ON METASTABLE PATTERNS IN PARABOLIC SYSTEMS

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0. INTRODUCTION

Much work has been done in the past few years on the large-time asymptotic behavior of the solutions of a system of reaction-diffusion equations. We wish to show that in some cases such results can be misleading in the sense that important phenomena of pattern formation which occur on a relatively long time scale can be missed.

We shall treat a semilinear parabolic system of the form

$$(0.1)\qquad \begin{aligned} v_t &= \varepsilon^2 v_{xx} + G(v,w) \\ w_t &= w_{xx} + H(v,w) \end{aligned}$$

in which the diffusion constant ε^2 in the first equation is much smaller than that in the second one. We shall show that when G and H satisfy suitable conditions and ε is sufficiently small, such a system displays the behavior which is usually associated with metastable states. By this we mean that there is a long time interval on which the component v appears to converge to a stable discontinuous steady state of the degenerate system

$$(0.2)\qquad \begin{aligned} v_t &= G(v,w) \\ w_t &= w_{xx} + H(v,w), \end{aligned}$$

while in the very long run the solution goes away from this state.

Our interest in such degenerate parabolic systems stems from joint work with Donald Aronson and Alberto Tesei [2] on a quasilinear system of the general form (0.3). The system comes from a model of N. Shigesada, K. Kawasaki, and E. Teramoto [5] for the growth and spread of two competing biological species, one of which remains stationary while the other migrates to get away from the first. It was found in [8] that the system (0.2) has a continuum of stable steady states in which v displays discontinuous patterns. The present work is intended to relate these patterns to the behavior of solutions of (0.1) when ε is small but not zero.

When a model such as (0.2) has many steady states, it is common practice to eliminate all but a few of them by means of a "selection principle" which accepts only those steady states which are limits as $\varepsilon \to 0$ of steady states of a singularly perturbed system such as (0.1).

Our results will show that this practice can lead one to ignore a multitude of interesting metastable states.

We shall introduce the phenomenon we wish to study by discussing a very simple scalar system in Section 1.

In Section 2 we construct a family of steady state solutions (p,q) of the system (0.2) with p discontinuous which are stable but not asymptotically stable. We then find a large class of initial data near one of these solutions, for each of which the solution of (0.2) converges to a (possibly different) discontinuous steady state (r,s).

In Section 3 we show that when ε in (0.1) is positive and very small, the solutions with such initial data appears to converge to (r,s) for a rather long time, but that it must eventually depart from this behavior. That is, (r,s) is a metastable state but not a steady state of (0.1).

1. A SCALAR PROBLEM

The phenomenon we wish to discuss is most easily analyzed for the scalar initial-boundary value problem

$$
(1.1)\qquad
\begin{aligned}
& v_t = \varepsilon^2 v_{xx} + v(1 - v)(v - a) \quad \text{for} \quad 0 < x < 1, \quad t > 0 \\
& \varepsilon^2 v_x(0,t) = \varepsilon^2 v_x(1,t) = 0 \\
& v(x,0) = \phi(x).
\end{aligned}
$$

Here a is a constant in the interval $(0,1/2)$, ε^2 is small, and the initial function ϕ is continuous on $[0,1]$. The long-time asymptotic behavior is completely described by the following theorem.

THEOREM 1.1. (a) If $\phi > a$ *on some open subinterval of* $(0,1)$, *then there exists a positive constant* $e_1[\phi]$ *such that if* $0 < \varepsilon < e[\phi]$, *the solution* $v(x,t)$ *of the problem* (1.1) *approaches* 1, *uniformly for* $x \in [0,1]$, *as* $t \to \infty$.

(b) *If* $\phi(x) \leq a$ *and* $\phi \not\equiv a$, *then* v *approaches* 0 *uniformly as* $t \to \infty$.

(c) *If* $\phi \equiv a$, *then* $v \equiv a$.

Proof. Define the initial function $\phi(x)$ for all x by continuing it as an even function about $x = 0$ and $x = 1$. Introduce the new independent variable $y = x/\varepsilon$. Then the problem (1.1) is equivalent to the initial value problem

$$
(1.2)\qquad
\begin{aligned}
& v_t = v_{yy} + v(1 - v)(v - a) \qquad \text{for} \quad t > 0 \\
& v(y,0) = \phi(\varepsilon y).
\end{aligned}
$$

Statement (a) now follows from Theorem 6.2 of [8], which states that if, for some $b > a$, $v(y,0) \geq b$ on a sufficiently long interval, then $v(y,t)$ approaches 1 uniformly on bounded sets (*).

Statement (b) comes from applying Theorem 3.1 of [1] to the function $a - v$. Statement (c) is obvious.

(*) On the other hand, for any $\varepsilon > 0$ there is a ϕ which exceeds a on a short interval such that $v(x,t)$ approaches zero. Thus $e_1[\phi]$ is not independent of ϕ.

Both of the constructions make essential use of maximum principles and comparison principles for parabolic equations, which were pioneered by Mauro Picone [3,4,5].

It is to be expected that the behavior of solutions of (1.1) when ε is small should be related to the behavior of solutions of the problem

$$\begin{aligned} &u_t = u(1 - u)(u - a) \qquad \text{for } 0 \leqslant x \leqslant 1, \quad t > 0 \\ &u(x,0) = \phi(x), \end{aligned} \tag{1.3}$$

which arises when $\varepsilon = 0$. Since this problem deals with an ordinary differential equation for each fixed x, it is easy to find its large-time asymptotics:

$$\lim_{t\to\infty} u(x,t) = p(x) \equiv \begin{cases} 0 & \text{if } \phi(x) < a \\ a & \text{if } \phi(x) = a \\ 1 & \text{if } \phi(x) > a. \end{cases} \tag{1.4}$$

Because ϕ, and hence v, is continuous and p is, in general, discontinuous, one cannot expect uniform convergence. Because ϕ is continuous, the sets $\{x: p(x) = 0\}$ and $\{x: p(x) = 1\}$ are relatively open in $[0,1]$. The following result is easy to prove.

THEOREM 1.2. *Let* S *be the union of three closed sets on each of which* p *is constant. Then* $u(x,t)$ *converges to* $p(x)$ *as* $t \to \infty$, *uniformly for* x *in* S.

We define the epigraph $E[q]$ of a function $q(x)$ as

$$E[q] = \{(x,y): y \geqslant q(x)\},$$

and we define a neighborhood base $N_\delta[q]$ of a function q by

$$N_\delta[q] = \{r(x): d(E[r], E[q]) < \delta\}. \tag{1.5}$$

Here $d(M,N)$ is the ℓ_∞ distance between sets. That is, it is the infimum of numbers D such that every point of M is within distance D of N and every point of N is within distance D of M, where

$$d[(x_1,y_1),(x_2,y_2)] = \max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

We observe that the set S in Theorem 1.2 can be chosen to be the set of x which lie at distance at least δ from the points of discontinuity of p. Moreover, the range of the solution $u(x,t)$ converges to a subinterval of $[0,1]$. Thus we have the following convergence result:

THEOREM 1.3 _The solution_ $u(x,t)$ _of_ (1.3) _converges to_ $p(x)$ _in the sense of the epigraph topology defined by_ (1.5).

Suppose that $p(x)$ has the property that the value a is never a local maximum or minimum value. Then if $\delta < a$, $\phi(x) \in N_\delta[p]$ implies that $v(x,t) \in N_\delta[p]$ for all t. Thus, such a p is stable in the epigraph topology. However, if p is discontinuous, then any epigraph neighborhood contains discontinuous steady states of (1.3) with other discontinuities. Therefore such a p cannot be asymptotically stable.

Convergence to a continuous function ϕ in the epigraph topology is equivalent to uniform convergence. Since $u(x,t)$ depends continuously on $\phi(x)$ for each t, we see that _if the solution of_ (1.3) _converges to a stable_ p (that is, if a does not appear as a local minimum or minimum value of ϕ), _then for any_ $\delta > 0$ _there is a_ $\delta' > 0$ _such that_ $|\overline{\phi} - \phi| < \delta'$ _on_ $[0,1]$ _implies that the limit_ $\overline{p}$ _corresponding to initial data_ $\overline{\phi}$ _lies in_ $N_\delta[p]$.

The following lemma shows that for any finite time interval, the solutions of (1.1) and (1.3) are close when ε is small.

LEMMA 1.1. Let ϕ be a given continuous initial function. For any $T > 0$ and any $\delta > 0$ there exists a positive constant $e_2[\phi,T,\delta]$ such that if $0 < \varepsilon \leq e_2[\phi,T,\delta]$, then

$$|v(x,t) - u(x,t)| < \delta \quad \text{for} \quad 0 \leq t \leq T.$$

Proof. Suppose for the moment that the extension of ϕ as an even function about $x = 0$ and $x = 1$ lies in C^2. It is easy to see that then the solution $u(x,t)$ of (1.3) lies in C^2 for each t. Differentiating with respect to x, we find that

$$u_{xt} = G'(u)u_x,$$

where $G(u) = u(1 - u)(u - a)$. Moreover,

$$\min(\min \phi(x),0) \leq u(x,t) \leq \max(\max \phi(x),1).$$

Therefore if

$$G'(u) \leq M \quad \text{for} \quad \min(\min \phi(x),0) \leq u \leq \max(\max \phi(x),1),$$

then

$$|u_x| \leq |\phi'|e^{Mt}.$$

Differentiate again to find that

$$u_{xxt} = G'(u)u_{xx} + G''(u)u_x^2.$$

Then if also $|G''| \leq M$, we have

$$|u_{xx}| \leq |\phi''|e^{Mt} + \phi'^2 e^{Mt}(e^{Mt} - 1) \leq Ke^{2Mt}.$$

Thus we obtain the inequality

$$[e^{-3Mt}(v - u)^2 - \varepsilon^4K^2e^{Mt}]_t - [e^{-3Mt}(v - u)^2 - \varepsilon^4K^2e^{Mt}]_{xx} \leq 0.$$

The maximum principle shows that the function in brackets cannot have a positive maximum in $[0,1] \times (0,T]$. Therefore

$$(v - u)^2 \leq \varepsilon^4K^2e^{4Mt}.$$

Thus if $\phi \in C^2$, the result follows.

If ϕ is only continuous, we first approximate ϕ by a C^2 function $\overline{\phi}$ such that

$$|\phi - \overline{\phi}| \leq \frac{1}{3}\,\delta e^{-MT}.$$

If $\overline{v}$ and $\overline{u}$ are the solutions of (1.2) and (1.3) with initial values $\overline{\phi}$, it is easily seen that

$$|v - \overline{v}| \leq \frac{1}{3}\,\delta e^{-M(T-t)},$$

$$|u - \overline{u}| \leq \frac{1}{3}\,\delta e^{-M(T-t)}.$$

We now apply the above argument to find that

$$|\overline{v} - \overline{u}| \leq \varepsilon^2Ke^{2Mt}.$$

Thus if we define

$$e_2 \equiv (\delta/3K)^{1/2}e^{-MT},$$

the statement of the Theorem is valid.

Theorem 1.3 states that for any given $\delta > 0$ there exists a time $T_1(\delta)$ such that the solution u of (1.3) lies in the neighborhood $N_{\delta/2}[p]$ for $t \geq T_1(\delta)$. Lemma 1.1 states that for any given $T_2 > T_1(\delta)$ v is within $\delta/2$

of u, provided $\varepsilon < e_2[\phi,T_2,\delta/2]$. We conclude that if $\varepsilon < e_2[\phi,T_2,\delta/2]$, the solution v lies in the neighborhood $N_\delta[p]$ for $T_1(\delta) \le t \le T_2$.

If δ is very small and if T_2 is the duration of a laboratory or computer experiment, any reasonable scientist would say that the solution v has converged to the limit p. If ϕ oscillates about the value a, the function p displays a very sharp pattern of the values 0 and 1. Thus a pattern forms in time T_1 and remains through time T_2. As we have pointed out earlier, if a is not a local extremum value of p, the solution $\bar{u}$ of (1.3) corresponding to a $\bar{\phi}$ near ϕ also goes near p. Hence the same is true for the solution $\bar{v}$ of (1.2) corresponding to $\bar{\phi}$ for $t \le T_2$. Thus such a p appears to be a stable equilibrium state.

On the other hand, Theorem 1.1 states that if also $0 < \varepsilon < e_1[\phi]$, the function v converges uniformly to a constant state so that in the long run there is no pattern.

Engineers and physicists call a state which acts like a stable equilibrium for a long time but not forever a metastable state. The above discussion shows that a discontinuous stable limit state p of (1.3) is a metastable state of the singularity perturbed problem (1.1) when ε is sufficiently small.

2. A CLASS OF DEGENERATE REACTION-DIFFUSION SYSTEMS

We shall show that under some conditions on the functions G and H the system

$$\begin{aligned} v_t &= \varepsilon^2 v_{xx} + G(v,w) \\ w_t &= w_{xx} + H(v,w) \end{aligned} \qquad \text{for } 0 < x < 1, \quad t > 0 \tag{2.1}$$

with the Neumann boundary conditions

(2.2) $$\varepsilon^2 v_x = w_x = 0 \quad \text{at} \quad x = 0 \text{ and } x = 1$$

has metastable states for small positive ε. We shall always suppose that G and H are twice continuously differentiable.

We begin by constructing a family of equilibrium solutions of the system which is obtained by setting $\varepsilon = 0$. That is, we shall solve the system

(2.3) $$\begin{aligned} G(p,q) &= 0 \\ p_{xx} + H(p,q) &= 0 \end{aligned}$$

LEMMA 2.1 *Supose that there are two points* (v_1^*, w^*) and (v_2^*, w^*) *with* $v_1^* < v_2^*$ *in the* v-w *plane with the following properties:*

(2.4)
$$(a) \quad G(v_1^*, w^*) = G(v_2^*, w^*) = 0$$
$$(b) \quad G_v(v_1^*, w^*) \neq 0, \; G_v(v_2^*, w^*) \neq 0$$
$$(c) \quad H(v_1^*, w^*) H(v_2^*, w^*) < 0.$$

There exist positive numbers γ_0 *and* M *such that whenever the constants* α *and* γ *satisfy*

$$0 < M\alpha^2/2 \leq \gamma \leq \gamma_0,$$

there is a solution (p,q) *of* (2.3) *on the interval* $-\alpha < x < \alpha$ *which satisfies both the boundary conditions*

(2.5) $$q(-\alpha) = q(\alpha) = w^*$$

and

(2.6) $$q'(-\alpha) = q'(\alpha) = 0.$$

<u>The function</u> p <u>has two jump discontinuities in the interval, and</u> q <u>is continuously differentiable and piecewise twice continuously differentiable on the interval</u> $[-\alpha,\alpha]$. <u>Moreover</u>

$$|q - w^*| \le \gamma$$

and

$$|q'| \le (M\gamma/2)^{1/2}.$$

<u>Proof</u>. By the implicit function theorem there are a positive γ_1 and two functions $v_1(w)$ and $v_2(w)$ such that when $i = 1$ or 2

$$G(v_i(w),w) = 0 \qquad \text{for } |w - w^*| < \gamma_1$$
$$v_i(w^*) = v_i^*.$$

Choose $\gamma_0 \le \gamma_1$ and M such that

$$0 < |H(v_i(w),w)| \le M \quad \text{for } |w - w^*| \le \gamma_0, \quad i = 1,2.$$

Let $0 < \gamma \le \gamma_0$ and $\alpha \le (2\gamma/M)^{1/2}$. Choose $\beta \in (0,1)$ and define

$$\hat{H}(x,w) = \begin{cases} H(v_1(w),w) & \text{for } 0 \le |x| < \beta\alpha \\ H(v_2(w),w) & \text{for } \beta\alpha \le |x| \le \alpha. \end{cases} \tag{2.7}$$

We wish to solve the problem

$$\begin{aligned} &q'' + \hat{H}(x,q) = 0 \qquad \text{for } -\alpha < x < \alpha, \\ &q(-\alpha) = q(\alpha) = w^*. \end{aligned} \tag{2.8}$$

Since $|\hat{H}| \le M$ for $|w - w^*| \le \gamma_0$ and since $M\alpha^2/2 \le \gamma < \gamma_0$, we see that $q = w^* + M(\alpha^2 - x^2)/2$ is a supersolution and $q = w^* - M(\alpha^2 - x^2)/2$ is a subsolution. Therefore the solution of the initial value problem

$$
\begin{aligned}
&r_t = r_{xx} + \tilde{H}(x,r) \qquad \text{for} \quad -\alpha < x < \alpha\ , \quad t > 0 \\
&r(0,t) = r(1,t) = w^* \\
&r(x,0) = w^* - M(\alpha^2 - x^2)
\end{aligned}
$$

is nondecreasing in t and bounded above by $w^* + M(\alpha^2 - x^2)/2$. Hence as $t \to \infty$ r converges to a solution $q(x;\beta)$ of (2.9), and q satisfies

$$|q - w^*| \leq M\alpha^2/2 \leq \gamma.$$

The solution $q(x;\beta)$ depends continuously upon the parameter β through (2.7).

Suppose without loss of generality that $H(v_1^*,w^*) > 0$ and $H(v_2^*,w^*) < 0$. Then also $H(v_1(w),w) > 0$ and $H(v_2(w),w) < 0$ for $|w - w^*| \leq \gamma_0$.

If $\beta = 0$, we see from (2.7) that $\tilde{H} < 0$ so that $q(x;0) < w^*$ in $(-\alpha,\alpha)$. It follows that $q'(\alpha;0) > 0$. Similarly, we see that $q(x;1) > w^*$ so that $q'(\alpha;1) < 0$. Since $\tilde{H}$ is increasing in β, $q'(\alpha;\beta)$ is decreasing in β. By continuity, there is a unique $\beta = \beta(\alpha)$ such that $q'(\alpha;\beta(\alpha)) = 0$. Since (2.8) is invariant under the transformation $x \to -x$, the solution is even in x. Therefore $q'(-\alpha;\beta(\alpha)) = -q'(\alpha;\beta(\alpha)) = 0$.

Because $|q''| \leq M$ and each point is within distance $\alpha/2$ of either an end point or the midpoint, where $q' = 0$, we see that $|q'| \leq M\alpha/2 \leq (M\gamma/2)^{1/2}$. If we define

$$
p(x) = \begin{cases} v_1(q(x;\beta(\alpha))) & \text{for} \quad 0 \leq |x| < \beta(\alpha)\alpha \\ v_2(q(x;\beta(\alpha))) & \text{for} \quad \beta(\alpha)\alpha \leq |x| \leq \alpha \end{cases}
$$

$$q(x) = q(x;\beta(\alpha)),$$

we see that (p,q) has the properties stated in the Lemma.

We can now construct a family of discontinuous solutions of the problem

$$
(2.9) \qquad \begin{aligned} &G(p,q) = 0 \qquad \text{for } 0 < x < 1 \\ &q'' + H(p,q) = 0 \\ &q'(0) = q'(1) = 0. \end{aligned}
$$

LEMMA 2.2 *If the hypotheses (2.4) are satisfied, then for any* $0 < \gamma \leq \gamma_0$ *there exist solutions of the problem (2.9) in which* p *has arbitrarily many jump discontinuities while* q *is of class* C^1 *and piecewise* C^2,

$$
(2.10) \qquad |q(x) - w^*| \leq \gamma,
$$

and

$$
(2.11) \qquad |q'(x)| \leq (M\gamma/2)^{1/2} .
$$

Proof. Divide the interval [0,1] into any finite number of disjoint intervals, each of which has length at most $(8\gamma/M)^{1/2}$. Let the length of one of these intervals be 2α. In this interval define $(p(x),q(x))$ to be the translation of the solution constructed in Lemma 2.1 by an amount which takes $x = 0$ into the midpoint of the interval. Because of the conditions (2.5) and (2.6), the resulting function q is in C^1 and piecewise C^2. Because of the definition of p, p is continuous at the ends of the intervals, but it has two jumps in each interval.

Under some additional hypotheses we can show that the solutions constructed in the preceding lemma are linearly stable when γ is small.

LEMMA 2.3 *Suppose that the hypotheses (2.4) are satisfied and that, in addition,*

$$(2.12)\quad \begin{aligned} &\text{(a)}\quad G_v < 0 \quad \text{at} \quad (v_1^*,w^*) \text{ and } (v_2^*,w^*),\\ &\text{(b)}\quad \text{For some constant } \beta \text{ and } i = 1,2\\ &\qquad H_w - \beta H < 0 \quad \text{at} \quad (v_i^*,w^*),\\ &\qquad \frac{d}{dw} H(v_i(w),w) - \beta H < 0 \quad \text{at} \quad w^*. \end{aligned}$$

Then if (p,q) is the solution in Lemma 2.2 and γ is sufficiently small, the spectrum of the linearized operator

$$(2.13)\qquad L \equiv \begin{pmatrix} G_v(p,q) & G_w(p,q) \\ H_v(p,q) & \frac{d^2}{dx^2} + H_w(p,q) \end{pmatrix}$$

on the Hilbert space $L_2([0,1]) \times L_2([0,1])$ lies in the left half-plane and is bounded away from the imaginary axis. Moreover, the imaginary part of the spectrum is bounded.

Proof. By (2.12) and continuity there are a $\gamma_1 \leq \gamma_0$ and a positive constant ν such that

$$(2.14)\quad \begin{aligned} &\text{(a)}\quad G_v(v,w) < -\nu < 0,\\ &\text{(b)}\quad -\frac{\beta H(v,w)}{1+2\beta(w-w^*)} + H_w(v,w) + (-G_v(v,w))^{-1}[H_v(v,w)G_w(v,w)]_+ < -\nu < 0 \end{aligned}$$

for $|w - w^*| \leq \gamma_1$ and $v = v_i(w)$ with $i = 1$ or 2.

(As usual,

$$[z]_+ = \begin{cases} z & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z \leq 0 \end{cases} \quad .)$$

We choose $\gamma < \gamma_1$. It is easily seen that the continuous spectrum of the operator (2.13) consists of the range of the function $G_v(p,q)$, which lies in the half-plane $\mathrm{Re}\,\lambda < -\nu$. To examine the discrete spectrum, we consider the eigenvalue equation

$$G_v\eta + G_w\zeta = \lambda\eta$$
$$H_v\eta + \zeta'' + H_w\zeta = \lambda\zeta.$$

We solve the first equation for η and substitute in the second to find

$$\zeta'' + (H_w - \lambda + \frac{H_vG_w}{\lambda - G_v})\zeta = 0$$

We multiply this equation by $[1 + 2\beta(q - w^*)]\bar{\zeta}$, integrate from 0 to 1, integrate by parts, and take the real part to see that if $\mathrm{Re}\,\lambda > 0$, then by (2.14)

$$0 = \int_0^1 [1 + 2\beta(q - w^*)][-|\zeta'|^2 + \{-\frac{\beta H}{1 + 2\beta(q - w^*)} + H_w - \mathrm{Re}\lambda + \frac{(\mathrm{Re}\lambda - G_v)H_vG_w}{|\lambda - G_v|^2}\}\ |\zeta|^2]dx$$
$$< \int_0^1 \{-\frac{[H_vG_w]_+}{-G_v} - (\nu + \mathrm{Re}\lambda) + \frac{(\mathrm{Re}\lambda - G_v)[H_vG_w]_+}{|\lambda - G_v|^2}\}|\zeta|^2dx$$
$$< -\int [H_vG_w]_+ (\frac{1}{-G_v} - \frac{1}{\mathrm{Re}\,\lambda - G_v})|\zeta|^2dx < 0.$$

This contradiction shows that $\mathrm{Re}\lambda < 0$ on the spectrum.

Multiplying the equation for ζ by $\bar{\zeta}$, integrating, and taking the imaginary part gives a uniform bound for the eigenvalues with nonzero imaginary part. (See [8, p. 256.]) Thus the Lemma is proved.

REMARKS. 1. The hypothesis (2.12b) with $\beta = z'(w^*)/z(w^*)$ permits one to treat a system in which a function of the form $z(w)H(v,w)$ has local monotonicity properties. In particular, the choice $\beta = 1/w^*$ allows one to deal with the specific growth function H/w.

2. One can show that there is a β for which the hypothesis (2.12b) is valid if and only if

$$H(v_2^*,w^*)^2\{H_w(v_1^*,w^*) + [G_w(v_1^*,w^*)H_v(v_1^*,w^*)]_+/(-G_v(v_1^*,w^*))\}$$
$$- H(v_1^*,w^*)H(v_2^*,w^*)\{H_w(v_2^*,w^*) + [G_w(v_2^*,w^*)H_v(v_2^*,w^*)]_+/(-G_v(v_2^*,w^*))\} < 0.$$

We shall now show that, for possibly somewhat smaller values of γ, the solutions (p,q) which were shown to be linearly stable in the preceding Lemma are actually stable steady states of the system

$$\begin{aligned} v_t &= G(v,w) && \text{in } (0,1)\times(0,\infty) \\ w_t &= w_{xx} + H(v,w) \\ w_x(0,t) &= w_x(1,t) = 0 \end{aligned} \tag{2.15}$$

in a topology which allows discontinuous v and which is at least as strong as the epigraph topology for v.

LEMMA 2.4 <u>Suppose that the conditions</u> (2.4) and (2.12) <u>are satisfied. There exist positive constants</u> A, A', A'', γ_2, <u>and</u> δ_1 <u>with the following property</u>: <u>Let</u> $(p(x),q(x))$ <u>be one of the steady state solutions of the system</u> (2.9) <u>constructed in Lemma</u> 2.2, <u>with</u> $\gamma < \gamma_2$. <u>If</u> $\delta < \delta_1$, <u>and</u> S <u>is any closed set of points of continuity of</u> p <u>with complement</u> S' <u>whose measure</u> $|S'|$ <u>satisfies</u> $|S'| \le \delta^2$, <u>the inequalities</u>

$$\begin{aligned} v_1(q(x)) - \delta \le v(x,0) &\le v_2(q(x)) + \delta, \\ \|v(x,0) - p(x)\|_{L_\infty(S)} &\le \delta, \\ \|w(x,0) - q(x)\|_{L_\infty} &\le \delta \end{aligned} \tag{2.16}$$

<u>imply that the solution</u> (v,w) <u>of</u> (2.15) <u>satisfies</u>

$$(2.17)\qquad \begin{aligned} v_1(q(x)) - A''\delta < v(x,t) < v_2(q(x)) + A''\delta, \\ \|v(x,t) - p(x)\|_{L_\infty(S)} < A'\delta, \\ \|w(x,t) - q(x)\|_{L_\infty} < A\delta. \end{aligned}$$

<u>Moreover there is a constant</u> A''' <u>such that if, in addition,</u>

$$\|w_x(x,0) - q'(x)\|_{L_\infty} < \delta,$$

then

$$(2.18)\qquad \|w_x(x,t) - q'(x)\|_{L_\infty} < A'''\delta.$$

(We write L_∞ for $L_\infty([0,1])$.)

<u>Proof.</u> A simple continuity argument shows that the hypothesis (2.12a) implies that there exist positive constants B and $\bar{\gamma} < \gamma_1$ such that

$$(2.19)\qquad \begin{aligned} &G_v(v,w) < -\nu \quad \text{for } |w - w^*| < \bar{\gamma},\ |v - v_i^*| < B, \quad i = 1,2, \\ &|v_i(w) - v_0^*| < B \quad \text{for } |w - w^*| < \bar{\gamma}, \quad i = 1,2. \end{aligned}$$

Choose $\gamma_2 < \bar{\gamma}$ and a parameter $A > 1$. Since $\|w(x,0) - q(x)\|_{L_\infty} < \delta$, there is a $T > 0$ such that

$$(2.20)\qquad \|w(x,t) - q(x)\|_{L_\infty} < A\delta \quad \text{for } t < T.$$

We choose $\delta_1 > 0$ so that

$$(2.21)\qquad \begin{aligned} &\gamma_2 + A\delta_1 < \bar{\gamma}, \\ &|v_i(w) - v_i^*| + \delta_1 < B \quad \text{for } |w - w^*| < \bar{\gamma}, \quad i = 1,2. \end{aligned}$$

Since $|q - w^*| < \gamma$, the first inequality shows that if $\gamma < \gamma_2$ and $\delta < \delta_1$, then

$$(2.22)\qquad |w - w^*| < \bar{\gamma} \quad \text{for } t < T.$$

Since $p(x) = v_i(q(x))$ for $i = 1$ or 2 at each x, the second inequality together with the first line of (2.16) shows that at $t = 0$

$$v_1^* - B < v < v_2^* + B. \tag{2.23}$$

Since (2.19) shows that $G(v_1^* - B, w) > 0$ and $G(v_2^* + B, w) < 0$ when $|w - w^*| < \gamma$, we conclude from the first equation of (2.15) that the inequality (2.23) remains valid for $t < T$.

At a point $x \in S$ where $p = v_i(q)$, the second condition in (2.21) and the second inequality in (2.16) show that at $t = 0$

$$|v - v_i^*| < B \quad \text{for} \quad x \in S. \tag{2.24}$$

The reasoning which showed that (2.23) continues to hold for $t < T$ proves the same for (2.24).

Thus as long as the inequality (2.20) is valid we have (2.23) and (2.24). We now introduce the variables

$$\begin{aligned} \eta(x,t) &= v(x,t) - p(x) \\ \zeta(x,t) &= w(x,t) - q(x). \end{aligned} \tag{2.25}$$

The first equations of (2.9) and (2.15) show that

$$\begin{aligned} \eta_t &= G(p + \eta, q + \zeta) - G(q, \eta) \\ &= G_v \eta + G_w \zeta \end{aligned}$$

by the intermediate value theorem. Because the set $\{(v,w): |v - v_i^*| < B, \ |w - w^*| < \gamma\}$ in convex, the intermediate point again lies in this set when x lies in S. Thus $G_v < -\nu$. We solve the differential equation for η to see that

$$\begin{aligned} |\eta(x,t)| &< |\eta(x,0)| e^{-\nu t} + c_1 \nu \int_0^t |\zeta| e^{-\nu(t-\tau)} d\tau \\ &< (1 + c_1 A)\delta \end{aligned}$$

where c_1 is a bound for $|G_w|/\nu$. Thus if we let

$$A' = 1 + c_1 A, \tag{2.26}$$

we see that (2.20) implies

$$|\eta| < A'\delta \quad \text{for} \quad x \in S, \quad t \leq T. \tag{2.27}$$

We now write the difference between the equations for (v,w) and for (p,q) in the form

$$\left(\frac{\partial}{\partial t} - L\right)\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \rho \\ \sigma \end{pmatrix} \tag{2.28}$$

where L is defined by (2.13) and

$$\rho = G(p + \eta,\ q + \zeta) - G(p,q) - G_v(p,q)\eta - G_w(p,q)\eta,$$

$$\sigma = H(p + \eta,\ q + \zeta) - H(p,q) - H_v(p,q)\eta - H_w(p,q)\eta.$$

By Lemma 2.3 there is an $s > 0$ such that

$$\|\eta(t)\|^2_{L_2} + \|\zeta(t)\|^2_{L_2} \leq e^{-2st}[\|\eta(0)\|^2_{L_2} + \|\zeta(0)\|^2_{L_2}] + \int_0^t e^{-2s(t-\tau)}[\|\rho(\tau)\|^2_{L_2} + \|\sigma(\tau)\|^2_{L_2}]d\tau. \tag{2.29}$$

Since G and H are twice continuously differentiable, there is a constant c_2 such that

$$\|\rho\|^2_{L_2} + \|\sigma\|^2_{L_2} \leq 2s\ c_2 \int_0^1 (|\eta|^4 + |\zeta|^4)dx$$

$$\leq 2s\ c_2[(1 + c_1 A)^4\delta^4 + (v_2^* - v_1^* + 2B)^4|S'| + A^4\delta^4]$$

$$\text{for} \quad t \leq T.$$

Moreover,

$$\|\eta(0)\|^2_{L_2} + \|\zeta(0)\|^2_{L_2} \leq 2\delta^2 + (v_2^* - v_1^* + 2B)^2|S'|.$$

Since $|S'| < \delta^2$, (2.29) shows that

$$\|\zeta(t)\|^2_{L_2} + \|\eta(t)\|^2_{L_2} < \delta^2\{2 + (v_2^* + v_1^* + 2B)^2 + + c_2[(v_2^* - v_1^* + 2B)^4 + (1 + c_1A)^4\delta^2 + A^4\delta^2]\}. \tag{2.30}$$

We now write the equation for ζ in the form

$$\zeta_t - \zeta_{xx} = H(p + \eta, q + \zeta) - H(p,q)$$
$$= H_v\eta + H_w\zeta,$$
$$\zeta_x(x,0) = \zeta_x(x,1) = 0.$$

The Green's function for the heat operator with no-flux boundary conditions can be written in the form

$$g(x,t;y,\tau) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2(t-\tau)}\cos n\pi x \cos n\pi y.$$

We see that

$$\int_0^1 g(x,t;y,\tau)^2dy = 1 + 2\sum e^{-2n^2\pi^2(t-\tau)} < 1 + [2\pi(t - \tau)]^{-1/2}. \tag{2.31}$$

For $t < 1$ we write the representation

$$\zeta(x,t) = \int_0^1 g(x,t;y,0)\zeta(y,0)dy + \int_0^t\int_0^1 g(x,t;y,\tau)[H_v\eta + H_w\zeta]dyd\tau.$$

The first term is bounded by the maximum of $|\zeta(x,0)|$, which is bounded by δ. We apply Schwarz's inequality, (2.30), and (2.31) to the second term to find a bound of the form

$$|\zeta| \leq \delta(1+c_3\{2 + (v_2^* - v_1^* + 2B)^2+c_2[(v_2^* - v_1^* + 2B)^4+ (1+c_1A)^4\delta^2 + A^4\delta^2]\}^{1/2}) \tag{2.32}$$

for $t < 1$.

If $T > 1$, we apply Schwarz's inequality, (2.30), and (2.31) to the representation

$$\zeta(x,t) = \int_0^1 g(x,t;y,t-1)\zeta(y,t-1)dy + \int_{t-1}^{t}\int_0^1 g(x,t;y,\tau)\zeta(y,\tau)d\tau$$

to find a bound of the form

$$(2.33) \quad |\zeta| \leq \delta(c_3 + c_4)\{2 + (v_2^* - v_1^* + 2B)^2 + c_2[(v_2^* - v_1^* + 2B)^4 + (1+c_1A)^4\delta^2 + A^4\delta^2]\}^{1/2}$$

$$\text{for } 1 \leq t \leq T.$$

We now choose A so that

$$A < 1 + c_3\{2 + (\nu_2^* - \nu_1^* + 2B)^2 + c_2[(v_2^* - v_1^* + 2B)^4]\}^{1/2},$$

$$A < (c_3 + c_4)\{2 + (\nu_2^* - \nu_1^* + 2B)^2 + c_2[(v_2^* - v_1^* + 2B)^4]\}^{1/2}.$$

We decrease δ_1 if necessary so that the right-hand sides of (2.32) and (2.33) at $\delta = \delta_1$ are strictly less than $A\delta_1$. Since these right-hand sides are increasing in δ, we see that when $\delta \leq \delta_1$, the inequality (2.20) implies that $\|w(x,T) - q(x)\|_{L_\infty} < A\delta$. We conclude from continuity that the set of t where (2.20) is valid is open as well as closed. Therefore the inequality (2.20), and hence also (2.27), holds for all $t \geq 0$.

We now note that if r is a bound for $|v_1'(w)|$ and $|v_2'(w)|$ when $|w - w^*| \leq \bar{\gamma}$, then

$$v_1(q(x)) - rA\delta \leq v_1(w(x,t)),$$
$$v_2(q(x)) + rA\delta \leq v_2(w(x,t)).$$

We let

$$A'' = \max(1, rA)$$

and take δ_1 so small that

$$|v_i(w) - v_i^*| + A'''\delta_1 < B \quad \text{for} \quad i = 1,2.$$

Then when $\delta \leq \delta_1$

$$G(v_1(q) - A''\delta, w) > 0,$$
$$G(v_2(q) + A''\delta, w) < 0,$$

for all t, and

$$v_1(q) - A''\delta < v < v_2(q) + A''\delta$$

at $t = 0$. The equation $v_t = G$, shows that this inequality is valid for all t. Thus we have established the inequalities (2.17). The inequality (2.18) is established by differentiating the two representation formulas for ζ and proceeding as in the derivations of (2.32) and (2.33).

We note that Lemma 2.4 shows the stability in the epigraph topology for v and the uniform topology for w of the steady states constructed in Lemma 2.2 when γ is sufficiently small. However, those states in which v is discontinuous are not asymptotically stable, because in any neighborhood there are other states with jumps at slightly different locations.

Under the somewhat stronger hypotheses

(a) The function $G(v,w^*)$ has exactly three zeros $v_1^* < v_3^* < v_2^*$ on the interval $[v_1^*, v_3^*]$, and $G_v(v_1^*,w^*) < 0$, $G_v(v_3^*,w^*) > 0$, $G_v(v_2^*,w^*) < 0$;

(2.34) (b) $H(v_1^*,w^*)H(v_2^*,w^*) < 0$;

(c) for some constant β and for $i = 1$ and 2

$$H_w(v_i^*,w^*) - \beta H(v_i^*,w^*) < 0 \quad \text{and}$$

$$\frac{d}{dw} H(v_i(w),w)\Big|_{w = w^*} - \beta H(v_i^*,w^*) < 0$$

we can show that many solutions of (2.15) which start near a (p,q) with small γ converge to some steady state.

THEOREM 2.1 *There exist positive constants* γ_3, δ_2, C_1, and C_2 *with the following property: Let* (p,q) *be one of the steady states of* (2.15) *constructed in Lemma* 2.2 *with* $\gamma < \gamma_3$. *Let the hypotheses* (2.34) *be satisfied. Let* $\delta < \delta_2$, *and suppose that for some set* S *of points of continuity of* p *with* $|S'| < \delta^2$

$$(2.35)\qquad \begin{aligned} &\|v(x,0) - p(x)\|_{L_\infty(S)} < \delta \\ &\|w(x,0) - q(x)\|_{L_\infty} < \delta \\ &\|w_x(x,0) - q'(x)\|_{L_\infty} < \delta \\ &v_1(q) - \delta < v(x,0) < v_2(q) + \delta. \end{aligned}$$

If $v(x,0)$ *and* $w(x,0)$ *are continuously differentiable, if the set* $\{x: |v(x,0) - v_3^*| < C_1(\gamma + \delta)\}$ *has finitely many components, and if*

$$(2.36)\qquad |v_x(x,0)| > C_2(\gamma^{1/2} + \delta) \quad \text{whenever} \quad |v(x,0) - v_3^*| < C_1(\gamma + \delta),$$

then the solution $(v(x,t),w(x,t))$ *of the problem* (2.15) *converges to a steady state solution.*

Proof. We note that there are now three branches of solutions $v = v_i(w)$ of the equation $G(v,w) = 0$, with $v_i(w^*) = v_i^*$.

The second part of the hypothesis (2.31c) is equivalent to the condition

$$G_v\{H_w - \beta H\} - G_w H_v > 0 \quad \text{at} \quad (v_i^*, w^*), \quad i = 1,2.$$

For $i = 1$ and 2 we define the constants

$$(2.37)\qquad \mu_i = \begin{cases} [2\, G_v(v_i^*,w^*)\{H_w(v_i^*,w^*) - \beta H(v_i^*,w^*)\} - \\ \quad G_w(v_i^*,w^*)H_v(v_i^*,w^*)]/G_w(v_i^*,w^*)^2 & \text{if } G_w(v_i,w^*) \neq 0 \\ [1 + H_v(v_i^*,w^*)^2]/ \\ \quad [4\, G_v(v_i^*,w^*)\{H_w(v_i^*,w^*) - \beta H(v_i^*,w^*)\}] & \text{if } G_w(v_i,w^*) = 0. \end{cases}$$

It is easily verified that $\mu_i > 0$ and that with this definition $\mu_i G_v\{H_w - \beta H\} - \frac{1}{4}(\mu_i G_w + H_v)^2 > 0$ at (v_i^*, w^*), so that the quadratic form

$$\mu_i G_v \eta^2 + (\mu_i G_w + H_v)\eta\zeta - (H_w - \beta H)\zeta^2$$

is negative definite at (v_i^*, w^*).

We choose positive constants $B' < B$ and then $\gamma' < \overline{\gamma}$ so that

$$\text{(a)} \quad \begin{aligned} &G_v(v,w) \leq -\nu < 0 \quad \text{for} \quad |v - v_i^*| \leq B', \ |w - w^*| \leq \gamma', \quad i = 1,2 \\ &G_v(v,w) \geq \nu > 0 \quad \text{for} \quad |v - v_3^*| \leq B', \ |w - w^*| \leq \gamma'. \end{aligned}$$

$$\text{(b)} \quad |v_i(w) - v_i^*| < B \quad \text{for} \quad |w - w^*| \leq \gamma'.$$

(2.38)

$$\text{(c)} \quad [1 + 2\beta(\overline{w} - w^*)]H_w(v,w) - \beta H(v,w) < 0 \quad \text{and}$$

$$\mu_i G_v(v,w)\{[1 + 2\beta(\overline{w} - w^*)]H_w(v,w) - \beta H(v,w)\} - \frac{1}{4}\{\mu_i G_w(v,w) + [1 + 2\beta(\overline{w} - w^*)]H_v(v,w)\}^2 > 0$$

$$\text{for} \quad |v - v_i| \leq B', \ |w - w^*| \leq \gamma', \ |\overline{w} - w^*| \leq \gamma'.$$

We see from Lemmas 2.2 and 2.4 that if $\gamma \leq \gamma_2$ and $\delta \leq \delta_1$, then

$$\|w(x,t) - w^*\|_{L_\infty} \leq \gamma + A\delta. \tag{2.39}$$

We choose a $\gamma_3 \leq \gamma_2$ and a $\delta_2 \leq \delta_1$ so that

$$\gamma_3 + A\delta_2 \leq \gamma',$$

$$|v_i(w) - v_i^*| + \delta_2 \leq B' \quad \text{for} \quad |w - w^*| \leq \gamma', \quad i = 1,2,3.$$

We shall assume that $\delta \leq \delta_2$ and $\gamma \leq \gamma_3$. Then by (2.39) $|w(x,t) - w^*| \leq \gamma'$.

We now define

$$b = \max\{|v_3(w) - v_3^*| + \delta: |w - w^*| \leq \gamma + A\delta\}.$$

Because $|v_3'|$ is bounded for $|w - w^*| \leq \gamma'$, b has a bound of the form

$$0 < b \leq C_1(\gamma + \delta). \tag{2.40}$$

We see from (2.38) that when $|w - w^*| \leq \gamma + A\delta$

$$G < 0 \quad \text{for} \quad v_1^* + B' \leq v \leq v_3^* - b.$$
$$G > 0 \quad \text{for} \quad v_3^* + b \leq v \leq v_2^* - B'. \tag{2.41}$$

We now define the two open sets

$$S_1(t) = \{x\colon v(x,t) < v_3^* - b\},$$

$$S_2(t) = \{x\colon v(x,t) > v_3^* + b\}.$$

It follows from the differential equation for v that $x \in S_i(t_1)$, $t_2 > t_1 \Rightarrow x \in S_i(t_2)$. Define the limit sets

$$S_1 = \bigcup_{t>0} S_1(t)$$

$$S_2 = \bigcup_{t>0} S_2(t).$$

Let x be a point in the complement $[S_1(t) \cup S_2(t)]'$. Then $|v(x,\tau) - v_3^*| \leq b$ for $\tau \leq t$, so that $G_v(v,w) \geq \nu$ for $\tau \leq t$. We differentiate the first equation of (2.15) and solve for v_x to find that

$$|v_x(x,t)| \geq e^{\nu t}\left|\,|v_x(x,0)| - c_1 \int e^{-\nu\tau}\, |w_x(x,\tau)|d\tau\right| \tag{2.42}$$

where c_1 is a bound for $|G_w|$.

We see from Lemmas 2.2 and 2.4 that there is a constant c_2 such that

$$|w_x(x,t)| \leq c_2(\gamma^{1/2} + \delta).$$

Thus the second integral on the right of (2.42) is bounded by $c_1c_2(\gamma^{1/2} + \delta)/\nu$. If we choose

$$C_2 > c_1 c_2/\nu,$$

and use the bound (2.40), we see that the "transversality hypothesis" (2.36) implies that (2.42) gives a lower bound of the form $|v_x(x,t)| \geq c_3 e^{\nu t}$ with $c_3 > 0$. In particular, v_x cannot change sign while $|v - v_3^*| \leq b$.

We conclude that the set $[S_1(t) \cup S_2(t)]'$ has no more components than $[S_1(0) \cup S_2(0)]'$ and that the length of each component is exponentially decreasing in t. Therefore the limit set $(S_1 \cup S_2)'$ consists of finitely many points.

We differentiate the equations (2.15) with respect to t to find that

$$(2.43)\quad \begin{matrix} w_{tt} \\ w_{tt} \end{matrix} - L \begin{matrix} v_t \\ w_t \end{matrix} = \begin{matrix} G_v(v,w) - G_v(p,q) & G_w(v,w) - G_w(p,q) \\ H_v(v,w) - H_v(p,q) & H_w(v,w) - H_w(p,q) \end{matrix} \begin{matrix} v_t \\ w_t \end{matrix}$$

Because G and H have bounded second derivatives, we see from Lemma 2.3 that

$$(2.44)\quad \begin{aligned} \|v_t\|_{L_2}^2 + \|w_t\|_{L_2}^2 &\leq [\|v_t(x,1)\|_{L_2}^2 + \|w_t(x,1)\|_{L_2}^2]e^{-2s(t-1)} \\ &+ c_3 \int_1^t er^{-2s(t-1)} \int_0^1 [(v - p)^2 + (w - q)^2][v_t^2 + w_t^2]dx\, d\tau. \end{aligned}$$

It is easily seen from (2.43) that since $|v_t| = |G|$ is bounded, $|w_t|$ is bounded on any time interval of the form $1 \leq t \leq T$. For a given $T > 2$ let m be a number such that

$$(2.45)\quad \begin{aligned} &|G| \leq m \quad \text{for } v_1^* - B' \leq v \varepsilon v_2^* + B',\ |w - w^*| \leq \gamma', \\ &|w_t| \leq m \quad \text{for } 1 \leq t \leq T. \end{aligned}$$

Then Lemma 2.4 and (2.44) yield a bound of the form

$$\|v_t\|_{L_2}^2 + \|w_t\|_{L_2}^2 \leq m^2\{2e^{-2s(t-1)} + c_4\delta^2\}.$$

The method used to derive the bound (2.33) now yields a bound of the form

$$|w_t| \leq mc_5\{e^{-2s(t-1)} + c_4\delta^2\}^{1/2} \tag{2.46}$$

for $t \geq 2$. Reduce δ_2 if necessary so that

$$\delta_2 < 1/(2c_5c_4^{1/2}) \tag{2.47}$$

and choose $T > 2$ so large that

$$c_5e^{-2s(T-1)} < 1/2.$$

Then if $\delta \leq \delta_2$ and if (2.45) is valid for $1 \leq t \leq T$, we see from (2.46) that

$$\|w_t(x,T)\|_{L_\infty} < m.$$

We conclude that (2.45) continues to hold for all larger t. That is, $|w_t|$ is uniformly bounded for $t \geq 1$.

We now define a continuous positive function $\mu(v,w)$ in $[v_1^* - B', v_2^* + B'] \times [w^* - \gamma', w^* + \gamma']$ in such a way that

$$\mu(v,w) = \mu_i \quad \text{for} \quad |v - v_i^*| \leq B', \ |w - w^*| \leq \gamma'. \tag{2.48}$$

We multiply the first equation of (2.43) by μv_t, the second by $[1 + 2\beta(q - w^*)]w_t$, add, and integrate to find that

$$\int_0^t \int_0^1 (\mu v_t^2 + w_t^2)dx\,dt + [\int_0^1 [1 + 2\beta(q - w^*)]w_x^2\,dt]_{\tau=0}^t = \tag{2.49}$$

$$= \int_0^t \int_0^1 [\mu G_v v_t^2 + (\mu G_w + [1 + 2\beta(q - w^*)]H_v)v_t w_t +$$

$$([1 + 2\beta(q - w^*)]H_w - \beta H)w_t^2]dxdt.$$

Because of (2.38c) and (2.48) the integrand on the right is nonpositive when $|v - v^*| \leq B'$ for $i = 1$ or 2. We see from (2.41) that there is a number

$T_1 \geq 1$ such that if $x \in S_i(\sigma)$, then $|v(x,\tau) - v^*| < B$ for $\tau \geq \sigma + T_1$. Since $|v_t|$ and $|w_t|$ are bounded for $t > 1$, the integral on the right is bounded by a constant (the integral from $t = 0$ to T_1) plus a multiple of

$$\int_{T_1}^{\infty} [S_1(\tau - T_1) \cup S_2(\tau - T_1)]' d\tau.$$

We have already shown that the set $[S_1(t) \cup S_2(t)]'$ consists of finitely many intervals, the length of each of which decreases like $e^{-\nu t}$. We conclude that this integral converges, so that the right-hand side of (2.49) remains bounded as $t \to \infty$. Therefore

$$\int_0^{\infty} \int_0^1 (v_t^2 + w_t^2) dx\, dt$$

is finite. By multiplying the first equation of (2.43) by v_{tt} and the second by w_{tt} and integrating, we find that the integral of $v_{tt}^2 + w_{tt}^2$ is also finite. It follows that

$$\text{(2.50)} \qquad \lim_{t \to \infty} \int_0^1 [v_t(x,t)^2 + w_t(x,t)^2] dx = 0.$$

Because $|w_t|$ remains bounded, the second equation in (2.15) shows that $|w_{xx}|$ also remains bounded. Since w is bounded, it follows that w and w_x are bounded and equicontinuous for $t \geq 1$.

We wish to show that as $t \to \infty$ the solution (v,w) converges to a steady state solution on the set $S_1 \cup S_2$.

Let I be a closed subinterval of S_1. Then there exists a T_2 such that

$$|v(x,t) - v_1^*| < B \quad \text{for } x \in I, \quad t > T_2.$$

Hence for $t > T_2$, $G_v(v,w) < -\nu$.

By Lemmas 2.2 and 2.4 $|w_x|$ is uniformly bounded. We now differentiate the first equation in (2.15) with respect to x and integrate to find that for $x \in I$

$$|v_x(x,t)| \leq e^{-\nu(t-T_2)}|v_x(x,T_2)| + (M/\nu)\sup |w_x|.$$

Thus $|v_x|$ is also uniformly bounded in I.

Hence the restriction of v to I is bounded and equicontinuous. We showed above that w and w_x are bounded and equicontinuous for $t \geq 1$. Thus any sequence $t_n \to \infty$ has a subsequence $t_n' \to \infty$ such that

$$v(x,t_n') \to r(x) \qquad \text{uniformly on } I$$

$$\left.\begin{aligned} w(x,t_n') &\to s(x) \\ w_x(x,t_n) &\to s'(x) \end{aligned}\right\} \qquad \text{uniformly on } [0,1].$$

The same argument can be used when I is replaced by any closed subset of $S_1 \cup S_2$. We exhaust $S_1 \cup S_2$ by such subsets and use a diagonal process to obtain a subsequence t_n'' such that

$$v(x,t_n'') \to r(x) \qquad \text{in } S_1 \cup S_2, \text{ uniformly on compact subsets}$$

$$\left.\begin{aligned} w(x,t_n'') &\to s(x) \\ w_x(x,t_n'') &\to s'(x) \end{aligned}\right\} \qquad \text{uniformly on } [0,1].$$

We take limits in the equations (2.15) and use the convergence property (2.50) to show that (r,s) is a weak solution, and hence a solution, of the steady state equations (2.9).

Since $|r - v_i^*| \leq B'$ on S_i, we have

$$r = \begin{cases} v_1(s) & \text{for } x \in S_1 \\ v_2(s) & \text{for } x \in S_2. \end{cases}$$

Thus s is a solution of the problem

$$s'' + \overline{H}(x,s) = 0$$

$$s'(0) = s'(1) = 0$$

where

$$\overline{H}(x,s) = \begin{cases} H(v_1(s),s) & \text{for } s \in S_1 \\ H(v_2(s),s) & \text{for } x \in S_2. \end{cases}$$

By (2.14b) $\overline{H}/[1 + 2\beta(s - w^*)]^{1/2}$ is decreasing in s. Since the complement of $S_1 \cup S_2$ is a finite set of points at which s and s' are continuous, a comparison argument applied to the variable $u = \beta^{-1}[(1 + 2\beta(s - w^*))^{1/2} - 1]$ shows that there is only one solution s with $|s - w^*| < \gamma'$. That is, every sequence t_n has a subsequence t_n' such that

$$v(x,t_n') \to r(x)$$
$$w(x,t_n') \to s(x)$$

where r and s are independent of the sequences t_n and t_n'. We conclude that

$$v(x,t) \to r(x)$$
$$w(x,t) \to s(x)$$

as $t \to \infty$, uniformly in closed subsets of $S_1 \cup S_2$. This is the statement of the Theorem.

3. EQUATIONS WITH SMALL DIFFUSION

To relate Theorem 2.1 to solutions of the nondegenerate system (2.1), (2.2) we state the following lemma, whose proof is essentially that of Lemma 1.

LEMMA 3.1 <u>For any given smooth initial values</u> $\phi(x)$, $\psi(x)$ <u>and any</u> $T > 0$ and $\delta > 0$ <u>there exists an</u> $\varepsilon_0 > 0$ <u>such that when</u> $0 < \varepsilon \leq \varepsilon_0$ <u>the solution</u> (v,w) <u>of the problem</u>

$$\text{(3.1)}\qquad \begin{aligned} v_t &= \varepsilon^2 v_{xx} + G(v,w) \\ w_t &= w_{xx} + H(v,w) \\ v_x &= w_x = 0 \quad \text{at} \quad x = 0,1 \\ v(x,0) &= \phi \\ w(x,0) &= \psi \end{aligned}$$

<u>and the solution</u> $(\bar{v},\bar{w})$ <u>of the corresponding degenerate problem</u>

$$\begin{aligned} \bar{v}_t &= G(\bar{v},\bar{w}) \\ \bar{w}_t &= \bar{w}_{xx} + H(\bar{v},\bar{w}) \\ \bar{w}_x &= 0 \quad \text{at} \quad x = 0,1 \\ \bar{v}(x,0) &= \phi \\ \bar{w}(x,0) &= \psi \end{aligned}$$

<u>satisfy</u>

$$\|v - \bar{v}\|_{L_\infty} + \|w - \bar{w}\|_{L_\infty} < \delta \quad \text{for} \quad t \leq T.$$

Lemma 3.1 with T large and δ small together with Theorem 2.1 shows that when ε is very small the solutions of (3.1) corresponding to a certain class of initial values appear to converge to a solution with discontinuous v if the observations are confined to $t \leq T$.

The following result shows that if

$$\text{(3.2)}\qquad \int_{v_1^*}^{v_2^*} G(r,w^*)\,dr \neq 0,$$

this apparent convergence is only temporary.

THEOREM 3.1 *Let the hypotheses (2.34) be satisfied. Let (p,q) be one of the discontinuous solutions of (2.9) constructed in Lemma 2.2. Let the hypotheses of Theorem 2.1 be satisfied. Suppose that the subset S of $[0,1]$ contains both points where $p = v_1(q)$ and points where $p = v_2(q)$. If the inequality (3.2) is valid and if γ, δ, and ε are sufficiently small, then the solution of (3.1) must leave any neighborhood of the form*

$$(3.3)\qquad \begin{aligned} &\|v - p\|_{L_\infty(S)} < A'\delta \\ &\|w - q\|_{L_\infty} < A\delta \\ &v_1^* - B' < v < v_2^* + B'. \end{aligned}$$

with A, A', B', γ, and δ as in Theorem 2.1 at some time.

Proof. Suppose, without loss of generality, that the integral on the left of (3.2) is positive. Suppose that $\gamma + A\delta < \gamma'$ so that $|w - w^*| < \gamma'$. Define $\overline{G}(v) = \min\{G(v,w): |w - w^*| < \gamma'\}$.

By (2.38a) $\overline{G}$ is strictly monotone for $|v - v_i^*| < B'$, $i = 1,2,3$, $\overline{G}(v_1^* - B') > 0$, $\overline{G}(v_1^* + B') < 0$, $\overline{G} < 0$ on the interval $[v_1^* + B', v_3^* - B']$ and $\overline{G} > 0$ on $[v_3^* + B', v_2^* - B']$. Thus $\overline{G}$ has the properties of the function $v(1 - v)(v - a)$ in Theorem 1.1. Therefore we can use the reasoning used in this Theorem to prove that if ε is sufficiently small the solution of the problem

$$(3.4)\qquad \begin{aligned} z_t &= \varepsilon^2 z_{xx} + \overline{G}(z) \\ z_x(0,t) &= z_x(1,t) = 0 \\ z(x,0) &= \begin{cases} p(x) - A'\delta & \text{in } S \\ v_1^* - B' & \text{in } S' \end{cases} \end{aligned}$$

approaches a constant which lies in the interval $[v_2^* - B', v_2^* + B']$. (Note that at those points where $p = v_2(q)$, $v(x,0)$ lies above $v_3^* + B'$.)

Because of the definition of $\overline{G}$ and (3.4)

$$v_t \geq \varepsilon^2 v_{xx} + \overline{G}(v)$$

$$v_x = 0 \quad \text{at} \quad x = 0 \text{ and } 1$$

$$v(x,0) \geq z(x,0).$$

By the maximum principle

$$v(x,t) \geq z(x,t).$$

This eventually violates the inequality $|v - p| \leq A'\delta$ at those points of S where $p = v_1(q)$.

If the integral on the left of (3.2) is negative, one replaces the variable v by $-v$ and carries out the same proof. Thus the Theorem is proved.

The proof shows that the admissible sizes of γ, δ, and ε may depend upon the state (p,q) but not upon the particular solution (v,w) which is to satisfy (3.3).

An immediate corollary of this Theorem is the fact that the system (3.1) has no steady state in the set (3.3).

We have shown that for small ε the system (3.1), like the scalar equation (1.1), displays metastable states which cannot be predicted by examining its steady states.

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