

# A variational method for acoustic scattering from a thin penetrable shell

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## 1 Description of problem

In this paper, we investigate the behaviour of solutions of a transmission problem in acoustics exterior to a thin domain, as the thickness of this domain approaches zero. The problem under consideration is : *Find*  $(u_\epsilon, U_\epsilon) \in (C^2(\Omega_\epsilon) \cap C^1(\overline{\Omega_\epsilon})) \times (C^2(\Omega_\infty) \cap C^1(\overline{\Omega_\infty}))$  such that

$$(1) \quad \begin{cases} -\Delta u_\epsilon + k_0^2 u_\epsilon = 0 \text{ in } \Omega_\epsilon, & \Delta U_\epsilon + k^2 U_\epsilon = 0 \text{ on } \Omega_\infty, \\ u_\epsilon = 0 \text{ on } \Gamma_\epsilon, & u_\epsilon = U_\epsilon + f, \quad \frac{\partial u_\epsilon}{\partial n} = \lambda U_\epsilon + g \text{ on } \Gamma_0, \\ \frac{\partial U_\epsilon}{\partial r} - ikU_\epsilon = o\left(\frac{1}{\sqrt{r}}\right) & \text{as } r \rightarrow \infty. \end{cases}$$

The geometry is described schematically in Fig.1, with the positive orientations of the normals indicated. The wave numbers  $k$  and  $k_0$  are real and positive, and the parameter  $\lambda$  is a positive constant. We are interested in the asymptotic behaviour of  $(u_\epsilon, U_\epsilon)$  as  $\epsilon \rightarrow 0^+$ . This problem has been well-studied in the literature before (eg. [2], [3], [5] and the references therein), and has several applications, especially in electromagnetic scattering from coated objects. The analysis presented here is for two-dimensional domains, and can easily be extended to scatterers in  $\mathbf{R}^3$ . Our approach is similar in flavour to that of [1], but uses different techniques, and the details are presented in [4].

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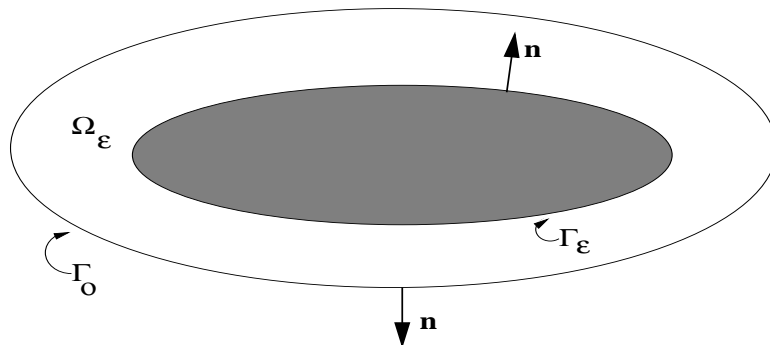


FIG. 1. *The parameter  $\epsilon$  is the thickness of the annular region. The curve  $\Gamma_0$  stays fixed as  $\epsilon \rightarrow 0^+$ .*

## 2 Non-local boundary value problems

We truncate the (infinite) computational domain of (1) by reducing the exterior Helmholtz problem to an integral equation on  $\Gamma_0$ , using direct methods. Precisely, by letting  $x$  approach  $\Gamma_0$  from inside  $\Omega_\epsilon$  in the representation formula for the exterior Helmholtz equation, we obtain the integral equation

$$(2) \quad U_\epsilon = \left( \frac{1}{2} \mathbf{I} + \mathbf{K} \right) U_\epsilon - \mathbf{V} \left( \frac{\partial U_\epsilon}{\partial n} \right) \quad \text{on } \Gamma_0,$$

where  $U_\epsilon^+$  and  $\frac{\partial U_\epsilon}{\partial n}$  are Cauchy data for the solution of the Helmholtz equation. The single and double layer boundary integral operators  $\mathbf{V}, \mathbf{K}$  are defined as usual for smooth boundary functions  $\sigma, \mu$ , which are related to the Cauchy data.:

$$\mathbf{V}\sigma(x) := \int_{\Gamma_0} \frac{i}{4} H_0^1(k|x-y|) \sigma(y) dy; \quad \mathbf{K}\mu(x) := \int_{\Gamma_0} \frac{\partial}{\partial n} \frac{i}{4} H_0^1(k|x-y|) \mu(y) dy, \quad x \in \Gamma_0.$$

The mapping properties of these operators are well known, and in particular, if  $\Gamma_0$  is smooth and  $k$  is not an exceptional value, then  $\mathbf{K}$  is compact as a mapping from  $H^{1/2}(\Gamma_0) \rightarrow H^{1/2}(\Gamma_0)$ , and  $\mathbf{V}$  is an isomorphism from  $H^s(\Gamma_0) \rightarrow H^{s+1}(\Gamma_0)$  (see[7]). The integral equation leads us to the coupled system: *For given  $f, g$ , find  $(u_\epsilon, \sigma_\epsilon)$  such that*

$$(3) \quad \begin{cases} \Delta u_\epsilon + k_0^2 u_\epsilon = 0 & \text{in } \Omega_\epsilon, & u_\epsilon|_{\Gamma_\epsilon} = 0, & \frac{\partial u_\epsilon^-}{\partial n}|_{\Gamma_0} = \lambda \sigma_\epsilon + g, \\ \mathbf{V}\sigma_\epsilon + \left( \frac{1}{2} \mathbf{I} - \mathbf{K} \right) u_\epsilon = & \left( \frac{1}{2} \mathbf{I} - \mathbf{K} \right) f & \text{on } \Gamma_0. \end{cases}$$

The system (3) is the nonlocal boundary value problem equivalent to (1).

In order to write a variational formulation for this system, we introduce the product space  $\mathcal{H}_\epsilon := \left\{ (u, \sigma) \mid u \in H_{\Gamma_\epsilon}^1(\Omega_\epsilon), \sigma \in H^{-1/2}(\Gamma_0) \right\}$  where  $H^1(\Omega_\epsilon)$  is the usual energy space. The space  $\mathcal{H}_\epsilon$  is equipped with the natural product norm. Let us denote by  $\langle \cdot, \cdot \rangle$  the  $H^{-1/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)$  duality pairing, and by  $(\cdot, \cdot)_\epsilon$  and  $[\cdot, \cdot]_\epsilon$ , the inner products on  $L^2(\Omega_\epsilon)$  and  $\mathcal{H}_\epsilon$ . With this notation, the weak formulation of (3) becomes: *Find  $(u_\epsilon, \sigma_\epsilon) \in \mathcal{H}_\epsilon$  such that for all  $(v, \chi) \in \mathcal{H}_\epsilon$*

$$(4) \quad \begin{cases} a_\epsilon(u_\epsilon, v) - k_0^2(u_\epsilon, v)_\epsilon + \langle \lambda \sigma_\epsilon, \bar{v} \rangle = \langle g, \bar{v} \rangle, \\ \langle \chi, \overline{2\mathbf{V}\sigma_\epsilon} \rangle + \langle \chi, \overline{(\mathbf{I} - 2\mathbf{K})u_\epsilon} \rangle = \langle \chi, \overline{(\mathbf{I} - 2\mathbf{K})f} \rangle, \end{cases}$$

where  $a_\epsilon(u, v) := \int_{\Omega_\epsilon} \nabla u \cdot \nabla \bar{v} dx$  is a sesquilinear operator. We use the subscript  $\epsilon$  to highlight the fact that the sesquilinear form depends on the thickness of the obstacle. It is easily seen that the system (4) admits a unique solution, provided that  $k$  is not an exceptional value for the domain. For the remainder of this paper, we make this assumption. To show existence, we first combine the two variational equations in (4) in an operator equation

$$(5) \quad [(\mathcal{A}_\epsilon + \mathcal{K}_0)(u_\epsilon, \sigma_\epsilon), (v, \chi)]_\epsilon = [\mathcal{F}, (v, \chi)]_\epsilon, \quad \forall (v, \chi) \in \mathcal{H}_\epsilon,$$

where, for fixed  $(u, \sigma) \in \mathcal{H}_\epsilon$ ,

$$\begin{aligned} [(\mathcal{A}_\epsilon)(u, \sigma), (v, \chi)]_\epsilon &:= a_\epsilon(u, v) - k_0^2(u, v)_\epsilon - \lambda \langle \sigma, \bar{v} \rangle + \lambda \langle \chi, \bar{u} \rangle + 2\lambda \langle \chi, \overline{\mathbf{V}\sigma} \rangle \\ [\mathcal{K}_0(u, \sigma), (v, \chi)]_\epsilon &:= -2\lambda \langle \chi, \overline{\mathbf{K}u} \rangle \end{aligned}$$

for all  $(v, \chi)$  in  $\mathcal{H}_\epsilon$ . We now show (5) is a Fredholm equation for fixed  $\epsilon > 0$ , which proves the existence of the solution.

**THEOREM 2.1.** *The linear operator  $\mathcal{A}_\epsilon$  is an isomorphism on  $\mathcal{H}_\epsilon$  for all  $0 < \epsilon < \epsilon_o$ . Further, there exist positive constants  $C_1, C_2$ , independent of  $\epsilon \in (0, \epsilon_o)$  such that*

$$(6) \quad C_1 \|(u, \sigma)\|_{\mathcal{H}_\epsilon} \leq \|\mathcal{A}_\epsilon(u, \sigma)\|_{\mathcal{H}_\epsilon} \leq C_2 \|(u, \sigma)\|_{\mathcal{H}_\epsilon}, \quad \forall (u, \sigma) \in \mathcal{H}_\epsilon.$$

Here,  $\epsilon_o > 0$  is a positive threshold thickness which depends on  $k_0$ . Moreover, the linear operator  $\mathcal{K}_0$  is a compact sesquilinear form on  $\mathcal{H}_\epsilon \times \mathcal{H}_\epsilon$ .

We remark that  $\mathcal{A}_\epsilon$  is *not* invertible in general, and that  $k$  is not an exceptional value. Indeed, the offending term  $-k_0^2 \|u\|_{L^2(\Omega_\epsilon)}^2$  is one of the key complications in the study of the Helmholtz equation.

*Proof.* The continuity of  $\mathcal{A}_\epsilon$  is clear, and establishes the right hand inequality in (6). The left hand estimate for  $\mathcal{A}_\epsilon$  is established from the definition of  $\mathcal{A}_\epsilon$ , the isomorphism property of  $\mathbf{V}$ , and Poincaré's inequality for a thin domain,  $\|u\|_{L^2(\Omega_\epsilon)}^2 \leq c_o \epsilon^2 \|\nabla u\|_{L^2(\Omega_\epsilon)}^2$ ,  $\forall u \in H_{\Gamma_\epsilon}^1(\Omega_\epsilon)$ , where  $c_o$  is a constant independent of  $\epsilon$ . After some manipulations, we get for  $\epsilon \in (0, \epsilon_o)$ ,

$$|[\mathcal{A}_\epsilon(u, \sigma), (u, \sigma)]_\epsilon| \geq \frac{1 - c_o \epsilon^2 k_0^2}{2} \|u\|_{H^1(\Omega_\epsilon)}^2 + \alpha_v \|\sigma\|_{H^{-1/2}(\Gamma_0)}^2 \geq C_1 \|(u, \sigma)\|_{\mathcal{H}_\epsilon}^2$$

where  $C_1 := \min(\frac{1 - c_o \epsilon_o^2 k_0^2}{2}, \alpha_v) > 0$ . The threshold thickness  $\epsilon_o$  is chosen so that  $0 < c_o \epsilon_o^2 < 1, c_o \epsilon_o^2 k_0 < 1$ . This establishes the left-hand inequality in (6).

The compactness of  $\mathcal{K}_0$  follows from standard arguments.

Under certain conditions, we are, in fact, able to get a stronger result:

**THEOREM 2.2.** *There is a threshold thickness  $\epsilon_0$  such that*

$$c \|(u, \sigma)\|_{\mathcal{H}_\epsilon} \leq \|(\mathcal{A}_\epsilon + \mathcal{K}_0)(u, \sigma)\|_{\mathcal{H}_\epsilon} \forall \epsilon \in (0, \epsilon_0),$$

when  $(u, \sigma) \in \mathcal{H}_\epsilon$  satisfy (5). Moreover,  $c$  is independent of  $\epsilon$ , under the assumption that there exists a finite constant  $\beta > 0$  independent of  $\epsilon \in (0, \bar{\epsilon}_o)$ , such that

$$(7) \quad 0 \leq \lambda \operatorname{Re} \langle \mathbf{V}^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{K})u, \bar{u} \rangle + \frac{k_o^2 \beta}{\epsilon} \|u\|_{L^2(\Omega_\epsilon)}^2, \quad \forall u \in \mathcal{H}_\epsilon.$$

*Proof.* The proof follows by first estimating  $u$ , and then  $\sigma$ . The details of this proof are interesting, and use, among other things, properties of the Stekhlov-Poincaré map  $\mathbf{V}^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{K})$ . The complete details are presented elsewhere ([4]).

### 3 Asymptotic Analysis

In this section, the parameter  $\epsilon$  will be allowed to vary, and in particular, approach zero. Unfortunately, the function space  $\mathcal{H}_\epsilon$  has a norm which depends on  $\epsilon$ , and therefore is not suitable for a careful analysis in the present case. We first scale the annular region  $\Omega_\epsilon$  as  $\Omega_\epsilon \ni x := X(s) + \epsilon(t - 1)\mathbf{n}$ , where  $X(s)$  is a point on the curve  $\Gamma_0$ , and  $t$  is a parameter which varies along the thickness. The variable  $s$  is an arclength parameter on  $\Gamma_0$ . We denote by  $\Omega := \{(s, t) | s \in [0, L], t \in (0, 1)\}$ , and introduce the following spaces:

$$\mathcal{H}_0 := \left\{ \tilde{u} \in H^1(\Omega) \mid \tilde{u}(s, t) = \tilde{u}(s + L, t), \tilde{u}^+(s, 0) = 0 \right\}, \quad \mathcal{H} := \left\{ (\tilde{u}, \sigma) \mid \tilde{u} \in \mathcal{H}_0, \sigma \in H^{-1/2}(\Gamma_0) \right\}.$$

The norms are defined by

$$\|u\|_{\mathcal{H}_0} := \left( \int_0^L \int_0^1 |u|^2 + \left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 dt ds \right)^{1/2}, \quad \|(u, \sigma)\|_{\mathcal{H}} := \left( \|u\|_{\mathcal{H}_0}^2 + \|\sigma\|_{H^{-1/2}(\Gamma_0)}^2 \right)^{1/2}.$$

A careful scaling of the differential operators involved allows us to rewrite (4) as: *For given  $f, g$ , find  $(u_\epsilon, \sigma_\epsilon) \in \mathcal{H}$  such that*

$$(8) \quad \begin{cases} \langle \lambda \sigma, \bar{v} \rangle + \sum_{i=0}^n \epsilon^{i-1} a_i(u, v) + \mathcal{Q}_n(u, v) = \langle g, \bar{v} \rangle, \\ \langle \chi, 2\bar{\mathbf{V}}\sigma \rangle + \langle \chi, \overline{(\mathbf{I} - 2\mathbf{K})u} \rangle = \langle \chi, \overline{(\mathbf{I} - 2\mathbf{K}_2)f} \rangle. \end{cases}$$

where the bilinear forms  $a_i(u, v)$  are defined over  $\Omega$ . If  $\kappa$  denotes the curvature of  $\Gamma_0$ , the first two terms  $a_0(u, v)$  and  $a_1(u, v)$  are defined by

$$a_0(u, v) := \int_0^L \int_0^1 \frac{\partial u}{\partial t} \frac{\partial \bar{v}}{\partial t} dt ds, \quad a_1(u, v) := \int_0^L \int_0^1 (t-1)\kappa \frac{\partial u}{\partial t} \frac{\partial \bar{v}}{\partial t} dt ds.$$

We note that neither the wave number  $k_0$ , nor any tangential derivatives of  $u_\epsilon$ , appear in these forms. Further,  $\mathcal{Q}_n$  is the remainder term from the bilinear form  $a_\epsilon(u, v)$ , defined by

$$(9) \quad \mathcal{Q}_n(u, v) := \epsilon^n \int_0^L \int_0^1 \frac{[(1-t)\kappa]^{n-1}}{1 + \epsilon(t-1)\kappa} \frac{\partial u}{\partial s} \frac{\partial \bar{v}}{\partial s} dt ds.$$

We see that  $\mathcal{Q}_n(u, v) = O(\epsilon^n)$  for fixed  $u, v \in \mathcal{H}_0$ . The boundary integral equation does not change, since the curve  $\Gamma_0$  does not move or change with varying  $\epsilon$ , and always corresponds to  $t = 1$ .

An examination of examples over simple domains shows that, at least to leading order, the solution pair  $(u_\epsilon, \sigma_\epsilon)$  should behave as if the wave number  $k_0$  were zero. This leads us to believe that the solution pairs can be approximated by regular asymptotic series, and with no boundary layer in the problem. We therefore use formal expansions in (8):

$$(10) \quad u_\epsilon(s, t; \epsilon) = \sum_{j=0}^n \epsilon^j u_j(s, t) + R_n(s, t), \quad \sigma_\epsilon(s; \epsilon) = \sum_{j=0}^n \epsilon^j \sigma_j(s) + S_n(s, t),$$

and collect like powers of  $\epsilon$  to obtain the sequence of problems: *Find  $(u_j, \sigma_j) \in \mathcal{H}$ ,  $j = 0, 1, 2, \dots, n$  such that for all  $(v, \chi) \in \mathcal{H}$*

$$(11) \quad (a) \ a_0(u_j, v) = F_j(v); \quad (b) \ \langle \chi, 2\bar{\mathbf{V}}\sigma_j \rangle = \langle \chi, \overline{(\mathbf{I} - 2\mathbf{K})S_j} \rangle.$$

Here, the linear functional  $F_j$  is a function of  $u_0, u_1, \dots, u_{j-1}$  and  $\sigma_0, \sigma_1, \dots, \sigma_{j-1}$ , and the exact forms are easily computed.  $S_0 = u_0|_{\Gamma_0} - f$  while  $S_j = u_j|_{\Gamma_0}$ ,  $j \geq 1$ . We notice that the problem for each solution pair  $(u_k, \sigma_k)$  involves two *uncoupled* variational equations, (11 a-b). We first solve (11a) for  $u_j$  over the domain  $\Omega$ , and then use the trace of  $u_j$  to solve (11b) for  $\sigma_j$ .

We now need to examine issues of unique solvability for these equations at each order. Uniqueness follows directly when  $k$  is not an exceptional value. The following

theorem provides a general existence result for  $u_j$  in our formal asymptotic scheme. A similar theorem, with different hypotheses on the functions, was first stated in [1].

**THEOREM 3.1.** *Let  $F, p, q, r$  be scalar functions such that  $F \in H^1(\Omega)$ , and for fixed  $t \in (0, 1)$ ,  $F(\cdot, t), F_s(\cdot, t), F_{ss}(\cdot, t) \in L^2(\Gamma_0)$ ;  $p \in H^1(\Omega)$ ,  $\frac{\partial^2}{\partial t^2} p \in L^2(\Omega)$  for all  $s$ ,  $q \in L^2(\Omega)$ ,  $r \in H^{1/2}(\Gamma_0)$  and finally,  $F, p, q, r$  satisfy the periodicity requirement in the  $s$  variable. Then the variational equation*

$$(12) \quad \int_0^L \int_0^1 \frac{\partial U}{\partial t} \frac{\partial \bar{v}}{\partial t} dt ds = - \int_0^L \int_0^1 \left[ \frac{\partial F}{\partial s} \frac{\partial \bar{v}}{\partial s} + (t-1) \frac{\partial p}{\partial t} \frac{\partial \bar{v}}{\partial t} + qv \right] dt ds + \langle r, \bar{v} \rangle,$$

$\forall v \in \mathcal{H}$ , determines a unique  $U \in \mathcal{H}$ , given explicitly by

$$(13) \quad U(s, t) = \int_0^t \int_\tau^1 [F_{ss}(s, y) - q(s, y)] dy d\tau - \int_0^t (\tau-1) p_\tau(s, \tau) d\tau + t r(s).$$

The proof is detailed in [4]. Once  $u_k$  is known, the existence of the solution  $\sigma_k$  of the problem (11b) is given by the isomorphism properties of  $\mathbf{V}$ , and smoothness of the data  $f, g$ . In particular, for  $\sigma_k \in H^s(\Gamma_0)$ ,  $s \in \mathbf{R}$ , we require that  $(\frac{1}{2}\mathbf{I} - \mathbf{K})u_k \in H^{s+1}(\Gamma_0)$ . In order for the formal asymptotic scheme to be justified, we examine the remainder terms by studying the variational problem satisfied by  $(R_n, S_n)$ , and using the estimate proved in Theorem 2.2. We also make use of the norm equivalence between  $\|\cdot\|_{\mathcal{H}_\epsilon}$  and  $\|\cdot\|_{\mathcal{H}}$ , to obtain

$$\|(R_n, S_n)\|_{\mathcal{H}} = O(\epsilon^{n-2}), \quad \forall n \geq 2.$$

## 4 A simple illustration

If  $\Gamma_0$  is the unit circle, and  $S_j$  is a smooth function of  $\theta$ , we can explicitly write the solution of (11b) as  $\sigma(\theta) = \sum_{n=1}^{\infty} F_n D_n e^{in\theta}$  with  $D_n := k \left( \frac{n}{k} - \frac{H_{n+1}^1(k)}{H_n^1(k)} \right) e^{in\theta}$ , and  $F_n$  the Fourier coefficients of  $F$ . To illustrate our method, we choose  $f(\theta) := f_m e^{im\theta}$ ,  $g(\theta) := g_n e^{in\theta}$ . This allows us to compute the solutions analytically. The first 3 terms in the asymptotic sequence are then given by

$$\begin{aligned} (u_0, \sigma_0) &= (0, -f_m D_m), & (u_1, \sigma_1) &= (t(g + \lambda\sigma_0), g_n D_n - \lambda f_m D_m^2), \\ (u_2, \sigma_2) &= \left( (g + \lambda\sigma_0) \left( t - \frac{t^2}{2} \right) + t\lambda\sigma_1, \frac{1}{2} g_n D_n \left( \frac{1}{2} + \lambda D_n \right) - \lambda f_m D_m^2 \left( \frac{1}{2} + \lambda D_m \right) \right). \end{aligned}$$

## 5 Conclusion

The analysis presented in this paper demonstrates that the transmission problem (1) can be solved using asymptotic techniques. The computational domain was reduced by using integral equations, leading to a coupled system of equations. In the asymptotic setting, however, the variational formulations of the equations decouple, and can be solved rapidly. An exact representation formula for the solution in the thin region is presented, and the formal asymptotic procedure is justified. Finally, we explicitly compute some terms in the sequence. Typically, the integral equation will need to be solved numerically. However, since the curve  $\Gamma_0$  is fixed, the stiffness matrices need to be built only once for a large range of  $\epsilon$ , and data  $f, g$ .

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