SMALL-ENERGY ASYMPTOTICS OF THE SCATTERING MATRIX FOR THE MATRIX SCHRÖDINGER EQUATION ON THE LINE

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Abstract: The one-dimensional matrix Schrödinger equation is considered when the matrix potential is selfadjoint with entries that are integrable and have finite first moments. The small-energy asymptotics of the scattering coefficients are derived, and the continuity of the scattering coefficients at zero energy is established. When the entries of the potential have also finite second moments, some more detailed asymptotic expansions are presented.

PACS Numbers (1999): 2.30.Hq, 03.65.Nk

Mathematics Subject Classification (2000): 34B25, 34L25, 34L40, 47A40, 81U05, 81U20

Keywords: Matrix Schrödinger equation on the line, Scattering matrix, Selfadjoint matrix potential, Small-energy asymptotics

Short title: Matrix Schrödinger equation on the line

I. INTRODUCTION

Consider the matrix Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=Q(x)\,\psi(k,x),\qquad x\in\mathbf{R}, \tag{1.1}$$

where $x\in\mathbf{R}$ is the spatial coordinate, the prime denotes the derivative with respect to x, k^2 is the energy, $Q(x)$ is an $n\times n$ selfadjoint matrix potential, i.e. $Q(x)^\dagger=Q(x)$ with the dagger standing for the matrix conjugate transpose, and $\psi(k,x)$ is either an $n\times 1$ or an $n\times n$ matrix function. We use $||\cdot||$ to denote the (Euclidean) norm of a vector or the operator norm of a matrix. Let $L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ with $m\ge 0$ denote the Banach space of all measurable $n\times n$ matrix functions f for which $(1+|x|)^m\|f(x)\|$ is integrable on $\mathbf{R}$. If $n=1$, we denote this space by $L^1_m(\mathbf{R})$. In this paper we always assume that Q is selfadjoint and belongs to $L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Certain results will be obtained under the assumption that $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$, but we will clearly indicate when this stronger assumption is needed. We use $\mathbf{C}^+$ to denote the upper-half complex plane and write $\overline{\mathbf{C}^+}$ for $\mathbf{C}^+\cup\mathbf{R}$.

Among the $n\times n$ solutions of (1.1) are the so-called Jost solution from the left, $f_l(k,x)$, and the Jost solution from the right, $f_r(k,x)$, satisfying the asymptotic boundary conditions

$$e^{-ikx}f_l(k,x)=I_n+o(1)\ \text{ and }\ e^{-ikx}f_l'(k,x)=ikI_n+o(1),\qquad x\to+\infty, \tag{1.2}$$

$$e^{ikx}f_r(k,x)=I_n+o(1)\ \text{ and }\ e^{ikx}f_l'(k,x)=-ikI_n+o(1),\qquad x\to-\infty, \tag{1.3}$$

where I_n denotes the identity matrix of order n. For each $k\in\mathbf{R}\setminus\{0\}$ we have

$$f_l(k,x)=a_l(k)\,e^{ikx}+b_l(k)\,e^{-ikx}+o(1),\qquad x\to-\infty, \tag{1.4}$$

$$f_r(k,x)=a_r(k)\,e^{-ikx}+b_r(k)\,e^{ikx}+o(1),\qquad x\to+\infty, \tag{1.5}$$

where $a_l(k)$, $b_l(k)$, $a_r(k)$, and $b_r(k)$ are some $n \times n$ matrix functions of k. These matrix functions enter the scattering matrix $\mathbf{S}(k)$ defined in (2.22), and our primary aim is the analysis of the small-k behavior of $\mathbf{S}(k)$.

The motivation for this paper comes from our interest in the inverse scattering problem for (1.1), namely the recovery of Q from an appropriate set of data involving the scattering matrix. As is known from the scalar case $n = 1$, it is important to have detailed information about the behavior of $\mathbf{S}(k)$ for small k. For example[1,2], this information is used to characterize the scattering data, so as to ensure that the potential Q constructed from the data at hand belongs to a certain class of functions such as $L_1^1(\mathbf{R})$ or $L_2^1(\mathbf{R})$. The inverse scattering problem for (1.1) when $n > 1$ has been considered by several authors,[3–8] but we are not aware of any in-depth study of the small-k behavior of $\mathbf{S}(k)$. Not even the continuity of the scattering matrix at $k = 0$ seems to have been established when $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$; for example, in Ref. 4 (p. 294), the continuity at $k = 0$ of the transmission coefficients is *assumed*. In the scalar case it is well known[1,2,9,10] that the continuity of $\mathbf{S}(k)$ at $k = 0$ is easy to establish if $Q \in L_2^1(\mathbf{R})$, but not if only $Q \in L_1^1(\mathbf{R})$. In the matrix case, the situation is somewhat different. The decay of Q as $x \to \pm\infty$ plays an important role, but there are further complications due to the particular structure of the solution space of (1.1) at $k = 0$. From the scalar case it is known[1,2,9] that the behavior of the solutions of (1.1) at $k = 0$ makes it necessary to distinguish between two cases, the *generic case* and the *exceptional case,* and that the small-k behavior of $\mathbf{S}(k)$ is different in each case. If $n > 1$, the situation is more complicated because the exceptional case gives rise to a variety of possibilities depending on the Jordan structure of a certain matrix associated with the solution space of (1.1) at $k = 0$. In this paper we clarify the connection between the solutions of (1.1) at $k = 0$ and the behavior of $\mathbf{S}(k)$ near $k = 0$. As a result, we are able to prove the continuity of the scattering matrix at $k = 0$ when $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$ and to obtain more detailed asymptotic expansions when $Q \in L_2^1(\mathbf{R}; \mathbf{C}^{n\times n})$. The inverse problem is not considered here; we may report on it elsewhere.

This paper is organized as follows. In Section II we establish our notations and review some basic known results on the solutions of (1.1). Since this material is standard, we refer the reader to the literature for proofs and more details. In Section II we also give characterizations of the generic and exceptional cases. In Section III we prove the continuity of the scattering matrix at $k = 0$ in the generic case, and we obtain some more detailed asymptotic results when $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$. The exceptional case is treated in Section IV. The main results are contained in Theorem 4.9 when $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$ and in Theorem 4.10 when $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$, where we prove the continuity and differentiability of $\mathbf{S}(k)$ at $k = 0$, respectively. In Section V we discuss some special cases that illustrate the results of Section IV. Finally, the Appendix contains the proof of Proposition 4.3, which is a key result needed to establish Theorems 4.9 and 4.10.

II. JOST SOLUTIONS AND SCATTERING COEFFICIENTS

In this section we review some basic results about those solutions of (1.1) that are relevant to scattering theory, and we define the scattering coefficients and some related quantities which will be studied in the subsequent sections. The proofs for these results can be found in Refs. 3 and 5 or are simple generalizations of those in the scalar case.[2,9]

We define the Faddeev functions $m_l(k,x)$ and $m_r(k,x)$ by

(2.1) $$m_l(k,x) = e^{-ikx}\, f_l(k,x), \qquad m_r(k,x) = e^{ikx}\, f_r(k,x).$$

From (1.2), (1.3), and (2.1) it follows that

(2.2) $$m_l(k,x) = I_n + \frac{1}{2ik}\int_x^\infty dy\, [e^{2ik(y-x)} - 1]\, Q(y)\, m_l(k,y),$$

(2.3) $$m_r(k,x) = I_n + \frac{1}{2ik}\int_{-\infty}^x dy\, [e^{2ik(x-y)} - 1]\, Q(y)\, m_r(k,y).$$

Some properties of the matrix functions $m_l(k,x)$ and $m_r(k,x)$ are summarized in the next theorem. Throughout the paper we use C to denote a generic constant that does not depend on x or k and that does not necessarily assume the same value at different appearances. Differentiation with respect to k is denoted by an overdot.

Theorem 2.1 If $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$, then, for each $x \in \mathbf{R}$, the functions $m_l(k,x)$, $m_r(k,x)$, $m_l'(k,x)$, and $m_r'(k,x)$ are analytic in $k \in \mathbf{C}^+$, continuous in $k \in \overline{\mathbf{C}^+}$, and

$$(2.4) \qquad m_l(k,x) = I_n + o(1), \qquad m_l'(k,x) = o(1/x), \quad x \to +\infty,$$

$$m_r(k,x) = I_n + o(1), \qquad m_r'(k,x) = o(1/x), \quad x \to -\infty,$$

$$(2.5) \quad \|m_l(k,x)\| \le C[1 + \max\{0,-x\}], \qquad \|m_r(k,x)\| \le C[1 + \max\{0,x\}], \qquad k \in \overline{\mathbf{C}^+}.$$

In addition, if $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$, then $\dot{m}_l(k,x)$ and $\dot{m}_r(k,x)$ exist, are analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and satisfy the estimates

$$\|\dot{m}_l(k,x)\| \le C\,(1+x^2), \qquad \|\dot{m}_r(k,x)\| \le C\,(1+x^2), \qquad k \in \overline{\mathbf{C}^+}.$$

In the following an asterisk will be used to denote complex conjugation. From (2.1) and Theorem 2.1 we get

Corollary 2.2 Assume $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, for each fixed $x \in \mathbf{R}$, the matrix functions $f_l(-k^*,x)^\dagger$, $f_r(-k^*,x)^\dagger$, $f_l'(-k^*,x)^\dagger$, and $f_r'(-k^*,x)^\dagger$ are analytic in $k \in \mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Moreover, if $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$, then these functions are differentiable with respect to $k \in \overline{\mathbf{C}^+}$.

The scattering coefficients to be defined below will be given in terms of certain Wronskians involving the Jost solutions. We first state a standard result about such Wronskians, which is a consequence of the selfadjointness of Q. Let $[F;G] = FG' - F'G$ denote the Wronskian of two square matrix functions $F(x)$ and $G(x)$.

Proposition 2.3 For $k \in \mathbf{C}$, let $\phi(k,x)$ be any $n\times p$ solution and $\psi(k,x)$ any $n\times q$ solution of (1.1). Then the $p\times q$ Wronskian matrix $[\phi(\pm k^*,x)^\dagger; \psi(k,x)]$ is independent of x.

The matrices $a_l(k)$, $b_l(k)$, $a_r(k)$, and $b_r(k)$ appearing in (1.4) and (1.5) can be expressed in terms of certain Wronskians of the Jost solutions as follows:

$$a_l(k) = \frac{1}{2ik}[f_r(-k^*,x)^\dagger; f_l(k,x)], \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \tag{2.6}$$

$$a_r(k) = -\frac{1}{2ik}[f_l(-k^*,x)^\dagger; f_r(k,x)], \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \tag{2.7}$$

$$b_l(k) = -\frac{1}{2ik}[f_r(k,x)^\dagger; f_l(k,x)], \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.8}$$

$$b_r(k) = \frac{1}{2ik}[f_l(k,x)^\dagger; f_r(k,x)], \qquad k \in \mathbf{R} \setminus \{0\}. \tag{2.9}$$

Alternatively, it is sometimes convenient to use the integral representations

$$a_l(k) = I_n - \frac{1}{2ik}\int_{-\infty}^{\infty} dx\, Q(x)\, m_l(k,x), \tag{2.10}$$

$$a_r(k) = I_n - \frac{1}{2ik}\int_{-\infty}^{\infty} dx\, Q(x)\, m_r(k,x), \tag{2.11}$$

$$b_l(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} dx\, e^{2ikx}\, Q(x)\, m_l(k,x), \tag{2.12}$$

$$b_r(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} dx\, e^{-2ikx}\, Q(x)\, m_r(k,x), \tag{2.13}$$

which follow from (1.4), (1.5), and (2.1)-(2.5). Also, by using (1.2)-(1.5) and some Wronskian relations for the Jost solutions, we obtain

$$a_r(-k^*)^\dagger = a_l(k), \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \tag{2.14}$$

$$b_r(k) = -b_l(k)^\dagger, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.15}$$

$$a_l(k)^\dagger\, a_l(k) = b_l(k)^\dagger\, b_l(k) + I_n, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.16}$$

$$a_r(k)^\dagger\, a_r(k) = b_r(k)^\dagger\, b_r(k) + I_n, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.17}$$

$$a_l(-k)^\dagger\, b_l(k) = b_l(-k)^\dagger\, a_l(k), \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.18}$$

$$a_r(-k)^\dagger\, b_r(k) = b_r(-k)^\dagger\, a_r(k), \qquad k \in \mathbf{R} \setminus \{0\}. \tag{2.19}$$

We define the transmission coefficient from the left, $T_l(k)$, and the transmission coefficient from the right, $T_r(k)$, by

$$T_l(k) = a_l(k)^{-1}, \qquad T_r(k) = a_r(k)^{-1}, \tag{2.20}$$

provided the inverses on the right-hand side exist, and we define the reflection coefficient from the left, $L(k)$, and the reflection coefficient from the right, $R(k)$, by

$$L(k) = b_l(k)\, a_l(k)^{-1}, \qquad R(k) = b_r(k)\, a_r(k)^{-1}. \tag{2.21}$$

From (2.16) and (2.17) we see that $a_l(k)$ and $a_r(k)$ are nonsingular for $k \in \mathbf{R} \setminus \{0\}$. In $\mathbf{C}^+$, $a_l(k)$ and $a_r(k)$ are nonsingular except possibly at a finite number of points on the positive imaginary axis where[5] both $\det a_l(k) = 0$ and $\det a_r(k) = 0$; at these points, $T_l(k)$ and $T_r(k)$ have simple poles corresponding to the bound states of (1.1). The finiteness of the number of bound states when $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$ follows from the (operator) inequality $Q(x) \geq -\|Q(x)\|\, I_n$ and the fact that in one dimension a scalar potential in $L^1_1(\mathbf{R})$ can support at most a finite number of bound states. Alternatively, the finiteness of the number of bound states follows from the continuity of the transmission coefficients at $k = 0$ (cf. Theorems 3.2 and 4.9); for $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$ this was already noted in Ref. 5. Therefore, it is meaningful to study the asymptotic behavior of the transmission coefficients as $k \to 0$ through values in $\overline{\mathbf{C}^+}$. The reflection coefficients, on the other hand, in general do not have analytic extensions off the real axis, so their asymptotics will be studied for real k only. The $n \times n$ matrix functions $T_l(k)$, $T_r(k)$, $R(k)$, and $L(k)$ are referred to as scattering coefficients, and the $2n \times 2n$ matrix

$$\mathbf{S}(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}, \tag{2.22}$$

is called the scattering matrix.

From (2.15), (2.16), and (2.17), we get

$$T_l(k)^\dagger \, R(k) + L(k)^\dagger \, T_r(k) = 0, \qquad k \in \mathbf{R} \setminus \{0\},$$

$$T_l(k)^\dagger \, T_l(k) + L(k)^\dagger \, L(k) = I_n, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.23}$$

$$T_r(k)^\dagger \, T_r(k) + R(k)^\dagger \, R(k) = I_n, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.24}$$

and hence, for $k \in \mathbf{R} \setminus \{0\}$, $\mathbf{S}(k)$ is unitary. Using (2.14) we obtain

$$T_r(k) = T_l(-k^*)^\dagger, \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \tag{2.25}$$

whenever $a_r(k)$ is nonsingular, and from (2.18) and (2.19) we get

$$L(-k)^\dagger = L(k), \qquad R(-k)^\dagger = R(k), \qquad k \in \mathbf{R} \setminus \{0\}.$$

In the next section we will begin our study of $\mathbf{S}(k)$ in the small-energy limit. We will distinguish between the generic case and the exceptional case. These two cases can be characterized in several ways. We choose the following definition as our starting point and then arrive at some other characterizations as we go along. Let

$$\mathcal{N} = \{\xi \in \mathbf{C}^n : \; f_l(0,x)\,\xi \; \text{ is bounded on } \mathbf{R}\}. \tag{2.26}$$

Then we say that the generic case occurs if $\mathcal{N} = \{0\}$ and that the exceptional case occurs if $\mathcal{N} \neq \{0\}$. This can also be phrased in the following way. Consider (1.1) with $k = 0$, i.e.

$$\psi''(0,x) = Q(x)\,\psi(0,x), \qquad x \in \mathbf{R}. \tag{2.27}$$

Since, every bounded vector solution of (2.27) is of the form $f_l(0,x)\,\xi$ for some vector ξ in $\mathbf{C}^n$, the generic case occurs if and only if (2.27) has no bounded nontrivial solution. The exceptional case occurs if and only if there exists at least one nontrivial bounded solution. Alternatively, we can characterize the two cases by means of the subspace

$$\mathcal{M} = \{\chi \in \mathbf{C}^n : f_r(0,x)\,\chi \text{ is bounded on } \mathbf{R}\}. \tag{2.28}$$

Then the generic (exceptional) case occurs if and only if $\mathcal{M} = \{0\}$ ($\mathcal{M} \neq \{0\}$). The precise connection between the subspaces $\mathcal{N}$ and $\mathcal{M}$ will be established in the subsequent sections.

We end this section with a brief look at the scalar case. When $n = 1$ the exceptional case occurs if and only if $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent, i.e. the Wronskian $[f_r(0,x); f_l(0,x)]$ is zero. In our paper we generalize this characterization to the matrix case. In the scalar case it is also known that the generic (exceptional) case occurs if $T_l(0) = 0$ ($T_l(0) \neq 0$). This will also turn out be true in the matrix case, but we do not use this property as our primary characterization because it is implicitly based on the assumption that $T_l(k)$ is continuous at $k = 0$, something we first need to prove.

III. SMALL-k BEHAVIOR IN THE GENERIC CASE

In this section we analyze the behavior of the scattering coefficients near $k = 0$ in the generic case. We will see that the distinction between the generic case and the exceptional case is closely related to the invertibility, resp. noninvertibility, of the Wronskian matrices

$$\Delta_l = [f_r(0,x)^\dagger; f_l(0,x)], \qquad \Delta_r = -[f_l(0,x)^\dagger; f_r(0,x)], \tag{3.1}$$

which are related by

$$\Delta_l = \Delta_r^\dagger. \tag{3.2}$$

By Proposition 2.3, Δ_l and Δ_r are independent of x. The importance of these Wronskians lies in the fact that they are related to the transmission coefficients via (2.20) and

$$\Delta_l = \lim_{k\to 0} 2ik\, a_l(k), \qquad \Delta_r = \lim_{k\to 0} 2ik\, a_r(k), \tag{3.3}$$

where the limits are taken from within $\overline{\mathbf{C}^+}$; (3.3) follows from (2.6), (2.7), and Corollary 2.2.

The Wronskians Δ_l and Δ_r in (3.1) can be expressed in other ways by using the integral representations for $f_l(0,x)$ and $f_r(0,x)$, namely

$$f_l(0,x) = I_n + \int_x^\infty dy\,(y-x)\,Q(y)\,f_l(0,y), \tag{3.4}$$

$$f_r(0,x) = I_n - \int_{-\infty}^x dy\,(y-x)\,Q(y)\,f_r(0,y). \tag{3.5}$$

Evaluating the first Wronskian in (3.1) as $x\to -\infty$ and using (3.4) we obtain

$$\Delta_l = \lim_{x\to-\infty} f_l'(0,x) = -\int_{-\infty}^\infty dy\,Q(y)\,f_l(0,y). \tag{3.6}$$

Similarly, from (3.1) and (3.5), letting $x\to+\infty$, we get

$$\Delta_r = -\lim_{x\to+\infty} f_r'(0,x) = -\int_{-\infty}^\infty dy\,Q(y)\,f_r(0,y). \tag{3.7}$$

From (3.4) and (3.5) we also infer

$$f_l(0,x) = x\,\Delta_l + o(x), \qquad x\to-\infty, \tag{3.8}$$

$$f_r(0,x) = -x\,\Delta_r + o(x), \qquad x\to+\infty,$$

from which we conclude that (2.27) can have bounded nontrivial vector solutions only if Δ_l (resp. Δ_r) has a nontrivial kernel. This observation has the following consequence.

Theorem 3.1 Assume $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. Then the generic case occurs if and only if Δ_l, or equivalently Δ_r, is nonsingular.

PROOF: First, if Δ_l is nonsingular, then (3.8) shows that every solution of (2.27) having the form $f_l(0,x)\,\eta$ with some nonzero vector $\eta \in \mathbf{C}^n$ becomes unbounded as $x \to -\infty$. Recall that the asymptotic properties of the solutions to (2.27) are determined in leading order by the unperturbed equation (1.1) with $Q = 0$. By the standard asymptotic theory[11] every bounded vector solution of (2.27) is of the form $f_l(0,x)\,\eta$ for some $\eta \in \mathbf{C}^n$. Hence, if Δ_l is nonsingular, then (2.27) has no bounded nontrivial solutions; so the generic case occurs. Conversely, suppose the generic case occurs and Δ_l is singular. Then for any nonzero $\xi \in \operatorname{Ker}\Delta_l$, we have $f_l(0,x)\,\xi = o(x)$ as $x \to -\infty$. However, every solution with this property must be bounded; this again follows from the general theory. Thus (2.27) has a bounded solution, which is a contradiction. Hence, in the generic case, Δ_l cannot be singular. This proves the theorem for Δ_l. In view of (3.2), our theorem also holds if Δ_l is replaced by Δ_r. ∎

In order to state the next theorem, which is the main result of this section, we introduce the quantities

$$E_l = \int_{-\infty}^{\infty} dx\, x\, Q(x)\, m_l(0,x), \qquad E_r = \int_{-\infty}^{\infty} dx\, x\, Q(x)\, m_r(0,x), \tag{3.9}$$

$$G_l = \int_{-\infty}^{\infty} dx\, Q(x)\, \dot{m}_l(0,x), \qquad G_r = \int_{-\infty}^{\infty} dx\, Q(x)\, \dot{m}_r(0,x).$$

The quantities E_l and E_r will also play a role in Section IV.

Theorem 3.2 Assume Q is a generic potential in $L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ for $m = 1$ or 2. Then the scattering coefficients satisfy the following.

(i) If $m = 1$, then

$$T_l(k) = 2ik\,\Delta_l^{-1} + o(k), \qquad T_r(k) = 2ik\,\Delta_r^{-1} + o(k), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$R(k) = -I_n + o(1), \qquad L(k) = -I_n + o(1), \qquad k \to 0 \text{ in } \mathbf{R}.$$

(ii) If $m = 2$, then

$$T_l(k) = 2ik\,\Delta_l^{-1} + k^2\,\Delta_l^{-1}[4I_n + 2i\,G_l]\Delta_l^{-1} + o(k^2), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$T_r(k) = 2ik\,\Delta_r^{-1} + k^2\,\Delta_r^{-1}[4I_n + 2i\,G_r]\Delta_r^{-1} + o(k^2), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$L(k) = -I_n + 2ik\,[I_n + E_l]\,\Delta_l^{-1} + o(k), \qquad k \to 0 \text{ in } \mathbf{R},$$

$$R(k) = -I_n + 2ik\,[I_n - E_r]\,\Delta_r^{-1} + o(k), \qquad k \to 0 \text{ in } \mathbf{R}.$$

PROOF: Using the fact that in the generic case Δ_l and Δ_r are invertible, (i) is a consequence of (2.6)-(2.9), (2.20), (2.21), (3.3), and Corollary 2.2. When $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$, expanding the integrals in (2.10)-(2.13) as

$$a_l(k) = \frac{1}{2ik}\Delta_l + I_n + \frac{i}{2}G_l + o(1), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{3.10}$$

$$b_l(k) = -\frac{1}{2ik}\Delta_l + E_l - \frac{i}{2}G_l + o(1), \qquad k \to 0 \text{ in } \mathbf{R}, \tag{3.11}$$

$$a_r(k) = \frac{1}{2ik}\Delta_r + I_n + \frac{i}{2}G_r + o(1), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{3.12}$$

$$b_r(k) = -\frac{1}{2ik}\Delta_r - E_r - \frac{i}{2}G_r + o(1), \qquad k \to 0 \text{ in } \mathbf{R}, \tag{3.13}$$

and using (2.20) and (2.21) we obtain (ii). ∎

For later use we remark that when $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$, E_l and E_r can be expressed in terms of certain Wronskians, namely

$$I_n + E_l = i\,[\dot{f}_r(0,x)^\dagger; f_l(0,x)], \qquad I_n - E_r = -i\,[\dot{f}_l(0,x)^\dagger; f_r(0,x)]. \tag{3.14}$$

Note that the Wronskians in (3.14) are independent of x because $\dot{f}_l(0,x)$ and $\dot{f}_r(0,x)$ are also solutions of (2.27). The expressions in (3.14) follow easily from (3.4), (3.5), and the corresponding integral equations for $\dot{f}_l(0,x)$ and $\dot{f}_r(0,x)$ (cf. (A.20)). We also have $G_r = -G_l^\dagger$ and $E_r = E_l^\dagger + i\,G_l^\dagger$, as can be seen by using (2.14), (2.15), and (3.10)-(3.13).

Theorem 3.2 shows that if the generic case occurs, then $T_l(0)=0$. In the next section we will see that the converse is also true.

IV. SMALL-k BEHAVIOR IN THE EXCEPTIONAL CASE

Recall that in the exceptional case (2.27) has at least one bounded nontrivial solution. In this section we analyze how this affects the small-k properties of $\mathbf{S}(k)$, and we prove in the exceptional case the continuity of $S(k)$ at $k=0$ when $Q\in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$ and its differentiability when $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$. It turns out that when $Q\in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$ the exceptional case gives rise to technical complications that require a careful study of certain asymptotic expansions. Since the proof of one result, Proposition 4.3, is especially long, that proof is given in the Appendix.

The first two propositions provide more information about the subspaces $\mathcal{N}$ and $\mathcal{M}$ defined in (2.26) and (2.28), respectively, and their relation with the Wronskians in (3.1).

Proposition 4.1 Assume $Q\in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Then we have

(i) $\mathcal{N}=\operatorname{Ker}\Delta_l$ and $\mathcal{M}=\operatorname{Ker}\Delta_r$.

(ii) $\dim\mathcal{N}=\dim\mathcal{M}$.

PROOF: If $\xi\in\operatorname{Ker}\Delta_l$, then, by (3.8), $f_l(0,x)\,\xi=o(x)$ as $x\to-\infty$. Hence $f_l(0,x)\,\xi$ is bounded and so $\xi\in\mathcal{N}$. Conversely, if $\xi\in\mathcal{N}$, then $f_l(0,x)\,\xi$ is bounded and therefore, again by (3.8), $\Delta_l\,\xi=0$. This proves the first equality in (i). The second equality is proved similarly. We obtain (ii) directly from (3.2). ∎

There is a natural mapping from $\mathcal{N}$ to $\mathcal{M}$, which we denote by Γ, defined as follows. For every $\xi\in\mathcal{N}$, let

$$\chi=\lim_{x\to-\infty}f_l(0,x)\,\xi, \tag{4.1}$$

and put

$$\chi = \Gamma\,\xi. \tag{4.2}$$

To see that Γ maps $\mathcal{N}$ into $\mathcal{M}$, we note that (4.1) implies

$$\lim_{x\to-\infty}[f_l(0,x)\,\xi - f_r(0,x)\,\chi] = 0.$$

Hence $f_l(0,x)\,\xi - f_r(0,x)\,\chi$ is a solution of (2.27) which approaches zero as $x \to -\infty$; therefore, it must be identically zero and we have

$$f_l(0,x)\,\xi = f_r(0,x)\,\chi, \qquad x \in \mathbf{R}. \tag{4.3}$$

Hence $f_r(0,x)\,\chi$ is bounded, which implies $\chi \in \mathcal{M}$.

Proposition 4.2 Assume $Q \in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Then Γ is a bijection between $\mathcal{N}$ and $\mathcal{M}$.

PROOF: We have already seen that Γ maps $\mathcal{N}$ into $\mathcal{M}$. The map Γ is injective, for if $\Gamma\,\xi = 0$, then, by (4.2) and (4.3), $f_l(0,x)\,\xi = 0$ for all $x \in \mathbf{R}$ and hence $\xi = 0$. It is also onto, because for every $\chi \in \mathcal{M}$, $\lim_{x\to+\infty} f_r(0,x)\,\chi = \xi$ exists, and hence (4.3) holds; thus $\chi = \Gamma\,\xi$. ∎

The mapping Γ will make its appearance as a restriction to $\mathcal{N}$ of certain linear transformations defined on all of $\mathbf{C}^n$. One such representation immediately follows from (4.3). We can pick any x_0 for which $f_r(0,x_0)$ is invertible and write

$$\Gamma = [f_r(0,x_0)^{-1}\,f_l(0,x_0)]|_{\mathcal{N}}, \tag{4.4}$$

where the symbol $|_{\mathcal{N}}$ denotes the restriction to the subspace $\mathcal{N}$. Recall that when $n = 1$, Γ is a constant, so that (4.4) expresses the fact that, in the exceptional case, the two Jost solutions at $k = 0$ are linearly dependent. Clearly, (4.4) is valid whenever $Q \in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Another representation of Γ that will play a role in this section is only valid when $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$. It follows from (3.4) which, for any $\xi \in \mathcal{N}$, implies

$$\chi = \lim_{x\to-\infty} f_l(0,x)\,\xi = \xi + \int_{-\infty}^{\infty} dy\, y\, Q(y)\ [f_l(0,y)\,\xi], \tag{4.5}$$

where we have also used (3.6) and the fact that $\Delta_l\,\xi = 0$. Note that the integral on the right-hand side of (4.5) exists when $Q \in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$ because $f_l(0,y)\,\xi$ is bounded. However, without the vector ξ in the integrand, the integral in general does not exist as a matrix-valued integral, because some column vectors of the matrix $f_l(0,y)$ may grow linearly as $y \to -\infty$. In fact, according to (3.8), this is always the case unless $\Delta_l = 0$. On the other hand, if $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$, then the integral in (4.5) without having ξ in it exists as a matrix-valued integral and, in view of (3.9), we can write $\chi = (I_n + E_l)\,\xi$. In other words, we have

$$\Gamma = (I_n + E_l)|_{\mathcal{N}} \quad \text{provided} \quad Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n}).$$

We will also need representations for Γ^{-1}. To this end we assume, without loss of generality, that $f_l(0,0)$ is invertible. If not, we can perform a shift of the origin and use the fact that $f_l(0,x)$ is invertible for x sufficiently large. We define

$$\mathcal{R} = f_l(0,0)^{-1} f_r(0,0), \tag{4.6}$$

and note that, by (4.3),

$$\mathcal{R}|_{\mathcal{M}} = \Gamma^{-1}. \tag{4.7}$$

Another representation for Γ^{-1} is obtained by using the integral relation for $f_r(0,x)$ given in (3.5). If $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$, then, for any $\chi \in \mathcal{M}$, by using (3.5), (3.7), and the fact that $\Delta_r\chi = 0$, we obtain

$$\xi = \lim_{x\to+\infty} f_r(0,x)\,\chi = \chi - \left[\int_{-\infty}^{\infty} dy\, y\, Q(y)\, f_r(0,y)\right]\chi,$$

and thus, by (3.9), $\xi = (I_n - E_r)\,\chi$. Therefore

$$\Gamma^{-1} = (I_n - E_r)|_{\mathcal{M}} \quad \text{provided} \quad Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n}).$$

After these preparations we are ready to begin the analysis of the small-k asymptotics of $\mathbf{S}(k)$ in the exceptional case. We first consider the Wronskian

$$W(k) = [f_r(-k^*,x)^\dagger; f_l(k,x)], \qquad k \in \overline{\mathbf{C}^+},$$

which appears in (2.6) and, as seen from (2.20), is related to the transmission coefficient $T_l(k)$ by

$$T_l(k) = 2ik\, W(k)^{-1}. \tag{4.8}$$

The method used here to study $W(k)$ is patterned after that used in Ref. 10 in the scalar case. Unless otherwise stated, we will assume k is real. This suffices for all the auxiliary results leading up to our main result given in Theorem 4.9. There we will extend the asymptotics from the real axis to $\mathbf{C}^+$ with the help of a Phragmén-Lindelöf theorem.

Using $[f_l(0,x)^\dagger; f_l(0,x)] = 0$ we first write $W(k)$ in the form

$$W(k) = \ f_r(-k,0)^\dagger\, (f_l(0,0)^\dagger)^{-1}\Omega_1 + \Omega_2\, f_l(0,0)^{-1} f_l(k,0),$$

where we have defined

$$\Omega_1 = f_l(0,0)^\dagger\, f_l'(k,0) - f_l'(0,0)^\dagger\, f_l(k,0),$$

$$\Omega_2 = f_r(-k,0)^\dagger\, f_l'(0,0) - f_r'(-k,0)^\dagger\, f_l(0,0).$$

The quantities Ω_1 and Ω_2 can be written as Wronskians by means of a new solution, $\varphi(k,x)$, of (1.1), which is defined by the initial conditions

$$\varphi(k,0) = f_l(0,0), \qquad \varphi'(k,0) = f_l'(0,0), \tag{4.9}$$

so that

$$\varphi(0,x) = f_l(0,x). \tag{4.10}$$

Then we have

(4.11)

$$W(k) = f_r(-k,0)^\dagger (f_l(0,0)^\dagger)^{-1} [\varphi(k,x)^\dagger; f_l(k,x)] + [f_r(-k,x)^\dagger; \varphi(k,x)]\, f_l(0,0)^{-1} f_l(k,0).$$

We mention that the particular choice of the solution $\varphi(k,x)$ is motivated by the fact that there is a crucial estimate, namely (A.8) of the Appendix, for the difference $[\varphi(k,x) -$

$\varphi(0,x)]\,\xi$ with $\xi \in \mathcal{N}$, which plays a key role in the proof of the next proposition. Since the proof of this proposition is lengthy, it is given in the Appendix.

Proposition 4.3 Assume $Q \in L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ for $m=1$ or 2. Then, as $k\to 0$ in $\mathbf{R}$ we have

$$[\varphi(k,x)^\dagger; f_l(k,x)] = \sum_{j=1}^{m} k^j Y_j + o(k^m), \tag{4.12}$$

where

$$Y_1 = iI_n, \qquad Y_2 = \int_0^\infty dz\,[f_l(0,z)^\dagger\, f_l(0,z) - I_n],$$

$$[f_r(-k,x)^\dagger; \varphi(k,x)] = \sum_{j=0}^{m-1} k^j X_j + o(k^{m-1}), \tag{4.13}$$

$$X_0 = \Delta_l, \qquad X_1 = i[I_n + E_l].$$

For $\xi \in \mathcal{N}$ we have

$$[f_r(-k,x)^\dagger; \varphi(k,x)]\,\xi = \sum_{j=1}^{m} k^j \check{X}_j\,\xi + o(k^m), \tag{4.14}$$

$$\check{X}_1 = i\Gamma, \qquad \check{X}_2 = \int_{-\infty}^0 dz\,[f_r(0,z)^\dagger\, f_r(0,z) - I_n]\,\Gamma.$$

The notational differences between (4.13) and (4.14) are justified by the fact that in (4.14) the coefficient $\check{X}_1$ is used when $m=1$, while in (4.13) the corresponding coefficient X_1 is used only when $m=2$. Of course, if $m=2$, then $\check{X}_1 = X_1|_{\mathcal{N}}$, by (4.5).

Our first goal is to find the leading terms in the asymptotics of $W(k)^{-1}$ as $k\to 0$. For this purpose it is convenient to temporarily replace the factors multiplying the Wronskians in (4.11) by their limits as $k\to 0$. That is, we consider the simpler expression

$$Z(k) = \mathcal{R}^\dagger\,[\varphi(k,x)^\dagger; f_l(k,x)] + [f_r(-k,x)^\dagger; \varphi(k,x)], \tag{4.15}$$

where we have used (4.6) via its adjoint. In order to further motivate the use of $Z(k)$, we note that on account of (4.11) and (4.15) we can write

$$W(k)^{-1} = f_l(k,0)^{-1}\, f_l(0,0)\, [Z(k) + \Theta_1(k) + \Theta_2(k)]^{-1}, \tag{4.16}$$

where

$$\Theta_1(k) = \mathcal{R}^\dagger\, [\varphi(k,x)^\dagger; f_l(k,x)]\, [f_l(k,0)^{-1}\, f_l(0,0) - I_n], \tag{4.17}$$

$$\Theta_2(k) = [f_r(-k,0)^\dagger\, (f_l(0,0)^\dagger)^{-1} - \mathcal{R}^\dagger]\, [\varphi(k,x)^\dagger; f_l(k,x)]\, f_l(k,0)^{-1}\, f_l(0,0), \tag{4.18}$$

provided the second inverse on the right-hand side of (4.16) exists. The existence of this inverse will be established below, where we show that $Z(k)^{-1}$ exists for sufficiently small k and satisfies $Z(k)^{-1} = O(1/k)$ as $k \to 0$. This, together with the fact that, in view of (4.12) and Corollary 2.2, $\Theta_1(k)$ and $\Theta_2(k)$ are both $o(k)$ as $k \to 0$, implies that

$$W(k)^{-1} = f_l(k,0)^{-1}\, f_l(0,0)\, Z(k)^{-1} \left\{ I_n + [\Theta_1(k) + \Theta_2(k)]\, Z(k)^{-1} \right\}^{-1}, \tag{4.19}$$

where the inverse of the matrix inside the braces exists provided k is sufficiently small. This explains why we focus on $Z(k)$ in the next result, which is an immediate consequence of (4.15) and Proposition 4.3.

Corollary 4.4 Suppose that $Q \in L^1_m(\mathbf{R}; \mathbf{C}^{n\times n})$ for $m = 1$ or 2. Then

$$Z(k) = \sum_{j=0}^{m-1} k^j V_j + o(k^{m-1}), \tag{4.20}$$

$$V_0 = \Delta_l, \qquad V_1 = i[I_n + E_l + \mathcal{R}^\dagger].$$

Moreover, for $\xi \in \mathcal{N}$, we have

$$Z(k)\, \xi = \sum_{j=1}^{m} k^j \check{V}_j\, \xi + o(k^m), \tag{4.21}$$

$$\check{V}_1 = i[\Gamma + \mathcal{R}^\dagger],$$

$$\check{V}_2 = \mathcal{R}^\dagger \int_0^\infty dz\,[f_l(0,z)^\dagger\, f_l(0,z) - I_n] + \int_{-\infty}^0 dz\,[f_r(0,z)^\dagger\, f_r(0,z) - I_n]\,\Gamma.$$

Now our task is to identify those matrix elements of $Z(k)^{-1}$ that dominate as $k \to 0$. To do this we choose a Jordan basis for Δ_l as follows. We assume that there are κ Jordan chains indexed by α for $\alpha = 1, \dots, \kappa$, each consisting of n_α vectors $u_{\alpha j}$, with $j = 1, \dots, n_\alpha$, satisfying the relations

$$\begin{cases} (\Delta_l - \lambda_\alpha) u_{\alpha 1} = 0, \\ (\Delta_l - \lambda_\alpha)\, u_{\alpha j} = u_{\alpha(j-1)}, \qquad j = 2, \dots, n_\alpha. \end{cases} \tag{4.22}$$

Here λ_α is an eigenvalue of Δ_l, $u_{\alpha 1}$ is the corresponding eigenvector belonging to the αth chain, and the vectors $u_{\alpha j}$ with $j \neq 1$ are the generalized eigenvectors. The vectors $u_{\alpha j}$ with $\alpha = 1, \dots, \kappa$ and $1 \le j \le n_\alpha$ form the Jordan basis for Δ_l. We assume that the eigenvalue 0 of Δ_l has geometric multiplicity μ and algebraic multiplicity ν; thus $\sum_{\alpha=1}^{\mu} n_\alpha = \nu$ and $\mu = \dim \mathcal{N} \ge 1$. We arrange the vectors of the Jordan basis in a list which is ordered according to the rule that $u_{\alpha j}$ comes before $u_{\beta s}$ if and only if $\alpha < \beta$ or $\alpha = \beta$ and $j < s$. In other words, this is the "dictionary order" of the two two-letter words αj and βs. It is necessary to specify an order on the Jordan basis because later we will have to perform certain permutations on these basis vectors. The transition matrix from the standard basis to the Jordan basis will be denoted by $\mathcal{S}$; it is formed from the column vectors $u_{\alpha j}$ as

$$\mathcal{S} = [\,u_{11} \quad \cdots \quad u_{1n_1} \quad \cdots \quad u_{\kappa 1} \quad \cdots \quad u_{\kappa n_\kappa}\,]. \tag{4.23}$$

We also assume that the basis vectors are enumerated in such a way that the first μ Jordan chains belong to the eigenvalue 0 of Δ_l and the set $\{u_{11}, u_{21}, \dots, u_{\mu 1}\}$ forms a basis for the kernel of Δ_l. Hence, by Proposition 4.1, we have

$$\mathcal{N} = \text{Span}\,\{u_{11}, u_{21}, \dots, u_{\mu 1}\}.$$

Given any $n \times n$ matrix M in the standard basis, we use $\tilde{M}$, where $\tilde{M} = \mathcal{S}^{-1} M\, \mathcal{S}$, to denote the matrix representation of M in the Jordan basis $\{u_{\alpha j}\}$. Then from (4.22) it follows that $\tilde{\Delta}_l$ has the appearance

$$\tilde{\Delta}_l = \bigoplus_{\alpha=1}^{\kappa} J_{n_\alpha}(\lambda_\alpha), \tag{4.24}$$

where

$$J_{n_\alpha}(\lambda_\alpha) = \begin{bmatrix} \lambda_\alpha & 1 & \dots & 0 \\ 0 & \lambda_\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & \lambda_\alpha \end{bmatrix}, \tag{4.25}$$

with $\lambda_\alpha = 0$ for $\alpha = 1, \dots, \mu$ and $\lambda_\alpha \neq 0$ for $\alpha = \mu + 1, \dots, \kappa$.

In the notation introduced above we can view the pair αj as a "block index" in the sense that α indicates the Jordan block (resp. the Jordan chain) to which the vector $u_{\alpha j}$ belongs, and j indicates the position within that block. Note that the "address" αj corresponds to the $(n_1 + \dots + n_{\alpha-1} + j)$th basis vector in the list defined above (4.23). Generalizing this notation, we will sometimes use block indices to designate the matrix elements of matrices represented in the Jordan basis $\{u_{\alpha j}\}$. So $\tilde{M}_{\beta s;\alpha j}$ designates the matrix element in row $n_1 + \dots + n_{\beta-1} + s$ and column $n_1 + \dots + n_{\alpha-1} + j$ of $\tilde{M}$. In order to evaluate $\tilde{M}_{\beta s;\alpha j}$, it is convenient to use the adjoint basis $\{w_{\alpha j}\}$ associated with the Jordan basis $\{u_{\alpha j}\}$. The vectors $\{w_{\alpha j}\}$ satisfy

$$w^\dagger_{\alpha j} u_{\rho t} = \delta_{\alpha\rho}\delta_{jt}, \tag{4.26}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker delta, and for $\alpha = 1, \dots, \kappa$ we have

$$\begin{cases} (\Delta_r - \lambda^*_\alpha)\, w_{\alpha n_\alpha} = 0, \\ (\Delta_r - \lambda^*_\alpha)\, w_{\alpha j} = w_{\alpha(j+1)}, \qquad j = 1, \dots, n_\alpha - 1. \end{cases}$$

Thus the eigenvectors for the eigenvalue 0 of Δ_r are given by w_{sn_s} for $s = 1, \dots, \mu$, and so by Proposition 4.1 we have

$$\mathcal{M} = \mathrm{Span}\,\{w_{1n_1}, w_{2n_2}, \dots, w_{\mu n_\mu}\}.$$

Moreover, (4.23) and (4.26) imply that the matrix $\mathcal{S}^{-1}$ is given by

$$\mathcal{S}^{-1} = [\, w_{11} \quad \cdots \quad w_{1n_1} \quad \cdots \quad w_{\kappa 1} \quad \cdots \quad w_{\kappa n_\kappa} \,]^\dagger .$$

Thus $\mathcal{S}^{-1}$ is the matrix whose rows are $w_{11}^\dagger, \ldots, w_{\kappa n_\kappa}^\dagger$. Then the matrix elements of $\tilde{M} = \mathcal{S}^{-1} M\, \mathcal{S}$ in block index notation are given by

$$\tilde{M}_{\beta s;\alpha j} = w_{\beta s}^\dagger M u_{\alpha j}. \tag{4.27}$$

Proposition 4.5 Assume $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then the matrix elements of $\tilde{Z}(k)$, where $Z(k)$ is the matrix given in (4.15), have the following asymptotic behaviors as $k \to 0$:

(i) All the elements in columns $\alpha 1$ for $\alpha = 1, \ldots, \mu$ are of the form $ck + o(k)$ for suitable constants c (which may depend on the particular element considered).

(ii) The elements in column αj for $\alpha = 1, \ldots, \mu$ and $j = 2, \ldots, n_\alpha$ are $o(1)$ except for the element in row $\alpha(j-1)$ which is $1 + o(1)$.

(iii) The elements in column $\alpha 1$ for $\alpha = \mu + 1, \ldots, \kappa$ are all $o(1)$ except for the element in row $\alpha 1$ which is $\lambda_\alpha + o(1)$.

(iv) The elements in column αj for $\alpha = \mu + 1, \ldots, \kappa$ and $j = 2, \ldots, n_\alpha$ are all $o(1)$ except for the element in row $\alpha(j-1)$ which is $1 + o(1)$, and the element in row αj which is $\lambda_\alpha + o(1)$.

If, in addition, $Q \in L_2^1(\mathbf{R}; \mathbf{C}^{n\times n})$, then the $O(k)$ terms in (i) are of the form $c_1 k + c_2 k^2 + o(k^2)$ for suitable constants c_1 and c_2, and the $o(1)$ terms in (ii)-(iv) are $O(k)$ and of the form $ck + o(k)$ for suitable constants c.

PROOF: The proof follows from Corollary 4.4 using (4.27). In particular, (i) follows from (4.21) on setting $\xi = u_{\alpha 1}$, because $u_{\alpha 1} \in \mathcal{N}$ for $1 \le \alpha \le \mu$. To prove (ii), we use (4.20), (4.22), (4.26), and $\lambda_\alpha = 0$, which together imply

$$\tilde{Z}(k)_{\beta s;\alpha j} = w_{\beta s}^\dagger\, \Delta_l\, u_{\alpha j} + o(1) = w_{\beta s}^\dagger\, u_{\alpha(j-1)} + o(1) = \delta_{\beta\alpha}\, \delta_{s(j-1)} + o(1).$$

Looking at column αj of $\tilde{Z}(k)$, we see that the only matrix element which is not $o(1)$ occurs when $\beta = \alpha$ and $s = j - 1$, and this element is of the form $1 + o(1)$. Since this is the element in row $\alpha(j-1)$, the assertion follows. Parts (iii) and (iv) follow in a similar way; the details are omitted. Finally, the result when $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$ is a consequence of the expansions in (4.20) and (4.21), using $m = 2$ there. ∎

An important observation about $\tilde{Z}(k)$ is that it has μ columns, namely those with "addresses" $\alpha 1$ for $\alpha = 1, \ldots, \mu$ which are $O(k)$, and these are the only columns with this property. Any other column contains at least one element that tends to a nonzero limit as $k \to 0$. Now, as we shall see below, the entries of $\tilde{Z}(k)$ which determine the leading asymptotic behavior of $\tilde{Z}(k)^{-1}$ as $k \to 0$ form a submatrix of $\tilde{Z}(k)$ consisting of the columns $\alpha 1$ and the rows βn_β, where α and β both belong to $\{1, \ldots, \mu\}$. It is therefore convenient to perform suitable permutations of the columns and rows of $\tilde{Z}(k)$ in order to collect these particular matrix elements in a $\mu \times \mu$ diagonal block of a new matrix, called $\mathcal{Z}(k)$. We briefly explain the action of these permutations; first for the columns. Column 11, which is the first column, is already in the right place because, by Proposition 4.5(i), its elements are $O(k)$. Then we move column 21 (the (n_1+1)st column) into second place next to the first column by means of a sequence of column-interchanges. Similarly, column 31 is moved into third place, etc. The last column to be shifted is column $\mu 1$ which is moved into μth place. The columns αj with $\alpha > \mu$ or $\alpha = \mu$ and $j > 1$ stay put during this process. With the rows we proceed similarly, first moving row $1n_1$ into first place, then moving row $2n_2$ into second place, etc. We do this with all the rows βn_β for $1 \leq \beta \leq \mu$. The formal definition of these permutations and their implementation are as follows. Let π_1 be the permutation

$$\pi_1 : (1, \ldots, \nu) \mapsto (q_1, \ldots, q_\nu),$$

where

$$q_\tau = \begin{cases} n_1 + \cdots + n_{\tau-1} + 1, & \tau = 1, \ldots, \mu, \\ \tau - \mu + \alpha, & \tau = \mu + 1, \ldots, \nu, \end{cases} \tag{4.28}$$

and $\alpha \in \{1, \ldots, \mu\}$ is the unique integer such that, for given τ and μ,

$$n_1 + n_2 + \cdots + n_{\alpha-1} - \alpha + j = \tau - \mu,$$

for some $j \in \{2, \ldots, n_\alpha\}$. Note that, since $n_\alpha \geq 1$, the quantity $n_1 + n_2 + \cdots + n_{\alpha-1} - \alpha$ is a nondecreasing function of α. Similarly, let π_2 be the permutation

$$\pi_2 : (1, \ldots, \nu) \mapsto (\sigma_1, \ldots, \sigma_\nu),$$

where

$$\sigma_\alpha = \begin{cases} n_1 + \cdots + n_\alpha, & \alpha = 1, \ldots, \mu, \\ \alpha - \mu + \rho - 1, & \alpha = \mu + 1, \ldots, \nu, \end{cases} \tag{4.29}$$

and $\rho \in \{1, \ldots, \mu\}$ is the unique integer such that, for given α and μ

$$n_1 + n_2 + \cdots + n_{\rho-1} - \rho + s = \alpha - \mu,$$

for some $s \in \{2, \ldots, n_\rho\}$. Next we explain how these permutations are implemented. Let $\hat{e}_j$ for $j = 1, \ldots, \nu$ denote the column vectors of the standard basis in $\mathbf{C}^\nu$. Let Π_1 be the $\nu \times \nu$ permutation matrix whose jth column vector is $\hat{e}_{q_j}$, and let Π_2 be the $\nu \times \nu$ permutation matrix whose kth row vector is $\hat{e}^\dagger_{\sigma_k}$. Now observe that, if M is any $\nu \times \nu$ matrix, then the matrix $\Pi_2\, M\, \Pi_1$ can be thought as obtained from M by a permutation of the columns according to π_1 and a permutation of the rows according to π_2. In order to apply these operations to $\tilde{Z}(k)$ we define

$$P_1 = \begin{bmatrix} \Pi_1 & 0 \\ 0 & I_{n-\nu} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} \Pi_2 & 0 \\ 0 & I_{n-\nu} \end{bmatrix},$$

$$\mathcal{Z}(k) = P_2\, \tilde{Z}(k)\, P_1 = P_2\, \mathcal{S}^{-1}\, Z(k)\, \mathcal{S}\, P_1, \tag{4.30}$$

and we partition $\mathcal{Z}(k)$ as

$$\mathcal{Z}(k) = \begin{bmatrix} \mathcal{A}(k) & \mathcal{B}(k) \\ \mathcal{C}(k) & \mathcal{D}(k) \end{bmatrix}, \tag{4.31}$$

where $\mathcal{A}(k)$ has size $\mu \times \mu$ and, consequently, $\mathcal{D}(k)$ has size $(n-\mu) \times (n-\mu)$. Then $\mathcal{A}(k)$ coincides with the submatrix of $\tilde{Z}(k)$ consisting of the elements in columns $\alpha 1$ and rows sn_s, where $1 \leq \alpha \leq \mu$ and $1 \leq s \leq \mu$. As we have already indicated above, the matrix $\mathcal{A}(k)$ determines the leading asymptotic behavior of $\mathcal{Z}(k)^{-1}$ as $k \to 0$. The next two propositions provide the necessary information about the behavior of the four matrix blocks in (4.31).

Proposition 4.6 Assume $Q \in L^1_m(\mathbf{R}; \mathbf{C}^{n\times n})$ for $m = 1$ or 2. Then the matrices $\mathcal{A}(k)$, $\mathcal{B}(k)$, $\mathcal{C}(k)$, and $\mathcal{D}(k)$ appearing in (4.31) behave near $k = 0$ like

$$\mathcal{A}(k) = \sum_{j=1}^{m} k^j \mathcal{A}_j + o(k^m), \qquad \mathcal{B}(k) = \sum_{j=1}^{m-1} k^j \mathcal{B}_j + o(k^{m-1}), \tag{4.32}$$

$$\mathcal{C}(k) = \sum_{j=1}^{m} k^j \mathcal{C}_j + o(k^m), \qquad \mathcal{D}(k) = \sum_{j=0}^{m-1} k^j \mathcal{D}_j + o(k^{m-1}), \tag{4.33}$$

where in the expansion for $\mathcal{B}(k)$ the sum is absent when $m = 1$.

PROOF: We give the proof for $\mathcal{B}(k)$; the proofs for the other matrices are similar. Let e_j for $j = 1, \ldots, n$ denote the standard basis vectors in $\mathbf{C}^n$. Let $s \in \{1, \ldots, \mu\}$ and first suppose that $p \in \{1, \ldots, \nu - \mu\}$. Then we have

$$\begin{aligned}
\mathcal{B}(k)_{sp} &= e_s^\dagger\, \mathcal{Z}(k)\, e_{\mu+p} = e_s^\dagger\, P_2\, \tilde{Z}(k)\, P_1\, e_{\mu+p} \\
&= [\,\hat{e}_{\sigma_s}^\dagger \quad 0\,]\; \tilde{Z}(k) \begin{bmatrix} \hat{e}_{q_{\mu+p}} \\ 0 \end{bmatrix} = e_{\sigma_s}^\dagger\, \tilde{Z}(k)\, e_{q_{\mu+p}} \\
&= \tilde{Z}(k)_{\sigma_s q_{\mu+p}} = \tilde{Z}(k)_{sn_s;\alpha j},
\end{aligned}$$

where α and j are determined by (4.28) with $\tau = \mu + p \leq \nu$; hence $2 \leq j \leq n_\alpha$ and $1 \leq \alpha \leq \mu$. Thus it follows from parts (ii) and (v) of Proposition 4.5 that $\mathcal{B}(k)_{sp} = o(1)$ if $m = 1$ and $\mathcal{B}(k)_{sp} = k\,\mathcal{B}_{1,sp} + o(k)$ if $m = 2$. Specifically, we have $\mathcal{B}_{1,sp} = w_{sn_s}^\dagger\, V_1\, u_{\alpha j}$, where V_1 is given in (4.20). Note that, since the rows have block indices sn_s, they do not include any rows with block indicies $\alpha(j-1)$. For if $s = \alpha$, then $n_\alpha \neq j - 1$, because $j \leq n_\alpha$. Hence the possibility that $\mathcal{B}_{1,sp}$ is of the form $1 + o(1)$ is excluded. It remains to consider the matrix elements with $p \in \{n-\nu+1,\dots,\nu-\mu\}$. Since $P_1\, e_{\mu+p} = e_{\mu+p}$, we obtain

$$\mathcal{B}(k)_{sp} = \tilde{Z}(k)_{\sigma_s(\mu+p)} = \tilde{Z}(k)_{sn_s;\alpha j} = w^\dagger_{sn_s}\, Z(k)\, u_{\alpha j},$$

where α and j are determined by the equation $n_1+\cdots+n_{\alpha-1}+j = \mu+p$; note that $\mu+p>\nu$ and thus $\alpha\ge\mu+1$. Since $s\le\mu$, by (iii)-(v) of Proposition 4.5, we conclude that $\mathcal{B}(k)_{sp} = o(1)$ if $m=1$, and $\mathcal{B}(k)_{sp} = k\,\mathcal{B}_{1,sp}+o(k)$ with $\mathcal{B}_{1,sp} = w^\dagger_{sn_s}\,V_1\,u_{\alpha j}$ if $m=2$. Note that the row indices sn_s cannot be equal to $\alpha 1$, αj, or $\alpha(j-1)$ because $s\le\mu<\alpha$. ∎

Proposition 4.7 Assume $Q\in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Then

(i) $\mathcal{A}_1$ in (4.32) is invertible and hence $\mathcal{A}(k)$ is invertible for k near 0, $k\ne 0$.

(ii) $\mathcal{D}_0$ defined in (4.33) is invertible.

PROOF: To prove (i) we proceed as follows. Note that for s and $j\in\{1,\dots\mu\}$,

$$\mathcal{A}(k)_{sj} = \tilde{Z}(k)_{\sigma_s q_j} = \tilde{Z}(k)_{sn_s;j1} = w^\dagger_{sn_s} Z(k)\, u_{j1},$$

and thus, by (4.20),

$$\mathcal{A}_{1,sj} = w^\dagger_{sn_s}\, V_1\, u_{j1} = i\, w^\dagger_{sn_s}[\Gamma+\mathcal{R}^\dagger]\, u_{j1}. \tag{4.34}$$

We show that the kernel of the transformation $\mathcal{A}_1:\mathbf{C}^\mu\mapsto\mathbf{C}^\mu$ is trivial. Suppose there is a vector $(c_1,\dots,c_\mu)$ such that $\sum_{j=1}^{\mu}\mathcal{A}_{1,sj}\,c_j = 0$ for $s=1,\dots,\mu$. Let $\xi = \sum_{j=1}^{\mu} c_j\, u_{j1}$ and $\chi = \Gamma\,\xi$ (cf. (4.2)). Since $\chi\in\mathcal{M}$, it is a linear combination of the vectors $w_{1n_1},\dots,w_{\mu n_\mu}$ and hence $\chi^\dagger\, V_1\,\xi = 0$. On the other hand, by using (4.6), we obtain

$$\chi^\dagger V_1\,\xi = i\,\chi^\dagger[\Gamma+\mathcal{R}^\dagger]\,\xi = i\,(\|\chi\|^2+\|\xi\|^2),$$

which is nonzero unless $c_1=\cdots=c_\mu=0$. Hence $\mathcal{A}_1$ is invertible. To prove (ii), we use (4.31) and Proposition 4.5 to derive

$$\mathcal{D}_0 = \begin{bmatrix} I_{\nu-\mu} & 0 & \dots & 0 \\ 0 & J_{n_{\mu+1}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{n_\kappa} \end{bmatrix}, \tag{4.35}$$

where J_{n_α} are the matrices defined in (4.25). Clearly, $\mathcal{D}_0$ is invertible. ∎

We remark that (4.35) is obvious if we recall how the permutation matrices P_1 and P_2 act. The block $I_{\nu-\mu}$ comes from the unit entries in the superdiagonal of the Jordan blocks in (4.24) that belong to the eigenvalue 0. The remaining blocks in (4.35) correspond to those in (4.24) with $\alpha \geq \mu+1$; these have not been affected by P_1 and P_2.

Next we study the behavior of the inverse of the matrix defined in (4.31) near $k=0$.

Proposition 4.8 Assume $Q \in L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ for $m=1$ or 2. Then as $k\to 0$ in $\mathbf{R}$ we have the following.

(i) If $m=1$, then

$$\mathcal{Z}(k)^{-1} = \begin{bmatrix} (1/k)\,\mathcal{A}_1^{-1} + o(1/k) & o(1/k) \\ -\mathcal{D}_0^{-1}\,\mathcal{C}_1\,\mathcal{A}_1^{-1} + o(1) & \mathcal{D}_0^{-1} + o(1) \end{bmatrix}. \tag{4.36}$$

(ii) If $m=2$, then

$$\mathcal{Z}(k)^{-1} = \frac{1}{k}\,\mathcal{Z}_{-1} + \mathcal{Z}_0 + o(1), \tag{4.37}$$

where

$$\mathcal{Z}_{-1} = \begin{bmatrix} \mathcal{A}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.38}$$

$$\mathcal{Z}_0 = \begin{bmatrix} -\mathcal{A}_1^{-1}\,\mathcal{A}_2\,\mathcal{A}_1^{-1} + \mathcal{A}_1^{-1}\,\mathcal{B}_1\,\mathcal{D}_0^{-1}\,\mathcal{C}_1\,\mathcal{A}_1^{-1} & -\mathcal{A}_1^{-1}\,\mathcal{B}_1\,\mathcal{D}_0^{-1} \\ -\mathcal{D}_0^{-1}\,\mathcal{C}_1\,\mathcal{A}_1^{-1} & \mathcal{D}_0^{-1} \end{bmatrix}. \tag{4.39}$$

PROOF: We exploit the fact that

$$\begin{bmatrix} I_\mu & -\mathcal{B}(k)\,\mathcal{D}(k)^{-1} \\ 0 & I_{n-\mu} \end{bmatrix} \mathcal{Z}(k) \begin{bmatrix} I_\mu & 0 \\ -\mathcal{D}(k)^{-1}\,\mathcal{C}(k) & I_{n-\mu} \end{bmatrix} = \begin{bmatrix} \mathcal{U}(k) & 0 \\ 0 & \mathcal{D}(k) \end{bmatrix}, \tag{4.40}$$

where

$$\mathcal{U}(k) = \mathcal{A}(k) - \mathcal{B}(k)\,\mathcal{D}(k)^{-1}\,\mathcal{C}(k).$$

Since, by (4.32), (4.33), and Proposition 4.7,

$$\mathcal{B}(k)\,\mathcal{D}(k)^{-1}\,\mathcal{C}(k)=o(k),\qquad \mathcal{A}(k)=k\,\mathcal{A}_1+o(k),$$

with $\det\mathcal{A}_1\neq 0$, we conclude that, for small enough nonzero k, $\mathcal{U}(k)$ is invertible and

$$\mathcal{U}(k)^{-1}=\begin{cases}(1/k)\,\mathcal{A}_1^{-1}+o(1/k), & m=1,\\ (1/k)\,\mathcal{A}_1^{-1}-\mathcal{A}_1^{-1}\,\mathcal{A}_2\,\mathcal{A}_1^{-1}+\mathcal{A}_1^{-1}\,\mathcal{B}_1\,\mathcal{D}_0^{-1}\,\mathcal{C}_1\,\mathcal{A}_1^{-1}+o(1), & m=2.\end{cases}\tag{4.41}$$

As a result, from (4.40) we obtain

$$\mathcal{Z}(k)^{-1}=\begin{bmatrix}\mathcal{U}(k)^{-1} & -\mathcal{U}(k)^{-1}\,\mathcal{B}(k)\,\mathcal{D}(k)^{-1}\\ -\mathcal{D}(k)^{-1}\,\mathcal{C}(k)\,\mathcal{U}(k)^{-1} & \mathcal{D}(k)^{-1}\,\mathcal{C}(k)\,\mathcal{U}(k)^{-1}\,\mathcal{B}(k)\,\mathcal{D}(k)^{-1}+\mathcal{D}(k)^{-1}\end{bmatrix},$$

and hence (4.36)-(4.39) follow by using (4.32), (4.33), and (4.41). ∎

The main conclusion of Proposition 4.8 is that $\mathcal{Z}(k)^{-1}$ has a $1/k$-singularity at $k=0$ if $\dim\mathcal{N}\geq 1$. Therefore, $\tilde{Z}(k)^{-1}$ and $Z(k)^{-1}$ have a similar behavior. Indeed, from (4.30) and (4.37) we infer that

$$Z(k)^{-1}=\sum_{j=0}^{m-1}k^{j-1}Z_{j-1}+o(k^{m-2}),\qquad k\to 0 \text{ in } \mathbf{R},\tag{4.42}$$

where

$$Z_{-1}=\mathcal{S}\,P_1\,\mathcal{Z}_{-1}\,P_2\,\mathcal{S}^{-1},\qquad Z_0=\mathcal{S}\,P_1\,\mathcal{Z}_0\,P_2\,\mathcal{S}^{-1}.\tag{4.43}$$

This leads us to the main result of this section. We will lift the restriction that k be real and allow $k\in\overline{\mathbf{C}^+}$ in the asymptotics of the transmission coefficients.

Theorem 4.9 Assume $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ and $\dim\mathcal{N}\geq 1$. Then the scattering coefficients are continuous at $k=0$, and we have

$$T_l(k)=2i\,Z_{-1}+o(1),\qquad T_r(k)=-2i\,Z_{-1}^\dagger+o(1),\qquad k\to 0 \text{ in } \overline{\mathbf{C}^+},\tag{4.44}$$

$$\operatorname{Im}T_l(0)=\operatorname{Ker}\Delta_l,\qquad \operatorname{Ker}T_l(0)=\operatorname{Im}\Delta_l,\tag{4.45}$$

$$\text{(4.46)} \qquad \operatorname{Im} T_r(0) = \operatorname{Ker} \Delta_r, \qquad \operatorname{Ker} T_r(0) = \operatorname{Im} \Delta_r,$$

$$\text{(4.47)} \quad L(k) = -I_n + \Gamma\, T_l(0) + o(1), \qquad R(k) = -I_n + \Gamma^{-1}\, T_l(0)^\dagger + o(1), \qquad k \to 0 \text{ in } \mathbf{R},$$

$$\text{(4.48)} \qquad \operatorname{Ker}\{I_n + L(0)\} = \operatorname{Ker} T_l(0), \qquad \operatorname{Ker} T_r(0) = \operatorname{Ker}\{I_n + R(0)\},$$

$$\text{(4.49)} \qquad \operatorname{Im}\{I_n + L(0)\} = \operatorname{Im} T_r(0), \qquad \operatorname{Im}\{I_n + R(0)\} = \operatorname{Im} T_l(0).$$

PROOF: For $k \in \mathbf{R}$ the continuity of the transmission coefficients and (4.44) follow immediately from (2.25), (4.8), (4.19), (4.42), and (4.43). To extend the asymptotic formulas in (4.44) to $k \in \overline{\mathbf{C}^+}$ we first note that

$$\det W(k) = [\det Z(k)]\,[1 + o(1)] = [\det \mathcal{Z}(k)]\,[1 + o(1)] = C_0\, k^\mu\, [1 + o(1)], \qquad k \to 0 \text{ in } \mathbf{R},$$

where $C_0 = (-1)^{\nu-\mu}\,(\det \mathcal{A}_1)\,(\det \mathcal{D}_0) \neq 0$. This follows from (4.19), (4.30), (4.32), (4.38), (4.40), Proposition 4.7, and the fact that $(\det P_1)\,(\det P_2) = (-1)^{\nu-\mu}$. It follows that $k^{-\mu} \det W(k) \to C_0$ as $k \to 0$ along the real axis. Since $\det W(k)$ extends as an analytic function to $\mathbf{C}^+$, there is a constant C such that $|k^{-\mu} \det W(k)| \leq C\,|k|^{-\mu}$ for k near 0 in $\overline{\mathbf{C}^+}$. Appealing to some theorems of Phragmén-Lindelöf (see Theorems 1.4.1-1.4.4 in Ref. 12) we conclude that $k^{-\mu} \det W(k) \to C_0$ as $k \to 0$ in $\overline{\mathbf{C}^+}$. Thus there is a set $\Sigma_\epsilon = \{k \in \overline{\mathbf{C}^+} : 0 < |k| < \epsilon\}$, with ϵ sufficiently small, on which $|\det W(k)| \geq C_1 |k|^\mu$ for some constant C_1. Recalling the cofactor representation of the inverse of a matrix we conclude that

$$\|W(k)^{-1}\| \leq C_2\,|k|^{-\mu}, \qquad k \in \Sigma_\epsilon,$$

for some constant C_2. Since $T_l(k) \to T_l(0)$ as $k \to 0$ along the real axis, we can apply a Phragmén-Lindelöf theorem to $2ik\,W(k)^{-1}$ and conclude that, by (4.8), $T_l(k) \to T_l(0)$ as $k \to 0$ in $\overline{\mathbf{C}^+}$. This, together with (2.25) completes the proof of (4.44).

To prove (4.45) we note that (4.24) and (4.28) imply

$$\operatorname{Ker}\tilde{\Delta}_l = \operatorname{Span}\{e_{q_1}, \dots, e_{q_\mu}\}.$$

Thus, in view of the form of $\mathcal{Z}_{-1}$ given in (4.38), we have

$$\begin{aligned}
\operatorname{Im}\{P_1\,\mathcal{Z}_{-1}\,P_2\} &= P_1\,\left\{\begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \mathbf{C}^\mu\right\} \\
&= P_1\,\operatorname{Span}\{e_1, \dots, e_\mu\} \\
&= \operatorname{Ker}\tilde{\Delta}_l.
\end{aligned}$$

Since $\Delta_l = \mathcal{S}\,\tilde{\Delta}_l\,\mathcal{S}^{-1}$, the first equality in (4.45) follows from (4.43) and (4.44). To prove the second equality we note that

$$\operatorname{Im}\tilde{\Delta}_l = \operatorname{Span}\{e_k : k \notin \{\sigma_1, \dots, \sigma_\mu\}\},$$

which follows from (4.24) and (4.29). Therefore,

$$\begin{aligned}
\operatorname{Ker}\{\mathcal{Z}_{-1}\,P_2\} &= \{w \in \mathbf{C}^n : P_2\,w = \begin{bmatrix} 0 \\ v \end{bmatrix}, \quad v \in \mathbf{C}^{n-\mu}\} \\
&= \{w \in \mathbf{C}^n : e_k^\dagger\,P_2\,w = 0, \quad k = 1, \dots, \mu\} \\
&= \{w \in \mathbf{C}^n : e_{\sigma_k}^\dagger\,w = 0, \quad k = 1, \dots, \mu\} \\
&= \operatorname{Im}\tilde{\Delta}_l.
\end{aligned}$$

This implies $\operatorname{Ker}T_l(0) = \operatorname{Im}\Delta_l$ and thus the second equality in (4.45) is proved. The equalities (4.46) follow from (4.45) by taking adjoints and using $(\operatorname{Ker}M)^\perp = \operatorname{Im}M^\dagger$ for any $n \times n$ matrix M.

To prove the remaining assertions we go back to (1.4) and (1.5), which we write as

$$f_l(k,x)\,T_l(k) = f_r(-k,x) + f_r(k,x)\,L(k), \qquad k \in \mathbf{R} \setminus \{0\}, \tag{4.50}$$

$$f_r(k,x)\,T_r(k) = f_l(-k,x) + f_l(k,x)\,R(k), \qquad k \in \mathbf{R} \setminus \{0\}. \tag{4.51}$$

From (4.50) and the continuity of $T_l(k)$ it immediately follows that $L(k)$ is continuous at $k=0$ and that

$$f_r(0,x)\,[I_n + L(0)] = f_l(0,x)\,T_l(0). \tag{4.52}$$

Now choose x such that $f_r(0,x)$ is invertible and multiply (4.52) from the left by $f_r(0,x)^{-1}$. Owing to (4.4) and the first equation in (4.45), we can replace $f_r(0,x)^{-1}f_l(0,x)$ by Γ. Hence the first relation in (4.47) follows. Similarly, the second relation in (4.47) follows from (4.51). The two equalities in (4.48) are immediate consequences of (4.47). Finally, (4.49) follows from (4.45), (4.47), and Proposition 4.2. ∎

We mention that there is a shorter alternate proof of (4.45), which follows directly from (4.52). Indeed, taking the derivative in (4.52), letting $x \to -\infty$, and using (3.6), we obtain $\Delta_l\, T_l(0) = 0$. Thus $\operatorname{Im} T_l(0) \subset \operatorname{Ker} \Delta_l$. However, it is seen from (4.38), (4.43), and (4.44) that $\operatorname{Im} T_l(0)$ has dimension μ; hence $\operatorname{Im} T_l(0) = \operatorname{Ker} \Delta_l$. The other relation in (4.45) follows similarly by using (3.7) and (4.51).

From (4.45), (4.46), and Theorems 3.1 and 3.2 we infer that the exceptional case occurs if and only if $T_l(0) \neq 0$. Moreover, (4.45), (4.48), and Theorem 3.2 show that $L(0)$ and $R(0)$ have eigenvalue -1 if and only if $\Delta_l \neq 0$. In view of (2.23) and (2.24) we also have $\|L(0)\| = \|R(0)\| = 1$ if and only if $\Delta_l \neq 0$. The case $\Delta_l = 0$ can be called the purely exceptional case because then we have $\mathcal{N} = \mathcal{M} = \mathbf{C}^n$. This case is further analyzed in Example 5.4 of the next section.

Theorem 4.10 Assume $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$ and $\dim \mathcal{N} \geq 1$. Then the scattering coefficients are differentiable at $k=0$ and

$$T_l(k) = T_l(0) + k\,\dot{T}_l(0) + o(k), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{4.53}$$

with

$$\dot{T}_l(0) = 2i\left[Z_0 - f_l^{-1}(0,0)\,\dot{f}_l(0,0)\,Z_{-1} + iH_1 + iH_2\right], \tag{4.54}$$

where Z_{-1} and Z_0 are given in (4.43) and

$$H_1 = Z_{-1}\, \mathcal{R}^\dagger\, f_l^{-1}(0,0)\, \dot{f}_l(0,0)\, Z_{-1}, \qquad H_2 = Z_{-1}\, \dot{f}_r(0,0)^\dagger\, (f_l(0,0)^\dagger)^{-1}\, Z_{-1}.$$

Moreover,

$$T_r(k) = T_l(0)^\dagger - k^*\, \dot{T}_l(0)^\dagger + o(k), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$L(k) = -I_n + (I_n + E_l)\, T_l(0) + k\, \dot{L}(0) + o(k), \qquad k \to 0 \text{ in } \mathbf{R},$$

$$R(k) = -I_n + (I_n - E_r)\, T_r(0) + k\, \dot{R}(0) + o(k), \qquad k \to 0 \text{ in } \mathbf{R},$$

where E_l and E_r are as in (3.9) and

$$\dot{L}(0) = [I_n + E_l]\, \dot{T}_l(0) + i\, [\dot{f}_r(0,x)^\dagger; \dot{f}_l(0,x)]\, T_l(0),$$

$$\dot{R}(0) = [I_n - E_r]\, \dot{T}_r(0) - i\, [\dot{f}_l(0,x)^\dagger; \dot{f}_r(0,x)]\, T_r(0).$$

PROOF: To prove (4.53) and (4.54) for $k \to 0$ in $\mathbf{R}$, we first note the expansions

$$f_l(k,0)^{-1}\, f_l(0,0) = I_n - k\, f_l(0,0)^{-1}\, \dot{f}_l(0,0) + o(k),$$

$$f_r(-k,0)^\dagger\, (f_l(0,0)^\dagger)^{-1} = \mathcal{R}^\dagger - k\, \dot{f}_r(0,0)^\dagger\, (f_l(0,0)^\dagger)^{-1} + o(k),$$

$$\Theta_1(k) = -ik^2\, \mathcal{R}^\dagger\, f_l(0,0)^{-1}\, \dot{f}_l(0,0) + o(k^2),$$

$$\Theta_2(k) = -ik^2\, \dot{f}_r(0,0)^\dagger\, (f_l(0,0)^\dagger)^{-1} + o(k^2),$$

which follow from (4.17), (4.18), together with (4.6) and Proposition 4.3. Inserting these expansions in (4.19) and using (4.8) we obtain (4.53) and (4.54). As with (4.44) we can use a Phragmén-Lindelöf argument to extend the result to $\overline{\mathbf{C}^+}$. To find the expansions for $L(k)$ and $R(k)$ we first note that the existence of $\dot{T}_l(0)$, together with (4.50) and (4.51), implies the existence of $\dot{L}(0)$ and $\dot{R}(0)$. Differentiating (4.50) with respect to k and taking the Wronskian with $\dot{f}_r(0,x)^\dagger$, we obtain

$$[\dot{f}_r(0,x)^\dagger; f_r(0,x)]\, \dot{L}(0) = [\dot{f}_r(0,x)^\dagger; \dot{f}_l(0,x)]\, T_l(0) + [\dot{f}_r(0,x)^\dagger; f_l(0,x)]\, \dot{T}_l(0), \tag{4.55}$$

where we have used $[\dot{f}_r(0,x)^\dagger;\dot{f}_r(0,x)]=0$. Using the integral relation (3.5) and that for $\dot{f}_r(0,x)$ (cf. (A.20)) we obtain $[\dot{f}_r(0,x)^\dagger;f_r(0,x)]=-i\,I_n$. Inserting this together with (3.14) in (4.55) and using (4.49) we get the expansion for $L(k)$. The proof of the expansion for $R(k)$ is similar. ∎

V. EXAMPLES

In this section we consider some special cases that illustrate various details of the analysis in Section IV. With the exception of Example 5.4 we only consider $T_l(k)$.

Example 5.1 Let $n=1$ with $Q\in L^1_2(\mathbf{R})$ and assume the exceptional case occurs. Then $Z(k)=\tilde{Z}(k)=\mathcal{Z}(k)=\mathcal{A}(k)$ are scalar functions. We choose $\xi=1=w$ and put $\gamma=\Gamma=f_l(0,0)/f_r(0,0)$, where now γ is a real nonzero constant. Since $T_l(k)=T_r(k)$, we denote the transmission coefficient by $T(k)$. By (4.34) we have

$$\mathcal{A}_1=i\frac{\gamma^2+1}{\gamma},$$

$$\mathcal{A}_2=\gamma^{-1}\int_0^\infty dz\,[f_l(0,z)^\dagger\,f_l(0,z)-I_n]+\gamma\int_{-\infty}^0 dz\,[f_r(0,z)^\dagger\,f_r(0,z)-I_n],$$

so that

$$T(k)=\frac{2\gamma}{\gamma^2+1}+\frac{2ik\gamma\Xi}{(\gamma^2+1)^2}+o(k),\qquad k\to 0\text{ in }\overline{\mathbf{C}^+},\tag{5.1}$$

where

$$\Xi=\gamma\,[\dot{f}_r(0,x);\dot{f}_l(0,x)]+\int_0^\infty dz\,[f_l(0,z)^2-1]+\gamma^2\int_{-\infty}^0 dz\,[f_r(0,z)^2-1].$$

In deriving (5.1) we have used the identity

$$\frac{\dot{f}_l(0,x)}{f_r(0,x)}+\frac{\dot{f}_r(0,x)}{f_l(0,x)}=-i[\dot{f}_r(0,x);\dot{f}_l(0,x)]\tag{5.2}$$

which can be verified as follows. Since $f_r(0,x)$ and $\dot{f}_r(0,x)$ are linearly independent solutions of (2.27), we can write

$$\dot{f}_l(0,x)=c_1\,\dot{f}_l(0,0)+c_2\,\dot{f}_r(0,x).$$

Using Wronskians we obtain

$$c_1 = -i[\dot{f}_r(0,0); \dot{f}_l(0,0)], \qquad c_2 = -\frac{1}{\gamma},$$

so that (5.2) follows. It seems that the expansion (5.1) is new under the assumption $Q \in L_2^1(\mathbf{R})$.

Example 5.2 Assume $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$ and suppose that $\tilde{\Delta}_l$ consists of one single Jordan block of size $n \geq 2$ associated with the eigenvalue 0. Thus $\kappa = 1$, $\mu = 1$, and $n = n_1 = \nu$. In this case we can simplify the notation by setting $u_{1j} = u_j$, for $j = 1, \ldots, n$. Then u_1 is the eigenvector for the eigenvalue 0 of Δ_l, that is $\mathcal{N} = \mathrm{Span}\,\{u_1\}$. The adjoint basis is $\{w_1, \ldots, w_n\}$ and we have $\mathcal{M} = \mathrm{Span}\,\{w_n\}$. The mapping Γ maps u_1 to a multiple of w_n, i.e. $\Gamma u_1 = c_3 w_n$ for some $c_3 \neq 0$. Moreover, $\mathcal{A}(k)$ is a scalar function and from (4.34) we obtain

$$\mathcal{A}_1 = \frac{i}{c_3^*}\,(|c_3|^2\,\|w_n\|^2 + \|u_1\|^2),$$

where we have used (4.6) via $\mathcal{R}w_n = (1/c_3)\,u_1$. The permutation matrices appearing in (4.30) are given by

$$P_1 = I_n, \qquad P_2 = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Using (4.43) and (4.44) we obtain

$$\tilde{T}_l(0) = \begin{bmatrix} 0 & 0 & \ldots & 0 & 2/c_4 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

where

$$c_4 = \frac{1}{c_3^*}\,(|c_3|^2\,\|w_n\|^2 + \|u_1\|^2)$$

and $\tilde{T}_l(0) = \mathcal{S}^{-1}\,T_l(0)\,\mathcal{S}$.

Example 5.3 This example illustrates the situation where $\tilde{\Delta}_l$ in (4.24) consists of two Jordan blocks. We assume $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$ and let $n = 3$, $\mu = 2$, $n_1 = 1, n_2 = 2$, $\nu = 3$, $\kappa = 2$, so that Δ_l has the Jordan form

$$\tilde{\Delta}_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Jordan basis is $\{u_{11}, u_{21}, u_{22}\}$, where $\{u_{11}, u_{21}\}$ is a basis for $\mathcal{N}$, and the adjoint basis is $\{w_{11}, w_{21}, w_{22}\}$, where $\{w_{11}, w_{22}\}$ is a basis for $\mathcal{M}$. In this case the rows of $\tilde{Z}(k)$ need to be permuted according to $\pi_2 : (1, 2, 3) \mapsto (1, 3, 2)$, whereas no permutation of the columns is required. Thus we have

$$P_1 = I_3, \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $\mathcal{A}(k)$ is a 2×2 matrix and

$$\mathcal{A}_1 = \begin{bmatrix} w_{11}^\dagger V_1 u_{11} & w_{11}^\dagger V_1 u_{21} \\ w_{22}^\dagger V_1 u_{11} & w_{22}^\dagger V_1 u_{21} \end{bmatrix},$$

where V_1 is given in (4.20). Thus we obtain

$$\tilde{T}_l(0) = \frac{1}{\det \mathcal{A}_1} \begin{bmatrix} w_{22}^\dagger V_1 u_{21} & 0 & -w_{11}^\dagger V_1 u_{21} \\ -w_{22}^\dagger V_1 u_{11} & 0 & w_{11}^\dagger V_1 u_{11} \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 5.4 This is the purely exceptional case mentioned above Theorem 4.10. We assume $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$ with $n > 1$. We have $\Delta_l = 0$, which implies $\mu = \nu = \kappa = n$. Then $\mathcal{N} = \mathcal{M} = \mathbf{C}^n$, and thus no restrictions are necessary in (4.4) and (4.7); that is, we have $\mathcal{R} = \Gamma^{-1}$. Moreover, $P_1 = P_2 = I_n$. It follows that

$$\mathcal{A}_1 = i\, \mathcal{S}^{-1} (\Gamma + [\Gamma^{-1}]^\dagger)\, \mathcal{S},$$

and thus, since $\mathcal{Z}_{-1} = \mathcal{A}_1^{-1}$, we obtain

$$T_l(0) = 2i\, \mathcal{S}\, (\mathcal{S}\, \mathcal{A}_1)^{-1} = 2\, \Gamma^\dagger\, (\Gamma\, \Gamma^\dagger + I_n)^{-1}.$$

For the reflection coefficients we find after some straightforward manipulations

$$L(0) = (\Gamma\,\Gamma^\dagger - I_n)\,(\Gamma\,\Gamma^\dagger + I_n)^{-1}, \qquad R(0) = (I_n - \Gamma^\dagger\,\Gamma)\,(\Gamma^\dagger\,\Gamma + I_n)^{-1}.$$

Example 5.5 Suppose $Q(x)$ is even and belongs to $L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. This implies $f_r(0,x) = f_l(0,-x)$ and from (3.2), (3.6), and (3.7) we conclude that Δ_l is selfadjoint. Hence Δ_l is diagonalizable and there are no Jordan chains of length greater than 1. We have $\mu = \nu$ and $n_\alpha = 1$ for $1 \le \alpha \le \kappa$, and $\kappa = n$. We also have $P_1 = P_2 = I_n$. It may be that Δ_l has some nonzero eigenvalues, so $\mu < n$ in general. If $\xi \in \mathcal{N}$, then

$$f_l(0,x)\,\xi = f_r(0,-x)\,\xi,$$

which implies that $f_r(0,x)\,\xi$ is bounded. This means $\xi \in \mathcal{M}$ and hence $\mathcal{N} = \mathcal{M}$. Furthermore, using (4.2) and (4.3) we conclude that

$$f_l(0,x)\,\chi = f_r(0,-x)\,\chi = f_l(0,-x)\,\xi, \tag{5.3}$$

where $\xi \in \mathcal{N}$ and $\chi = \Gamma\,\xi$. Letting $x \to -\infty$, we see that $\Gamma\,\chi = \xi$, that is,

$$\Gamma^2 = I_\mu. \tag{5.4}$$

It follows that Γ is diagonalizable because $(I_\mu \pm \Gamma)^p = 2^{p-1}\,(I_\mu \pm \Gamma)$ for $p \ge 1$ and has eigenvalues ± 1. Let $\epsilon_\pm$ denote the corresponding multiplicities ($\epsilon_+ + \epsilon_- = \mu$). Since $n_\alpha = 1$, we put $u_{\alpha 1} = u_\alpha$ for the vectors of the Jordan basis for Δ_l and assume that they are normalized and arranged such that

$$\Gamma\,u_\alpha = u_\alpha, \qquad \alpha = 1,\ldots,\epsilon_+,$$

$$\Gamma\,u_\alpha = -u_\alpha, \qquad \alpha = \epsilon_+ + 1,\ldots,\mu.$$

We also set $w_{sn_s} = w_s$, so that $w_s^\dagger\,u_\alpha = \delta_{s\alpha}$ for $s = 1,\ldots,n$ and $\alpha = 1,\ldots,n$. Note that as a consequence of (5.3), ϵ_+ (ϵ_-) is the number of linearly independent bounded even (odd) solutions of (2.27). Then from (4.7) and (5.4) we conclude that

$$w_s^\dagger\,\mathcal{R}^\dagger\,u_\alpha = (\Gamma^{-1}\,w_s)^\dagger\,u_\alpha = (\Gamma\,w_s)^\dagger\,u_\alpha = w_s^\dagger\,\Gamma^\dagger\,u_\alpha, \tag{5.5}$$

where $\Gamma^\dagger$ is the adjoint of Γ as a mapping from $\mathcal{N}$ to itself. Using (5.5) in (4.34) we obtain

$$\mathcal{A}_{1,sj} = i\, w_s^\dagger\, [\Gamma + \Gamma^\dagger]\, u_j,$$

and therefore

$$(\mathcal{A}_1^{-1})_{sj} = -i\, w_s^\dagger\, [\Gamma + \Gamma^\dagger]^{-1}\, u_j.$$

As a result, from (4.38), (4.43), and (4.44) we deduce that

$$T_l(0) = (2\, [\Gamma + \Gamma^\dagger]^{-1}) \oplus 0,$$

where the direct sum refers to the direct decomposition $\mathbf{C}^n = \mathcal{N} \oplus \mathcal{N}'$ with

$$\mathcal{N}' = \mathrm{Span}\, \{u_{\mu+1}, \dots, u_n\}.$$

APPENDIX: PROOF OF PROPOSITION 4.3

PROOF: Since the assertions of Proposition 4.3 concern the small-k asymptotics, we assume that k lies in a fixed interval $[-\delta, \delta]$ with $\delta > 0$. In the following C is used to denote various constants that may depend on the choice of δ but do not depend explicitly on k or x.

The solution $\varphi(k,x)$ of (1.1) defined by the initial conditions (4.9) satisfies the integral equation

$$\varphi(k,x) = \cos kx\, f_l(0,0) + \frac{\sin kx}{k}\, f_l'(0,0) + \frac{1}{k} \int_0^x dy\, \sin[k(x-y)]\, Q(y)\, \varphi(k,y), \tag{A.1}$$

which can be solved by iteration. A standard Gronwall inequality shows that

$$\|\varphi(k,x)\| \le C\, (1 + |x|), \qquad x \in \mathbf{R}. \tag{A.2}$$

Therefore, by using (A.1) and (A.2), it follows that for each $k \in \mathbf{R} \setminus \{0\}$ the asymptotic behavior of $\varphi(k,x)$ is of the form

$$\varphi(k,x) = \alpha_\pm(k)\, e^{ikx} + \beta_\pm(k)\, e^{-ikx} + \varepsilon_\pm(k,x), \tag{A.3}$$

where $\varepsilon_\pm(k,x)$ and $\varepsilon'_\pm(k,x)$ are both $o(1)$ as $x\to\pm\infty$, and where

$$\text{(A.4)}\qquad \alpha_\pm(k)=\frac{1}{2}\,f_l(0,0)+\frac{1}{2ik}\,f'_l(0,0)+\frac{1}{2ik}\int_0^{\pm\infty}dy\,e^{-iky}\,Q(y)\,\varphi(k,y),$$

$$\beta_\pm(k)=\frac{1}{2}\,f_l(0,0)-\frac{1}{2ik}\,f'_l(0,0)-\frac{1}{2ik}\int_0^{\pm\infty}dy\,e^{iky}\,Q(y)\,\varphi(k,y).$$

From (A.3) and (A.4), together with (1.2) and (1.3), it follows that

$$\text{(A.5)}\quad [\varphi(k,x)^\dagger;f_l(k,x)]=2ik\,\alpha_+(k)^\dagger=ikf_l(0,0)^\dagger-f'_l(0,0)^\dagger-\int_0^\infty dz\,e^{ikz}\varphi(k,z)^\dagger\,Q(z),$$

$$\text{(A.6)}\quad [f_r(-k,x)^\dagger;\varphi(k,x)]=2ik\,\alpha_-(k)=ikf_l(0,0)+f'_l(0,0)-\int_{-\infty}^0 dz\,e^{-ikz}\,Q(z)\,\varphi(k,z).$$

In order to control the remainder terms in the subsequent asymptotic expansions, we will need the estimates

$$\text{(A.7)}\qquad \|\varphi(k,x)-\varphi(0,x)\|\le C\,(1+\max\{0,-x\})\left(\frac{k\,x}{1+|k||x|}\right)^2,$$

$$\text{(A.8)}\qquad \|[\varphi(k,x)-\varphi(0,x)]\xi\|\le C\left(\frac{k\,x}{1+|k||x|}\right)^2\|\xi\|,\qquad \xi\in\mathcal{N}.$$

The factor $1+\max\{0,-x\}$ in (A.7) accounts for the fact that $\varphi(0,x)$ is in general unbounded and $O(x)$ as $x\to-\infty$. In (A.8), this factor is absent because $\varphi(0,x)\,\xi$ is bounded when $\xi\in\mathcal{N}$. We omit the proofs of (A.7) and (A.8) here because (A.7) follows from (A.1) by some standard estimates and (A.8) can be proved by mimicking the proof in the scalar case (see Lemma 2.2 in Ref. 10).

Now consider the integral on the right-hand side of (A.5) and write it as

$$\text{(A.9)}\qquad \int_0^\infty dz\,e^{ikz}\,\varphi(k,z)^\dagger\,Q(z)=A_1(k)+A_2(k),$$

where

$$\text{(A.10)}\qquad A_1(k)=\int_0^\infty dz\,e^{ikz}\,\varphi(0,z)^\dagger\,Q(z),$$

$$A_2(k) = \int_0^\infty dz\, e^{ikz}\,[\varphi(k,z)^\dagger - \varphi(0,z)^\dagger]\,Q(z). \tag{A.11}$$

When $m = 1$, from (A.10) we get

$$\begin{cases} A_1(k) &= \displaystyle\int_0^\infty dz\,\varphi(0,z)^\dagger\,Q(z) + ik\int_0^\infty dz\,z\,\varphi(0,z)^\dagger\,Q(z) + \mathcal{F}(k) \\ &= -f_l'(0,0)^\dagger + ik[f_l(0,0)^\dagger - I_n] + \mathcal{F}(k), \end{cases} \tag{A.12}$$

where

$$\mathcal{F}(k) = \int_0^\infty dz\,(e^{ikz} - 1 - ikz)\,\varphi(0,z)^\dagger\,Q(z). \tag{A.13}$$

Note that $\mathcal{F}(k)$ is $o(k)$ by (4.10), the boundedness of $f_l(0,z)$ on $[0,+\infty)$, and the estimate

$$|e^{ikz} - 1 - ikz| \le \frac{C\,z^2}{1+z}, \qquad z \ge 0.$$

In deriving (A.12) we have also used the relations

$$\begin{cases} \displaystyle\int_0^\infty dz\,\varphi(0,z)^\dagger\,Q(z) = -f_l'(0,0)^\dagger, \\ \displaystyle\int_0^\infty dz\,z\,\varphi(0,z)^\dagger\,Q(z) = f_l(0,0)^\dagger - I_n, \end{cases} \tag{A.14}$$

which follow from (3.4). Using (A.7) in (A.11) we see that

$$A_2(k) = o(k). \tag{A.15}$$

Combining (A.5), (A.9), (A.12), and (A.15) we obtain

$$[\varphi(k,x)^\dagger; f_l(k,x)] = ikI_n + o(k),$$

which agrees with (4.12) for $m = 1$.

Now consider (A.5) for $m = 2$, that is, $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. In this case we can expand the remainder $\mathcal{F}(k)$ in (A.12) as

$$\mathcal{F}(k) = \int_0^\infty dz\,\frac{(ikz)^2}{2}\,\varphi(0,z)^\dagger\,Q(z) + o(k^2), \tag{A.16}$$

where we have used (A.7), (A.13), and the estimate

$$\left| e^{ikz} - 1 - ikz + \frac{k^2 z^2}{2} \right| \leq \frac{C\,(|k|\,z)^3}{1 + |k|\,z}, \qquad z \geq 0.$$

The integral in (A.16) can be expressed in a form that does not involve Q explicitly. To see this, substitute $\varphi''(0,z)^\dagger$ for $\varphi(0,z)^\dagger\, Q(z)$ and replace the upper limit of integration by N. Then integrate by parts twice and let $N \to +\infty$. This gives

$$-\frac{k^2}{2} \left[\int_0^\infty dz\, z^2\, \varphi''(0,z) \right]^\dagger = -\frac{k^2}{2} \lim_{N\to+\infty} \mathcal{G}_N^\dagger = -k^2 \int_0^\infty dz\, [f_l(0,z)^\dagger - I_n],$$

where we have defined

$$\mathcal{G}_N = N^2\, \varphi'(0,N) - 2\,N\,[\varphi(0,N) - I_n] + 2 \int_0^N dz\, [\varphi(0,z) - I_n].$$

Thus

$$\text{(A.17)} \qquad \mathcal{F}(k) = -k^2 \int_0^\infty dz\, [f_l(0,z)^\dagger - I_n] + o(k^2).$$

In the derivation of (A.17) we have also used

(A.18)

$$\varphi'(0,N) = o(1/N^2), \qquad \varphi(0,N) - I_n = o(1/N), \qquad \varphi(0,N) - I_n \in L^1(\mathbf{R}^+; \mathbf{C}^{n\times n}).$$

These properties follow directly from (3.4) and (4.10). The expression (A.17) for $\mathcal{F}(k)$ has the advantage that it allows us to combine $\mathcal{F}(k)$ with another term that arises from the expansion of $A_2(k)$. To see this we return to (A.11). In order to expand the difference $\varphi(k,x) - \varphi(0,x)$ we use the variation of parameters formula in the form

(A.19)

$$\varphi(k,x) = \varphi(0,x) + i\,k^2 f_l(0,x) \int_0^x dz\, \dot{f}_l(0,z)^\dagger\, \varphi(k,z) + i\,k^2 \dot{f}_l(0,x) \int_0^x dz\, f_l(0,z)^\dagger\, \varphi(k,z).$$

We briefly mention some details of the derivation of (A.19) because there is a useful identity that falls out in the process. In the usual way we write (2.27) as a first-order system with

$2n$ components and note that a fundamental matrix $\Psi(x)$ for this system and its inverse $\Psi(x)^{-1}$ are given by

$$\Psi(x) = \begin{bmatrix} f_l(0,x) & \dot{f}_l(0,x) \\ f_l'(0,x) & \dot{f}_l'(0,x) \end{bmatrix}, \qquad \Psi(x)^{-1} = i \begin{bmatrix} \dot{f}_l'(0,x)^\dagger & -\dot{f}_l(0,x)^\dagger \\ f_l'(0,x)^\dagger & -f_l(0,x)^\dagger \end{bmatrix}.$$

By taking $x \to +\infty$ and using (2.4), one can prove that $\det \Psi(x) = i$. For this and also later we need to use asymptotic information about the functions $\dot{f}_l(0,x)$ and $\dot{f}_l'(0,x)$. It suffices to mention that $\dot{f}_l(0,x)$ is the unique solution of the integral equation

$$\dot{f}_l(0,x) = ix\, I_n + \int_x^\infty dy\, (y-x)\, Q(y)\, \dot{f}_l(0,y), \tag{A.20}$$

which, incidentally, shows that $\dot{f}_l(0,x)$ is also a matrix solution of (2.27). Moreover, a Gronwall inequality gives

$$\|\dot{f}_l(0,x)\| \le C\,(1+|x|), \qquad x \in \mathbf{R}. \tag{A.21}$$

The identity $\Psi(x)^{-1}\,\Psi(x) = I_{2n}$ is easily verified by using the Wronskian relations

$$[f_l(0,x)^\dagger; f_l(0,x)] = [\dot{f}_l(0,x)^\dagger; \dot{f}_l(0,x)] = 0,$$

$$[f_l(0,x)^\dagger; \dot{f}_l(0,x)] = [\dot{f}_l(0,x)^\dagger; f_l(0,x)] = i\, I_n,$$

which follow from (3.4), (A.20), and the first formula in (A.18) which indicates that $f_l'(0,x) = o(1/x^2)$ as $x \to +\infty$. Then (A.19) is an easy consequence of the variation of parameters formula for first-order systems. The useful identity alluded to above appears when we write out the identity $\Psi(x)\,\Psi(x)^{-1} = I_{2n}$ (in this order!) in terms of the entries of the matrices involved. Among the resulting identities we find

$$f_l'(0,x)\,\dot{f}_l(0,x)^\dagger + \dot{f}_l'(0,x)\,f_l(0,x)^\dagger = i\, I_n,$$

which will be useful later.

By iterating (A.19) once and using (4.10) we obtain

$$\begin{aligned}\varphi(k,x) =&\varphi(0,x)+ik^2 f_l(0,x)\int_0^x dz\, \dot{f}_l(0,z)^\dagger\, f_l(0,z)\\ &+ik^2\dot{f}_l(0,x)\int_0^x dz\, f_l(0,z)^\dagger\, f_l(0,z)+\rho(k,x),\end{aligned} \tag{A.22}$$

where $\rho(k,x)$ obeys

$$\|\rho(k,x)\| \le C\,k^2\,(1+|x|)^2\left(\frac{kx}{1+|k|x|}\right)^2. \tag{A.23}$$

This estimate follows by using (A.7) and (A.21). Taking the adjoint of $A_2(k)$ given in (A.11) and expanding the exponential function there we get

$$A_2(k)^\dagger = \int_0^\infty dz\, Q(z)\,[\varphi(k,z)-\varphi(0,z)]+o(k^2), \tag{A.24}$$

where we have used (A.7) to determine the order of the error term. Now we insert (A.22) into (A.24) and proceed as in the derivation of (A.17), using $\dot{f}_l''(0,x)=Q(x)\,\dot{f}_l(0,x)$ and two integrations by parts. We also use (A.21), (A.23), and the property $\dot{f}_l'(0,N)-iI_n = o(1/N)$, which follows from (A.20). The result is

$$\begin{aligned}\int_0^\infty dz\, Q(z)\,[\varphi(k,z)-\ \varphi(0,z)] =&k^2\int_0^\infty dz\,[f_l(0,z)-I_n]\\ &-k^2\int_0^\infty dz\,[f_l(0,z)^\dagger\, f_l(0,z)-I_n]+o(k^2).\end{aligned} \tag{A.25}$$

Combining (A.9), (A.12), (A.17), (A.24), and (A.25) we obtain

$$[\varphi(k,x)^\dagger; f_l(k,x)] = i\,k\,I_n + k^2\int_0^\infty dz\,[f_l(0,z)^\dagger\, f_l(0,z)-I_n]+o(k^2),$$

which is the desired result in (4.12) for $m=2$.

To prove (4.13) we return to the Wronskian in (A.6). If $m=1$, we have

$$\begin{cases}\displaystyle\int_{-\infty}^0 dz\, e^{-ikz}\,Q(z)\,\varphi(k,z) & \displaystyle =\int_{-\infty}^0 dz\,Q(z)\,\varphi(0,z)+o(1)\\ & = -\Delta_l+f_l'(0,0)+o(1),\end{cases} \tag{A.26}$$

where we have used (3.4) and (A.14). The order of the error term is again a consequence of (A.7). Substituting (A.26) in (A.6) we get (4.13) for $m = 1$. If $m = 2$, we have

$$\int_{-\infty}^{0} dz\, e^{-ikz}\, Q(z)\, \varphi(k,z) = -\Delta_l + f_l'(0,0) - ik \int_{-\infty}^{0} dz\, z\, Q(z)\, \varphi(0,z) + o(k), \tag{A.27}$$

and, using (3.9), (A.14), and (A.27) we obtain

$$\int_{-\infty}^{0} dz\, z\, Q(z)\, \varphi(0,z) = I_n + E_l - f_l(0,0). \tag{A.28}$$

Substituting this in (A.6) we get

$$[f_r(-k,x)^\dagger; \varphi(k,x)] = \Delta_l + ik(I_n + E_l) + o(k),$$

proving (4.13) when $m = 2$.

It remains to prove (4.14). So pick $\xi \in \mathcal{N}$ and assume $m = 1$. Then $\varphi(0,x)\,\xi$ stays bounded as $x \to -\infty$, which has the same effect on the integral in (A.6), when it acts on ξ, as if m were 2. In particular, (A.28) now becomes

$$\int_{-\infty}^{0} dz\, z\, Q(z)\, [\varphi(0,z)\,\xi] = \Gamma\,\xi - f_l(0,0)\,\xi,$$

where we have used (3.4) and (4.5). Since $\Delta_l\,\xi = 0$, from (A.27) we obtain

$$\int_{-\infty}^{0} dz\, e^{-ikz}\, Q(z)\, [\varphi(k,z)\,\xi] = f_l'(0,0)\,\xi - ik\,\Gamma\,\xi + ik\, f_l(0,0)\,\xi + o(k).$$

Substituting this expression in (A.6) we get

$$[f_r(-k,x)^\dagger; \varphi(k,x)]\,\xi = ik\,\Gamma\,\xi + o(k),$$

which agrees with (4.14) for $m = 1$. If $m = 2$ and $\xi \in \mathcal{N}$, then we can carry the expansion in (A.27) further as in the case of (A.9) and (A.11). To obtain the corresponding coefficients in the expansion we could proceed by using variation of parameters in terms of the solutions $f_r(0,x)$ and $\dot{f}_r(0,x)$. However, there is a simpler approach that exploits the connection between the left and right Jost solutions for (1.1) under the substitution $x \mapsto -x$, that is, under the transformation $Q(x) \mapsto Q^{\#}(x)$, where $Q^{\#}(x) = Q(-x)$. We use a superscript $\#$ to indicate that a given quantity pertains to (1.1) with potential $Q^{\#}$. It is straightforward to show that

$$f_r(k,x) = f_l^{\#}(k,-x), \qquad f_l(k,x) = f_r^{\#}(k,-x). \tag{A.29}$$

We now introduce a solution $\omega(k,x)$ of (1.1) satisfying the initial conditions

$$\omega(k,0) = f_r(0,0), \qquad \omega'(k,0) = f_r'(0,0).$$

Then it follows from (4.3) and (4.9) that for $\xi \in \mathcal{N}$ we have

$$\varphi(k,x)\,\xi = \omega(k,x)\,\chi, \tag{A.30}$$

where $\chi = \Gamma\,\xi$. Since, by (4.9) and (A.29),

$$\varphi^{\#}(k,0) = f_l^{\#}(k,0) = f_r(k,0), \qquad \varphi^{\#\prime}(k,0) = f_l^{\#\prime}(k,0) = -f_r{}'(k,0),$$

we get $\varphi^{\#}(k,x) = \omega(k,-x)$, which, together with (A.30), yields $\varphi^{\#}(k,-x)\,\chi = \varphi(k,x)\,\xi$. In the following argument we use the more elaborate notation $[G(k,x); H(k,x)]_{(x_0)}$ to denote the Wronskian of two matrix functions $G(k,x)$ and $H(k,x)$ evaluated at $x = x_0$. Then we have

$$\begin{cases} [f_r(-k,x)^\dagger; \varphi(k,x)]_{(x)}\,\xi & = [f_l^{\#}(-k,-x)^\dagger; \varphi^{\#}(k,-x)]_{(x)}\,\chi \\ & = -[f_l^{\#}(-k,x)^\dagger; \varphi^{\#}(k,x)]_{(-x)}\,\chi \\ & = [\varphi^{\#}(k,x)^\dagger; f_l^{\#}(-k,x)]_{(-x)}^\dagger\,\chi \\ & = [\varphi^{\#}(-k,x)^\dagger; f_l^{\#}(-k,x)]_{(x)}^\dagger\,\chi, \end{cases} \tag{A.31}$$

where in the the last step we used the fact that the Wronskian is constant and that $\varphi(k,x)$ is an even function of k. The latter follows from the fact that the initial conditions in (4.9) are independent of k. Now the Wronskian on the right-hand side of (A.31) is of the same form as that in (4.12). We can therefore apply the expansion given there. Then the integrand of Y_2 involves $f_l^{\#}(0,z)$ which can be rewritten in terms of $f_r(0,z)$ by means of (A.29). Using also (4.2), we obtain (4.14). The proof of Proposition 4.3 is now complete. ∎

Acknowledgments. This material is based on work supported by the National Science Foundation under grant DMS-9803219, and by INDAM and MURST.

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