The Coulomb-Oscillator relation on the N-sphere

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Abstract

In this paper we establish a relation between Kepler-Coulomb and oscillator systems on the N-dimensional sphere. Using the solution of the Schrödinger equation for the oscillator system, we construct the energy spectrum and quasiradial wave function for the Kepler-Coulomb problem.

1 Introduction

In a previous article [1] we have constructed a series of mappings $S_{2C} \to S_2$, $S_{4C} \to S_3$ and $S_{8C} \to S_5$, which extend to spherical geometry the Levi-Civita, Kustaanheimo-Stiefel and Hurwitz transformations, well known for Euclidean space. We have shown that these transformations establish a correspondence between Kepler-Coulomb and oscillator problems in classical and quantum mechanics for (2,2), (3,4) and (5,8) dimensions.

In the present note we find the relation between the Schrödinger equations for Kepler-Coulomb and oscillator problems on the N-sphere.

2 The Coulomb-Oscillator relation

The Schrödinger equation describing the nonrelativistic quantum motion on the N-dimensional sphere: $s_0^2 + s_1^2 + \cdots + s_N^2 = R^2$, where s_i are Cartesian coordinates in ambient Euclidean $(N+1)$-space, has the following form ($\hbar = \mu = 1$)

$$\mathcal{H}\Psi = \left[-\frac{1}{2}\Delta_{LB} + V(\vec{s})\right]\Psi = E\Psi \tag{1}$$

where the Laplace-Beltrami operator in arbitrary curvilinear coordinate ξ_μ is

$$\Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi_\mu} g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial \xi_\nu}, \qquad g = det||g_{\mu\nu}||. \tag{2}$$

For any central potential $V(\chi)$ the Schrödinger equation admits separation of variables in hyperspherical coordinates

$$\begin{aligned}
s_0 &= R\cos\chi \\
s_1 &= R\sin\chi\cos\vartheta_1 \\
s_2 &= R\sin\chi\sin\vartheta_1\cos\vartheta_2 \\
&\cdots \\
s_{N-1} &= R\sin\chi\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{N-2}\cos\varphi \\
s_N &= R\sin\chi\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{N-2}\sin\varphi.
\end{aligned}$$

We can separate the angular part of the wave function using the ansatz

$$\Psi(\chi,\vartheta_1,\cdots\vartheta_{N-2},\varphi) = \mathcal{R}(\chi)\,Y_{Ll_1,l_2,l_{N-2}}(\vartheta_1,\cdots\vartheta_{N-2},\varphi) \tag{3}$$

where l_i are the angular hypermomenta and L is total angular momentum, and the hyperspherical function $Y_{L,l_1,l_2,l_{N-2}}(\vartheta_1,\cdots\vartheta_{N-2},\varphi)$ is the solution of the Laplace-Beltrami eigenvalue equation on the $N-1$ dimensional sphere. Then after separation of variables in (1) we obtain the quasiradial equation

$$\frac{1}{\sin^{N-1}\chi}\frac{d}{d\chi}\sin^{N-1}\chi\frac{d\mathcal{R}(\chi)}{d\chi} + \left[2R^2E - \frac{L(L+N-2)}{\sin^2\chi} - 2R^2V(\chi)\right]\mathcal{R}(\chi) = 0. \tag{4}$$

Using the substitution

$$Z(\chi) = (\sin\chi)^{\frac{N-1}{2}}\mathcal{R}(\chi) \tag{5}$$

we find

$$\frac{d^2Z}{d\chi^2} + \left[\tilde{E} - \frac{(2L+N-1)(2L+N-3)}{4\sin^2\chi} - 2R^2V(\chi)\right] Z = 0 \tag{6}$$

where $\tilde{E} = 2R^2E + (N-1)^2/4$ and the quasiradial wave function $Z(\chi)$ satisfies the normalization condition

$$\int_0^\pi Z(\chi)Z^*(\chi)\,R^N\,d\chi = 1. \tag{7}$$

Let us consider the oscillator potential [2, 3]

$$V(\chi) = \frac{\omega^2R^2}{2}\frac{s_1^2+s_2^2+.....s_N^2}{s_0^2} = \frac{\omega^2R^2}{2}\tan^2\chi. \tag{8}$$

For equation (6) we have

$$\frac{d^2Z}{d\chi^2} + \left[\epsilon - \frac{\nu^2-\frac{1}{4}}{\cos^2\chi} - \frac{(L+\frac{N-2}{2})^2-\frac{1}{4}}{\sin^2\chi}\right] Z = 0 \tag{9}$$

where $\nu = \sqrt{\omega^2R^4+1/4}$ and $\epsilon = \tilde{E} + \omega^2R^4$. The solution of the above equation, regular for $\theta \in [0,\pi]$ and expressed in terms of the hypergeometric function, is [4]

$$Z_{n_rL}(\theta) = \frac{\sqrt{2}}{\Gamma(L+\frac{N}{2})}\sqrt{\frac{(2n_r+L+\nu+\frac{N}{2})\Gamma(n_r+L+\nu+\frac{N}{2})\Gamma(n_r+L+\frac{N}{2})}{R^N\,\Gamma(n_r+\nu+1)(n_r)!}}$$

$$(\sin\chi)^{L+\frac{N-1}{2}}\,(\cos\chi)^{\nu+\frac{1}{2}}\,{}_2F_1(-n_r, n_r+L+\nu+\frac{N}{2};\,L+\frac{N}{2};\,\sin^2\chi). \tag{10}$$

The energy spectrum is

$$\epsilon = (2n_r + L + \nu + \frac{N}{2})^2 \tag{11}$$

where $n_r = 0, 1, 2, ...$ is a quasiradial quantum number.

The potential, which is the analogue of the the Kepler-Coulomb potential on the N-dimensional sphere, has the following form [2, 3]:

$$V(\chi) = -\frac{\alpha}{R}\frac{s_0}{\sqrt{s_1^2 + s_2^2 + ...s_N^2}} = -\frac{\alpha}{R}\cot\chi. \tag{12}$$

The Schrödinger equation for this potential is

$$\frac{d^2Z}{d\chi^2} + \left[\tilde{E} - \frac{(2L+N-1)(2L+N-3)}{4\sin^2\chi} + 2\alpha R\cot\chi\right] Z = 0. \tag{13}$$

We make now a transformation to the new variable $\theta \in [0, \frac{\pi}{2}]$

$$e^{i\chi} = \cos\theta, \tag{14}$$

which is possible if we continue the variable χ in the complex domain $\{$ Re $\chi = 0,\ 0 \leq$ Im $\chi < \infty\}$. We complexify also the coupling constant α by putting $k = i\alpha$ such that

$$\alpha\cot\chi = k(1 - 2\sin^{-2}\theta). \tag{15}$$

As result we obtain the equation

$$\frac{d^2W}{d\theta^2} + \left[(\tilde{E} + 2kR) - \frac{(\tilde{E} - 2kR) - \frac{1}{4}}{\cos^2\theta} - \frac{(2L+N-2)^2 - \frac{1}{4}}{\sin^2\theta}\right] W = 0 \tag{16}$$

where $W(\theta) = \sqrt{\cot\theta}Z(\theta)$. From this equation we see that up to the substitution

$$\epsilon = \tilde{E} + 2kR, \qquad \nu^2 = \tilde{E} - 2kR, \tag{17}$$

and transformation $L \to 2L$, the quasiradial equation (16) for the N-dimensional Coulomb problem coincides with the D-dimensional quasiradial oscillator equation (9) (which really is a Pöschl-Teller type equation) for dimensions $D = 2(N-1)$. This means that relations between these two systems are possible only for oscillators in even dimensions: $D = 2, 4, 6, 8....$

Comparing now the eqs. (11) with (17) we obtain the energy spectrum for the Coulomb problem described by the formula

$$E_n = \frac{n(n+N-1)}{2R^2} - \frac{\alpha^2}{2(n+\frac{N-1}{2})^2}, \qquad n = n_r + L = 0, 1, 2, ... \tag{18}$$

Returning to the variable χ, we see that the Coulomb quasiradial wave function has the form

$$Z_{n_rL}(\chi) = C_{n_rL}(\sigma)\,(\sin\chi)^{L+\frac{N-1}{2}}\ \exp[-i\chi(n - L - i\sigma)]$$
$$_2F_1(-n_r, L + \frac{N-1}{2} + i\sigma; 2L + N - 1; 1 - e^{2i\chi}), \tag{19}$$

where

$$\sigma = \frac{\alpha R}{n + \frac{N-1}{2}}$$

and the normalization constant $C_{n_r L}(\sigma)$ is

$$C_{n_r L}(\sigma) = 2^{L+\frac{N-1}{2}}\, e^{\frac{\pi\sigma}{2}}\, \frac{\mid \Gamma(L + \frac{N-1}{2} - i\sigma) \mid}{\Gamma(2L + N - 1)} \sqrt{\frac{[(n + \frac{N-1}{2})^2 + \sigma^2](n + L + N - 2)!}{2R^N \pi (n + \frac{N-1}{2})(n - L)!}}. \quad (20)$$

Thus by using the relation between Kepler-Coulomb and oscillator systems we have constructed the quasiradial wave functions and energy spectrum for a Kepler-Coulomb system on the N-dimensional sphere.

Finally, note that in the contraction limit $R \to \infty$ it is easy to recover the well known formulas for the flat space Kepler-Coulomb problem.

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