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THE VIEW POINT OF CHANGE IN THE NATURE OF
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By

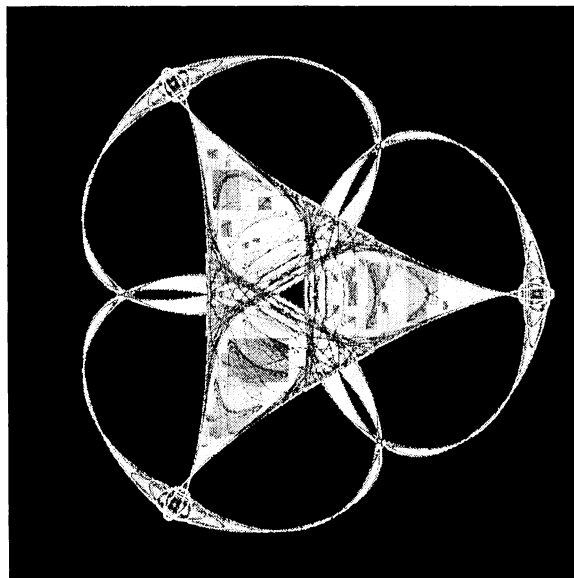
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GENERATING FUNCTION OF HYPERGEOMETRIC FUNCTIONS FROM THE VIEW POINT OF CHANGE IN THE NATURE OF HYPERGEOMETRIC SERIES

(Key Words: Summation theorem / product identity / hypergeometric series / integral representations)

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Abstract

In the present paper generating function has been obtained for a mixed hypergeometric series. It has been observed that the new generated function satisfies the properties required for falling under the area of special functions and so some of them have been investigated in the light of hypergeometric functions.

Introduction

There are many ways of extending the definition of the Gauss function. The idea of extending the number of parameters in the Gauss function gives rise to the proliferation of generalized Gauss functions while introduction of a base q produces basic series. The results of this generalization process have been discussed in detail in the book of Slater¹. Andrews² has proved Jacobi's triple product identity in a very simple manner and his this technique was modified further in order to prove a generalization of the identity due to Ramanujan. The work of Andrews² and Askey³ has been generalized in order to obtain a few bilateral summations theorem, by Singh⁴. A general class of mixed trilinear relations for Gauss hypergeometric functions has been investigated by Chakrabarty and Hazra⁵.

Workers of statistics dealing with hypergeometric distributions come across several hypergeometric series, where the parameters assume values of the mixed nature, parameters in numerator assume increasing values while these in denominator assume decreasing values in the following form of hypergeometric series.

$$1 - \left(\frac{a}{(b-1)}\right) \frac{x}{1!} + \left(\frac{a(a+1)}{(b-1)(b-2)}\right) \frac{x^2}{2!} - \left(\frac{a(a+1)(a+2)}{(b-1)(b-2)(b-3)}\right) \frac{x^3}{3!} + \dots \dots (1)$$

where b is not a positive integer

Our concern in the present paper is to generate functions for the above case and discuss their properties in the light of hypergeometric functions. Differential equations and integral representations for them have been also imparted.

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General formulation.

The generating function for the hypergeometric series of type given by (1) shall be expressed as,

$$B \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} a_p & b_r \\ ; & ; x \\ a'_q & b'_s \end{matrix} \right], \text{ such that } a'_q \text{ is not negative integer and } b'_r \text{ is not positive integer}$$

$$= \sum_{n=0}^{\infty} \frac{(a_p)_n (b_r)_{-n} x^n}{(a'_q)_n (b'_s)_{-n} n!} \quad \dots(2)$$

$$= \sum_{n=0}^{\infty} \frac{(a_p)_n (-1)^{rn} (1-b'_s)_n x^n}{(a'_q)_n (1-b_r)_n (-1)^{sn} n!}$$

$$= {}_{p+s}F_{q+r} \left[\begin{matrix} a_p & 1-b'_s \\ ; & ; (-1)^{r-s} x \\ a'_q & 1-b_r \end{matrix} \right] \quad \dots(3)$$

Which is quite an idiosyncratic function having the property that when the later parameters are zero, the function assumes the form of hypergeometric function and when the first two parameters are zero, the function acquires a sequence form.

Special cases

The equation (2) establishes the above defined B-function as a special function as it possesses the important property of these functions, being expressible in terms of other special functions, for example.

$$B \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \left[\begin{matrix} - & - \\ ; & ; x \\ - & - \end{matrix} \right] = e^x \quad \dots(4)$$

$$B \begin{matrix} 2 & 1 \\ 0 & 0 \end{matrix} \left[\begin{matrix} 1, 1 & -1 \\ ; & ; x \\ - & - \end{matrix} \right] = \frac{\log(1+x)}{x} \quad \dots(5)$$

$$B \begin{matrix} 2 & 1 \\ 0 & 0 \end{matrix} \left[\begin{matrix} 1, 1 & -1 \\ ; & ; -x \\ - & - \end{matrix} \right] = \frac{-\log(1-x)}{x} \quad \dots(6)$$

$${}_B \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \begin{bmatrix} 1 & - \\ & ; \\ - & - \end{bmatrix} ; x = (1-x)^{-1} \quad \dots(7)$$

$${}_B \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \begin{bmatrix} a & b \\ & ; \\ - & - \end{bmatrix} ; x = {}_1F_1(a; 1-b; -x) \quad \dots(8)$$

and

$${}_B \begin{matrix} p & 0 \\ q & 0 \end{matrix} \begin{bmatrix} a_p & - \\ & ; \\ b_q & - \end{bmatrix} ; x = {}_pF_q(a_p; b_q; x) \quad \dots(9)$$

Different forms of product of two B- functions

It has been seen that products of two special functions are always expressed in terms of special functions involving greater number of parameters. Whether the same trend shall be followed by our function can be a matter of concern, whence we observe that

$$\begin{aligned} & {}_B \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \begin{bmatrix} a & 1-b \\ & ; \\ - & - \end{bmatrix} ; -x \cdot {}_B \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \begin{bmatrix} a & 1-b \\ & ; \\ - & - \end{bmatrix} ; x \\ & = {}_B \begin{matrix} 2 & 3 \\ 0 & 0 \end{matrix} \begin{bmatrix} a, b-a & 1-b, 1-b/2, 1/2-b/2 \\ & ; \\ - & - \end{bmatrix} ; -x^2/4 \quad \dots(10) \end{aligned}$$

This functions can be further expressed in terms of other B- functions as depicted below.

$$= {}_B \begin{matrix} 2 & 2 \\ 1 & 0 \end{matrix} \begin{bmatrix} a, b-a & 1-b, 1-b/2 \\ & ; \\ 1/2+b/2 & - \end{bmatrix} ; x^2/4 \quad \dots(11)$$

$$= {}_B \begin{matrix} 2 & 1 \\ 2 & 0 \end{matrix} \begin{bmatrix} a, b-a & 1-b \\ & ; \\ b/2, 1/2+b/2 & - \end{bmatrix} ; -x^2/4 \quad \dots(12)$$

$$= \begin{matrix} 2 & 0 \\ \text{B} \\ 3 & 0 \end{matrix} \left[\begin{matrix} a, b-a & - & \\ & ; & \\ b, b/2, 1/2+b/2 & - & \end{matrix} ; \quad x^2/4 \right] \dots(13)$$

$$= \begin{matrix} 1 & 3 \\ \text{B} \\ 0 & 1 \end{matrix} \left[\begin{matrix} b-a & 1-b, 1-b/2, 1/2-b/2 & \\ & ; & \\ - & & 1-a \end{matrix} ; \quad -x^2/4 \right] \dots(14)$$

$$= \begin{matrix} 0 & 3 \\ \text{B} \\ 0 & 2 \end{matrix} \left[\begin{matrix} - & 1-b, 1-b/2, 1/2-b/2 & \\ & ; & \\ - & 1+a-b, 1-a & \end{matrix} ; \quad x^2/4 \right] \dots(15)$$

$$= \begin{matrix} 0 & 2 \\ \text{B} \\ 1 & 2 \end{matrix} \left[\begin{matrix} - & 1-b/2, 1/2-b/2 & \\ & ; & \\ b & 1+a-b, 1-a & \end{matrix} ; \quad x^2/4 \right] \dots(16)$$

In a similar fashion other five results may be obtained as total number of results shall depend upon double partitions of 2 & 3 . One can infer from theory of numbers that total number of double partitions for 2 are 3 and for 3 they are 4. Therefore total number of results in above theorem shall be $3.4 = 12$.

Differential Equation For The Function

General differential equation for the function

$$w = \begin{matrix} p & r \\ \text{B} \\ q & s \end{matrix} \left[\begin{matrix} a_p & a'_r \\ & ; & \\ b_q & b'_s \end{matrix} ; \quad x \right]$$

can be obtained as

$$\left\{ \theta \prod_{i=1}^q (\theta + b_i - 1) \prod_{j=1}^r (\theta - a'_j) + x \prod_{i=1}^p (\theta + a_i) \prod_{j=1}^s (\theta - b'_j + 1) \right\} w = 0 \dots(17)$$

where $\theta = x d/dx$

The validity of above obtained differential equation can be tested for its special case, say when $p=r=1$; $q=s=0$; $a_1 = a$ and $a'_1 = b$, that is when the function assumes the form of confluent hypergeometric function, the equation becomes.

$$\left\{ \theta \prod_{j=1}^1 (\theta - a'_j) + x \prod_{j=1}^1 (\theta + a_j) \right\} w = 0$$

or $\{\theta^2 + (x-b)\theta + xa\} w = 0$

Which is comparable with the differential equation of confluent hypergeometric function.

Integral Representation

Let us consider

$$I = \int_0^x B \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} a_p & a'_r \\ b_q & b'_s \end{matrix} ; x \right] dx \quad \dots(18)$$

whose value shall be equal to

$$\frac{\pi \prod_{i=1}^r a'_i \prod_{i=1}^q (b_i - 1)}{s \prod_{i=1}^p (a_i - 1)} \left\{ B \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} a_p - 1 & a'_r + 1 \\ b_q - 1 & b'_s + 1 \end{matrix} ; x \right]^{-1} \right\} \dots(19)$$

The nth integral shall have its value equal to

$$\begin{aligned} I_n &= \int_0^x \int_0^x \dots \int_0^x B \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} a_p & a'_r \\ b_q & b'_s \end{matrix} ; x \right] (dx)^n \\ &= \frac{\pi \prod_{i=1}^r (a'_i)_n \prod_{i=1}^p (a_i)_{-n}}{s \prod_{i=1}^q (b_i)_{-n}} B \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} a_p - n & a'_r + n \\ b_q - n & b'_s + n \end{matrix} ; x \right] \\ &\quad - \sum_{k=1}^n \frac{\pi \prod_{i=1}^p (a_i)_{-k} \prod_{i=1}^r (a'_i)_k}{s \prod_{i=1}^q (b_i)_{-k} \prod_{i=1}^p (a_i)_{-k}} \frac{x^{n-k}}{(n-k)!} \quad \dots(20) \end{aligned}$$

In case $n = 1$ result (19) can be verified easily.

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