FURTHER ANALYSIS OF THE LIPPMANN-SCHWINGER EQUATION

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Abstract. In this article, we investigate the Lippmann-Schwinger equation for weak solutions via the Fredholm Alternative Theorem. We use estimates of operator norms other than the Hilbert-Schmidt norm, thereby enlarging the class of potentials with which we may analyze such solutions.

Introduction

In this article we will investigate the solvability of the so-called Modified Lippmann-Schwinger equation for certain classes of potentials; that is, we seek solutions ψ of the equation

$$\begin{aligned} \psi(x,\kappa) =& |V(x)|^{1/2} e^{i\kappa\cdot x} \\ & - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|V(x)|^{1/2} e^{i|\kappa||x-y|} V(y)^{1/2}}{|x-y|} \psi(y,\kappa)\, dy. \end{aligned} \tag{0.1}$$

with x, y and $\kappa \in \mathbb{R}^3$, and $V^{1/2} \stackrel{\text{def}}{=} |V|^{1/2}(\text{sgn}V)$ for certain classes of real-valued functions V defined on $\mathbb{R}^3$. The Lippmann-Schwinger equation arises in the study of Møller operators and in the computation of certain eigenfunction expansions of the operator $H = H_o + V = -\Delta + V$ on $L^2(\mathbb{R}^3)$ [P][Th][KK]. We want to know for which classes of potentials V and for which κ does equation (0.1) have an unique solution $\psi(x,\kappa) \in L^2(\mathbb{R}^3)$. It is known [RS][GW1][GW2][N] that (0.1) has an unique solution as such for each $\kappa \in \mathbb{R}^3$, except possibly those of a set of Lebesgue measure 0, provided $V \in L^1(\mathbb{R}^3)$

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satisfies

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\,dx\,dy < \infty. \tag{0.2}$$

We outline some key ingredients to the proof that $L^2(\mathbb{R}^3)$ solutions to (0.1) exist for the aforementioned types of potentials for almost every κ when V satisfies (0.2): Denote by $A_{|\kappa|}$ the integral operator defined by

$$A_{|\kappa|}\psi(x,\kappa) = -\frac{1}{4\pi}\int \frac{|V(x)|^{1/2}e^{i|\kappa||x-y|}V(y)^{1/2}}{|x-y|}\psi(y,\kappa)d\,y. \tag{0.3}$$

If V satisfies (0.2) then $A_{|\kappa|} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is a bounded operator. Indeed, it is an Hilbert-Schmidt operator and is, hence, compact. After rearrangement equation (0.1) can be written as

$$(I - A_{|\kappa|})\psi(x,\kappa) = |V(x)|^{1/2}e^{i\kappa\cdot x} \tag{0.4}$$

where I denotes the identity operator on $L^2(\mathbb{R}^3)$. The result then follows via the analytic Fredholm theorem. (See Theorem VI.41 of [RS].)

The condition (0.2) on V is satisfied if $V \in L^1_{\text{loc}}(\mathbb{R}^3)$ and $V(x) = O(e^{-\alpha|x|})$ as $|x| \to \infty$ for some positive α [GW1] [GW2]. Moreover, by Sobolev's inequality, this condition is also satisfied if $V(x) \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ [RS]. However, for some potentials the operators $A_{|\kappa|}$ may not be of Hilbert-Schmidt class yet may be bounded - even compact. Indeed, using estimates from [F] (see also [R]), we demonstrate locally bounded V for which the operator $A_{|\kappa|}$ is not Hilbert-Schmidt for any κ yet is compact for all κ.

1. Compactness of $A_{|\kappa|}$ for Some Bounded Potentials

A measurable function $V(x)$ defined on $\mathbb{R}^3$ is of Rollnik class [RS] [Si] if

$$||V||^2_{\text{Rollnik}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\,dx\,dy < \infty. \tag{1.1}$$

and, using an operator norm from [F], we will say V is of class $\mathrm{cl}(2\beta)$ if

$$||V||_{2\beta}^2 \stackrel{\text{def}}{=} \sup_{z\in\mathbb{R}^3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)|^\beta |V(y)||V(z)|^{1-\beta}}{|x-y|^{2\beta}|y-z|^{2-2\beta}}\,dx\,dy < \infty \tag{1.2}$$

with $0 < \beta \le 1$. We note that the Rollnik class is the same as the class $\mathrm{cl}(2)$. Furthermore, for an operator T on L^2 given by

$$T\phi(x) = \int K(x,y)\phi(y)\,dy$$

with integral kernel K, given $0 < \beta \le 1$, we introduce the norm

$$||T||_{2\beta}^2 \stackrel{\text{def}}{=} \sup_z \int |K(x,y)|^{2\beta}|K(y,z)|^{2-2\beta}\,dx\,dy. \tag{1.3}$$

T will be said to be 2β - bounded if (1.3) is finite. (So, a measureable function V is of class $\mathrm{cl}(2\beta)$ if and only if the associated operator $A_{|\kappa|}$ is of 2β-bounded.) Note also that $||T||_{\mathrm{HS}} = ||T||_2$. It follows from (20.14) of [F] that integrals (1.3) produce upper bounds on the $L^2(\mathbb{R}^3)$ operator norms of integral operators T since

$$||T|| \le ||T||_{2\beta}.$$

In the case $\beta = 1$ we have

$$||T|| \le ||T||_{\mathrm{HS}}$$

where $||\ ||_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. Finally, we denote by $||T||_{\mathrm{Hol}}$ the Holmgren norm of an integral operator T which is defined by

$$||T||_{\mathrm{Hol}} \stackrel{\text{def}}{=} \sup_z \int |K(x,z)|\,dx.$$

We will also denote for a positive number a

$$||T||_{\mathrm{Hol},a} \stackrel{\text{def}}{=} \sup_z \int |K(x,z)|^a\,dx.$$

It is easy to show that

$$||T||_{2\beta}^2 \le ||T||_{\mathrm{Hol},2\beta} \cdot ||T||_{\mathrm{Hol},2-2\beta}. \tag{1.4}$$

For bounded V, we find a range of p for which $V \in L^p(\mathbb{R}^3) \implies \exists\ 0 < \beta < 1$ such that $V \in \mathrm{cl}(2\beta)$:

Proposition 1.5. *Suppose V is a bounded measurable function such that $V \in L^p(\mathbb{R}^3)$ for some $p < \frac{5+\sqrt{131}}{16}$. Then, there are positive numbers, $\beta_1(p)$ and $\beta_2(p)$, such that $V \in cl(2\beta)$ for all $\beta_1(p) < \beta < \beta_2(p)$.*

Proof: Since V is bounded,

$$||V||_{2\beta}^2 \leq \sup_{z\in\mathbb{R}^3} |V(z)|^{1-\beta} \sup_{z\in\mathbb{R}^3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)|^{\beta}|V(y)|}{|x-y|^{2\beta}|y-z|^{2-2\beta}}\, dx\, dy$$

Write

$$\int \frac{|V(x)|^{\beta}}{|x-y|^{2\beta}}\, dx = \int_{|x-y|<1} \frac{|V(x)|^{\beta}}{|x-y|^{2\beta}}\, dx + \int_{|x-y|\geq 1} \frac{|V(x)|^{\beta}}{|x-y|^{2\beta}}\, dx.$$

We have

$$\int_{|x-y|<1} \frac{|V(x)|^{\beta}}{|x-y|^{2\beta}}\, dx \leq \sup_{x\in\mathbb{R}^3} |V(x)|^{\beta} \int_{|x-y|<1} |x-y|^{-2\beta}\, dx \tag{1.6}$$

and, using Hölder's inequality we have for appropriate $\beta < p < \infty$,

$$\int_{|x-y|\geq 1} \frac{|V(x)|^{\beta}}{|x-y|^{2\beta}}\, dx \leq \tag{1.7}$$

$$\Big[\int_{|x-y|\geq 1} |V(x)|^p\, dx\Big]^{\beta/p} \Big[\int_{|x-y|\geq 1} |x-y|^{\frac{-2\beta p}{p-\beta}}\, dx\Big]^{1-\beta/p}.$$

Likewise,

$$\int \frac{|V(y)|}{|y-z|^{2-2\beta}}\, dy = \int_{|y-z|<1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\, dy + \int_{|y-z|\geq 1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\, dy$$

with

$$\int_{|y-z|<1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\, dy \leq \sup_{y\in\mathbb{R}^3} |V(y)| \int_{|y-z|<1} |y-z|^{2\beta-2}\, dy \tag{1.8}$$

and, for appropriate $q > 1$

$$\int_{|y-z|\geq 1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\, dy \leq \tag{1.9}$$

$$\Big[\int_{|y-z|\geq 1} |V(y)|^q\, dy\Big]^{1/q} \Big[\int_{|y-z|\geq 1} |y-z|^{\frac{(2\beta-2)q}{q-1}}\, dy\Big]^{1-1/q}.$$

and for $q \leq 1$, then (1.9) is bounded for any $0 < \beta \leq 1$ if $V \in L^1(\mathbb{R}^3)$.

We now determine for which p and β are the quantities (1.6) - (1.9) finite for bounded functions $V \in L^p(\mathbb{R}^3)$. For any positive R

$$\int_{|x-y|<R} |x-y|^{-r}\, dx < \infty \tag{1.10}$$

for $r < 3$ and

$$\int_{|x-y|\geq R} |x-y|^{-r}\, dx < \infty \tag{1.11}$$

for $r > 3$. Now, $2\beta < 2$ and $2-2\beta < 2$ for $0 < \beta < 1$ so that (1.6) and (1.8) are finite for any p and q respectively. Moreover, from (1.10) and (1.11) quantities (1.7) and (1.9) are both finite provided that

$$\frac{2\beta p}{p-\beta} > 3 \tag{1.12}$$

for $p > \beta$ and

$$\frac{(2-2\beta)q}{q-1} > 3 \tag{1.13}$$

for $q > 1$. Simultaneous inequalities (1.12) and (1.13) give

$$\frac{3p}{2p+3} < \beta < \max_q[\min\{\frac{3-q}{2q}, q\}] \tag{1.14}$$

where the maximimum is taken over those q for which $V \in L^q(\mathbb{R}^3)$. Yet, since V is bounded, $V \in L^p(\mathbb{R})$ for $p < 1$ implies $V \in L^1(\mathbb{R})$ and, hence, the statement of the proposition holds for

$$\beta_1(p) \stackrel{\text{def}}{=} \frac{3p}{2p+3}$$

and

$$\beta_2(p) \stackrel{\text{def}}{=} \begin{cases} \frac{3-p}{2p} & : 1 < p < \frac{5+\sqrt{131}}{16} \\ 1 & : 0 < p \leq 1 \end{cases}$$

□

Remark 1.15. We note that in the case when V is bounded, an estimate on Riesz potentials [St] show that $A_{|\kappa|}$ is also bounded as a transformation from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ for $\frac{1}{q} = \frac{1}{p} - \frac{2}{3}$ where $1 < p < 3/2$.

To investigate the compactness of the operators $A_{|\kappa|}$ for possibly unbounded V, we will use the following

Lemma 1.16. *Suppose that* $V \in cl(2\beta) \bigcap L^2$ *for some* $0 < \beta < 1$. *Then, the associated operator* $A_{|\kappa|}$ *is compact.*

Proof: Let $0 \leq g_{_R}(x) \leq 1$ be the function defined by

$$g_{_R}(x) = \begin{cases} 1 & : |x| \leq R \\ 0 & : |x| > R. \end{cases}$$

Then, for each $R > 0$, we define the operators $A_{|\kappa|,R}$ by

$$4\pi A_{|\kappa|,R}\phi(x) = \int \frac{e^{i|\kappa||x-y|}|V(x)|^{1/2}V(y)^{1/2}}{|x-y|} g_{_R}(x-y)\phi(y)\,dy.$$

Now, using changes variables $u = y - x$ and $r = |u|$, we obtain

$$\begin{aligned}
&\int \frac{|V(x)||V(y)|}{|x-y|^2} g_{_R}(x-y)\,dy\,dx \\
&\leq \int_{\mathbb{R}^3}\int_{|u|\leq R} \frac{|V(x)||V(u+x)|}{|u|^2}\,du\,dx \\
&\leq \int_{\mathbb{R}^3}\int_{S^2}\int_0^R |V(x)||V(x+r\omega)|\,dr\,d\omega\,dx.
\end{aligned}$$

Since

$$|V(x)||V(x+r\omega)| \leq \frac{(V(x))^2 + (V(x+r\omega))^2}{2},$$

we have for each $r > 0$ and for each $\omega \in S^2$,

$$\int |V(x)||V(x+r\omega)|\,dx \leq ||V||^2_{_{L^2}}.$$

So, by the Fubini-Tonelli Theorem $\forall R > 0$,

$$||A_{|\kappa|,R}||_{\text{HS}} \leq ||V||_{_{L^2}}\sqrt{R}.$$

Therefore, for each $R > 0$ $A_{|\kappa|,R}$ is of Hilbert-Schmidt class and is, hence, compact. Clearly, $||A_{|\kappa|} - A_{|\kappa|,R}||_{2\beta} \leq ||A_{|\kappa|}||_{2\beta}$ so that by the Lebesgue Dominated Convergence Theorem

$$\lim_{R\to\infty} ||A_{|\kappa|} - A_{|\kappa|,R}||_{2\beta} = 0$$

and, hence,

$$\lim_{R\to\infty} ||A_{|\kappa|} - A_{|\kappa|,R}||_{L^2} = 0.$$

This shows that $A_{|\kappa|}$ is the strong-L^2 limit of compact operators which proves that it is compact. □

We now provide a necessary condition for bounded central potentials to be class cl(2β): V is said to be a central potential if there is a function $\mathcal{V}$ defined on $\mathbb{R}^+$ such that $V(x) = \mathcal{V}(|x|)$. For $r = |x|$ we state the following

Proposition 1.17. *: A bounded central potential $V \in$ cl(β) only if the associated function $\mathcal{V}$ satisfies*

$$\mathcal{V}(r) \in L^1(\mathbb{R}^+;\, dr) \cap L^\beta(\mathbb{R}^+; r^{2-2\beta}\, dr)$$

Proof: For each z, we use the Fubini-Tonelli Theorem and a change of coordinates to obtain

$$\begin{aligned} ||V||_{2\beta}^2 &\geq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)|^\beta |V(y)||V(z)|^{1-\beta}}{|x-y|^{2\beta}|y-z|^{2-2\beta}}\, dx\, dy \\ &\geq 4\pi |V(z)|^{1-\beta} \int_0^\infty \frac{|\mathcal{V}(r)|^\beta r^2}{(R+r)^{2\beta}}\, dr \int_{|y|<R} \frac{|V(y)|}{|y-z|^{2-2\beta}}\, dy \end{aligned} \tag{1.18}$$

Likewise,

$$\begin{aligned} ||V||_{2\beta}^2 &\geq 4\pi \int_{\mathbb{R}^3} \frac{|V(z)|^{1-\beta}|V(y)|\, dy}{(R+|y|)^{2\beta}(|z|+|y|)^{2-2\beta}} \int_{|x|<R} |V(x)|^\beta\, dx \\ &= 4\pi |V(z)|^{1-\beta} ||V||_{L^1} \int_0^\infty \frac{\mathcal{V}(r) r^2}{(R+r)^{2\beta}(|z|+r)^{2-2\beta}}\, dr \end{aligned} \tag{1.19}$$

Choosing z not a root of V and sufficiently large R, it is clear that since $\mathcal{V}$ is bounded (1.18) is finite only if $\mathcal{V}(r) \in L^\beta(\mathbb{R}^+; r^{2-2\beta}\, dr)$ and that (1.19) is finite only if $\mathcal{V}(r) \in L^1(\mathbb{R}^+;\, dr)$ □

To motivate the results of the next section, we demonstrate that for a simple class of bounded potentials, $V \in$ cl(2β) implies V is Rollnik. Indeed, for potentials

$$V_\gamma(x) \stackrel{\text{def}}{=} (1+|x|)^{-\gamma}$$

we have the following

Proposition 1.20. *: For $\gamma > 3$, $V_\gamma(x) \in cl(2\beta)$ if and only if $\gamma > \frac{3}{\beta} - 1$. Hence, $V_\gamma \in cl(\beta)$ for each $\beta > \frac{3}{\gamma+1}$.*

Proof:

For $0 < \beta \leq 1$ $V_\gamma \in L^p$ if and only if $\gamma > 3/p$ So, by Proposition 1.1 $V_\gamma \in \mathrm{cl}(2\beta)$ for

$$\gamma > \max\{\frac{3}{\beta} - 2, 1 + 2\beta\}.$$

Yet, by hypothesis $\gamma > 2\beta + 1$ for any $0 < \beta \leq 1$ so that for $V \in \mathrm{cl}(2\beta)$ it suffices that $\beta > \frac{3}{\beta} - 1$.

Since $\mathcal{V} \in L^1(\mathbb{R}^+; dr)$ for $\gamma > 3$, by Proposition 1.17 $V_\gamma \in \mathrm{cl}(2\beta)$ only if $\mathcal{V} \in L^\beta(\mathbb{R}^+; dr)$ and, hence, $-\gamma\beta + 2 - 2\beta < -1$ which gives $\gamma > \frac{3}{\beta} - 1$ and the proof is complete. □

Remark 1.21. The associated operator $A_{|\kappa|}$ is already known to be compact, indeed Hilbert-Schmidt, for V_γ as in Proposition 1.20 via Sobolev's Inquality.

Remark 1.22. It follows immediately from Proposition 1.17 that for $\gamma \leq 3$ $V_\gamma \notin \mathrm{cl}(\beta)$ for any $0 < \beta \leq 1$. However, the operators $A_{|\kappa|}$ are known to be bounded for $\gamma > 1$ [Ka][Ku].

2. Compactness of $A_{|\kappa|}$ for Some Unbounded Potentials

We consider another class of potentials V which includes unbounded functions and we determine which of these are of class $\mathrm{cl}(2\beta)$ for some β. We will consider functions that are supported on $\bigcup_{k=1}^\infty E_k$ for certain Lebesgue measurable sets E_k which have the following properties:

(i) The sets E_k are disjoint and the distance $\mathrm{dist}(E_k, E_l)$ between two such sets E_k and E_l satisfies

$$c_1|k - l| \leq \mathrm{dist}(E_k, E_l) \leq c_2|k - l|$$

for some positive constants c_1 and c_2 independant of k and l.

(ii) There are positive constants C_1, C_2, and b such that for each k the Lebesgue measure $\mu(E_k)$ of E_k satisfies

$$C_1 k^{-b} \leq \mu(E_k) \leq C_2 k^{-b}.$$

(iii) For every $1/2 \leq \beta' < 1$ there is a positive constant $C_{\beta'}$, depending on β', such that for each k

$$\int_{E_k} |x-y|^{-2\beta'}\,dx \leq C_{\beta'}\mu(E_k)$$

uniformly for $y \in E_k$.

(iv) There is a positive constant D such that the diameter, $\operatorname{diam}(E_k)$, of each E_k satisfies $\operatorname{diam}(E_k) \leq D$.

Given such sets E_k and constants $a \geq 0$ and $b > 0$ as above, we define the functions

$$V_{a,b}(x) = \sum_{k=1}^{\infty} \chi_k(x) k^a$$

where χ_k is the chacteristic function of the set E_k.

As a simple example, we note that the sets

$$E_k = \{(x_1,x_2,x_3) : \sqrt{x_2^2+x_3^2} < 1, k < x_1 < k + \frac{1}{2k^b}\}$$

satisfy the above criterion.

We procede to determine for which parameters a and b and for which β is $V_{a,b}$ of class $\operatorname{cl}(2\beta)$, but first we intoduce some notation: Given two functions f and g, the expression

$$f \lesssim g \tag{2.1}$$

means that there is a positive constant c such that $|g| \leq c|f|$ uniformly on the domain of both f and g. We use the following estimates:

Proposition 2.2. *Given constants $0 < \beta < 1$, and $\alpha < 2\beta - 1$ we have the various estimates for $l \in \mathbb{Z}^+$:*

$$\sum_{\substack{k>0\\k\neq l}} \frac{k^\alpha}{|k-l|^{2\beta}} \lesssim \begin{cases} l^{\alpha-2\beta+1} & : 0<\beta<1/2 \\ l^\alpha \ln l & : \beta = 1/2 \\ l^\alpha & : 1/2 < \beta < 1 \end{cases} \qquad (l \to \infty).$$

In each of the above estimates the constants as in (2.1) depend only on α.

We note that these sums diverge for every l when $\alpha \geq 2\beta - 1$.
Proof : From standard sum and integral estimates along with a change of variables we find for $0 < \beta < 1$

$$\sum_{\substack{k>0 \\ k \neq l}} \frac{k^\alpha}{|k-l|^{2\beta}} \lesssim \int_1^\infty \frac{(t+l)^\alpha}{|t|^{2\beta}}\, dt$$

$$= l^{\alpha+1-2\beta} \int_{\frac{1}{l}}^\infty \frac{(w+1)^\alpha}{w^{2\beta}}\, dw$$

$$\lesssim l^{\alpha+1-2\beta}\Big[\int_{\frac{1}{l}}^1 w^{-2\beta}\, dw + \int_1^\infty w^{\alpha-2\beta}\, dw\Big] \tag{2.3}$$

The second integral of (2.3) is finite for $\alpha < 2\beta - 1$. Yet,

$$\int_{\frac{1}{l}}^1 w^{-2\beta}\, dw \lesssim \begin{cases} 1 & : 0 < \beta < 1/2 \\ \ln l & : \beta = 1/2 \\ l^{2\beta-1} & : 1/2 < \beta < 1 \end{cases} \qquad (l \to \infty).$$

The combined estimates prove the claim. ☐

Now, supposing that $z \in E_l$ for some l (for otherwise $V(z) = 0$) we have

$$|V_{a,b}(x)|^\beta = \sum_{k=1}^\infty \chi_k(x) k^{a\beta}$$

so that for $k \neq l$

$$\int \chi_k(x) \frac{k^{a\beta}}{|x-z|^{2\beta}}\, dx \leq d_1 \cdot \frac{k^{a\beta}\mu(E_k)}{|k-l|^{2\beta}}$$

$$\lesssim \frac{k^{a\beta-b}}{|k-l|^{2\beta}} \tag{2.4}$$

and that for $k = l$

$$\int \chi_l(x) \frac{l^{a\beta}}{|x-z|^{2\beta}}\, dx \leq d_2 \cdot l^{a\beta}\mu(E_l)$$

$$\lesssim l^{a\beta-b} \tag{2.5}$$

for some positive constants d_1 and d_2, depending on β but independent of l.

Now, we apply estimates (2.4) and (2.5) along with Proposition 2.2 to estimate $||A_{|\kappa|}||_{Hol,\beta}$ and $A_{|\kappa|}||_{Hol,1-\beta}$ for $0<\beta\leq 1/2$; estimates for $1/2<\beta<1$ are thereby immediate.

We have the following estimates uniform for $z\in E_l$ and for $l\in\mathbb{Z}^+$: For $0<\beta<1/2$ and $a\beta-b\ -2\beta<-1$

$$\begin{aligned}\int\frac{V_{a,b}^{\beta}(x)V_{a,b}^{\beta}(z)}{|x-z|^{2\beta}}\,dx &\lesssim l^{a\beta-b}\cdot l^{a\beta}+\sum_{\substack{k>0\\k\neq l}}\frac{k^{a\beta-b}}{|k-l|^{2\beta}}l^{a\beta}\\ &\lesssim l^{2a\beta-b}+l^{1+a\beta-b-2\beta}\cdot l^{a\beta}\\ &\lesssim l^{2a\beta-b+1-2\beta};\end{aligned}$$

For $a(1-\beta)-b<-1$ (noting that $1/2<1-\beta<1$)

$$\begin{aligned}\int\frac{V_{a,b}^{1-\beta}(x)V_{a,b}^{1-\beta}(z)}{|x-z|^{2-2\beta}}\,dx &\lesssim l^{a(1-\beta)-b}\cdot l^{a(1-\beta)}+\sum_{\substack{k>0\\k\neq l}}\frac{k^{a(1-\beta)-b}}{|k-l|^{2(1-\beta)}}l^{a(1-\beta)}\\ &\lesssim l^{2a(1-\beta)-b}+l^{a(1-\beta)-b}\cdot l^{a(1-\beta)}\\ &\lesssim l^{2a(1-\beta)-b}\end{aligned}$$

Finally, for $\beta=1/2$ and $a/2-b<-1$

$$\begin{aligned}\int\frac{V_{a,b}^{1/2}(x)V_{a,b}^{1/2}(z)}{|x-z|}\,dx &\lesssim l^{a/2-b}\cdot l^{a/2}+\sum_{\substack{k>0\\k\neq l}}\frac{k^{a/2-b}}{|k-l|}l^{a/2}\\ &\lesssim l^{a-b}+l^{a/2-b}\cdot l^{a/2}\cdot\ln l\\ &\lesssim l^{a-b}\ln l.\end{aligned}$$

We now define $A_{|\kappa|}$ to be the associated operator with $V=V_{a,b}$. Therefore, considering the supremum in l of the above estimates over $\mathbb{Z}^+$, we see that for $0<\beta<1/2$ the norms $||A_{|\kappa|}||_{\text{Holm},1-\beta}$ and $||A_{|\kappa|}||_{\text{Holm},\beta}$ are both finite if $2a\beta-b+1-2\beta\leq 0$ and $2a(1-\beta)-b\leq 0$ and that $||A_{|\kappa|}||_{\text{Holm},1/2}$ is finite if $a-b<0$.

In the same fashion in which we define $A_{|\kappa|}$, for $m=1,2,\ldots$ we likewise define the operators B_m and T_m but with $V=\sum_{k=1}^m\chi_k(x)k^a$

and with $V = \sum_{k=m+1}^{\infty} \chi_k(x)k^a$, respectively. Now, to proof the main result of this section, we will use the following

Lemma 2.6. *Suppose that for some* $0 < \beta \leq 1/2$*, the operators* T_m *satisfies both*

$$\lim_{m\to\infty} ||T_m||_{\text{Holm},\beta} = 0 \tag{2.7}$$

$$\lim_{m\to\infty} ||T_m||_{\text{Holm},1-\beta} = 0. \tag{2.8}$$

Then, $T_0 = A_{|\kappa|}$ *is a compact operator on* $L^2(\mathbb{R}^3)$*.*

Proof : By Lemma 1.16 the operators B_m are compact and by inequality (1.4) our hypothesis gives

$$\lim_{m\to\infty} ||B_m - A_{|\kappa|}||_{2\beta} = 0$$

and, hence,

$$\lim_{m\to\infty} ||B_m - A_{|\kappa|}||_{L^2} = 0$$

which proves the claim. □

We note that although the conclusion of the above lemma also follows if one merely requires that the product $||T_m||_{\text{Holm},\beta}\ ||T_m||_{\text{Holm},1-\beta}$ tends to zero as $m \to \infty$, the present formulatation is best suited for the applications which follow.

Theorem 1. *Given* $0 < \beta < 1$ *there are functions* $V_{a,b}$ *which are of class* $cl(2\beta)$ *but are not of Rollnik class. Indeed, for each such* β *numbers* $a \geq 0$ *and* $b > 0$ *may be chosen so that the associated operator* A_κ *is compact but not Hilbert-Schmidt.*

Proof: First we will show that for any $a \geq 0$ and $b > 0$ for which $a - b > -1/2$ the function $V_{a,b}$ is not of Rollnik class. We note that for D as in property *iv*) for each $y \in E_k$ the set

$$\{u + y : |u| \leq D, y \in E_k\}$$

contains E_k. So,

$$\begin{aligned}
\int\int \frac{|V_{a,b}(x)||V_{a,b}(y)|}{|x-y|^2}\,dx\,dy &= \int\int \frac{|V_{a,b}(u+y)||V_{a,b}(y)|}{|u|^2}\,du\,dy \\
&\geq \int \int_{|u|\leq D} \frac{|V_{a,b}(u+y)||V_{a,b}(y)|}{|u|^2}\,du\,dy \\
&\geq \int \int_{|u|\leq D} \frac{\sum_{k,l\geq 1} \chi_k(u+y)\chi_l(y)(kl)^a}{D^2}\,du\,dy \\
&\geq \frac{1}{D^2}\int \frac{\sum_{k\geq 1} \chi_k(u+y)\chi_k(y)(k)^{2a}}{D^2}\,du\,dy \\
&\geq \frac{1}{D^2}\sum_{k=1}^{\infty}(\mu(E_k))^2 k^{2a} \\
&= \frac{1}{D^2}\sum_{k=1}^{\infty} k^{2(a-b)}
\end{aligned}$$

Now, to find non-Rollnik functions $V_{a,b}$ for which Lemma 1 applies, for $0 < \beta \leq 1/2$ we seek pairs of non-negative numbers a and b which satisfy the following simultaneous inequalities:

$$\begin{aligned}
2a\beta - b + 1 - 2\beta &< 0 \\
2a(1-\beta) - b &< 0 \\
a - b &> -\frac{1}{2}.
\end{aligned}$$

We have the following solutions:

1.) For $1 - \frac{1/2}{1-2\beta} < a < 1/2$ and $0 < \beta \leq \frac{1}{4}$ we have

$$2\beta(a-1) + 1 < b < a + 1/2.$$

2.) For $0 < a < 1/2$, we have

$$2\beta(a-1) + 1 < b < a + 1/2,$$

provided $\frac{1}{4} \geq \beta < 1/2$.

3.) For $a = \frac{1}{2}$ and $0 < \beta < 1/2$ we have

$$1 - \beta < b < 1;$$

4.) And, for $\frac{1}{2} < a < \frac{1/2}{1-2\beta}$ and $0 < \beta < 1/2$ we have

$$2a(1 - \beta) < b < a + \frac{1}{2}.$$

5.) Finally, for $\beta = 1/2$, we have for any $a \geq 1/2$,

$$a - \frac{1}{2} < b < a + \frac{1}{2}.$$

□

Remark 2.9. We note that for a and b as above, $V_{a,b} \notin L^1$.

Remark 2.10. In case 5.), above, the associated operator $A_{|kappa|}$ is bound-ed in Holmgren norm but not in Hilbert-Schmidt norm.

3. Weak Solutions to the Lippmann-Schwinger Equation

In this section we will analyze solutions to equation (0.1) for potentials V as in Theorem 1 as limits of the form $(A_{|\kappa|} + I)^{-1}(f_m)$ for appropriate functions f_m. First, we will establish the invertibility of the operator $A_{|\kappa|} + I$ in $L^2(\mathbb{R}^3)$. The proof of the following result follows closely part II of the proof of Theorem XI.41 from [RS].

Theorem 2. *The operator $A_{|\kappa|} + I$ is invertible on $L^2(\mathbb{R}^3)$ for all $|\kappa|$ except on a set $\mathcal{E}$ of Lebesgue measure zero.*

Proof: Define for complex numbers λ the operator A_λ to be the same as (0.3) but with $|\kappa|$ replaced by λ. From the estimate (20.8) of [F] we find

$$(4\pi)||A_\lambda||^2 \leq$$
$$\sup_{z\in\mathbb{R}^3} \int_{\mathbb{R}^6} \frac{|V(x)|^\beta e^{-2\beta \mathrm{Im}\lambda|x-y|}|V(y)||V(z)|^{1-\beta}e^{(-2+2\beta)\mathrm{Im}\lambda|y-z|}}{|x-y|^{2\beta}|y-z|^{2-2\beta}}\, dx\, dy$$
$$\leq ||V||_{2\beta}^2$$

So, by Fubini's Theorem and Morera's Theorem A_λ is an analytic, operator-valued function in the upper half-plane $\mathrm{Im}\lambda > 0$. (See the first two paragraphs of section 4 in [GW1] for details.) Furthermore, since for any real numbers λ_1 and λ_2

$$||A_{\lambda_1} - A_{\lambda_2}||_{2\beta} \leq 2||A_0||_{2\beta}$$

we have, by the Lebesgue Dominated Convergence Theorem and the Mean Value Theorem that $||A_\lambda||_{2\beta}$ is continuous for λ on the real axis, $\mathrm{Im}\lambda = 0$, and, hence, so is $||A_\lambda||_{L^2}$. Similarly, one can show that

$$\lim_{\mathrm{Im}\lambda \to +\infty} ||A_\lambda||_{L^2} = 0$$

where the limit is independent of $\mathrm{Re}\lambda$. Therefore, there is a positive number γ_o for which $(A_\lambda + I)^{-1}$ is analytic whenever $\mathrm{Im}\lambda > \gamma_o$. Now, the main result of the theorem follows from a version of the analytic Fredholm theorem (see Proposition of page 101 in [RS] and the two paragraphs which follow) wherefore the exceptional set $\mathcal{E} \subset \mathbb{R}$ is closed and of Lebesgue measure 0. □

Remark 3.1. As in [RS] we likewise note the following: It follows by the Riemann-Lebesgue Lemma that the set $\mathcal{E}$ is bounded; and, if $\sup_{\mathbb{R}^3} |V(x)|$ is sufficiently small, $||A_{|\kappa|}||$ can be made so small that $(A_{|\kappa|} + I)^{-1}$ exists for all $|\kappa|$ so that $\mathcal{E}$ is empty.

In the next theorem we consider, for certain measure spaces, solutions to equation (0.1) as weak limits $(A^*_{|\kappa|} + I)^{-1}(f_m)(x)$ for sequences of functions $f_m \in L^2(\mathbb{R}^3)$ for $m = 1, 2, 3\ldots$ converging pointwise to $V^{1/2}(x)e^{ik\cdot x}$ with $|f_m(x)| \leq |V^{1/2}(x)|$ $\forall x$ for each m. (Here, $*$ denotes the adjoint of the operator.)

Theorem 3. *Suppose that $A_{|\kappa|}$ is a compact operator. Then, for all $|\kappa| \notin \mathcal{E}$, the sequence of functions $(A_{|\kappa|} + I)^{-1}(f_m)$ converges weakly to an unique solution to equation (0.1) in the following sense. Given, $g \in L^2(\mathbb{R}^3)$ such that $V^{1/2} \in L^1(\mathbb{R}^3; g(x)\,dx)$, $\exists w \in L^2(\mathbb{R}^3)$ such that*

$$\lim_{m\to\infty} \int_{\mathbb{R}^3} (A^*_{|\kappa|} + I)^{-1}(f_m)(x) w(x)\,dx = \int_{\mathbb{R}^3} V^{1/2}(x) g(x)\,dx$$

Proof : For each m as above,

$$\int_{\mathbb{R}^3} (A_{|\kappa|} + I)^{-1}(f_m)(x)w(x)\,dx = \int_{\mathbb{R}^3} f_m(x)(A^*_{|\kappa|} + I)^{-1}(w)(x)\,dx$$
$$= \int_{\mathbb{R}^3} f_m(x)g(x)\,dx$$

where $w \stackrel{\text{def}}{=} (A^*_{|\kappa|} + I)g$. Now, the result of the theorem follows from the Lebesgue Dominated Convergence Theorem. □

Before we state the next result, we make the following definitions: We will denote by $\mathcal{C}_{\eta,\delta}$ the open cone given by

$$\mathcal{C}_{\eta,\delta} \stackrel{\text{def}}{=} \{x : \frac{x \cdot \eta}{|x|} > \delta\}$$

for some $-1 < \delta < 1$ and for some unit vector $\eta \in \mathbb{R}^3$. Given δ, a function $\phi(x)$ will be said to be rapidly decreasing on a cone $\mathcal{C}_{\eta,\delta}$ if for every positive integer j,

$$\lim_{|x|\to\infty} \sup_{\frac{x\cdot\eta}{|x|}>\delta} |x|^j|\phi(x)| = 0.$$

Finally, the expression $f \sim h$ on $\mathcal{C}_{\eta,\delta}$ will mean that the difference $f - h$ is rapidly decreasing on $\mathcal{C}_{\eta,\delta}$.

The following result characterizes the relationship between g and w as in Theorem 3 for large $r \stackrel{\text{def}}{=} |x|$ with $F \stackrel{\text{def}}{=} V^{1/2}g$:

Theorem 4. *Let V and g be as in Theorem 3 for $|\kappa| \notin \mathcal{E}.\bigcup\{0\}$ Suppose that F is supported in the complement of a cone $\mathcal{C}_{\eta,\delta}$, and that F satisfies*

$$|\frac{d^j}{d\,r^j}F| \lesssim (1 + r^2)^{-\frac{\gamma+j}{2}}$$

on $\mathbb{R}^3$ for each $j = 0, 1, 2, \ldots$ for some (fixed) positive $\gamma > 3$. Then, $w(x) \sim g(x)$ on $\mathcal{C}_{-\eta,\delta'}$ for any δ' such that $\delta < \delta' < 1$.

Proof: For a given cone $\mathcal{C}_{\eta,\delta}$ we will show that $w(x) = g(x) + \phi(x)$ where $\phi(x) = A_{|\kappa|}(g)(x)$ is rapidly decreasing on $\mathcal{C}_{-\eta,\delta'}$. To this end, it suffices to show that for $|\kappa| \notin \mathcal{E}$

$$T_{|\kappa|}(g)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \frac{e^{i|\kappa||x-y|}}{|x-y|}F(y)\,dy$$

is rapidly decreasing on $\mathcal{C}_{-\eta,\delta'}$.

For $\omega \stackrel{\text{def}}{=} \frac{x}{r}$ $(x \neq 0)$, fixed, we introduce the variable $u = \frac{y}{r} - \omega$, define $s = |u|$ and $\nu \stackrel{\text{def}}{=} \frac{u}{s}$, and write

$$\begin{aligned} T_{|\kappa|}(g)(x) =& T_{|\kappa|}(g)(r\omega) \\ =& \int_{\mathbb{R}^3} \frac{e^{ir|\kappa||\omega - \frac{y}{r}|}}{r|\omega - \frac{y}{r}|} F(y)\, dy \\ =& \int \frac{e^{ir|\kappa||u|}}{|u|} F(r(\omega + u)) r^2\, du \\ =& r^2 \int_{S^2 \setminus \mathcal{C}_{\eta,\delta}} \int_0^\infty e^{ir|\kappa|s} sF(r(\omega + s\nu))\, ds\, d\Omega(\nu) \end{aligned}$$

Now, by the Lebesgue Dominated Convergence Theorem and the Fubini-Tonelli Theorem it suffices to show that

$$\int_0^\infty e^{ir|\kappa|s} sF(r(\omega + s\nu))\, ds \tag{3.2}$$

rapidly decreases (as $r \to \infty$) uniformly in ν. Supposing $r \geq 1$, it follows by induction using the chain rule that for each $j = 0, 1, 2, \ldots$ with $\mathfrak{d} \stackrel{\text{def}}{=} \delta' - \delta$,

$$\begin{aligned} \frac{d^j}{d\,s^j}[sF(r(\omega + s\nu)] \lesssim& \frac{r^j(s^{j+2} + 1)}{(1 + r^2(s^2 + 2s\omega\cdot\nu + 1))^{\frac{\gamma+j}{2}}} \\ \lesssim& \frac{r^j(s^{j+2} + 1)}{(1 + r^2(s-1)^2 + 2\mathfrak{d}r^2 s))^{\frac{\gamma+j}{2}}} \\ \lesssim& \begin{cases} \frac{r^j s^{j+2}}{(rs)^{\gamma+j}} & : s \geq 2 \\ \frac{r^j}{(1+r^2 s))^{\frac{\gamma+j}{2}}} & : 0 \leq s < 2 \end{cases} \\ \lesssim& (s+1)^{2-\gamma} \end{aligned} \tag{3.3}$$

uniformly for $r \geq 1$. In (3.3) we use that $\omega \cdot \nu \geq 1 - d$ for $\omega \in \mathcal{C}_{-\eta,\delta}$ and $s\nu \in \operatorname{supp} F$. Therefore, since each derivative

$$\frac{d^j}{d\,s^j}[sF(r(\omega + s\nu)] \in L^1(\mathbb{R},\, ds),$$

it follows from the Riemann-Lebesgue Lemma that the integral (3.2) is indeed rapidly decreasing (as $r \to \infty$) and the result follows. $\square$

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