

PERSISTENCE AND SMOOTHNESS OF HYPERBOLIC INVARIANT MANIFOLDS
FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

The study of invariant sets is of fundamental importance in the geometric theory of dynamical systems. Since, in general, these sets can have a wild topological structure, it is of interest to identify situations where they have the particularly simple structure of differentiable manifolds. Especially important in this context are those manifolds that persist under small perturbations, a situation that leads naturally to the study of hyperbolic invariant manifolds.

There is an extensive literature on hyperbolic invariant manifolds for Ordinary Differential Equations, c.f. [1-3,6,8,9,13,14]. In the context of infinite dimensional dynamical systems, the study of hyperbolic invariant manifolds was pursued for certain Parabolic Partial Differential Equations [7] and for particular cases of Functional Differential Equations [10,11,12].

The studies of Kurzweil [10,11] on hyperbolic invariant manifolds for Functional Differential Equations (FDE) rely on establishing fixed points for maps which correspond to discrete dynamical systems obtained by discretization of the semiflow induced by the equations. The approach in this paper is, in contrast, based directly on the semiflow and uses techniques of exponential dichotomies for obtaining bounded solutions, following the work of Hale, c.f. [3], for hyperbolic invariant manifolds of Ordinary Differential Equations

(ODE). It is believed that this approach is simpler than methods based on discretization.

The paper begins by establishing the concept of exponential dichotomies in the context of skew-product semiflows in Banach vector bundles, and the concept of hyperbolicity using the notion of spectrum of an invariant manifold, developed by Sacker and Sell for flows [15-19]. Here, the numerous discussions with George Sell, while this work was in progress, played an important role, in particular in connection to work of him in collaboration with Sacker [18]. A short reference is then made to FDEs on manifolds, based on the work of Oliva, c.f. [5], referring the reduction of the general situation to the case of FDEs on euclidean space R^q . These are basically the contents of Sections 2,3 and 4.

The main difficulty in establishing the persistence and properties of hyperbolic manifolds for FDEs is related to the impossibility of extending backwards in time all the solutions. It is useful to linearize the equation around the original invariant manifold and to consider the corresponding tangential, unstable and stable manifolds, which are themselves invariant under the linearized semiflow. As it turns out, one only needs to extend the solutions backwards in time along the tangential and unstable directions. That involves no difficulties because both the tangential and unstable manifolds are finite-dimensional and, therefore, the semiflow for the linearized equation can be extended on them to a flow of an ODE. This leaves out only the evolution on the stable manifold to be considered as an FDE, but we do not need to extend the solutions backwards there. These ideas are pursued in Section 5 by the introduction of a moving system of coordinates centered at solutions of the unperturbed equation lying on the original invariant manifold. The perturbed equation in these coordinates is given by a family of systems of two perturbed ODEs describing the evolution of the tangential and unstable variables, and an FDE for the stable variable. These systems are written as perturbations of the

linear variational equation around the given hyperbolic manifold, using the variation of constants formula introduced by Hale for FDEs, c.f. [4]. The analysis is actually based on the existence of exponential dichotomies for the linear variational equation which split the solutions with initial conditions in the tangential, unstable, or stable spaces.

In Section 6, we study the persistence and the smoothness of integral manifolds for FDEs of the form obtained by application of the system of coordinates around the original invariant manifold, and we also study the structure of the semiflow around the integral manifolds, under general assumptions of global lipschitzian nonlinearities with sufficiently small lipschitz constants. The persistence of the integral manifolds is established using techniques of exponential dichotomies for the determination of bounded solutions, following an approach introduced by Hale, c.f. [3], and also applied by Sell [19] in the context of ODEs. The smoothness properties of the integral manifolds are established by a modification of the method developed by Fenichel [2] also in the context of ODEs.

Finally, the results obtained for systems in coordinate form are applied, in Section 7, to prove the persistence and smoothness of hyperbolic invariant manifolds for FDEs, and to study the local geometric structure of the orbits around them. Using "cut off" functions around the original invariant manifold, in a similar way as it is usually done for center manifold theory, one can get a system that agrees with the original system inside a neighborhood of the invariant manifold, and whose nonlinear terms satisfy the global lipschitz conditions assumed in Section 6. Thus, finding an invariant manifold for the perturbed equation, in such a neighborhood of the original manifold, amounts to finding an invariant manifold for the auxiliar system which is obtained through the application of the "cut off" functions mentioned above. The results obtained for systems in coordinate form are only good to get patches of the

invariant manifolds which occur under perturbations. These pieces have to be patched up in order to obtain the perturbed manifold, but this can be easily done using the uniqueness properties established in Section 6 for the integral manifolds of systems in coordinate form.

The system of coordinates which is introduced in this paper, around the original hyperbolic invariant manifold, is redundant, in the sense that each point close to the manifold can be represented by infinitely many combinations of the coordinates. In fact, the system of coordinates is centered at points of the invariant manifold, which therefore account for some of the coordinates used, and also involves coordinates along the tangential, unstable and stable spaces of the linearized equation around the manifold. The redundancy comes from the fact that the coordinates giving the point on the manifold where the coordinate frame is centered and the tangential coordinates eventually play against each other when describing neighborhoods of a particular point on the manifold. This situation is not different from many other cases where moving systems of coordinates have been used with the intent of simplifying the analysis, as for instance in certain situations describing the movement of bodies in the context of Newtonian mechanics. It is, of course, true that a nonredundant system of coordinates could be introduced leaving out the tangential components. However, the use of the redundant coordinates described above simplifies the analysis for two reasons: i) it allows the separation of the dynamics of the linearized equations around the original manifold from the dynamics on the manifold itself, in particular the tangent bundle is invariant under the linearized equation, and ii) it avoids the introduction of abstract equations to describe the evolution of the different coordinates, because, in this way, they can be given by FDEs in euclidean space.

Some of the ideas introduced in this paper are new and useful even in the case of ODEs. In particular, the use of the redundant system of coordinates described above, although most useful in the case of FDEs, is also natural in the context of ODEs since it simplifies the description of the linearized equation around the original invariant manifold and considerably simplifies the analysis.

2. Spectrum of a linear skew-product semiflow

Let W be a topological space. A flow on W is a continuous mapping $\pi: \mathbb{R} \times W \rightarrow W$ such that $\pi(0, w) = w$ and $\pi(t, \pi(s, w)) = \pi(t + s, w)$ for all $w \in W$ and $t, s \in \mathbb{R}$. A semiflow on W is a continuous mapping $\pi: [0, \infty) \times W \rightarrow W$ satisfying the preceding conditions for $t, s \geq 0$.

Let X be a smooth Banach manifold without boundary and let E be a Banach vector bundle over X with fiber projection $p: E \rightarrow X$, i.e., E is a vector bundle over X with each fiber $E(x) = p^{-1}(x)$, $x \in X$, being a Banach space. Points in E can be represented by ordered pairs (x, z) with $x \in X$, and z a vector in the fiber $E(x)$. A semiflow π on E is said to be a skew-product semiflow on E if there is a flow ϕ on X such that the fiber projection p commutes with π and ϕ , i.e., π can be represented as

$$\pi(t, x, z) = (\phi(t, x), \psi(t, x, z)), \quad t \geq 0$$

and $\psi(t, x, z)$ is in the fiber $E(\phi(t, x))$. Such a skew-product semiflow π is a linear skew-product semiflow if the mapping $z \rightarrow \Psi(t, x)z = \psi(t, x, z)$ is a linear mapping from the fiber $E(x)$ to the fiber $E(\phi(t, x))$. One defines analogously skew-product flow and linear skew-product flow. When π is a semiflow on E ,

$\pi(t,x,z)$, $\psi(t,x,z)$ and $\Psi(t,x)$ are only defined for $t \geq 0$. However, they can be extended for $t \leq 0$ at those points (x,z) through which there is a backwards continuation defined for all $t \leq 0$. Let us define the set B by

$$B = \{(x,z) \in E: \text{there is exactly one continuous function } (u,v): (-\infty,0] \rightarrow E \text{ such that } u(0) = x, v(0) = z \text{ and } \pi(t,u(s),v(s)) = (u(t+s),v(t+s)) \text{ for all } s \leq 0 \text{ and all } t \in [0,-s]\}.$$

The set

$$S = \{(x,z) \in E: |\psi(t,x,z)| \rightarrow 0 \text{ as } t \rightarrow +\infty\}$$

is called the stable set of X under π , and the set

$$U = \{(x,z) \in B: |\psi(t,x,z)| \rightarrow 0 \text{ as } t \rightarrow -\infty, \text{ for the continuation of } \psi(t,x,z) \text{ for } t \leq 0\}$$

is called the unstable set of X under π . A set $I \subset E$ is said to be positively invariant under π if $\pi(t,x,z) \in I$ for all $t \geq 0$ and $(x,z) \in I$, and I is said to be invariant under π if $I \subset B$ and the preceding condition holds for all $t \in \mathbb{R}$, for the continuation of $\pi(t,x,z)$. The sets S and U are both positively invariant under π , and the sets B , $S \cap B$ and U are all invariant under π . It is easy to see that these sets are vector subbundles of E .

The linear skew-product semiflow $\pi = (\phi, \psi)$ on E is said to admit an exponential dichotomy on X if there exist linear projections $P(x)$ defined on $E(x)$ and depending continuously on $x \in X$, and there exist constants $K, \alpha > 0$ such that

- (i) $N(P) = \{(x,z) \in E: P(x)z = 0\} \subset B,$
- (ii) $|\Psi(t,x)P(x)| \leq Ke^{-\alpha t}, \quad t \geq 0, \quad x \in X$
- (iii) $|\Psi(t,x)[I - P(x)]| \leq Ke^{+\alpha t}, \quad t \leq 0,$ for the continuation
of $\Psi(t,x)z$ for $t \leq 0$ with $(x,z) \in B.$

We note that condition (iii) makes sense because (i) implies that the range of the mapping $(x,z) \rightarrow (x,[I - P(x)]z)$ is contained in B . Whenever π admits an exponential dichotomy on X we have $U = N(P) = \{(x,z) \in E: P(x)z = 0\}$ and $S = R(P) = \{(x,z) \in E: z = P(x)z' \text{ for some } z' \in E(x)\}$. Then, the stable and unstable sets are complementary subbundles of E .

Given a linear skew-product semiflow $\pi = (\phi, \psi)$ on a vector bundle E , and a real number λ , we define a mapping π_λ by

$$\pi_\lambda(t,x,z) = (\phi(t,x), e^{-\lambda t} \psi(t,x,z)).$$

It is easily seen that π_λ is also a linear skew-product semiflow on E and that the invariant sets under π and under π_λ agree for all $\lambda \in \mathbb{R}$. We also define

$$\Psi_\lambda(t,x)z = e^{-\lambda t} \psi(t,x,z) = e^{-\lambda t} \Psi(t,x)z.$$

The stable and the unstable sets of X under π_λ are denoted by S_λ and U_λ , respectively. Clearly, if $\mu < \lambda$ then $S_\mu \subset S_\lambda$ and $U_\mu \supset U_\lambda$. The set of all $\lambda \in \mathbb{R}$ for which π_λ admits an exponential dichotomy on X is called the resolvent set of π on E and is denoted by $\rho(E, \pi)$. The complement of the resolvent set on \mathbb{R} is called the spectrum of π on E and is denoted by $\Delta(E, \pi)$.

A skew-product semiflow $\pi = (\phi, \psi)$ on the vector bundle E is said to be uniformly completely continuous if for each $x \in X$ there is a neighborhood V_x of x in X and a real number $r_x > 0$ such that, for all $t \geq r_x$, the mapping $(y,z) \rightarrow \pi(t,y,z)$ maps bounded subsets of $E_{V_x} = \{(y,z) \in E: y \in V_x\}$ into rela-

tively compact subsets of E .

The following theorem contains the properties of the spectrum $\Sigma(E,\pi)$ which are used in this paper.

Theorem 2.1

Let $\pi = (\varphi, \psi)$ be a uniformly completely continuous linear skew-product semiflow on a Banach vector bundle E defined over a compact, connected, smooth Banach manifold X . Then the spectrum $\Sigma(E,\pi)$ is a closed set of real numbers which is bounded above, and, consequently, it is a union of closed intervals, the spectral intervals.

Associated with each spectral interval there is a spectral subbundle V of E , which satisfies the following properties:

- 1) If $\mu, \lambda \in \rho(E,\pi)$ and $(\mu, \lambda) \cap \Sigma(E,\pi) = [a, b]$, then the spectral subbundle V associated with $[a, b]$ has finite dimension, satisfies $V = U_\mu \cap S_\lambda$, and is invariant under π ,
- 2) If $\lambda \in \rho(E,\pi)$ and $(-\infty, \lambda) \cap \Sigma(E,\pi) = (-\infty, b]$, then the spectral subbundle V associated with $(-\infty, b]$ satisfies $V = S_\lambda$ and is positively invariant under π .

Moreover, if $\lambda \in \rho(E,\pi)$, then the number of spectral intervals included in $(\lambda, +\infty)$ is finite.

Proof

Let $\lambda \in \rho(E,\pi)$. Then π_λ admits an exponential dichotomy on X . There exist a linear projection $P_\lambda(x)$ on $E(x)$ and constants $K, \alpha > 0$ such that $N(P_\lambda) \subset B$, and

$$\begin{aligned} |\Psi_\lambda(t,x)P_\lambda(x)| &\leq Ke^{-\alpha t}, & t > 0, \quad x \in X \\ |\Psi_\lambda(t,x)[I - P_\lambda(x)]| &\leq Ke^{+\alpha t}, & t < 0, \quad x \in X \end{aligned}$$

where the last inequality holds for some backwards extension of $\Psi_\lambda(t,x)$. If μ satisfies $|\lambda - \mu| < \beta = \alpha/2$, then

$$\begin{aligned} |\Psi_\mu(t,x)P_\lambda(x)| &\leq Ke^{-\beta t}, & t > 0, \quad x \in X, \\ |\Psi_\mu(t,x)[I - P_\lambda(x)]| &\leq Ke^{+\beta t}, & t < 0, \quad x \in X. \end{aligned}$$

Therefore, with $P_\mu = P_\lambda$, π_μ admits an exponential dichotomy on X . It follows that $\mu \in \rho(E,\pi)$, $U_\mu = U_\lambda$ and $S_\mu = S_\lambda$, for all μ such that $|\lambda - \mu| < \alpha/2$. This implies that $\rho(E,\pi)$ and $\{\mu \in \mathbb{R} : U_\mu = U_\lambda, S_\mu = S_\lambda\}$, for any $\lambda \in \rho(E,\pi)$, are open sets. Thus $\Sigma(E,\pi)$ is a closed set.

The fact that $\Sigma(E,\pi)$ is bounded above results from the compactness of X and the semigroup property of π . In fact, let $k = \sup\{|\Psi(t,x)| : x \in X \text{ and } 0 \leq t \leq 1\}$. Since Ψ is continuous and the supremum is taken over a compact set, the constant k is finite. Fix $t > 0$ and let m be the largest integer smaller or equal to t . Then, with $x_i = \phi(i,x)$, we have

$$\Psi(t,x) = \Psi(t - m, x_m) \Psi(1, x_{m-1}) \dots \Psi(1, x)$$

and, consequently

$$|\Psi(t,x)| \leq k^{m+1} \leq kk^t = ke^{at}, \quad t > 0,$$

where $a = \log k$. Therefore, π_λ admits an exponential dichotomy on X for $\lambda > 0$, with $\alpha = \lambda - a$ and $P_\lambda(x)$ equal to the identity on $E(x)$. This proves that $(a, +\infty) \subset \rho(E,\pi)$ and $\Sigma(E,\pi) \subset (-\infty, a]$.

Since $\Sigma(E,\pi)$ is a closed set of real numbers which is bounded above, it is a countable union of compact intervals with, possibly an interval of the form

$(-\infty, b]$. Let $\{\lambda_i\}$ be a decreasing sequence of points in $\rho(E, \pi)$ which separate the closed intervals that make up $\Sigma(E, \pi)$, i.e., the intersection of the interval $(\lambda_i, \lambda_{i-1})$ with $\Sigma(E, \pi)$ is precisely one spectral interval $[a_i, b_i]$. With each one of the spectral intervals we can associate the spectral bundle $V_i = U_{\lambda_i} \cap S_{\lambda_{i-1}}$. In order to prove the properties 1) and 2) we need to show that V_i is independent of the sequence $\{\lambda_i\}$, provided this sequence satisfies the properties indicated. More precisely, we need to show that for $\mu, \lambda \in \rho(E, \pi)$ with $\mu < \lambda$ we have $U_\mu = U_\lambda$ and $S_\mu = S_\lambda$ if and only if $(\mu, \lambda) \subset \rho(E, \pi)$. Assume that $U_\mu \neq U_\lambda$ or $S_\mu \neq S_\lambda$ and define $\bar{\sigma} = \sup\{\sigma \in \rho(E, \pi) : U_\sigma = U_\mu, S_\sigma = S_\mu\}$. Because the set $\{\mu \in \mathbb{R} : U_\mu = U_\lambda, S_\mu = S_\lambda\}$, for any $\lambda \in \rho(E, \pi)$, is an open set, and because U_σ depends monotonically on σ , we get a contradiction. This shows that $U_\mu = U_\lambda$ and $S_\mu = S_\lambda$. To prove the converse, let $\mu, \lambda \in \rho(E, \pi)$ and assume that $U_\mu = U_\lambda$ and $S_\mu = S_\lambda$. Then π_μ and π_λ admit exponential dichotomies on X , with projections P_μ, P_λ and constants K_μ, K_λ and $\alpha_\mu, \alpha_\lambda$, respectively. Let $K = \max\{K_\mu, K_\lambda\}$ and $\alpha = \min\{\alpha_\mu, \alpha_\lambda\}$. For either $\sigma = \mu$ or $\sigma = \lambda$, we have

$$\begin{aligned} |\Psi_\sigma(t, x)P_\sigma(x)| &\leq Ke^{-\alpha t}, & t > 0, & x \in X \\ |\Psi_\sigma(t, x)[I - P_\sigma(x)]| &\leq Ke^{+\alpha t}, & t < 0, & x \in X. \end{aligned}$$

Since $U_\mu = U_\lambda$ and $S_\mu = S_\lambda$, we have $P_\mu = P_\lambda = P$. Consequently

$$\begin{aligned} e^{-\alpha t} |\Psi(t, x)P(x)| &\leq Ke^{-\alpha t}, & t > 0, & x \in X \\ e^{-\alpha t} |\Psi(t, x)[I - P(x)]| &\leq Ke^{+\alpha t}, & t < 0, & x \in X, \end{aligned}$$

for σ equal to either μ or λ . This implies that these inequalities must also hold for all $\sigma \in [\mu, \lambda]$, proving that each one of the π_σ , for $\sigma \in [\mu, \lambda]$, admits an exponential dichotomy, and, therefore, $[\mu, \lambda] \subset \rho(E, \pi)$.

The invariance properties of the spectral subbundles follow from the

positive invariance of the S_λ and the invariance of the U_λ and $S_\lambda \cap B$, for every $\lambda \in \rho(E, \pi)$.

It remains to prove that the spectral bundles associated with compact spectral intervals have finite dimension. For this we use the complete continuity of the semiflow. The semiflow π_λ is uniformly completely continuous for each $\lambda \in \rho(E, \pi)$. Thus, for each $x \in X$ there is a neighborhood V_x of x in X and a real number $r_x > 0$ such that, for $t > r_x$, the mapping $(x, z) \rightarrow \pi(t, x, z)$ maps bounded subsets of $E_{V_x} = \{(y, z) \in E: y \in V_x\}$ into relatively compact subsets of E . Since X is compact, we can extract a finite subcovering V_{x_i} of X and define $r = \max\{r_{x_i}\}$. Then, for all $x \in X$, $t > r$, the mapping $z \rightarrow \Psi_\lambda(t, x)z$ maps bounded subsets of $E(x)$ into relatively compact subsets of $E(\phi(t, x))$. Since $\lambda \in \rho(E, \pi)$, π_λ admits an exponential dichotomy on X with projection P and constants $K, \alpha > 0$. Then $U_\lambda(x) = N(P(x))$ is a closed linear subspace of $E(x)$. Let us denote $S = \{z \in U_\lambda(x): |z| \leq 1\}$. For each $z \in S$, the mapping $\Psi_\lambda(t, x)z$ has one backwards extension defined for all $t \leq 0$ and such that

$$|\Psi_\lambda(t, x)z| \leq Ke^{+\alpha t}, \quad t \leq 0.$$

For each $z \in S$, define the set $S' = \{z' \in E(\phi(-r, x)): z' = \Psi_\lambda(-r, x)z \text{ with } z \in S\}$. For each $z' \in S'$ we have $|z'| \leq Ke^{-\alpha r}$ and, consequently, the set S' is bounded. Therefore, the mapping $z' \rightarrow \Psi_\lambda(r, \phi(-r, x))z'$ maps S' into a compact subset of $E(x)$. Since S is the image of S' under this map, it follows that S is a compact subset of the Banach space $U_\lambda(x)$. The only Banach spaces which have the closed unit ball compact, are the finite dimensional spaces. Consequently, $\dim U_\lambda(x) < \infty$. Clearly, $V_i(x) = U_{\lambda_i}(x) \cap S_{\lambda_i-1}(x)$ is also finite dimensional and, because X is connected, the dimensions of all fibers $V_i(x)$ are the same.

Finally, if $\lambda \in \rho(E, \pi)$, then $\dim U_\lambda$ is finite and the preceding properties of U_λ, S_λ , with the monotone dependence of S_λ on λ , imply that the union of the spectral subbundles associated with spectral intervals contained in the interval $(\lambda, +\infty)$ is equal to U_λ and, consequently, the number of such spectral intervals is finite. Q.E.D.

The evolution on the spectral subbundles associated with compact spectral intervals can be given by Ordinary Differential Equations (ODE).

Proposition 2.2

Let π be a skew-product semiflow on a Banach vector bundle E defined over a connected, smooth Banach manifold X . If V is a finite-dimensional subbundle of E which is invariant under π , then the restriction of π to V can be extended to a flow on V .

Moreover, if the mapping $t \rightarrow \pi(t, x, z)$ is differentiable at $t = 0$, for every $(x, z) \in V$, and its derivative at $t = 0$ is locally lipschitzian in $(x, z) \in V$, then the flow of π on V can be given by an ODE.

Proof.

Since V is invariant under π , through every point of V there is one backwards continuation of π . Consequently, $\pi(t, x, z)$ is well-defined and belongs to V , for all $t \in \mathbb{R}$, $(x, z) \in V$. It is clear from the definition of backwards continuation that π is a flow on V .

If the mapping $t \rightarrow \pi(t, x, z)$ is differentiable at $t = 0$ with the derivative being locally lipschitzian in $(x, z) \in V$, then we can define a vector field on V by assigning to each point $(x, z) \in V$ the derivative $\partial\pi(t, x, z)/\partial t|_{t=0}$. Initial value problems for the ODE defined by this vector field have unique solutions and define a flow which coincides with π . Q.E.D.

3. Hyperbolic invariant manifolds

Let X be a smooth Banach manifold without boundary and let ϕ be a continuously differentiable semiflow on X . Let Y be a smooth, compact, connected submanifold of X and assume that Y is positively invariant under the semiflow ϕ . Denote by E the subset of the tangent bundle TX defined by $E = \bigcup_{y \in Y} T_y X$ and suppose that there exists a subbundle N of E which is complementary to the tangent bundle $T = TY$. Then E , T and N are vector bundles over Y . Since ϕ is continuously differentiable, we can define

$$\psi(t, y, z) = \Psi(t, y)z = D_2 \phi(t, \phi(t, y))z, \quad t > 0$$

for all $y \in Y$ and $z \in E(y)$, and

$$\pi(t, y, z) = (\phi(t, y), \psi(t, y, z)) \quad , \quad t > 0.$$

Then π is a linear skew-product semiflow on E . It is called the linearized skew-product semiflow around Y induced by the semiflow ϕ . The vector bundle T is positively invariant under the semiflow $\pi = (\phi, \psi)$. The semiflow π induces, by restriction, a semiflow on T which is denoted by π^T and is called the tangential flow induced by π on T

$$\pi^T(t, y, z) = (\phi(t, y), \psi^T(t, y, z)), \quad t > 0$$

defined for $(y, z) \in T$. Analogously, π induces a semiflow π^N on N . In fact, if $P(y)$ denote projections on $E(y)$ which depend continuously on y and are such that $T(y) = \text{null space of } P(y)$ and $N(y) = \text{range space of } P(y)$, we can set for $(y, z) \in N$

$$\pi^N(t, y, z) = (\phi(t, y), P(\phi(t, y))\psi(t, y, z)), \quad t > 0.$$

Since T is a positively invariant set for π , the mapping π^N is a linear skew-product semiflow on N . It is called the normal flow induced by π on N . Let $\Sigma(E, \pi)$, $\Sigma(T, \pi^T)$ and $\Sigma(N, \pi^N)$ denote the spectra of the semiflows π , π^T and π^N on the vector bundles E , T and N , respectively. We say that Y is a k -hyperbolic invariant manifold under ϕ if there exists an $\alpha > 0$ such that $\Sigma(T, \pi^T) \subset (-\alpha, \alpha)$ and $\Sigma(N, \pi^N) \cap (-k\alpha, k\alpha) = \emptyset$. An hyperbolic invariant manifold under ϕ is simply a 1-hyperbolic invariant manifold.

Theorem 3.1

Let X be a smooth Banach manifold without boundary and let ϕ be a continuously differentiable semiflow on X . If $Y \subset X$ is a connected k -hyperbolic invariant manifold under ϕ and $\pi = (\phi, \psi)$ is the linearized skew-product semiflow around Y induced by the semiflow ϕ , then the tangent bundle of Y , $T = TY$, the unstable set U and the stable set S of Y under π decompose the tangent bundle of X as a Whitney sum $TX = T + U + S$, and the restrictions of π to U , of π to T and of ϕ to Y can be extended to flows on U , T and Y , respectively. Furthermore, if $\pi^U = (\phi, \psi^U)$, $\pi^T = (\phi, \psi^T)$ denote the skew-product flows on U , T obtained by extension of the restrictions of π to U , T , respectively, and if $\pi^S = (\phi, \psi^S)$ denotes the restriction of π to S , then there exist $K, \alpha > 0$ such that

$$\begin{aligned} |\psi^U(t, x, z)| &\leq Ke^{k\alpha t} |z|, & t < 0, & (x, z) \in U \\ |\psi^S(t, x, z)| &\leq Ke^{-k\alpha t} |z|, & t > 0, & (x, z) \in S \\ |\psi^T(t, x, z)| &\leq Ke^{\alpha|t|} |z|, & t \in \mathbb{R}, & (x, z) \in T. \end{aligned} \tag{3.1}$$

Proof

We can write

$$\pi(t,y,z) = (\phi(t,y), \psi(t,y, P(y)z)) + (\phi(t,y), \psi(t,y, [I - P(y)]z)).$$

Consequently, if $\psi(t,y,z) = \Psi(t,y)z$, $\psi^T(t,y,z) = \Psi^T(t,y)z$ and $\psi^N(t,y,z) = \Psi^N(t,y)z$ we have

$$\Psi(t,y)z = \Psi^N P(y)z + \Psi^T [I - P(y)]z.$$

Since $\Sigma(N, \pi^N) \cap \Sigma(T, \pi^T) = \emptyset$, it follows directly from the definitions of spectrum and dichotomy that

$$\Sigma(E, \pi) = \Sigma(N, \pi^N) \cup \Sigma(T, \pi^T).$$

If we denote $V_0 = U_{-\alpha} \cap S_\alpha$, $V_+ = U_{k\alpha}$ and $V_- = S_{-k\alpha}$ it follows from Theorem 2.1 that V_0 and V_+ are unions of a finite number of compact spectral subbundles and V_- is a countable union of spectral subbundles such that $TX = V_0 + V_- + V_+$, as a Whitney sum. Since $\Sigma(T, \pi^T) \subset (-\alpha, \alpha)$, we have $V_0 = T$, and it is also clear that $V_- = S$ and $V_+ = U$. Thus, T and U are finite-dimensional, and Proposition 2.2 implies that the restrictions of π to T and U can be extended to flows on T and U , respectively. It is clear that $\Sigma(U, \pi^U) \subset (k\alpha, +\infty)$, $\Sigma(T, \pi^T) \subset (-\alpha, \alpha)$, $\Sigma(S, \pi^S) \subset (-\infty, -k\alpha)$, and, therefore, the inequalities (3.1) are valid.

The dimensions of Y and $TY = T$ are the same. Consequently, Y is a finite dimensional manifold invariant under ϕ . It follows that the restriction of ϕ to Y can be extended to a flow on Y . Q.E.D.

4. Functional differential equations on manifolds

Let M be a separable smooth finite dimensional connected manifold without boundary, and let TM be the tangent bundle of M , that is, TM is the union of the tangent spaces $T_y M = TM(y)$ of points $y \in M$, with $p_M: TM \rightarrow M$ denoting

the projection that maps each $TM(y)$ onto the base point y . If I denotes the closed interval $I = [-r, 0]$ for $r > 0$, $C^0(I, M)$ denotes the set of continuous functions from I to M and $\rho: C^0(I, M) \rightarrow M$ is the evaluation map $\rho(\phi) = \phi(0)$, then a retarded functional differential equation (RFDE) on M is a continuous function $F: C^0(I, M) \rightarrow TM$ such that $p_M \circ F = \rho$.

The tangent bundle TM can be identified with $M \times R^m$ where $m = \dim M$. Then, for any RFDE F , there exists a function $f: C^0(I, M) \rightarrow R^m$, such that $F(\phi)$ can be identified with $(\xi(0), f(\xi))$, for all $\xi \in C^0(I, M)$. The RFDE (F) is frequently represented as $(x(t), \dot{x}(t)) = F(x_t) = (x(t), f(x_t))$ or, simply, $\dot{x}(t) = f(x_t)$, where, given a function x of a real variable and with values in the manifold M , we denote $x_t(\theta) = x(t + \theta)$, $\theta \in I$, whenever the right-hand side is defined.

Given a locally lipschitzian RFDE (F) on M , its maximal solution $x(t)$ satisfying the initial condition ξ at $t = t_0$ (which necessarily exists and is unique) is sometimes denoted by $x(t; t_0, \xi, F)$ and x_t is denoted $x_t(t_0, \xi, F)$. The solution map or semiflow of F is then defined by $\phi(t, \xi) = x_t(0, \xi, F)$. The arguments ξ, F are dropped when confusion may not arise, and t_0 is dropped when it is equal to zero. If F is bounded and has bounded continuous derivative, then the solution map is a smoothing operator, in the sense that if it is uniformly bounded for t in compact sets of $[0, \infty)$, then for $t > r$, the function $\phi(t, \cdot): C^0(I, M) \rightarrow C^0(I, M)$ maps bounded sets into relatively compact sets.

We denote by BC^k the set of bounded continuous functions from $C^0(I, M)$ into TM which have bounded continuous derivatives up to order $k > 1$. The RFDEs on M we consider in the sequel will always be taken from BC^k for $k > 1$. Each such RFDE (F) induces, by linearization, another RFDE (L) on the tangent bundle TM , which is called the linear variational equation. Being an RFDE on TM , the linear variational equation is a map $L: C^0(I, TM) \rightarrow T^2M$. The

double tangent bundle T^2M can be identified with $M \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$, and, therefore, if the given RFDE (F) on M is represented as $\dot{x}(t) = f(x_t)$, as done before, then the linear variational equation of F can be represented, in an analogous fashion, as $(x(t), y(t), \dot{x}(t), \dot{y}(t)) = L(x_t, y_t) = (x(t), y(t), f(x_t), Df(x_t)y_t)$ or, simply, as a system of the two equations $\dot{x}(t) = f(x_t)$ and $\dot{y}(t) = Df(x_t)y_t$, where Df denotes the derivative of f . The solution maps ϕ of F and λ of the linear variational equation L are then related by $\lambda(t, \cdot) = D\phi(t, \cdot)$.

It is clear from the preceding discussion that we can, without loss of generality, restrict the study of the persistence of hyperbolic invariant manifolds, under small perturbations of RFDEs on a manifold M , to the particular case where the manifold is euclidean, i.e., $M = \mathbb{R}^q$, for some integer q .

5. System of coordinates around hyperbolic invariant manifolds for FDEs

For a fixed real number $r > 0$, let $C = C([-r, 0]; \mathbb{R}^q)$ denote the Banach space of continuous functions from the interval $[-r, 0]$ to \mathbb{R}^q , taken with the uniform norm. Given a Banach space B and positive integers k, p , the set

$$BC^k(B; \mathbb{R}^p) = \{f: B \rightarrow \mathbb{R}^p, f \text{ is continuously differentiable and has bounded derivatives up to order } k\}$$

taken with the usual addition and multiplication by scalars and the uniform C^k -norm, is a Banach space. By uniform C^k -norm we mean

$$\|f\|_k = \sup\{|D^i f(\xi)|: i = 1, \dots, k \text{ and } \xi \in B\}.$$

We are interested in discussing functional differential equations (FDE) defined by functions $f \in BC^k(C; \mathbb{R}^n)$ with $k > 1$, as

$$\dot{u}(t) = f(u_t), \tag{5.1}$$

where u_t denotes the segment of the function u defined over the interval $[t-r, t]$, i.e., $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$. The solutions of (5.1) define a semiflow $(t, \xi) \mapsto u_t(\xi)$ on C , with $u_0 = \xi$. The mapping $u_t(\cdot): C \rightarrow C$ is C^k for all $t \geq 0$ and is completely continuous for $t \geq r$.

Let $M \subset C$ be a compact, connected, C^k -manifold which is k -hyperbolic under the semiflow defined by the solutions of (5.1). The vector bundle $E = \bigcup_{\omega \in M} T_\omega C$ can be identified with $M \times C$, since C is infinite dimensional and M is a finite dimensional manifold.

We will also consider the linear variational equation around M

$$\dot{v}(t) = Df(u_t(\omega))v_t \quad (5.2)$$

for each $\omega \in M$. The linearized semiflow around M which is induced by (5.1) is the linear skew-product semiflow defined, for $(\omega, \xi) \in E$, by

$$\pi(t, \omega, \xi) = (u_t(\omega), v_t(\omega, \xi)), \quad t \geq 0,$$

where $v_t(\omega, \xi) \in C$ denotes points in the orbit of (5.2) which passes through the point ξ at $t = 0$.

Because M is a k -hyperbolic manifold under (5.1), the vector bundle TM is invariant under the skew-product semiflow π , and TM has a complementary subbundle N of E , i.e., $E = TM + N$. Let U, S denote, respectively, the unstable and stable subbundles of N . The fibers $T_\omega = T_\omega M$, U_ω and S_ω can be identified with linear subspaces of C , and one can write $C = T_\omega + U_\omega + S_\omega$. Because M is connected, the dimensions of these fibers are independent of $\omega \in M$, and because M is hyperbolic and the semiflow π is completely continuous for $t \geq r$, it follows that both T_ω and U_ω are finite dimensional, with dimensions that we denote $d_T = \dim T_\omega$ and $d_U = \dim U_\omega$. We can choose bases for T_ω and U_ω consisting of vectors of unit length which depend on ω

in a C^k fashion. These bases are arranged as columns of $q \times d_T$ matrices ψ_ω^T and $q \times d_U$ matrices ψ_ω^U . To each point $(\omega, \xi) \in E$, we can associate coordinates $y \in R^{d_U}$, $\zeta \in C$ by the relations $\xi = \xi_\omega^T + \xi_\omega^U + \xi_\omega^S$ where $\xi_\omega^T \in T_\omega$, $\xi_\omega^U \in U_\omega$, $\xi_\omega^S \in S_\omega$, $\xi^T = \psi_\omega^T x$, $\xi^U = \psi_\omega^U y$ and $\zeta = \xi_\omega^S$. These relations associate a unique quadruple $(\omega, x, y, \zeta) \in M \times R^{d_T} \times R^{d_U} \times C$ to each point $(\omega, \xi) \in E$. This system of coordinates (ω, x, y, ζ) around the hyperbolic invariant manifold M is redundant. In fact, the same point in a neighborhood of M can be represented in several ways in these coordinates, according to which point $\omega \in M$ is taken as origin of the coordinate system. In spite of this redundancy, the use of these coordinates facilitates the study of the persistence of hyperbolic invariant manifolds under perturbations.

Given an FDE

$$\dot{w}(t) = f(w_t) + g(w_t), \quad (5.3)$$

where $g \in BC^k$, and defining $v(t) = w(t) - u(t)$, we can write it as a perturbation of the linear variational equation (5.2) in the form

$$\dot{v}(t) = Df(u_t(\omega))v_t + G(u_t(\omega), v_t), \quad (5.4)$$

with $\omega \in M$, by defining

$$G(\phi, \psi) = f(\phi + \psi) - Df(\phi)\psi - f(\phi) + g(\phi + \psi).$$

Equation (5.4) defines a skew-product semiflow on E by

$$\tilde{\pi}(t, \omega, \xi) = (u_t(\omega), v_t(\omega, \xi)),$$

where $v_t(\omega, \xi)$ denotes points on the orbit of (5.4) which satisfies the initial condition $v_0 = \xi$. The variation of constants formula for (5.4) can be written (see [4]) as

$$v_t = T_\omega(t, \sigma)v_\sigma + \int_\sigma^t T_\omega(t, s)X_0G(u_s(\omega), v_s)ds, \quad t > \sigma, \quad (5.5)$$

where $T_\omega(t, \sigma)$ denotes the solution operator of the linear variational equation (5.2) and $X_0(\theta)$ is defined to be the identity I_q at $\theta = 0$ and to be zero for $\theta \in [-r, 0]$ (notice that the columns of X_0 do not belong to C , but the formula still makes sense if interpreted as suggested by Hale in [4]).

As M is an hyperbolic compact manifold under the semiflow defined by equation (5.1), which is completely continuous for $t > r$, it follows that (5.1) defines an ODE on M . The points $v_t(\omega)$, which must satisfy equation (5.5), can be represented in the system of coordinates introduced above as $(u_t(\omega), x(t), y(t), z_t)$, where $x(t), y(t), z_t$ satisfy the variation of constants formulas obtained by projecting both sides of (5.5) along the "coordinate directions". More precisely, we have the following result.

Theorem 5.1

Let $f \in BC^k(C, R^q)$, $k \geq 1$, and assume $M \subset C$ is a compact connected C^k -manifold which is k -hyperbolic under the semiflow defined by the solutions of equation

$$\dot{u}(t) = f(u_t). \quad (5.6)$$

Then there exists a system of local coordinates around M , $(\omega, x, y, \zeta) \in M \times R^{d_T} \times R^{d_U} \times C$, where $d_T = \dim M$ and d_U is the dimension of the unstable bundle associated with the linear variational equation (5.2), such that, for each $\omega \in M$, there exist two matrix-valued functions $\Psi_\omega^T(t, \sigma)$ and $\Psi_\omega^U(t, \sigma)$ which are continuously differentiable in $t, \sigma \in R$, a linear subspace L of C with codimension $d_T + d_U$, a linear operator $T_\omega^S(t, \sigma)$ acting on L which is continuously differentiable in $t > \sigma$, a $q \times q$ matrix-valued function

$\chi_0^{S,\omega,\tau}$ defined on $[-r,0]$ which is continuous on τ , and functions n,u,s defined from $M \times R^{d_T} \times R^{d_U} \times C \times BC^k(C,R)$ into, respectively, R^{d_T}, R^{d_U}, R^q which are bounded and, for each fixed $\omega \in M$, are of class BC^k in the remaining variables, such that

- (i) $\Psi_\omega^T(t,t) = I_{d_T}, \Psi_\omega^U(t,t) = I_{d_U}, T_\omega^S(t,t) = I_L$, for all $t \in R$,
(ii) there exist $K, \alpha, \alpha_0 > 0$ with $\alpha > K\alpha_0$ such that

$$\begin{aligned} e^{-\alpha_0|t|} |\Psi_\omega^T(t,\tau)| &\rightarrow 0 \text{ as } |t| \rightarrow \infty \text{ for all } \tau \in R \\ |\Psi_\omega^U(t,\tau)| &\leq Ke^{\alpha(t-\tau)}, t \leq \tau \\ |T_\omega(t,\tau)\phi| &\leq Ke^{-\alpha(t-\tau)}|\phi|, t > \tau, \phi \in S_{u_0}(\omega), \end{aligned}$$

where S denotes the stable bundle associated with the linear variational equation of (5.6) around M ,

(iii) $|\chi_0^{S,\omega,\tau}| \leq 1, \omega \in M, \tau \in R,$

- (iv) the functions n,u,s and their partial derivatives relative to x,y,ζ vanish at the points $(\omega,0,0,0,0) \in M \times R^{d_T} \times R^{d_U} \times L \times BC^k(C;R^q)$, and each one of them assumes related values at all points (ω,x,y,ζ) which represent the same point of C ,

- (v) for each $g \in BC^k(C;R^q)$, the perturbed equation

$$\dot{w}(t) = f(w_t) + g(w_t) \quad (5.7)$$

is, in the new coordinates and for $t > \sigma$, equivalent to the system

$$\begin{aligned} x(t) &= \Psi_\omega^T(t,\sigma)x(\sigma) + \int_\sigma^t \Psi_\omega^T(t,\tau)n(u_\tau(\omega),x(\tau),y(\tau),z_\tau,g)d\tau \\ y(t) &= \Psi_\omega^U(t,\sigma)y(\sigma) + \int_\sigma^t \Psi_\omega^U(t,\tau)u(u_\tau(\omega),x(\tau),y(\tau),z_\tau,g)d\tau \\ z_t &= T_\omega^S(t,\sigma)z_\sigma + \int_\sigma^t T_\omega^S(t,\tau)\chi_0^{S,\omega,\tau} s(u_\tau(\omega),x(\tau),y(\tau),z_\tau,g)d\tau. \end{aligned} \quad (5.8)$$

Proof

We need to project both sides of the variation of constants formula (5.5) in the tangential, unstable and stable directions along the points $u_t(\omega) \in M$, as indicated in the discussion preceding the theorem. Forgetting, for the moment, the differentiability properties of the functions involved, and recalling that T and U are invariant under the skew-product semi-flow associated with the linear variational equation, we see that the introduction of bases for the finite dimensional fibers $T_{u_t(\omega)}$ and $U_{u_t(\omega)}$ and the representation of the projected equations in terms of these bases lead to the first two equations in system (5.8).

The linear spaces $S_{u_t(\omega)}$ have codimension $d_T + d_U$ in C and can be one-to-one mapped onto a fixed subspace L of C of the same codimension. Projecting (5.5) onto S and representing this projection in the subspace L , we obtain an equation of the form of the last equation in system (5.8). The term $T_\omega(t, \tau) X_0^{S, \omega, \tau}$ needs some explanation. First, we notice that $T_\omega(\tau+r, \tau) X_0$ is a matrix with columns in C because of the smoothing action of $T_\omega(t, \tau)$. In fact, though $X_0(\theta)$ is a matrix valued function defined for $\theta \in [-r, 0]$ and discontinuous at $\theta = 0$, the solutions of the FDE with initial conditions equal to each one of the columns of X_0 are continuous for $t \geq 0$ and, consequently, after $t = r$ units of time all the segment of the solution from $t - r$ to t is continuous, showing that the columns of $T_\omega(\tau+r, \tau) X_0$ do, indeed, belong to C . This matrix can be projected onto $T_{u_{\tau+r}(\omega)}$ and $U_{u_{\tau+r}(\omega)}$ to give components $[T_\omega(\tau+r, \tau) X_0]_{u_{\tau+r}(\omega)}^T$ and $[T_\omega(\tau+r, \tau) X_0]_{u_{\tau+r}(\omega)}^U$, respectively. Since $T_\omega(\tau+r, \tau)$ is a homeomorphism from $T_{u_\tau(\omega)}$ to $T_{u_{\tau+r}(\omega)}$ and from $U_{u_\tau(\omega)}$ to $U_{u_{\tau+r}(\omega)}$, because T and U are invariant under the semiflow, it follows that there exist unique matrix-valued functions $X_0^{T, \omega, \tau}$ and $X_0^{U, \omega, \tau}$ whose columns belong to $T_{u_\tau(\omega)}$ and $U_{u_\tau(\omega)}$, respectively, and are such that

$$T_{\omega}(\tau+r, \tau)X_0^{T, \omega, \tau} = [T_{\omega}(\tau+r, \tau)X_0]_{u_{\tau+r}}^T(\omega)$$

$$T_{\omega}(\tau+r, \tau)X_0^{U, \omega, \tau} = [T_{\omega}(\tau+r, \tau)X_0]_{u_{\tau+r}}^U(\omega)$$

Now, we can define $X_0^{S, \omega, \tau} = X_0 - X_0^{T, \omega, \tau} - X_0^{U, \omega, \tau}$, and it becomes clear that the last equation in system (5.8) is correct, provided $T_{\omega}^S(t, \tau)X_0^{S, \omega, \tau}$ is understood in the same sense as $T_{\omega}(t, \tau)X_0$ was (notice the $X_0^{S, \omega, \tau}$ does not belong to L , as X_0 does not belong to C ; for an explanation of this notation refer to [4]).

Properties (i) and (iii) are easy to verify, property (ii) is a consequence of the hyperbolicity of M through Theorem 3.1, and property (iv) results from the positive invariance of T, U, L under the skew-product semiflow associated with the linear variational equation around M and the invariance of M under equation (5.6).

It remains to establish the smoothness properties of the functions n, u, s . For this we need to show that the normal bundle N , the projections associated with the decomposition $C = T_{u_t}(\omega) + U_{u_t}(\omega) + S_{u_t}(\omega)$, the vectors forming the bases for $T_{u_t}(\omega)$ and $U_{u_t}(\omega)$ and the one-to-one mapping from $S_{u_t}(\omega)$ onto L can all be chosen to be C^k -smooth in t . The possibility of choosing a C^k -smooth normal bundle N can be proved by a slight modification of the proof given by Whitney [20] for the case when the manifold M is modeled in a finite dimensional euclidean space. In fact, a "natural" choice of the normal bundle would only be C^{k-1} smooth, but the procedure introduced by Whitney in the cited paper can be used to smoothen it to be of class C^k . It follows that the projections associated with the decomposition $C = T_{u_t}(\omega) + U_{u_t}(\omega) + L_{u_t}(\omega)$ are of class C^k in t , provided $U_{u_t}(\omega)$ and $S_{u_t}(\omega)$ are C^k in t . These are defined in terms of the null space and the range, respectively, of the linear

projections $P(u_t(\omega))$, defined on $N_{u_t(\omega)}$, which are associated with the dichotomy of the linearized skew-product semi-flow around M induced by the given equation. Although these projections are, at the outset, only required to depend continuously on the points in the manifold M , they are in fact of class C^k in t because their null spaces are related, forwards and backwards in time, by a semiflow of class C^k . More precisely, the null space of $P(\omega)$ is mapped onto the null space of $P(u_t(\omega))$ by the map $\xi \rightarrow v_t(\omega, \xi)$ given by the solutions of equation (5.2). Since this map is of class C^k in t , due to the general results on smoothness of solutions of FDEs (see [4]), and N is a C^k vector bundle, it follows that $P(u_t(\omega))$ is C^k in t . The possibility of choosing the one-to-one mapping from $S_{u_t(\omega)}$ onto L to be C^k -smooth in t is a direct consequence of the C^k -smoothness of $P(u_t(\omega))$. In order to get the C^k -smoothness in t for the bases taken for $T_{u_t(\omega)}$ and $U_{u_t(\omega)}$, we only need to choose them to be mapped one to each other by the flows on these bundles, since these flows are of class C^k in t . Q.E.D.

6. Functional differential equations in coordinate form

Under certain general conditions discussed in the preceding section, the linearization of a given FDE around an hyperbolic compact manifold M , and the introduction of local coordinates around the manifold lead to a family of systems parametrized by $\omega \in M$ and of the form

$$\dot{x}(t) = N(t)x(t) + n(t, x(t), y(t), z_t, \lambda) \quad (6.1)$$

$$y(t) = \Psi(t, \sigma)y(\sigma) + \int_{\sigma}^t \Psi(t, \tau)u(\tau, x(\tau), y(\tau), z_{\tau}, \lambda)d\tau \quad (6.2)$$

$$z_t = T(t, \sigma)z_{\sigma} + \int_{\sigma}^t T(t, \tau)\lambda_0^{S, \tau} s(\tau, x(\tau), y(\tau), z_{\tau}, \lambda)d\tau, \quad (6.3)$$

where λ is a parameter in a Banach space Λ , $x(t) \in R^{d_N}$, $y(t) \in R^{d_U}$, $z_t \in L$, L

is a linear subspace of $C = C([-r,0];R^q)$ of codimension $(d_N + d_U)$, d_N, d_U, q are nonnegative integers with $d_N \geq 1$, $X_0^{S,\tau}$ is a $q \times q$ matrix-valued function defined on $[-r,0]$ and continuous in τ with its columns belonging to L and satisfying $|X_0^{S,\tau}| \leq 1$ for all $\tau \in R$, $N(t)$ and $\Psi(t,\tau)$ are matrix-valued functions defined for $t, \tau \in R$, $T(t,\tau)$ are linear operators acting on L for $t \geq \tau$, and the following hypotheses are satisfied:

- (H₁) The function N of t is bounded and continuous for all $t \in R$.
- (H₂) the functions n, u, s of $(t, x, y, \zeta, \lambda)$ are bounded, continuously differentiable in x, t, ζ and their partial derivatives relative to x, y, ζ as well as the functions n, u, x themselves are all bounded by some $B(\mu, \epsilon) > 0$ over the region $t \in R, |x|, |y|, |\zeta| \leq \mu \leq \mu_0, |\lambda| \leq \epsilon \leq \epsilon_0$, with the function $B(\mu, \epsilon)$ being nondecreasing in μ and ϵ , and approaching zero as $\mu, \epsilon \rightarrow 0$.
- (H₃) The matrix-valued function $\Psi(t,\tau)$ is continuously differentiable in $t, \tau \in R$, satisfies $\Psi(t,\tau)\Psi(\tau,\sigma) = \Psi(t,\sigma)$ for all $t, \tau, \sigma \in R$ and $\Psi(t,t)$ is the identity matrix for all $t \in R$. The linear operators $T(t,\tau)$ defined on L are continuously differentiable in t, τ such that $t \geq \tau$, satisfy $T(t,\tau)T(\tau,\sigma) = T(t,\sigma)$ for all $t \geq \tau \geq \sigma$, and $T(t,t)$ is the identity operator on L for all $t \in R$.
- (H₄) There exist $K \geq 1$ and $\alpha > \alpha_0 > 0$, such that

$$\begin{aligned} |\Psi(t,\tau)| &\leq Ke^{\alpha(t-\tau)} & , t \leq \tau \\ |T(t,\tau)\phi| &\leq Ke^{-\alpha(t-\tau)}|\phi| & , t \geq \tau, \phi \in L, \end{aligned}$$

and, the principal matrix solution $\Phi(t,\tau)$ of $\dot{x} = N(t)x$ satisfies

$$|\Phi(t,\tau)| \leq Ke^{\alpha_0|t-\tau|} \quad , \quad \text{for all } t, \tau \in R$$

In this section we consider a more restricted situation which will be used later on to establish the general result on the persistence of hyperbolic manifolds. More precisely, the hypothesis (H_2) is replaced by

(H'_2) The functions n, u, s of $(t, x, y, \zeta, \lambda)$ are bounded and continuous, vanish at all points where x, y, ζ are simultaneously zero, and are globally lipschitzian in the coordinates x, y, ζ , in the sense that there exists a $D > 0$ such that

$$|n(t, x, y, \zeta, \lambda) - n(t, \bar{x}, \bar{y}, \bar{\zeta}, \lambda)| \leq D(|x - \bar{x}| + |y - \bar{y}| + |\zeta - \bar{\zeta}|)$$

for all $t \in \mathbb{R}$, $x, \bar{x} \in \mathbb{R}^{d_N}$, $y, \bar{y} \in \mathbb{R}^{d_U}$, $\zeta, \bar{\zeta} \in L$, $\lambda \in \Lambda$, and similarly for u and s .

In the proofs of the results of this section on the persistence of hyperbolic invariant manifolds for system (6.1)-(6.3) with $|\lambda|$ small, we use the following property of solutions $(x(t), y(t), z_t)$, $t \in \mathbb{R}$, which have $y(t)$ and z_t bounded.

Lemma 6.1

Assume the hypotheses (H_1) - (H_4) hold. Then $(x(t), y(t), z_t), t \in \mathbb{R}$, is a solution of the system (6.1)-(6.3) with $y(t)$ and z_t bounded if and only if for some γ belonging to the interval (α_0, α) the function

$$w(t, x(0)) = e^{-\gamma|t|} (x(t), y(t), z_t), \quad t \in \mathbb{R} \tag{6.4}$$

agrees, when $x(0) = b \in \mathbb{R}^{d_N}$, with a fixed point of the transformation T defined on the set of bounded continuous functions $w: \mathbb{R} \times \mathbb{R}^{d_N} \rightarrow \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L$ by

$$Tw(t,b) = (e^{-\gamma|t|}\psi(t,0)b + e^{-\gamma|t|}\int_0^t \psi(t,\tau)n(\tau,e^{\gamma|t|}w(\tau,b),\lambda)d\tau,$$

$$e^{-\gamma|t|}\int_{+\infty}^t \Psi(t,\tau)u(\tau,e^{\gamma|t|}w(\tau,b),\lambda)d\tau, \quad (6.5)$$

$$e^{-\gamma|t|}\int_{-\infty}^t T(t,\tau)X_0^{S,\tau} s(\tau,e^{\gamma|t|}w(t,b),\lambda)d\tau).$$

If (H_2) is replaced by (H_2') the same holds.

Proof.

If $(x(t),y(t),z_t)$ is a solution of (6.1)-(6.3) which is defined for all $t \in R$ and has $y(t)$ bounded, it follows from hypothesis (H_4) that for $\sigma > t$

$$|y(t) - \int_{\sigma}^t \Psi(t,\tau)u(\tau,x(\tau),y(\tau),z_{\tau},\lambda)d\tau| = |\Psi(t,\sigma)y(\sigma)| \leq Ke^{\alpha(t-\sigma)}|y(\sigma)|$$

and, letting $\sigma \rightarrow +\infty$, we get

$$y(t) = \int_{+\infty}^t \Psi(t,\tau)u(\tau,x(\tau),y(\tau),z_{\tau},\lambda)d\tau, \quad (6.6)$$

where the improper integral converges because u is bounded and Ψ satisfies the exponential estimates in assumption (H_4) . Analogously, if

$(x(t),y(t),z_t)$, $t \in R$, is a solution of (6.1)-(6.3) with z_t bounded, then

$$z_t = \int_{-\infty}^t T(t,\tau)X_0^{S,\tau} s(\tau,x(\tau),y(\tau),z_{\tau},\lambda)d\tau. \quad (6.7)$$

Conversely, if $(x(t),y(t),z_t)$, $t \in R$, is a given continuous function satisfying equation (6.1) and (6.6)-(6.7) then it is a solution of the system (6.1)-(6.3), with $y(t)$ and z_t bounded, because of hypotheses (H_1) and (H_4) .

The variation of constants formula for equation (6.1) gives

$$x(t) = \psi(t,0)x(0) + \int_0^t \psi(t,\tau)n(\tau,x(\tau),y(\tau),z_{\tau},\lambda)d\tau. \quad (6.8)$$

From hypothesis (H_2) or (H_2') there exists $B > 0$ which bounds n , and from hypothesis (H_4) we get

$$|x(t)| \leq Ke^{\alpha_0|t|} |x(0)| + B \int_0^t Ke^{\alpha_0|t-\tau|} d\tau \leq (|x(0)| + \frac{B}{\alpha_0}) Ke^{\alpha_0|t|}.$$

Consequently (6.1) is equivalent to (6.8) and $e^{-\gamma|t|}|x(t)|$, $t \in R$, is bounded.

This finishes the proof of the statement.

Q.E.D.

Theorem 6.2

If the hypotheses $(H_1), (H_2'), (H_3), (H_4)$ are satisfied and the constants α_0, α, K of hypothesis (H_4) and D of hypothesis (H_2') satisfy the inequality

$$DK(K+1) \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] < 1 \quad (6.9)$$

for some γ in the interval (α_0, α) , then there exist continuous functions h_1, h_2 defined on $R \times R^{d_N} \times \Lambda$ and with values in R^{d_U} and L , respectively, which are bounded and such that the function $x \rightarrow (h_1(t, x, \lambda), h_2(t, x, \lambda))$ is lipschitzian with Lipschitz constant $(K+1)$ and with $h_1(t, 0, 0) = 0, h_2(t, 0, 0) = 0$ for all $t \in R, x \in R^{d_N}, \lambda \in \Lambda$, such that the set

$$M_\lambda = \{(t, x, y, \zeta) \in R \times R^{d_N} \times R^{d_U} \times L : y = h_1(t, x, \lambda), \zeta = h_2(t, x, \lambda)\}$$

is an integral manifold for system (6.1)-(6.3), in the sense that if

$(t_0, x(t_0), y(t_0), z_{t_0}) \in M_\lambda$ then the solution of (6.1)-(6.3) with this initial data stays in M_λ for all time.

Furthermore, M_λ is the maximal integral manifold for system (6.1)-(6.3) contained in $R \times R^{d_N} \times V$, for any bounded neighborhood of zero $V \subset R^{d_U} \times L$.

Proof.

The preceding lemma indicates that finding integral manifolds M_λ for (6.1)-(6.3), which belong to $R \times R^{d_N} \times V$ for some neighborhood of zero $V \subset R^{d_U} \times L$, amounts to finding fixed points of the mapping T in the lemma. These fixed points are studied, in the present proof, by an application of the contraction mapping principle to a specific set of continuous functions $w: R \times R^{d_N} \rightarrow R^{d_N} \times R^{d_U} \times L$ taken with a metric generated by a suitable family of pseudonorms.

Let us denote by W the set of bounded continuous functions $w: R \times R^{d_N} \rightarrow R^{d_N} \times R^{d_U} \times L$ which satisfy

$$|w(t,b) - w(t,\bar{b})| \leq (K+1)|b - \bar{b}| \quad \text{for all } b, \bar{b} \in R^{d_N}, t \in R,$$

where $K > 1$ is the constant in the hypothesis (H_4) . The set W is a complete metric space with the topology generated by the family of pseudonorms

$$\|w\|_n = \sup\{|w(t,b)| : t \in R, |b| \leq n\}, \quad n = 1, 2, \dots \quad (6.10)$$

It is clear that, for each $w \in W$, Tw is a continuous function from $R \times R^{d_N}$ into $R^{d_N} \times R^{d_U} \times L$. Using the hypotheses (H_2') and (H_4) we get

$$\begin{aligned} |Tw(t,b)| \leq & [e^{-\gamma|t|} e^{\alpha_0|t|} K|b| + e^{-\gamma|t|} \int_0^t K e^{\alpha_0|t-\tau|} De^{\gamma|\tau|} d\tau + \\ & + e^{-\gamma|t|} \int_t^{+\infty} K e^{\alpha(t-\tau)} De^{\gamma|\tau|} d\tau + e^{-\gamma|t|} \int_{-\infty}^t K e^{-\alpha(t-\tau)} De^{\gamma|\tau|} d\tau] \sup_{\tau \in R} |w(\tau,b)|. \end{aligned}$$

Consequently, as $\gamma \in (\alpha_0, \alpha)$ we obtain

$$|Tw(t,b)| \leq K|b| + DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] \sup_{\tau \in R} |w(\tau,b)|.$$

this shows that Tw is bounded for each $w \in W$. In a similar way, and using inequality (6.9), one obtains

$$|Tw(t,b) - Tw(t,\bar{b})| \leq K|b - \bar{b}| + DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] \sup_{\tau \in R} |w(\tau, \bar{b}) - \bar{w}(\tau, \bar{b})| < (K + 1)|b - \bar{b}|.$$

On the other hand, if $w, \bar{w} \in W$, we have

$$|Tw(t,b) - T\bar{w}(t,b)| \leq DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] \sup_{\tau \in R} |w(\tau, b) - \bar{w}(\tau, b)|$$

which implies that

$$\|Tw - T\bar{w}\|_n \leq DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] \|w - \bar{w}\|_n.$$

Since condition (6.9) is satisfied then T is a uniform contraction from W into itself, in the given family of pseudonorms. The contraction mapping principle implies that there exists a unique fixed point of T in the set W and that this fixed point depends continuously on λ .

Let $w^* = Tw^*$ be the fixed point of T in W and define the functions h_1 and h_2 by

$$(b, h_1(t, b, \lambda), h_2(t, b, \lambda)) = w^*(t, b).$$

It is clear that h_1, h_2 are defined for $t \in R, b \in R^{d_N}, \lambda \in \Lambda$, that they are continuous bounded functions which are lipschitzian in the variable b with lipschitz constant $(K + 1)$ and that they vanish at the points $(t, 0, 0)$.

Suppose $(x(t), y(t), z_t), t \in R$, is any solution of (6.1)-(6.3) with $y(t)$ and z_t bounded, and denote $b = x(0)$. Lemma 6.1 implies that $w(t) = e^{-\gamma|t|}(x(t), y(t), z_t)$ is a fixed point of the map T_b defined on the set

$B(R)$ of the continuous bounded functions $w: R \rightarrow R^{d_N} \times R^{d_U} \times L$ by the same formula (6.5) as in the definition of T , but replacing $w(t,b)$ by $w(t)$. The argument used above for T also shows that T_b maps $B(R)$ into itself and is a contraction in the supremum norm on $B(R)$. Therefore T_b has a unique fixed point in $B(R)$ which must satisfy $w(t) = w^*(t,b)$. It follows that M_λ is the maximal integral manifold in $R \times R^{d_N} \times V$, for any $V \subset R^{d_U} \times L$ which is a bounded neighborhood of zero. Q.E.D.

Remark: The functions h_1, h_2 of the previous theorem do not depend on the particular value of $\gamma \in (\alpha_0, \alpha)$, provided it satisfies inequality (6.9). In fact, if $\alpha_0 < \gamma_1 < \gamma_2 < \alpha$ and the function $w_1(t, x(0))$ is bounded and given as in (6.4) with $\gamma = \gamma_1$, then the function $w_2(t, x(0))$ also given as in (6.4) but with $\gamma = \gamma_2$ is also bounded. The uniqueness of the fixed point of the mapping T implies that $w_1 = w_2$.

The structure of the solutions around M_λ is preserved under small perturbations, as is illustrated by the following result.

Theorem 6.3

Assume the same hypotheses as for Theorem 6.2. Then the manifold M_λ is the intersection of two manifolds $S_\lambda, U_\lambda \subset R \times R^{d_N} \times R^{d_U} \times L$ which are positive integral manifolds for (6.1)-(6.3), and are such that solutions with initial data in S_λ approach M_λ as $t \rightarrow +\infty$ and solutions with initial data in U_λ are globally defined and approach M_λ as $t \rightarrow -\infty$. Moreover S_λ, U_λ are homeomorphic to $R \times L, R \times R^{d_U}$, respectively, and have the form

$$S_\lambda = \{(t, x, y, \zeta) \in R \times R^{d_N} \times R^{d_U} \times L: y = h^S(t, x, \zeta, \lambda)\},$$

and

$$U_\lambda = \{(t, x, y, \zeta) \in \mathbb{R} \times \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L : \zeta = h^U(t, x, y, \lambda)\},$$

where the functions $(x, \zeta) \rightarrow h^S(t, x, \zeta, \lambda)$ and $(x, y) \rightarrow h^U(t, x, y)$ are lipschitzian homeomorphisms from, respectively, $\mathbb{R}^{d_N} \times L$ to S_λ and $\mathbb{R}^{d_N} \times \mathbb{R}^{d_U}$ to U_λ , with lipschitz constant $(K+1)$ and satisfy $h^S(t, x, \zeta, 0) = 0, h^U(t, x, y, 0) = 0$. In addition, there exist $\epsilon, C > 0$ such that, if $(x(t), y(t), z_t)$ is a solution of (6.1)-(6.3) with initial condition in S_λ , then

$$|(x(t), y(t), z_t)| \leq C(|x(0)| + |z_0|)e^{\epsilon t}, \quad t > 0$$

and, if $(x(t), y(t), z_t)$ is a solution of (6.1)-(6.3) with initial condition in U_λ , then

$$|(x(t), y(t), z_t)| \leq C(|x(0)| + |y(0)|)e^{-\epsilon t}, \quad t < 0,$$

where $\epsilon \in (\alpha_0, \Upsilon)$.

Proof.

If $(x(t), y(t), z_t)$ is an arbitrary solution of (6.1)-(6.3) which is defined for all $t \geq 0$ and has $y(t)$ bounded for $t \geq 0$, we find, as in the proof of Lemma 6.1, that

$$y(t) = \int_{+\infty}^t \Psi(t, \tau) u(\tau, x(\tau), y(\tau), z_\tau, \lambda) d\tau.$$

Consequently, based on the discussion in Lemma 6.1 and Theorem 6.2, we expect that looking for the set S_λ will amount to finding fixed points of the transformation T^S defined on the set of bounded continuous functions

$w: \mathbb{R}^+ \times \mathbb{R}^{d_N} \times L \rightarrow \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L$, by

$$\begin{aligned}
 T^S w(t, b, \zeta) = & (e^{-\gamma t} \psi(t, 0) b + e^{-\gamma t} \int_0^t \psi(t, \tau) n(\tau, e^{\gamma \tau} w(\tau, b, \zeta), \lambda) d\tau, \\
 & e^{-\gamma t} \int_{+\infty}^t \Psi(t, \tau) u(\tau, e^{\gamma \tau} w(\tau, b, \zeta), \lambda) d\tau, \\
 & e^{-\gamma t} T(t, 0) \zeta + e^{-\gamma t} \int_0^t T(t, \tau) X_0^{S, \tau} s(\tau, e^{\gamma \tau} w(\tau, b, \zeta), \lambda) d\tau),
 \end{aligned} \tag{6.11}$$

where $\gamma \in (\alpha_0, \alpha)$.

We denote by W^S the set of bounded continuous function $w: R^+ \times R^{d_U} \times L \rightarrow R^{d_N} \times R^{d_U} \times L$ which satisfy

$$|w(t, b, \zeta) - w(t, \bar{b}, \bar{\zeta})| \leq (K + 1)(|b - \bar{b}| + |\zeta - \bar{\zeta}|)$$

for all $b, \bar{b} \in R^{d_N}$, $\zeta, \bar{\zeta} \in L$, $t \in R^+$, where $K > 1$ is the constant in the hypothesis (H_4) . The set W^S is a complete metric space with the topology generated by the family of pseudonorms

$$\|w\|_n = \sup\{|w(t, b, \zeta)| : t \in R^+, |b| \leq n, |\zeta| \leq n\}, \quad n = 1, 2, \dots$$

Similarly to what was done in the proof of Theorem 6.2 for the mapping T on W , it can be shown that T^S is a uniform contraction on W^S and, therefore, there exists a unique fixed point of T^S in W^S and it depends continuously on λ .

If we let $w^S = T^S w^S$ be the fixed point of T^S on W^S and define the function h^S by

$$(b, h^S(t, b, \zeta, \lambda), \zeta) = w^S(t, b, \zeta),$$

it is clear that the set S_λ , defined as in the statement of the theorem, is a positive integral manifold for system (6.1)-(6.3) and it is homeomorphic to $R^{d_N} \times L$.

Next, we will show that $|w^S(t, b, \zeta)| \rightarrow 0$ as $t \rightarrow +\infty$. Let $\mu = \limsup_{t \rightarrow +\infty} |w^S(t, b, \zeta)|$. Because of inequality (6.9), we can choose $\delta > 1$ so that

$$KD \left[\frac{1}{\gamma - \alpha_0} + \frac{1}{\gamma - \alpha} + \frac{1}{\gamma + \alpha} \right] \delta < 1.$$

If $\mu > 0$, then there is a $\sigma > 0$ so that $|w^S(t, b, \zeta)| \leq \mu\delta$ for $t \geq \sigma$. Then, using formula (6.11) and the estimates available for its terms, we get for $t \geq \sigma$

$$\begin{aligned} |w^S(t, b, \zeta)| &= |T^S w^S(t, b, \zeta)| \leq Ke^{-(\gamma - \alpha_0)t} |b| + Ke^{-(\alpha + \gamma)t} |\zeta| + \\ &+ KD \left[\frac{e^{-(\gamma - \alpha_0)(t - \sigma)}}{\gamma - \alpha_0} + \frac{e^{-(\gamma + \alpha)(t - \sigma)}}{\gamma + \alpha} \right] \sup_{\tau \geq 0} |w^S(\tau, b, \zeta)| + \\ &+ KD \left[\frac{1}{\gamma - \alpha_0} + \frac{1}{\gamma - \alpha} + \frac{1}{\gamma + \alpha} \right] \mu\delta. \end{aligned}$$

Letting $t \rightarrow +\infty$, we get

$$\mu \leq KD \left[\frac{1}{\gamma - \alpha_0} + \frac{1}{\gamma - \alpha} + \frac{1}{\gamma + \alpha} \right] \delta \mu < \mu$$

which is a contradiction. Hence $\mu = 0$. This proves that $|w^S(t, b, \zeta)| \rightarrow 0$ as $t \rightarrow +\infty$.

Now we derive the exponential rate of decay of w^S as $t \rightarrow +\infty$. Let $v(t, b, \zeta) = \sup_{\tau \geq t} |w^S(\tau, b, \zeta)|$. Since $|w^S(\tau, b, \zeta)| \rightarrow 0$ as $\tau \rightarrow +\infty$, for every $t > 0$ there is a $\sigma \geq t$ such that

$$v(\tau, b, \zeta) = v(\sigma, b, \zeta) = |w^S(\sigma, b, \zeta)|, \quad t \leq \tau \leq \sigma.$$

On the other hand, estimates using formula (6.11) give

$$|w^S(t, b, \zeta)| \leq Ke^{-(\gamma-\alpha_0)t} |b| + Ke^{-(\gamma+\alpha)t} |\zeta| + \int_0^t Ke^{-(\gamma-\alpha_0)(t-\tau)} D|w^S(\tau, b, \zeta)| d\tau \\ + \int_t^{+\infty} Ke^{-(\gamma-\alpha)(t-\tau)} D|w^S(\tau, b, \zeta)| d\tau + \int_0^t Ke^{-(\gamma+\alpha)(t-\tau)} D|w^S(\tau, b, \zeta)| d\tau.$$

Consequently,

$$v(t, b, \zeta) = v(\sigma, b, \zeta) \leq Ke^{-(\gamma-\alpha_0)t} |b| + Ke^{-(\gamma+\alpha)t} |\zeta| + \int_0^t Ke^{-(\gamma-\alpha_0)(t-\tau)} Dv(\tau, b, \zeta) \\ + \int_t^{\sigma} Ke^{-(\gamma-\alpha_0)(\sigma-\tau)} Dv(\tau, b, \zeta) d\tau + \int_t^{-\infty} Ke^{(\alpha-\gamma)(t-\tau)} Dv(\tau, b, \zeta) d\tau \\ + \int_0^t Ke^{-(\gamma+\alpha)(t-\tau)} Dv(\tau, b, \zeta) d\tau + \int_t^{\sigma} Ke^{-(\gamma+\alpha)(\sigma-\tau)} Dv(\tau, b, \zeta) d\tau$$

and, therefore,

$$v(t, b, \zeta) \leq Ke^{-(\gamma-\alpha_0)t} (|b| + |\zeta|) + \int_0^t Ke^{-(\gamma-\alpha_0)(t-\tau)} Dv(\tau, b, \zeta) d\tau + \\ + KD \left(\frac{1}{\gamma - \alpha_0} + \frac{1}{\alpha - \gamma} + \frac{1}{\gamma + \alpha} \right) v(t, b, \zeta).$$

Due to inequality (6.9), we can write

$$e^{(\gamma-\alpha_0)t} v(t, b, \zeta) \leq [1 - KD \left(\frac{1}{\gamma - \alpha_0} + \frac{1}{\alpha - \gamma} + \frac{1}{\gamma + \alpha} \right)]^{-1} K [|b| + |\zeta| + \\ + \int_0^t De^{(\gamma-\alpha_0)\tau} v(\tau, b, \zeta) d\tau].$$

Applying Gronwall inequality we obtain

$$|v(t, b, \zeta)| \leq C (|b| + |\zeta|) e^{-Bt},$$

where

$$C = K / [1 - KD \left(\frac{1}{\gamma - \alpha_0} + \frac{1}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right)] \quad \text{and} \quad B = \gamma - \alpha_0 - CD.$$

From inequality (6.9), noting that $K > 1$, it is easy to verify that C, B are positive. It follows that there exist $C, B > 0$ such that

$$|(x(t), y(t), z_t)| \leq C(|x(0)| + |z_0|)e^{(\gamma-B)t}, \quad t \geq 0 \quad (6.12)$$

for every solution $(x(t), y(t), z_t)$ of (6.1)-(6.3) which has initial data on S_λ .

In a similar way, we obtain the manifold U_λ . Now, we look for fixed points of the mapping defined on the set of bounded continuous functions $w: \mathbb{R}^- \times \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \rightarrow \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L$ by

$$\begin{aligned} T^U w(t, b, c) = & (e^{\gamma t} \psi(t, 0)b + e^{\gamma t} \int_0^t \varphi(t, \tau) n(\tau, e^{-\gamma \tau} w(\tau, b, c), \lambda) d\tau, \\ & e^{\gamma t} \psi(t, 0)c + e^{\gamma t} \int_0^t \Psi(t, \tau) u(\tau, e^{-\gamma \tau} w(\tau, b, c), \lambda) d\tau, \\ & e^{\gamma t} \int_{-\infty}^t \Gamma(t, \tau) \chi_0^{S, \tau} s(\tau, e^{-\gamma \tau} w(\tau, b, c), \lambda) d\tau). \end{aligned}$$

Similarly, we obtain for some constants $C, B > 0$

$$|(x(t), y(t), z_t)| \leq C(|x(0)| + |y(0)|) e^{-(\gamma-B)t}, \quad t \leq 0 \quad (6.13)$$

for every solution $(x(t), y(t), z_t)$ of system (6.1)-(6.3) which has initial data on U_λ .

It is clear, from the above construction and the proofs of Lemma 6.1 and Theorem 6.2, that $M_\lambda = S_\lambda \cap U_\lambda$ and the trajectory of any solution $(x(t), y(t), z_t)$ for which $e^{-\gamma|t|} |(x(t), y(t), z_t)|$ is bounded for $t \in \mathbb{R}$ lies necessarily on M_λ . Consequently, the inequalities (6.12) and (6.13) imply the exponential estimates in the statement of the theorem. Q.E.D.

The following result establishes the smoothness properties of M_λ, S_λ and U_λ .

Theorem 6.4

If the hypotheses $(H_1), (H_2^1), (H_3), (H_4)$ are satisfied, the functions n, u, s are continuously differentiable with bounded derivatives up to order $k \geq 1$, relative to x, y, ζ , and the constants α_0, α, K of hypothesis (H_4) , and D of hypothesis (H_2^1) satisfy the inequalities

$$\alpha > k \alpha_0 \tag{6.14}$$

and

$$DK(K + 1) \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\gamma - \alpha} + \frac{1}{\gamma + \alpha} \right] < 1, \tag{6.15}$$

for some γ in the interval $(k\alpha_0, \alpha)$, then the function

$x \rightarrow (h_1(t, x, \lambda), h_2(t, x, \lambda))$, defined as in Theorem 6.2, is of class C^k , and the same is true for the functions $(x, \zeta) \rightarrow h^S(t, x, \zeta, \lambda)$ and $(x, y) \rightarrow h^U(t, x, y, \lambda)$ of Theorem 6.3.

Proof

We recall from the proof of Theorem 6.2 that h_1, h_2 were defined in terms of the fixed point of the transformation T in the set W . Consequently, if w denotes this fixed point, in order to prove that $h_1(t, x, \lambda), h_2(t, x, \lambda)$ are C^k functions of x , we only need to show that the function $b \rightarrow w(t, b)$ is of class C^k . This will be done by induction.

Let us assume the hypothesis of the theorem is satisfied for $k = 1$. If the derivative $\partial w(t, b) / \partial b$ exists, it must satisfy the equation obtained by formally differentiating $w = Tw$ relative to b . In particular, $\partial w / \partial b$ must be a fixed point of the mapping F defined for functions taking $R \times R^{d_N}$ to linear transformations of R^{d_N} into $R^{d_N} \times R^{d_U} \times L$.

$$Fv(t,b) = (e^{-\gamma|t|} \psi(t,0) + e^{-\gamma|t|} \int_0^t \psi(t,\tau) \frac{\partial n}{\partial w} (\tau, e^{\gamma|\tau|} w(\tau,b), \lambda) e^{\gamma|\tau|} v(\tau,b) d\tau,$$

$$e^{-\gamma|t|} \int_{+\infty}^t \psi(t,\tau) \frac{\partial u}{\partial w} (\tau, e^{\gamma|\tau|} w(\tau,b), \lambda) e^{\gamma|\tau|} v(\tau,b) d\tau, \quad (6.16)$$

$$e^{-\gamma|t|} \int_{-\infty}^t T(t,\tau) \chi_0^{s,\tau} \frac{\partial s}{\partial w} (\tau, e^{\gamma|\tau|} w(\tau,b), \lambda) e^{\gamma|\tau|} v(\tau,b) d\tau).$$

We remark that the improper integrals converge because of hypothesis (H_4) , the boundedness of the partial derivatives of u, s and the assumption $\alpha > \alpha_0$. Let Z denote the set of all functions continuously differentiable in the second argument and defined from $R \times R^{d_N}$ to the set of all linear transformations of R^{d_N} into $R^{d_N} \times R^{d_U} \times L$ which satisfy $|v(t,b)| \leq K + 1$, for all $t \in R$, $b \in R^{d_N}$. If $v \in Z$ then

$$|Fv(t,b)| \leq Ke^{-(\gamma-\alpha_0)|t|} + e^{-\gamma|t|} \left| \int_0^t Ke^{\alpha_0|t-\tau|} De^{\gamma|\tau|} (K+1) d\tau \right| + \quad (6.17)$$

$$+ e^{-\gamma|t|} \left| \int_t^{+\infty} Ke^{\alpha(t-\tau)} De^{\gamma|\tau|} (K+1) d\tau + e^{-\gamma|\tau|} \int_{-\infty}^t Ke^{-\alpha(t-\tau)} De^{\gamma|\tau|} (K+1) d\tau \right|$$

$$\leq K + DK(K+1) \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] < K + 1.$$

Therefore, F maps the set Z into itself. Let us consider the sequence $\{v_i\}$ of functions, taking $R \times R^{d_N}$ into the linear transformations of R^{d_N} into $R^{d_N} \times R^{d_U} \times L$ which is defined recursively by

$$v_1 = 0 \quad \text{and} \quad v_{i+1} = F(v_i), \quad i \geq 1.$$

Since F maps Z into itself, we have $\{v_i\} \subset Z$. The set Z with the metric inherited from the usual uniform norm is a complete metric space. Similarly to inequality (6.17), we get

$$\|v_{i+1} - v_i\| = \|Fv_i - Fv_{i-1}\| \leq DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] \|v_i - v_{i-1}\|. \quad (6.18)$$

It follows from the inequality (6.9) that $\{v_i\}$ is a Cauchy sequence in Z and, consequently, converges to some $v \in Z$ as $i \rightarrow \infty$. Clearly v is a fixed point of F .

Now, we can prove that v is, indeed, the derivative $\partial w / \partial b$. Fix $t \in \mathbb{R}$, $b \in \mathbb{R}^{d_N}$ and let σ be a function defined for small values of $\varepsilon > 0$ by

$$\sigma(\varepsilon) = \sup_{\substack{t \in \mathbb{R} \\ |h| \leq \varepsilon}} \frac{|w(t, b+h) - w(t, b) - v(t, b)h|}{|h|} \quad (6.19)$$

In order to prove that $\partial w / \partial b$ exists and is equal to v we need to show that $\sigma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. From formulas (6.5) and (6.16), using the first order Taylor expansion and hypotheses (H_2') and (H_4) in a similar way as for inequalities (6.17) and (6.18), we get

$$|w(t, b+h) - w(t, b) - v(t, b)h| \leq DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right]$$

$$\cdot \sup_{\tau \in \mathbb{R}} [|w(\tau, b+h) - w(\tau, b) - v(\tau, b)h| + o(|w(\tau, b+h) - w(\tau, b)|)].$$

Recalling from Theorem 6.2 that $w(t, b)$ is lipschitzian in b with lipschitz constant equal to $(K + 1)$, we get

$$\sigma(\varepsilon) \leq DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] \sigma(\varepsilon) + o(\varepsilon)$$

as $\varepsilon \rightarrow 0$. Applying inequality (6.9) we get $\sigma(\varepsilon) = o(\varepsilon)$, proving that $w(t, b)$ is differentiable in b and the derivative $\partial w(t, b) / \partial b$ is the continuous function v defined above. The preceding reasoning shows that the function $b \rightarrow w(t, b)$ is continuously differentiable and, consequently, also $b \rightarrow (h_1(t, b, \lambda), h_2(t, b, \lambda))$ is.

Now, let us assume that $b \rightarrow w(t, b)$ is of class C^j for a certain $j \geq 1$ and the hypothesis of the theorem is satisfied for $k = j + 1$. Let $v = \partial w^j / \partial b^j$.

Differentiating j times both sides of equation $w = Tw$, we get

$$\begin{aligned}
 v(t,b) &= (e^{-\gamma|t|} \int_0^t \phi(t,\tau) \frac{\partial n}{\partial w} (\tau, e^{\gamma|\tau|} w(\tau,b), \lambda) e^{\gamma|\tau|} v(\tau,b) d\tau, \\
 &e^{-\gamma|t|} \int_{+\infty}^t \psi(t,\tau) \frac{\partial u}{\partial w} (\tau, e^{\gamma|\tau|} w(\tau,b), \lambda) e^{\gamma|\tau|} v(\tau,b) d\tau, \quad (6.20) \\
 &e^{-\gamma|t|} \int_{-\infty}^t T(t,\tau) \chi_0^{S,\tau} \frac{\partial s}{\partial w} (\tau, e^{\gamma|\tau|} w(\tau,b), \lambda) e^{\gamma|\tau|} v(\tau,b) d\tau) + \\
 &+ (\text{terms not involving } v).
 \end{aligned}$$

The terms not involving v contain derivatives of n, u, s relative to w up to order j , derivatives of w up to order $j - 1$ and exponential factors of the form $e^{i\gamma|\tau|}$ for $i = 1, 2, \dots, j$. The improper integrals in these terms converge because of hypothesis (H_4) , the boundedness of all the partial derivatives of n, u, s up to order j , and the assumption $\alpha > k\alpha_0$. We can let T be the mapping transforming v to the function $T(v)$ of (t, b) according to the right-hand side of (6.20) and define recursively the sequence $\{v_i\}$ by

$$v_i = 0 \quad \text{and} \quad v_{i+1} = T(v_i) \quad \text{for } i \geq 1.$$

As in the first part of the proof we have that $\{v_i\}$ is a Cauchy sequence and, consequently, it converges to a fixed point of T which must be $v = \partial^j w / \partial b^j$. Clearly, the functions $v_i(t, b)$ are differentiable in b , and $\partial v_{i+1} / \partial b$ is given by the right-hand side of (6.20) with v replaced by $\partial v_i / \partial b$. Proceeding as for inequality (6.18), we get

$$\left\| \frac{\partial v_{i+1}}{\partial b} - \frac{\partial v_i}{\partial b} \right\| \leq DK \left[\frac{1}{\gamma - \alpha_0} + \frac{2}{\alpha - \gamma} + \frac{1}{\alpha + \gamma} \right] \left\| \frac{\partial v_i}{\partial b} - \frac{\partial v_{i-1}}{\partial b} \right\|.$$

From inequality (6.9), we get that $\{\partial v_i / \partial b\}$ is a Cauchy sequence. Arguing as in the first part of the proof, where the first derivative was handled, we can show that $\partial v / \partial b = \partial^{j+1} w / \partial b^{j+1}$ exists and is equal to the limit of $\{\partial v_i / \partial b\}$ as $i \rightarrow \infty$. This completes the induction.

The smoothness of h^U and h^S can be handled in a similar way. Q.E.D.

7. Functional differential equations with hyperbolic invariant manifolds

As before, consider $r > 0$ and let $C = C([-r,0];R^q)$ denote the Banach space of continuous functions from the interval $[-r,0]$ into R^q , where q is a positive integer and C is taken with the uniform norm. Let us consider an FDE

$$\dot{u}(t) = f(u_t), \quad (7.1)$$

where $f \in BC^k(C,R^n)$ with $k > 1$, and suppose that $M \subset C$ is a compact, connected, C^k -manifold which is k -hyperbolic under the semiflow defined by the solutions of (7.1). It is known from Section 5 that the equation can be linearized around the manifold M and a system of coordinates can be introduced around M so that the equation becomes of the form discussed in Section 6. The aim of the present section is to show how the results on the persistence and smoothness of integral manifolds for systems in coordinate form, as presented in the previous section, can be applied to equation (7.1) yielding the persistence of an hyperbolic invariant manifold close to M , under small perturbations of equation (7.1).

Theorem 7.1

Let $f \in BC^k(C,R^q)$, $k > 1$, and assume that $M_0 \subset C$ is a compact, connected, C^k -manifold which is k -hyperbolic under the semiflow defined by the solutions of the equation

$$\dot{u}(t) = f(u_t). \quad (7.2)$$

If $g \in BC^k(C,R^q)$ and $\|g\|_1 = \sup\{|g(\phi)|, \|g'(\phi)\| : \phi \in C\}$ is sufficiently small, then there exists a C^k -manifold $M_g \subset C$ which is invariant under the

perturbed equation

$$\dot{u}(t) = f(u_t) + g(u_t). \quad (7.3)$$

There exists a neighborhood $0 \subset C$ of M_0 such that, for $\|g\|_1$ sufficiently small, the manifold M_g is the maximal invariant set for (7.3) which is contained in 0 . The manifold M_g depends continuously in g , in the sense that M_g can be made arbitrarily close to M_0 in the Hausdorff metric by choosing $\|g\|_1$ sufficiently close to zero. Furthermore, there exist C^k -manifolds U_g, S_g with $U_g \cap 0$ negatively invariant and $S_g \cap 0$ positively invariant under (7.3) such that $M_g = S_g \cap U_g \cap 0$ and

$$|u_t(\phi, g)| \leq C|\phi|e^{-\epsilon t}, \quad t > 0, \quad \text{for } \phi \in (S_g \cap 0)$$

$$|u_t(\phi, g)| \leq C|\phi|e^{\epsilon t}, \quad t < 0, \quad \text{for } \phi \in (U_g \cap 0)$$

for some constants $C, \epsilon > 0$.

Proof

Firstly, we introduce a system of coordinates around M_0 as indicated in Section 5. For each fixed $\omega \in M_0$, the system in coordinate form (5.8) can be written as an equation (6.1)-(6.3), where we take for Λ the Banach space of bounded continuously differentiable functions from C into R^q which have bounded first derivative, taken with the uniform C^1 -norm and take $\lambda = g$. As a consequence of Theorem 5.1, the hypotheses (H_1) to (H_4) of Section 6 are all satisfied with $\alpha > k\alpha_0$. The only hypothesis which is necessary for applicability of the results in Section 6 and is not necessarily fulfilled is that contained in (H_2') , namely that the functions n, u, s of $(t, x, y, \zeta, \lambda)$ are globally lipschitzian in x, y, ζ , and the requirement that they admit a lipschitz constant D satisfying inequality (6.9). Consequently these functions have to be

"cut-off" and replaced by functions $\bar{n}, \bar{u}, \bar{s}$ which agree with n, u, s for $|x|, |y|, |\zeta|$ sufficiently small and are globally lipschitzian in x, y, ζ with a lipschitz constant satisfying inequality (6.9).

Let α_0, α, K be as in Theorem 5.1 and let $D > 0$ be chosen to satisfy inequality (6.9). Assume $0 < \mu \leq \mu_0, 0 < \varepsilon \leq \varepsilon_0$ and $B(\mu, \varepsilon)$ is as in hypothesis (H_2) of Section 6. Let us consider a C^∞ function $v: \mathbb{R}^+ \rightarrow [0, 1]$ such that

$$v(\rho) \in \begin{cases} \{1\} & \text{if } \rho/[\mu^2(2+r)] \leq 1/4 \\ (0,1) & \text{if } 1/4 < \rho/[\mu^2(2+r)] < 1 \\ \{0\} & \text{if } 1 \leq \rho/[\mu^2(2+r)] \end{cases}$$

and $0 \leq -v'(\rho) \leq 2/[\mu^2(2+r)]$ for $\rho > 0$, and a C^∞ function $\sigma: \mathbb{R}^+ \rightarrow [0, 1]$ such that

$$\sigma(\rho) \in \begin{cases} \{1\} & \text{if } \rho \leq \varepsilon/2 \\ (0,1) & \text{if } \varepsilon/2 < \rho < \varepsilon \\ \{0\} & \text{if } \varepsilon \leq \rho. \end{cases}$$

We define the function $\bar{n}: \mathbb{R} \times \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L \times \{\lambda \in \Lambda: \|\lambda\| \leq \varepsilon_0\}$ so that it satisfies

$$\bar{n}(t, x, y, \zeta, \lambda) = \sigma(\varepsilon)v(|x|^2 + |y|^2 + \int_{-r}^0 |\zeta(\tau)|^2 d\tau)n(t, x, y, \zeta, \lambda)$$

for $t \in \mathbb{R}, |x|, |y|, |\zeta| \leq \mu, \|\lambda\| \leq \varepsilon$, vanishing outside this region, and define \bar{u}, \bar{s} in a similar way. Then, over the region $t \in \mathbb{R},$

$|x|, |y|, |\zeta| \leq \mu/2, \|\lambda\| \leq \varepsilon/2$, we have $\bar{n} = n, \bar{u} = u, \bar{s} = s$. It remains to show that $\bar{n}, \bar{u}, \bar{s}$ are globally lipschitzian in x, y, ζ with lipschitz constant D .

Let $t \in \mathbb{R}, \|\lambda\| \leq \varepsilon$ and $V = \{(x, y, \zeta) \in \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L: |x|, |y|, |\zeta| \leq \mu\}$. If $(x, y, \zeta), (\bar{x}, \bar{y}, \bar{\zeta}) \in V$, then

$$\begin{aligned}
 |\bar{n}(t, x, y, \zeta, \lambda) - \bar{n}(t, \bar{x}, \bar{y}, \bar{\zeta}, \lambda)| &\leq \nu(|x|^2 + |y|^2 + \int_{-r}^0 |\zeta(\tau)|^2 d\tau) - \\
 &- \nu(|\bar{x}|^2 + |\bar{y}|^2 + \int_{-r}^0 |\bar{\zeta}(\tau)|^2 d\tau) |n(t, x, y, \zeta, \lambda)| + \\
 &+ \nu(|\bar{x}|^2 + |\bar{y}|^2 + \int_{-r}^0 |\bar{\zeta}(\tau)|^2 d\tau) |n(t, x, y, \zeta, \lambda) - n(t, \bar{x}, \bar{y}, \bar{\zeta}, \lambda)| \leq \\
 &\leq \frac{2}{\mu^2(2+r)} 2\mu(2+r)(|x - \bar{x}| + |y - \bar{y}| + |\zeta - \bar{\zeta}|)B(\mu, \epsilon)3\mu + \\
 &+ B(\mu, \epsilon)(|x - \bar{x}| + |y - \bar{y}| + |\zeta - \bar{\zeta}|) = \\
 &= 13B(\mu, \epsilon)(|x - \bar{x}| + |y - \bar{y}| + |\zeta - \bar{\zeta}|).
 \end{aligned}$$

If $(x, y, \zeta) \in V$ and $(\bar{x}, \bar{y}, \bar{\zeta}) \notin V$ then there exists a point (x^*, y^*, ζ^*) lying in the intersection of the boundary of V and the straight line joining the points (x, y, ζ) and $(\bar{x}, \bar{y}, \bar{\zeta})$. Thus $\bar{n}(t, \bar{x}, \bar{y}, \bar{\zeta}, \lambda) = \bar{n}(t, x^*, y^*, \zeta^*, \lambda) = 0$ and

$$\begin{aligned}
 |\bar{n}(t, x, y, \zeta, \lambda) - \bar{n}(t, \bar{x}, \bar{y}, \bar{\zeta}, \lambda)| &= |\bar{n}(t, x, y, \zeta, \lambda) - \bar{n}(t, x^*, y^*, \zeta^*, \lambda)| \leq \\
 &\leq 13B(\mu, \epsilon)(|x - \bar{x}| + |y - \bar{y}| + |\zeta - \bar{\zeta}|).
 \end{aligned}$$

If $(x, y, \zeta) \notin V$ and $(\bar{x}, \bar{y}, \bar{\zeta}) \notin V$, then $\bar{n}(t, x, y, \zeta, \lambda) = \bar{n}(t, \bar{x}, \bar{y}, \bar{\zeta}, \lambda) = 0$. It follows that \bar{n} is globally lipschitzian in x, y, ζ with lipschitz constant D , provided μ and ϵ are taken so small that $13B(\mu, \epsilon) < D$.

The preceding reasoning also applies to \bar{u}, \bar{s} . Consequently, the functions $\bar{n}, \bar{u}, \bar{s}$ satisfy the hypothesis (H_2^1) of Section 6 with a global lipschitz constant D which satisfies inequality (6.9). We are now in a situation where the theorems of Section 6 can be applied to the system in coordinate form, with n, u, s replaced by $\bar{n}, \bar{u}, \bar{s}$. It remains to see what these theorems imply for the

system (7.3) in the phase space C . For this we need to take into account the redundancy built in the system of coordinates introduced around the manifold M_0 .

Each point of C lying close to M_0 is represented, in the system of coordinates around M_0 which was introduced in Section 5, by a set of points (ω, x, y, ζ) which contains exactly one element with the second coordinate equal to zero. Because the integral manifold introduced in Theorem 6.2,

$$M_\lambda = \{(t, x, y, \zeta) \in \mathbb{R} \times \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L : y = h_1(t, x, \lambda), \zeta = h_2(t, x, \lambda)\},$$

is, for the system in coordinate form, the maximal integral manifold contained in sets with the y, ζ -coordinates bounded, and because the functions n, u, s and $\bar{n}, \bar{u}, \bar{s}$ agree for x, y, ζ sufficiently small, it follows that there exists a neighborhood of zero $V \subset \mathbb{R} \times \mathbb{R}^{d_N} \times \mathbb{R}^{d_U} \times L$ such that $M_\lambda \cap V$ is the maximal integral manifold contained in V . Therefore $M_\lambda \cap V$ represents in coordinate form a patch of a submanifold M_g of C which is invariant under equation (7.3). We recall that the system (7.3) is represented in coordinate form by a family of systems of the form (6.1)-(6.3), one for each $\omega \in M$ which is taken as initial condition for the solution of $\dot{u}(t) = f(u_t)$ used as center of the moving coordinate system. Based on this and on the redundancy built in the system of coordinates used, we can consider a function H defined from $M_0 \times \{g \in BC^k(C; \mathbb{R}^n) : \|g\| < \epsilon\}$ into C so that $H(\omega, g)$ is the point of C represented in coordinate form by $(\omega, 0, h_1^\omega(0, 0, g), h_2^\omega(0, 0, g))$ where $h_1 = h_1^\omega$, $h_2 = h_2^\omega$ are the functions considered above for M_λ for the particular system in coordinate form which corresponds to take the moving coordinate system centered on the solution of $\dot{u}(t) = f(u_t)$, $u_0 = \omega$. Then $M_g = \{H(\omega, g) : \omega \in M_0\}$, and its properties stated in the theorem follow directly from the theorems of Section 6 about the properties of the functions h_1, h_2 , if we recall the redundancy built in the system of coordinates, namely that changes of $h_1^\omega(0, 0, g)$, $h_2^\omega(0, 0, g)$ with ω can be identified with changes of $h_1(0, x, g)$, $h_2(0, x, g)$

with ω fixed and x changing.

The manifolds U_g and S_g can be treated in a similar way. Q.E.D.

Remark

The persistence of hyperbolic invariant manifolds for FDEs was studied above for the case of retarded FDEs on euclidean space R^q . As indicated in Section 4, the same result for retarded FDEs on a smooth, finite dimensional, separable and connected manifold follows from the result in euclidean space.

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