underline{\mathcal{Q}}, \mathbf{w}, 0) \geq \kappa_1 \|q\|_0^2 - \kappa_2 |(\underline{\mathcal{T}}, \mathbf{v}, q)|_{\mathcal{A}}^2, \quad |(\underline{\mathcal{Q}}, \mathbf{w}, 0)|_{\mathcal{A}} = \Theta(\mathbf{w}, 0) \leq \|q\|_0. \quad (2.32)$$

Based in this inf-sup condition we obtain the following estimate for  $e_p$ .

**LEMMA 2.6.** *We have*

$$\|e_p\|_0 \leq \|p - \Pi p\|_0 + CK_B(\underline{\sigma}, \mathbf{u}, p),$$

with  $C$  independent of the meshsize.

*Proof.* We only have to find an estimate for  $\|\Pi e_p\|_0$ , since we trivially have  $\|p - p_N\|_0 \leq \|p - \Pi p\|_0 + \|\Pi e_p\|_0$ . To do that, we see that, by Proposition 2.5, there exists a test function  $\mathbf{w} \in \mathbf{V}_N$  such that (2.32) is satisfied for  $(\underline{\mathcal{T}}, \mathbf{v}, q) = (\underline{\Pi}\underline{e}_\sigma, \mathbf{\Pi}\mathbf{e}_u, \Pi e_p)$ . By (2.32), Galerkin orthogonality (2.27), assumption (2.29), estimate (2.30), the Cauchy-Schwarz inequality and the properties of  $\mathbf{w}$ , we obtain

$$\begin{aligned} \kappa_1 \|\Pi e_p\|_0^2 &\leq \mathcal{A}(\underline{\Pi}\underline{e}_\sigma, \mathbf{\Pi}\mathbf{e}_u, \Pi e_p; \underline{\mathcal{Q}}, \mathbf{w}, 0) + \kappa_2 |(\underline{\Pi}\underline{e}_\sigma, \mathbf{\Pi}\mathbf{e}_u, \Pi e_p)|_{\mathcal{A}}^2 \\ &= \mathcal{A}(\underline{\Pi}\underline{\sigma} - \underline{\sigma}, \mathbf{\Pi}\mathbf{u} - \mathbf{u}, \Pi p - p; 0, \mathbf{w}, 0) + \kappa_2 |(\underline{\Pi}\underline{e}_\sigma, \mathbf{\Pi}\mathbf{e}_u, \Pi e_p)|_{\mathcal{A}}^2 \\ &\leq C_1 |(\underline{\mathcal{Q}}, \mathbf{w}, 0)|_{\mathcal{A}} K_B(\underline{\sigma}, \mathbf{u}, p) + C_2 K_B(\underline{\sigma}, \mathbf{u}, p)^2 \\ &\leq \frac{C_1}{2\varepsilon} \|\Pi e_p\|_0^2 + (C_1 \frac{\varepsilon}{2} + C_2) K_B(\underline{\sigma}, \mathbf{u}, p)^2 \end{aligned}$$

for all  $\varepsilon > 0$ . We can now choose  $\varepsilon$  in such a way that

$$\|\Pi e_p\|_0 \leq CK_B(\underline{\sigma}, \mathbf{u}, p)$$

with a constant  $C$  depending on  $C_1$  and  $C_2$ . The assertion follows.  $\square$

**Error in the velocity.** The estimate for the error  $\|\mathbf{e}_u\|_0$  is based on a duality argument similar to the one used in [8].

**LEMMA 2.7.** *Assume that the elliptic regularity inequality (2.26) holds. Then, we have*

$$\|\mathbf{e}_u\|_0 \leq C \sup_{\boldsymbol{\lambda} \in L^2(\Omega)^d} \frac{K_{\mathcal{A}}(\underline{\sigma}, \mathbf{u}, p; \underline{\zeta}, \mathbf{z}, \tilde{q})}{\|\boldsymbol{\lambda}\|_0} + CK_B(\underline{\sigma}, \mathbf{u}, p) \sup_{\boldsymbol{\lambda} \in L^2(\Omega)^d} \frac{K_{\mathcal{B}}(\underline{\zeta}, \mathbf{z}, \tilde{q})}{\|\boldsymbol{\lambda}\|_0}, \quad (2.33)$$

with  $(\mathbf{z}, q)$  denoting the solution of (2.23)–(2.25) with right-hand side  $\boldsymbol{\lambda}$  and  $\underline{\zeta} = -\nabla \mathbf{z}$ ,  $\tilde{q} = -q$ .

*Proof.* We introduce the linear functional  $\Lambda(\mathbf{u}) = (\boldsymbol{\lambda}, \mathbf{u})$ , where  $(\cdot, \cdot)$  denotes the  $L^2(\Omega)^d$ -inner product. Then we have

$$\|e_u\|_0 = \sup_{\boldsymbol{\lambda} \in L^2(\Omega)^d} \frac{\Lambda(e_u)}{\|\boldsymbol{\lambda}\|_0}. \quad (2.34)$$

Now, let  $(\mathbf{z}, q)$  be the solution of the adjoint equation (2.23)–(2.25) with right-hand side  $\boldsymbol{\lambda}$ . It is easy to verify that, if we set  $\underline{\zeta} = -\nabla \mathbf{z}$ ,  $\tilde{q} = -q$ , we have

$$\mathcal{A}(-\underline{\zeta}, \mathbf{z}, -\tilde{q}; -\underline{\boldsymbol{\tau}}, \mathbf{w}, -r) = \Lambda(\mathbf{w}),$$

for all  $(\underline{\boldsymbol{\tau}}, \mathbf{w}, r) \in \underline{\Sigma} \times \mathbf{V} \times Q$ . Taking  $(\underline{\boldsymbol{\tau}}, \mathbf{w}, r) = (\underline{e}_\sigma, e_u, e_p)$ , we get by the definition of  $\mathcal{A}$  in (2.12) and by Galerkin orthogonality (2.27)

$$\begin{aligned} \Lambda(e_u) &= \mathcal{A}(-\underline{\zeta}, \mathbf{z}, -\tilde{q}; -\underline{e}_\sigma, e_u, -e_p) \\ &= \mathcal{A}(\underline{e}_\sigma, e_u, e_p; \underline{\zeta}, \mathbf{z}, \tilde{q}) \\ &= \mathcal{A}(\underline{e}_\sigma, e_u, e_p; \underline{\zeta} - \underline{\Pi}\underline{\zeta}, \mathbf{z} - \mathbf{\Pi}\mathbf{z}, \tilde{q} - \mathbf{\Pi}\tilde{q}) \\ &= \mathcal{A}(\underline{\Pi}e_\sigma, \mathbf{\Pi}e_u, \mathbf{\Pi}e_p; \underline{\zeta} - \underline{\Pi}\underline{\zeta}, \mathbf{z} - \mathbf{\Pi}\mathbf{z}, \tilde{q} - \mathbf{\Pi}\tilde{q}) \\ &\quad + \mathcal{A}(\underline{\sigma} - \underline{\Pi}\underline{\sigma}, \mathbf{u} - \mathbf{\Pi}\mathbf{u}, p - \mathbf{\Pi}p; \underline{\zeta} - \underline{\Pi}\underline{\zeta}, \mathbf{z} - \mathbf{\Pi}\mathbf{z}, \tilde{q} - \mathbf{\Pi}\tilde{q}). \end{aligned}$$

We obtain with assumption (2.29) and estimate (2.30)

$$\begin{aligned} &|\mathcal{A}(\underline{\Pi}e_\sigma, \mathbf{\Pi}e_u, \mathbf{\Pi}e_p; \underline{\zeta} - \underline{\Pi}\underline{\zeta}, \mathbf{z} - \mathbf{\Pi}\mathbf{z}, \tilde{q} - \mathbf{\Pi}\tilde{q})| \\ &= |\mathcal{A}(\underline{\zeta} - \underline{\Pi}\underline{\zeta}, -(\mathbf{z} - \mathbf{\Pi}\mathbf{z}), \tilde{q} - \mathbf{\Pi}\tilde{q}; \underline{\Pi}e_\sigma, -\mathbf{\Pi}e_u, \mathbf{\Pi}e_p)| \leq CK_B(\underline{\sigma}, \mathbf{u}, p) K_B(\underline{\zeta}, \mathbf{z}, \tilde{q}), \end{aligned}$$

and hence,

$$\begin{aligned} |\Lambda(e_u)| &\leq CK_B(\underline{\sigma}, \mathbf{u}, p) K_B(\underline{\zeta}, \mathbf{z}, \tilde{q}) \\ &\quad + |\mathcal{A}(\underline{\sigma} - \underline{\Pi}\underline{\sigma}, \mathbf{u} - \mathbf{\Pi}\mathbf{u}, p - \mathbf{\Pi}p; \underline{\zeta} - \underline{\Pi}\underline{\zeta}, \mathbf{z} - \mathbf{\Pi}\mathbf{z}, \tilde{q} - \mathbf{\Pi}\tilde{q})|. \end{aligned}$$

The estimate now follows from a simple application of assumption (2.28) and from the characterization (2.34) of the  $L^2$ -norm.  $\square$

**Conclusion.** Thus, in order to prove our a priori estimates, all we need to do is to obtain the functionals  $K_{\mathcal{A}}$  and  $K_B$ , as well as the stability estimate in Proposition 2.5; this will be carried out in the next section. Then, Theorems 2.2 and 2.3 will immediately follow after a simple application of Lemmas 2.4, 2.6 and 2.7.

**3. Proofs.** In this section, we prove our main results in the setting of section 2.5. We proceed as follows. After presenting some preliminary results, we obtain the functional  $K_{\mathcal{A}}$  for general projection operators  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\mathbf{\Pi}$ . To obtain the functional  $K_B$ , the projections  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\mathbf{\Pi}$  are chosen as  $L^2$ -projections.

**3.1. Preliminaries.** The following two lemmas contain all the information we actually use about our finite elements. The first one is a standard approximation result, valid for any linear continuous and polynomial preserving operator  $\mathbf{\Pi}$  from  $H^{s+1}(K)$  onto a finite dimensional space  $\mathcal{N}(K) \supset P^\kappa(K)$ ; it can be easily obtained by using the techniques of [10]. The second one is a standard inverse inequality.

**LEMMA 3.1.** *Let  $\mathbf{\Pi}$  be a linear continuous operator from  $H^{s+1}(K)$ ,  $s \geq 0$ , onto  $\mathcal{N}(K) \supset P^\kappa(K)$  such that  $\mathbf{\Pi}w = w$  for all  $w \in P^\kappa(K)$ ,  $\kappa \geq 0$ . Then we have*

$$\begin{aligned} |w - \mathbf{\Pi}w|_{r,K} &\leq Ch_K^{\min(s,\kappa)+1-r} \|w\|_{s+1,K}, \quad r = 0, 1, \\ \|w - \mathbf{\Pi}w\|_{0,\partial K} &\leq Ch_K^{\min(s,\kappa)+\frac{1}{2}} \|w\|_{s+1,K}, \end{aligned}$$

for some constant  $C$  that solely depends on  $\sigma_1$  in inequality (2.15), the dimension of  $\mathcal{N}(K)$ ,  $d$  and  $s$ .

LEMMA 3.2. *There exists a positive constant  $C_{\text{inv}}$  that solely depends on  $\sigma_1$  in inequality (2.15), the dimension of  $\mathcal{N}(K)$  and  $d$ , such that for all  $s \in \mathcal{N}(K)$  we have  $s_{0,\partial K} \leq C_{\text{inv}} h_K^{-\frac{1}{2}} \|s\|_{0,K}$ , for all  $K \in \mathcal{T}$ .*

Let  $\underline{\Pi} : \underline{\Sigma} \rightarrow \underline{\Sigma}_N$ ,  $\mathbf{\Pi} : \mathbf{V} \rightarrow \mathbf{V}_N$  and  $\Pi : Q \rightarrow Q_N$  be projection operators onto the corresponding FE spaces satisfying (componentwise) the assumptions in Lemma 3.1. We will make use of the following short-hand notation

$$\underline{\xi}_\sigma = \underline{\sigma} - \underline{\Pi}\sigma, \quad \xi_u = \mathbf{u} - \mathbf{\Pi}\mathbf{u}, \quad \xi_p = p - \Pi p,$$

for  $(\underline{\sigma}, \mathbf{u}, p) \in \underline{\Sigma} \times \mathbf{V} \times Q$ . We also define the quantities

$$\begin{aligned} \underline{C}_{11}^{\partial K} &:= \inf\{C_{11}(\mathbf{x}) : \mathbf{x} \in \partial K\}, & \overline{C}_{11}^{\partial K} &:= \sup\{C_{11}(\mathbf{x}) : \mathbf{x} \in \partial K\}, \\ \underline{D}_{11}^{\partial K} &:= \inf\{D_{11}(\mathbf{x}) : \mathbf{x} \in \partial K \setminus \partial\Omega\}, & \overline{D}_{11}^{\partial K} &:= \sup\{D_{11}(\mathbf{x}) : \mathbf{x} \in \partial K \setminus \partial\Omega\}. \end{aligned}$$

**3.2. The functional  $K_{\mathcal{A}}$ .** Using Cauchy-Schwarz's inequality, the approximation properties in Lemma 3.1 and the assumptions (2.15) and (2.16) on the meshes, we can prove, in exactly the same way as in [8, Section 3.2], the following approximation results for the LDG forms.

LEMMA 3.3. *Assume (2.15), (2.16) and (2.18). Let  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\Pi$  be projection operators satisfying (componentwise with  $\kappa = k$ ,  $\kappa = l$ , and  $\kappa = m$ , respectively) the assumptions in Lemma 3.1. Let  $\underline{\sigma} \in H^{r+1}(\Omega)^{d^2}$ ,  $\underline{\tau} \in H^{\bar{r}+1}(\Omega)^{d^2}$ ,  $\mathbf{u} \in H^{s+1}(\Omega)^d$ ,  $\mathbf{v} \in H^{\bar{s}+1}(\Omega)^d$ ,  $p \in H^{t+1}(\Omega)$  and  $q \in H^{\bar{t}+1}(\Omega)$  for  $r, \bar{r}, s, \bar{s}, t, \bar{t} \geq 0$ . Then we have*

$$\begin{aligned} |a(\underline{\xi}_\sigma, \underline{\xi}_\tau)| &\leq C \left( \sum_{K \in \mathcal{T}} h_K^{2\min(r,l)+2} \|\underline{\sigma}\|_{r+1,K}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}} h_K^{2\min(\bar{r},l)+2} \|\underline{\tau}\|_{\bar{r}+1,K}^2 \right)^{\frac{1}{2}}, \\ |b(\xi_u, \xi_\tau)| &\leq C \left( \sum_{K \in \mathcal{T}} h_K^{2\min(s,k)} \|\mathbf{u}\|_{s+1,K}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}} h_K^{2\min(\bar{r},l)+2} \|\underline{\tau}\|_{\bar{r}+1,K}^2 \right)^{\frac{1}{2}}, \\ |c(\xi_u, \xi_v)| &\leq C \left( \sum_{K \in \mathcal{T}} \overline{C}_{11}^{\partial K} h_K^{2\min(s,k)+1} \|\mathbf{u}\|_{s+1,K}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}} \overline{C}_{11}^{\partial K} h_K^{2\min(\bar{s},k)+1} \|\mathbf{v}\|_{\bar{s}+1,K}^2 \right)^{\frac{1}{2}}, \\ |d(\xi_u, \xi_q)| &\leq C \left( \sum_{K \in \mathcal{T}} h_K^{2\min(s,k)} \|\mathbf{u}\|_{s+1,K}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}} h_K^{2\min(\bar{t},m)+2} \|q\|_{\bar{t}+1,K}^2 \right)^{\frac{1}{2}}, \\ |e(\xi_p, \xi_q)| &\leq C \left( \sum_{K \in \mathcal{T}} \overline{D}_{11}^{\partial K} h_K^{2\min(t,m)+1} \|p\|_{t+1,K}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}} \overline{D}_{11}^{\partial K} h_K^{2\min(\bar{t},m)+1} \|q\|_{\bar{t}+1,K}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

with constants  $C$  independent of the meshsize.

For the special form of  $C_{11}$  and  $D_{11}$  proposed in (2.21) and (2.22), respectively, we have as a consequence of Lemma 3.3 and (2.16) the following result.

COROLLARY 3.4. *Under the same assumptions as in Lemma 3.3 and for coefficients  $C_{11}$  and  $D_{11}$  of the form (2.21) and (2.22), respectively, we have*

$$\begin{aligned} |a(\underline{\xi}_\sigma, \underline{\xi}_\tau)| &\leq C h^{\min(r,l)+\min(\bar{r},l)+2} \|\underline{\sigma}\|_{r+1} \|\underline{\tau}\|_{\bar{r}+1}, \\ |b(\xi_u, \xi_\tau)| &\leq C h^{\min(s,k)+\min(\bar{r},l)+1} \|\mathbf{u}\|_{s+1} \|\underline{\tau}\|_{\bar{r}+1}, \\ |c(\xi_u, \xi_v)| &\leq C h^{\min(s,k)+\min(\bar{s},k)+1+\gamma} \|\mathbf{u}\|_{s+1} \|\mathbf{v}\|_{\bar{s}+1}, \\ |d(\xi_u, \xi_q)| &\leq C h^{\min(s,k)+\min(\bar{t},m)+1} \|\mathbf{u}\|_{s+1} \|q\|_{\bar{t}+1}, \\ |e(\xi_p, \xi_q)| &\leq C h^{\min(t,m)+\min(\bar{t},m)+1+\delta} \|p\|_{t+1} \|q\|_{\bar{t}+1}, \end{aligned}$$

with constants  $C$  independent of the meshsize.

From Corollary 3.4 we immediately obtain a general expression for the functional  $K_{\mathcal{A}}$  since

$$\begin{aligned} \mathcal{A}(\underline{\xi}_\sigma, \underline{\xi}_u, \xi_p; \underline{\xi}_\tau, \underline{\xi}_v, \xi_q) &= a(\underline{\xi}_\sigma, \underline{\xi}_\tau) + b(\underline{\xi}_u, \underline{\xi}_\tau) - b(\underline{\xi}_v, \underline{\xi}_\sigma) \\ &\quad + c(\underline{\xi}_u, \underline{\xi}_v) + d(\underline{\xi}_v, \xi_p) - d(\underline{\xi}_u, \xi_q) + e(\xi_p, \xi_q). \end{aligned} \quad (3.1)$$

In the situations encountered in Lemmas 2.4 and 2.7 we obtain the following results.

**COROLLARY 3.5.** *Assume (2.15), (2.16) and (2.18) with approximation orders  $k \geq 1$ ,  $l, m \geq 0$ . Assume the coefficients  $C_{11}$  and  $D_{11}$  to be of the form (2.21) and (2.22), respectively. Let  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\Pi$  be projection operators as in Lemma 3.3. Let  $\underline{\sigma} \in H^{l+1}(\Omega)^{d^2}$ ,  $\mathbf{u} \in H^{k+1}(\Omega)^d$  and  $p \in H^{m+1}(\Omega)$ . Then we have in Lemma 2.4*

$$K_{\mathcal{A}}(\underline{\sigma}, \mathbf{u}, p; \underline{\sigma}, \mathbf{u}, p) \leq C \left[ h^{2l+2} \|\underline{\sigma}\|_{l+1}^2 + h^{2k+1+\gamma} \|\mathbf{u}\|_{k+1}^2 + h^{2m+1+\delta} \|p\|_{m+1}^2 \right].$$

Furthermore, assume the elliptic regularity inequality (2.26) and let  $(\mathbf{z}, q)$  denote the solution of (2.23)–(2.25) with right-hand side  $\boldsymbol{\lambda} \in L^2(\Omega)^d$ ,  $\underline{\zeta} = -\nabla \mathbf{z}$ ,  $\tilde{q} = -q$ . Then we have in Lemma 2.7

$$K_{\mathcal{A}}(\underline{\sigma}, \mathbf{u}, p; \underline{\zeta}, \mathbf{z}, \tilde{q}) \leq C \left[ h^{2l} \|\underline{\sigma}\|_{l+1} + h^{1+k} \|\mathbf{u}\|_{k+1} + h^{1+\delta+m} \|p\|_{m+1} \right] \|\boldsymbol{\lambda}\|_0.$$

*Proof.* The assertions follow immediately from Corollary 3.4, the identity (3.1), the choice of the coefficients  $C_{11}$  and  $D_{11}$ , and from the elliptic regularity estimate (2.26) which yields  $\|\underline{\zeta}\|_1 + \|\mathbf{u}\|_2 + \|\tilde{q}\|_1 \leq C \|\boldsymbol{\lambda}\|_0$ .  $\square$

**3.3. The functional  $K_{\mathcal{B}}$ .** In this subsection we determine the functional  $K_{\mathcal{B}}$  reflecting the approximation properties in equation (2.29). We start by investigating the forms  $a$ ,  $c$  and  $d$ . Lemma 3.1 and Cauchy-Schwarz's inequality immediately give the following estimates.

**LEMMA 3.6.** *Assume (2.15), (2.16) and (2.18). Let  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\Pi$  be projection operators satisfying (componentwise with  $\kappa = k$ ,  $\kappa = l$ , and  $\kappa = m$ , respectively) the assumptions in Lemma 3.1. Let  $\underline{\sigma} \in H^{r+1}(\Omega)^{d^2}$ ,  $\mathbf{u} \in H^{s+1}(\Omega)^d$  and  $p \in H^{t+1}(\Omega)$  for  $r, s, t \geq 0$ . Then we have*

$$\begin{aligned} |a(\underline{\xi}_\sigma, \underline{\mathcal{I}})| &\leq C \left( \sum_{K \in \mathcal{T}} h_K^{2\min(r,l)+2} \|\underline{\sigma}\|_{r+1,K}^2 \right)^{\frac{1}{2}} \|\underline{\mathcal{I}}\|_0, & \forall \underline{\mathcal{I}} \in \underline{\Sigma}, \\ |c(\underline{\xi}_u, \mathbf{v})| &\leq C \left( \sum_{K \in \mathcal{T}} \overline{C}_{11}^{\partial K} h_K^{2\min(s,k)+1} \|\mathbf{u}\|_{s+1,K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{v}, 0), & \forall \mathbf{v} \in \mathbf{V}, \\ |e(\xi_p, q)| &\leq C \left( \sum_{K \in \mathcal{T}} \overline{D}_{11}^{\partial K} h_K^{2\min(t,m)+1} \|p\|_{t+1,K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{0}, q), & \forall q \in Q, \end{aligned}$$

with constants  $C$  independent of the meshsize.

Next, we estimate the forms  $b$  and  $d$  in the case where  $\underline{\Pi} : \underline{\Sigma} \rightarrow \underline{\Sigma}_N$ ,  $\mathbf{\Pi} : \mathbf{V} \rightarrow \mathbf{V}_N$  and  $\Pi : Q \rightarrow Q_N$  are chosen to be  $L^2$ -projections. Note that these projections clearly satisfy the assumptions of Lemma 3.1. It is also important to note that this is the *only* part of our analysis in which we actually use the inclusion properties (2.17).

**LEMMA 3.7.** *Assume (2.15), (2.16) and (2.17), (2.18). Let  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\Pi$  be the (componentwise)  $L^2$ -projections onto the corresponding finite element spaces. Let  $\underline{\sigma} \in$*



$H^{r+1}(\Omega)^{d^2}$ ,  $\mathbf{u} \in H^{s+1}(\Omega)^d$  and  $p \in H^{t+1}(\Omega)$  for  $r, s, t \geq 0$ . Then we have

$$\begin{aligned} |b(\boldsymbol{\xi}_u, \boldsymbol{\tau})| &\leq C \left( \sum_{K \in \mathcal{T}} h_K^{2\min(s,k)} \|\mathbf{u}\|_{s+1,K}^2 \right)^{\frac{1}{2}} \|\boldsymbol{\tau}\|_0, & \forall \boldsymbol{\tau} \in \underline{\Sigma}_N, \\ |b(\mathbf{v}, \boldsymbol{\xi}_\sigma)| &\leq C \left( \sum_{K \in \mathcal{T}} \frac{1}{\underline{C}_{11}^{\partial K}} h_K^{2\min(r,l)+1} \|\boldsymbol{\sigma}\|_{r+1,K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{v}, 0), & \forall \mathbf{v} \in \mathbf{V}_N, \\ |d(\boldsymbol{\xi}_u, q)| &\leq C \left( \sum_{K \in \mathcal{T}} \frac{1}{\underline{D}_{11}^{\partial K}} h_K^{2\min(s,k)+1} \|\mathbf{u}\|_{s+1,K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{0}, q), & \forall q \in Q_N, \\ |d(\mathbf{v}, \boldsymbol{\xi}_p)| &\leq C \left( \sum_{K \in \mathcal{T}} \frac{1}{\underline{C}_{11}^{\partial K}} h_K^{2\min(t,m)+1} \|p\|_{t+1,K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{v}, 0), & \forall \mathbf{v} \in \mathbf{V}_N, \end{aligned}$$

with constants  $C$  independent of the meshsize.

*Proof.* We start by proving the estimates for the form  $b$ .

We note that  $\int_K (\mathbf{u} - \boldsymbol{\Pi}\mathbf{u}) \cdot \nabla \cdot \boldsymbol{\tau} \, dx = 0$  due to the properties of the  $L^2$ -projection and the inclusion property  $\partial_i \mathcal{S}(K) \subset \mathcal{V}(K)$  in (2.17). Therefore, using the fact that  $\mathbf{C}_{12}$  is of order one, a repeated application of Cauchy-Schwarz's inequality gives

$$\begin{aligned} |b(\boldsymbol{\xi}_u, \boldsymbol{\tau})| &= \left| \int_{\mathcal{E}_i} (\{\{\boldsymbol{\xi}_u\}\} + \llbracket \boldsymbol{\xi}_u \rrbracket \cdot \mathbf{C}_{12}) \cdot \llbracket \boldsymbol{\tau} \rrbracket \, ds \right| \\ &\leq C \left( \sum_{K \in \mathcal{T}} h_K^{-1} \|\boldsymbol{\xi}_u\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}} \hat{h}_K \|\boldsymbol{\tau}\|_{0,\partial K}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where  $\hat{h}_K = \sup\{h_{K'} : \langle K, K' \rangle \neq \emptyset\}$ . Assumption (2.16) implies that  $\hat{h}_K \leq \sigma_2^{-1} h_K$  and, therefore, the desired estimate follows from Lemma 3.1 and the inverse inequality in Lemma 3.2.

Furthermore, we also note that  $\int_K \nabla \mathbf{v} : (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}) \, dx = 0$ , since  $\boldsymbol{\Pi}|_K$  is the  $L^2$ -projection into  $\mathcal{S}(K)$  and  $\partial_i \mathcal{V}(K) \subset \mathcal{S}(K)$  in (2.17). Thus, we obtain

$$\begin{aligned} |b(\mathbf{v}, \boldsymbol{\xi}_\sigma)| &= \left| \int_{\mathcal{E}_i} (\{\{\boldsymbol{\xi}_\sigma\}\} - \llbracket \boldsymbol{\xi}_\sigma \rrbracket \otimes \mathbf{C}_{12}) : \llbracket \mathbf{v} \rrbracket \, ds + \int_{\mathcal{E}_D} \boldsymbol{\xi}_\sigma : (\mathbf{v} \otimes \mathbf{n}) \, ds \right| \\ &\leq \left( \int_{\mathcal{E}_i} \frac{1}{\underline{C}_{11}} (\{\{\boldsymbol{\xi}_\sigma\}\} - \llbracket \boldsymbol{\xi}_\sigma \rrbracket \otimes \mathbf{C}_{12})^2 \, ds + \int_{\mathcal{E}_D} \frac{1}{\underline{C}_{11}} \boldsymbol{\xi}_\sigma^2 \, ds \right)^{\frac{1}{2}} \Theta(\mathbf{v}, 0). \\ &\leq C \left( \sum_{K \in \mathcal{T}} \frac{1}{\underline{C}_{11}^{\partial K}} \|\boldsymbol{\xi}_\sigma\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{v}, 0). \end{aligned}$$

The second estimate for the form  $b$  follows from Lemma 3.1.

The estimates for  $d$  are obtained in a similar way from Lemma 3.1, observing again that  $\mathbf{D}_{12}$  is of order one and that the volume terms vanish due to the properties of the  $L^2$ -projections and the inclusions in (2.17). Thus, using the inclusion  $\partial_i \mathcal{Q}(K) \subset \mathcal{V}(K)$ , we have

$$\begin{aligned} |d(\boldsymbol{\xi}_u, q)| &= \left| \int_{\mathcal{E}_i} (\{\{\boldsymbol{\xi}_u\}\} + \mathbf{D}_{12} \llbracket \boldsymbol{\xi}_u \rrbracket) \cdot \llbracket \mathbf{q} \rrbracket \, ds \right| \\ &\leq \left( \int_{\mathcal{E}_i} \frac{1}{\underline{D}_{11}} (\{\{\boldsymbol{\xi}_u\}\} + \mathbf{D}_{12} \llbracket \boldsymbol{\xi}_u \rrbracket)^2 \, ds \right)^{\frac{1}{2}} \Theta(\mathbf{0}, q) \\ &\leq C \left( \sum_{K \in \mathcal{T}} \frac{1}{\underline{D}_{11}^{\partial K}} \|\boldsymbol{\xi}_u\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{0}, q), \end{aligned}$$

and using the inclusion  $\partial_i \mathcal{V}(K) \subset \mathcal{Q}(K)$ ,

$$\begin{aligned} |d(\mathbf{v}, \xi_p)| &= \left| \int_{\mathcal{E}_i} (\{\{\xi_p\}\} - \mathbf{D}_{12} \cdot \llbracket \xi_p \rrbracket) [\mathbf{v}] ds + \int_{\mathcal{E}_D} \xi_p \mathbf{v} \cdot \mathbf{n} ds \right| \\ &\leq C \left( \sum_{K \in \mathcal{T}} \frac{1}{C_{11}^{\partial K}} \|\xi_p\|_{0, \partial K}^2 \right)^{\frac{1}{2}} \Theta(\mathbf{v}, 0). \end{aligned}$$

The application of Lemma 3.1 completes the proof.  $\square$

For the special form of  $C_{11}$  and  $D_{11}$  proposed in (2.21) and (2.22), respectively, we have as a consequence of Lemma 3.6, Lemma 3.7 and (2.16) the following result.

**COROLLARY 3.8.** *Assume (2.15), (2.16) and (2.17), (2.18). Let the coefficients  $C_{11}$  and  $D_{11}$  be given by (2.21), (2.22), and let  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\Pi$  be the (componentwise)  $L^2$ -projections onto the corresponding finite element spaces. Let  $\underline{\sigma} \in H^{r+1}(\Omega)^{d^2}$ ,  $\mathbf{u} \in H^{s+1}(\Omega)^d$  and  $p \in H^{t+1}(\Omega)$  for  $r, s, t \geq 0$ . Then we have*

$$\begin{aligned} |a(\underline{\xi}_\sigma, \underline{\mathcal{I}})| &\leq Ch^{\min(r,l)+1} \|\underline{\sigma}\|_{r+1} \|\underline{\mathcal{I}}\|_0, & \forall \underline{\mathcal{I}} \in \underline{\Sigma}, \\ |b(\underline{\xi}_u, \underline{\mathcal{I}})| &\leq Ch^{\min(s,k)} \|\mathbf{u}\|_{s+1} \|\underline{\mathcal{I}}\|_0, & \forall \underline{\mathcal{I}} \in \underline{\Sigma}_N, \\ |b(\mathbf{v}, \underline{\xi}_\sigma)| &\leq Ch^{\min(r,l)+\frac{1-\gamma}{2}} \|\underline{\sigma}\|_{r+1} \Theta(\mathbf{v}, 0), & \forall \mathbf{v} \in \mathbf{V}_N, \\ |c(\underline{\xi}_u, \mathbf{v})| &\leq Ch^{\min(s,k)+\frac{1+\gamma}{2}} \|\mathbf{u}\|_{s+1} \Theta(\mathbf{v}, 0), & \forall \mathbf{v} \in \mathbf{V}, \\ |d(\underline{\xi}_u, q)| &\leq Ch^{\min(s,k)+\frac{1-\delta}{2}} \|\mathbf{u}\|_{s+1} \Theta(\mathbf{0}, q), & \forall q \in Q_N, \\ |d(\mathbf{v}, \xi_p)| &\leq Ch^{\min(t,m)+\frac{1-\gamma}{2}} \|p\|_{t+1} \Theta(\mathbf{v}, 0), & \forall \mathbf{v} \in \mathbf{V}_N, \\ |e(\xi_p, q)| &\leq Ch^{\min(t,m)+\frac{1+\delta}{2}} \|p\|_{t+1} \Theta(\mathbf{0}, q), & \forall q \in Q, \end{aligned}$$

with constants  $C$  independent of the meshsize.

From Corollary 3.8 we are able to derive the following estimate for  $K_B$ .

**COROLLARY 3.9.** *Assume (2.15), (2.16) and (2.17), (2.18), with approximation orders  $k \geq 1$ ,  $l, m \geq 0$ . Let the coefficients  $C_{11}$  and  $D_{11}$  be given by (2.21), (2.22), and let  $\underline{\Pi}$ ,  $\mathbf{\Pi}$ ,  $\Pi$  denote  $L^2$ -projections. For  $\underline{\sigma} \in H^{l+1}(\Omega)^{d^2}$ ,  $\mathbf{u} \in H^{k+1}(\Omega)^d$  and  $p \in H^{m+1}(\Omega)$  the error bound (2.29) is satisfied with*

$$K_B(\underline{\sigma}, \mathbf{u}, p) \leq C \left[ h^{l+\frac{1-\gamma}{2}} \|\underline{\sigma}\|_{l+1} + h^k \|\mathbf{u}\|_{k+1} + h^{m+\frac{1+\min(-\gamma, \delta)}{2}} \|p\|_{m+1} \right].$$

Furthermore, assume the elliptic regularity (2.26) and let  $(\mathbf{z}, q)$  denote the solution of (2.23)–(2.25) with right-hand side  $\lambda \in L^2(\Omega)^d$ ,  $\underline{\zeta} = -\nabla \mathbf{z}$ ,  $\tilde{q} = -q$ . Then we have in Lemma 2.7

$$K_B(\underline{\zeta}, \mathbf{z}, \tilde{q}) \leq Ch^{\frac{1+\min(-\gamma, \delta)}{2}} \|\lambda\|_0.$$

*Proof.* The first assertion follows from the fact that

$$\begin{aligned} \mathcal{A}(\underline{\xi}_\sigma, \pm \underline{\xi}_u, \xi_p; \underline{\mathcal{I}}, \pm \mathbf{v}, q) &= a(\underline{\xi}_\sigma, \underline{\mathcal{I}}) \pm b(\underline{\xi}_u, \underline{\mathcal{I}}) \mp b(\mathbf{v}, \underline{\xi}_\sigma) \pm c(\underline{\xi}_u, \mathbf{v}) \\ &\quad \pm d(\mathbf{v}, \xi_p) \mp d(\underline{\xi}_u, q) + e(\xi_p, q), \end{aligned}$$

from the definition of the  $\mathcal{A}$ -seminorm and from Corollary 3.8.

The second assertion follows similarly from Corollary 3.8, substituting  $(\underline{\sigma}, \mathbf{u}, p)$  by  $(\underline{\zeta}, \mathbf{z}, \tilde{q})$ , observing the special form of  $C_{11}$  and  $D_{11}$ , and (2.26) which gives  $\|\underline{\zeta}\|_1 + \|\mathbf{z}\|_2 + \|\tilde{q}\|_1 \leq C \|\lambda\|_0$ .  $\square$

**3.4. Proof of Proposition 2.5.** We prove the stability result in Proposition 2.5.

To do so, we fix  $(\underline{\mathcal{I}}, \mathbf{v}, q) \in \underline{\Sigma}_N \times \mathbf{V}_N \times Q_N$ . Then, by the continuous inf-sup condition for the standard Stokes forms (see, e.g., [6, 24]) there is a velocity field  $\mathbf{u} \in H_0^1(\Omega)^d = \{\mathbf{u} \in H^1(\Omega)^d : \mathbf{u}|_{\partial\Omega} = \mathbf{0}\}$  satisfying

$$-\int_{\Omega} q \nabla \cdot \mathbf{u} \, dx \geq \kappa \|q\|_0^2, \quad \|\mathbf{u}\|_1 \leq \|q\|_0, \quad (3.2)$$

with a constant  $\kappa > 0$  just depending on  $\Omega$ . Let  $\Pi \mathbf{u}$  be the  $L^2$ -projection of  $\mathbf{u}$  onto the FE space  $\mathbf{V}_N$ . By definition of  $\mathcal{A}$ , we have

$$\mathcal{A}(\underline{\mathcal{I}}, \mathbf{v}, q; \underline{\mathcal{Q}}, \Pi \mathbf{u}, 0) = -b(\Pi \mathbf{u}, \underline{\mathcal{I}}) + c(\mathbf{v}, \Pi \mathbf{u}) + d(\Pi \mathbf{u}, q) =: T_1 + T_2 + T_3.$$

We set  $\boldsymbol{\xi}_u := \mathbf{u} - \Pi \mathbf{u}$  and estimate each of the terms  $T_1$ – $T_3$  separately.

For  $T_1$  we have, by Corollary 3.8,

$$|T_1| \leq |b(\boldsymbol{\xi}_u, \underline{\mathcal{I}})| + |b(\mathbf{u}, \underline{\mathcal{I}})| \leq C \|\mathbf{u}\|_1 \|\underline{\mathcal{I}}\|_0 + \left| \int_{\Omega} \nabla \mathbf{u} : \underline{\mathcal{I}} \, dx \right| \leq C \|\mathbf{u}\|_1 \|\underline{\mathcal{I}}\|_0,$$

and, by (3.2),

$$T_1 \geq -\frac{C_1}{\varepsilon_1} \|q\|_0^2 - C_1 \varepsilon_1 \|\underline{\mathcal{I}}\|_0^2.$$

For the second term  $T_2$  we have, analogously,

$$T_2 = c(\mathbf{v}, \Pi \mathbf{u}) = c(\mathbf{v}, \boldsymbol{\xi}_u) \leq Ch^{\frac{1+\gamma}{2}} \|\mathbf{u}\|_1 \Theta(\mathbf{v}, 0),$$

and hence

$$T_2 \geq -\frac{C_2 h^{\frac{1+\gamma}{2}}}{\varepsilon_2} \|q\|_0^2 - C_2 \varepsilon_2 \Theta^2(\mathbf{v}, q).$$

For the third term, we write

$$T_3 = d(\Pi \mathbf{u}, q) = d(\mathbf{u}, q) - d(\boldsymbol{\xi}_u, q).$$

Since, by Corollary 3.8 and (3.2)

$$|d(\boldsymbol{\xi}_u, q)| \leq Ch^{\frac{1-\delta}{2}} \|\mathbf{u}\|_1 \Theta(\mathbf{0}, q) \leq \frac{Ch^{\frac{1-\delta}{2}}}{\varepsilon_3} \|q\|_0^2 + C\varepsilon_3 \Theta^2(\mathbf{v}, q),$$

and  $d(\mathbf{u}, q) = -\int_{\Omega} q \nabla \cdot \mathbf{u} \, dx$ , we obtain

$$T_3 \geq \kappa \|q\|_0^2 - \frac{C_3 h^{\frac{1-\delta}{2}}}{\varepsilon_3} \|q\|_0^2 - C_3 \varepsilon_3 \Theta^2(\mathbf{v}, q).$$

From the above estimates we conclude that

$$\begin{aligned} & \mathcal{A}(\underline{\mathcal{I}}, \mathbf{v}, q; \underline{\mathcal{Q}}, \Pi \mathbf{u}, 0) \\ & \geq \left( \kappa - \frac{C_1}{\varepsilon_1} - \frac{C_2 h^{\frac{1+\gamma}{2}}}{\varepsilon_2} - \frac{C_3 h^{\frac{1-\delta}{2}}}{\varepsilon_3} \right) \|q\|_0^2 - C_1 \varepsilon_1 \|\underline{\mathcal{I}}\|^2 - (C_2 \varepsilon_2 + C_3 \varepsilon_3) \Theta^2(\mathbf{v}, q). \end{aligned}$$

Since the exponents  $\gamma$  and  $\delta$  are in  $[-1, 1]$ , by the definition of the coefficients  $C_{11}$  and  $D_{11}$  in (2.21) and (2.22), respectively, we see that the parameters  $\{\varepsilon_i\}_{i=1}^3$  can be chosen in such a way that

$$\mathcal{A}(\mathcal{I}, \mathbf{v}, q; \mathcal{Q}, \mathbf{\Pi u}, 0) \geq K_1 \|q\|_0^2 - K_2 |(\mathcal{I}, \mathbf{v}, q)|_{\mathcal{A}}^2,$$

with constants  $K_i$  independent of the meshsize.

Furthermore, we have by Corollary 3.4 and by the choice of the coefficient  $C_{11}$

$$|(\mathcal{Q}, \mathbf{\Pi u}, 0)|_{\mathcal{A}}^2 = c(\mathbf{\Pi u}, \mathbf{\Pi u}) = c(\boldsymbol{\xi}_u, \boldsymbol{\xi}_u) \leq Ch^{1+\gamma} \|\mathbf{u}\|_1^2 \leq K_3^2 \|q\|_0^2.$$

The function  $\mathbf{w} = \mathbf{\Pi u}/K_3$  then satisfies the assertion in Proposition 2.5, with  $\kappa_1 = K_1/K_3$  and  $\kappa_2 = K_2/K_3$ . This completes the proof.

**3.5. Proof of the main results.** Theorems 2.2 and 2.3 follow now immediately by choosing the projection operators  $\underline{\Pi}$ ,  $\mathbf{\Pi}$  and  $\Pi$  as  $L^2$ -projections, by combining Corollaries 3.5, 3.9 with Lemmas 2.4, 2.6 and 2.7 and by taking into account the form of the coefficients  $C_{11}$  and  $D_{11}$ .

**4. Numerical results.** The numerical experiments we present in this section are devised to verify our theoretical error estimates. We also explore the effect of the use of several combinations of polynomial spaces on the efficiency of the resulting LDG methods. The numerical tests are carried out by using the finite element library `deal.II` by Bangerth and Kanschat [3].

We consider the Stokes system (1.1) with  $\Omega = (-1, 1)^2$  and right-hand side  $\mathbf{f}$  and Dirichlet boundary condition  $\mathbf{g}_{\mathcal{D}}$  chosen such that the exact solution is

$$\begin{aligned} u_1(x_1, x_2) &= -e^{x_1}(x_2 \cos x_2 + \sin x_2), \\ u_2(x_1, x_2) &= e^{x_1}x_2 \sin x_2, \\ p(x_1, x_2) &= 2e^{x_1} \sin x_2. \end{aligned}$$

In all our experiments, we use uniform triangulations made of squares; the grid whose squares have size  $h = 2^{-\nu+1}$  is called a grid of level  $\nu$ .

**4.1. Verifying the sharpness of the theoretical error estimates.** We begin by considering LDG methods with the same polynomial spaces for  $\underline{\sigma}$ ,  $\mathbf{u}$  and  $p$ , and take  $C_{11} = h^{-1}$ ,  $D_{11} = h$ ,  $\mathbf{C}_{12} = \mathbf{0}$  and  $\mathbf{D}_{12} = \mathbf{0}$ . The results are shown in Tables 4.1 and 4.2 for  $P^k$ - and  $Q^k$ -elements, respectively. The tables confirm that the orders of convergence predicted by the theory are sharp since they are actually achieved. However, one exception needs to be pointed out: The pressure converges better than expected for linear and bilinear shape functions since super-linear convergence is observed. This phenomenon is particularly well accentuated in the case of bilinear functions for which the order of convergence of  $3/2$  can be clearly seen. The same order of convergence has recently been observed by Berrone [5] for the stabilized  $P^1$ - $P^1$  SUPG method.

**4.2. The effect of the use of different polynomial spaces.** To get an idea of what is the effect of the use of  $P^k$ - versus  $Q^k$ -spaces on quadrilateral elements, the errors of quadratic and biquadratic elements are compared in relation to the numerical effort in Figure 4.1. We use the number of non-zero elements in the stiffness matrix as a measure of the solution cost of a discretization. The graphs show that it is possible to compute the velocities  $\mathbf{u}$  with the same accuracy and effort with  $P^2$ - and  $Q^2$ -shape functions; however, the pressures are computed more efficiently with  $P^2$ -elements.

TABLE 4.1  
Convergence rates for  $P^k$ -elements.

degree $k$	grid level	$ e _{\mathcal{A}}$		$\ \underline{e}_\sigma\ _0$		$\ \underline{e}_u\ _0$		$\ \underline{e}_p\ _0$	
		error	order	error	order	error	order	error	order
1	3	3.4e-1	0.94	2.1e-1	0.69	8.4e-3	2.07	2.0e-2	1.57
	4	1.7e-1	0.96	1.2e-1	0.85	2.1e-3	2.03	8.2e-3	1.27
	5	8.8e-2	0.98	6.2e-2	0.93	5.1e-4	2.01	3.4e-3	1.25
2	3	1.2e-2	1.87	9.1e-3	1.73	2.0e-4	3.07	5.1e-4	2.46
	4	3.2e-3	1.84	2.5e-3	1.88	2.4e-5	3.06	1.2e-4	2.08
	5	9.2e-4	1.80	6.4e-4	1.94	2.9e-6	3.03	3.0e-5	2.00
3	2	1.8e-3	2.86	1.4e-3	2.66	5.8e-5	3.98	2.4e-4	2.80
	3	2.4e-4	2.91	1.9e-4	2.82	3.6e-6	4.02	3.9e-5	2.65
	4	3.0e-5	2.96	2.5e-5	2.91	2.2e-7	4.02	5.3e-6	2.87

TABLE 4.2  
Convergence rates for  $Q^k$ -elements.

degree $k$	grid level	$ e _{\mathcal{A}}$		$\ \underline{e}_\sigma\ _0$		$\ \underline{e}_u\ _0$		$\ \underline{e}_p\ _0$	
		error	order	error	order	error	order	error	order
1	3	2.4e-1	0.94	2.2e-1	0.73	5.6e-3	2.06	2.9e-2	1.50
	4	1.3e-1	0.96	1.2e-1	0.86	1.4e-3	2.04	1.0e-2	1.52
	5	6.4e-2	0.97	6.2e-2	0.93	3.4e-4	2.01	3.8e-3	1.43
2	3	2.6e-3	2.00	6.3e-4	2.10	6.5e-5	3.01	4.5e-4	1.90
	4	6.4e-4	2.00	1.6e-4	2.02	8.1e-6	3.00	1.2e-4	1.94
	5	1.6e-4	2.00	3.9e-5	2.00	1.0e-6	3.00	3.0e-5	1.97
3	2	6.1e-4	2.76	3.8e-4	2.37	1.9e-5	3.82	2.4e-4	2.24
	3	8.1e-5	2.92	6.4e-5	2.55	1.1e-6	4.12	3.8e-5	2.63
	4	1.0e-5	2.98	9.3e-6	2.80	6.0e-8	4.19	5.2e-6	2.88

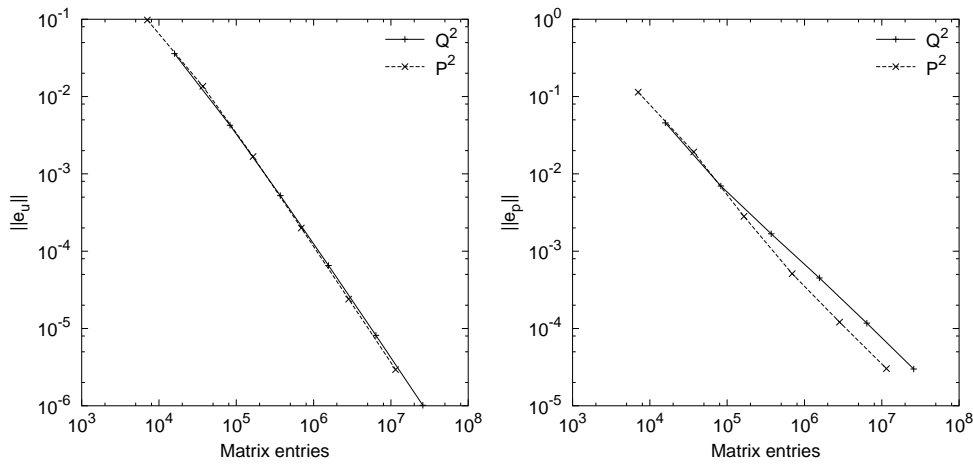
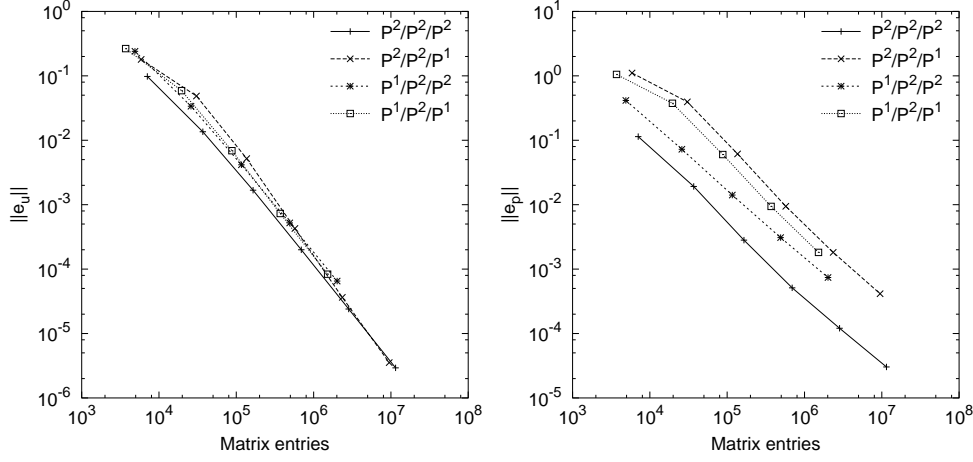
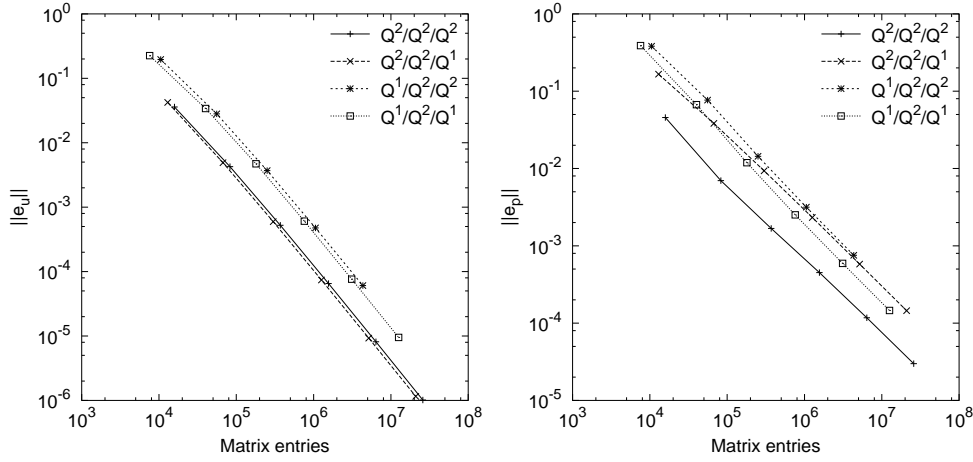


FIG. 4.1. Comparison of quadratic  $P^2$ - and biquadratic  $Q^2$ -elements.

FIG. 4.2. Comparison of mixed spaces for quadratic  $P^2$ -velocities.FIG. 4.3. Comparison of mixed spaces for biquadratic  $Q^2$ -velocities.

Finally, since the theoretical results predict the same orders of convergence for all quantities if we take lower order  $P$ -elements for  $\underline{\sigma}$  and  $p$ , we compare the efficiency of LDG methods obtained with several combinations of local spaces  $\mathcal{S}(K)/\mathcal{V}(K)/\mathcal{Q}(K)$  in Figures 4.2 and 4.3.

We can see that all these LDG discretizations converge with the same order, as expected and proved for  $P$ -elements, and that, in most cases, it is more efficient to use the same local approximating spaces for all quantities. In fact, only the velocities in the  $Q^2$ -case are computed slightly more efficient using a lower degree for the pressure. On the other hand, using lower order polynomials for  $\underline{\sigma}$  and/or  $p$  increases the error in  $p$  such that at least one additional refinement is necessary to recover the accuracy corresponding to an LDG method using the same local spaces.

**5. Extensions and concluding remarks.** In this paper, we have introduced LDG methods for the Stokes system and have carried out an a priori error analysis. We have shown that if polynomial approximations of degree  $k - 1$  are used for the pressure  $p$  and the stress tensor  $\underline{\sigma}$  and polynomial approximations of degree  $k$  for the

TABLE 5.1  
Orders of convergence for “superconvergent” fluxes.

Level	$P^2$ -elements			$Q^2$ -elements		
	$\ \underline{e}_\sigma\ _0$	$\ e_u\ _0$	$\ e_p\ _0$	$\ \underline{e}_\sigma\ _0$	$\ e_u\ _0$	$\ e_p\ _0$
1	1.77	2.85	2.38	2.28	2.80	2.05
2	1.86	2.94	2.49	2.43	2.90	2.33
3	1.92	2.98	2.60	2.48	2.95	2.45
4	1.96	2.99	2.66	2.49	2.98	2.48
5	1.98	3.00	2.68	2.50	2.99	2.49

velocity  $\mathbf{u}$  then optimal error estimates are obtained when the stabilization parameters  $C_{11}$  and  $D_{11}$  are taken to be of order  $h^{-1}$  and  $h$ , respectively. Future work will be devoted to the extension of the LDG method to the incompressible Navier-Stokes equations.

Extensions of our analysis to curvilinear elements and to (nonconvex) polygonal domains as well as to error estimates in negative-order norms for both the velocity and the pressure can easily be carried out; see [8] for details of the corresponding extensions for the Laplacian. Here, we simply must note that, to take into account the presence of the pressure, we have to consider the following modified adjoint problem

$$\begin{aligned} -\Delta \mathbf{z} + \nabla q &= \boldsymbol{\lambda} && \text{in } \Omega, \\ \nabla \cdot \mathbf{z} &= g && \text{in } \Omega, \\ \mathbf{z} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned}$$

where  $g$  is in  $H^1(\Omega)$ . The elliptic regularity result we have used in (2.26) is a particular case of the above more general case; see, for example, Proposition 3.14 in [2] and the references therein.

The technique of analysis employed is an extension of that used in [8] for the simpler case of the Laplacian. This same technique was then used in [13] to get improved convergence estimates for a special LDG method on Cartesian grids by changing some auxiliary projections used in the analysis. In a forthcoming paper, we shall carry out a similar study for the Stokes system. The numerical results in Table 5.1 already suggest a similar improvement as obtained for the Laplacian. Indeed, the use of these special fluxes with quadratic shape functions increases the order of convergence of the pressure by  $1/2$ ; moreover, they improve the order of convergence of the pressure *and* the stresses by  $1/2$  when biquadratic finite elements are used.

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