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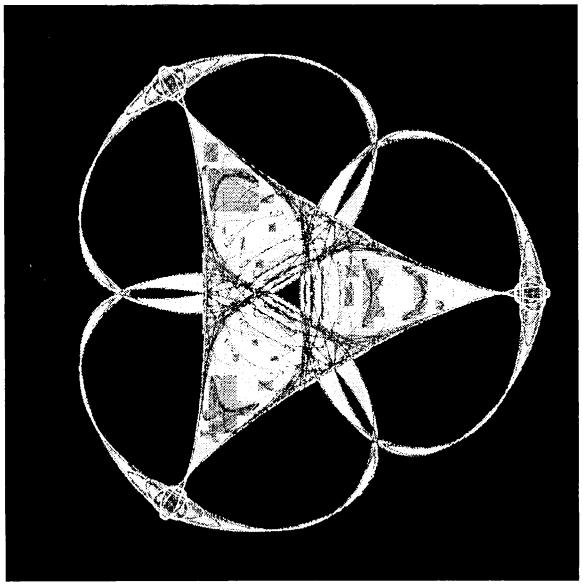
IMA Preprint Series # 1633

August 1999



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GENERATION OF LINEAR ALGEBRAIC SPACES FOR MULTIPLE HYPERGEOMETRIC FUNCTIONS

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The present paper deals with the generation of linear algebraic spaces for the multiple hypergeometric functions by taking parameters as matrices of order (n×n). Applications of these spaces in tensor algebra have been also discused.

***Key Words* :-** Vector space / Contravariant tensor / Covariant tensor / Matrices / Parameters.

1. INTRODUCTION

The process of multiplicative group formation has been discussed in our earlier works[1-3]. Motivated by a desire to go ahead in the field of abstract algebra with the aim of finding linear algebraic expressions for special functions, one may extend the theory of scalar function of scalar veriables to those of matrix variables. The technique developed for different functions by Mathai[7] Kumbhat[10], Kabe[8] John Greene[9] Saxena et al[11] Mathai & Pederzoli[7] can be cited in this context. The matrix arguments of such function may generate multiplicative group but our interest of obtaining linear algebraic forms can be served only if the parameters are taken as matrices of order (n×n) so that the multivariate functions

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$$\sum_{n=0}^{\infty} \left[\frac{(a_p)_n (b_r)_{-n}}{(a'_q)_n (b'_s)_{-n}} \right] \frac{x^n}{n!} = {}_{p+s}F_{q+r} \left[\begin{matrix} a_p & 1-b'_s \\ & ; & ; & (-1)^{r-s} x \\ a'_q & 1-b'_r \end{matrix} \right] \quad .... (1.1)$$

which for the sake of convience in our studies [4,5] has been represented as

$$B^{p\ r}_{q\ s} \left[\begin{matrix} a_p & b_r \\ & ; & ; & x \\ a'_q & b'_s \end{matrix} \right]$$

shall take the form

$$B^{p\ r}_{q\ s} \left[\begin{matrix} [A_p] & [C_r] \\ & ; & ; & x \\ [B_q] & [D_s] \end{matrix} \right]$$

$$= I + \frac{[A_p]([D_s] - [I])}{[B_q]([C_r] - [I])} x + \frac{[A_p]([A_p]+[I])([D_s] - [I])([D_s] - [2I])}{[B_q]([B_q]+[I])([C_r] - [I])([C_r] - [2I])} x^2 + \quad .... (1.2)$$

where $[B_q] + [rI]$ and $[C_r] - [rI]$ shall be non singular matrices $\forall \quad r \geq 0$. Appell, Lauricella, Kempe' de Fe'riet functions may also be generalised to matrix parameters in the same manner. If the matrices are replaced by their respective determinants then the function defined by (1.2) get converted in the ordinary classical B- function.

2. GENERAL FORMULATION

We shall assume

$$[A]_n = [A]([A] + [I])([A] + [2I]) .... ([A] + [(n-1)I]) \quad .... (2.1)$$

$$[A]_0 = I_{n\times n} \quad \text{(unit matrix)} \quad .... (2.2)$$

and

$$[A]_{-n} = \frac{I}{([A]-[I])([A]-[2I]).....([A]-[nI])} \quad ......(2.3)$$

Let $S = \{ B^{p\ r}_{q\ s}$ / parameters are $n \times n$ matrices} ..... (2.4)

$$B_1 \oplus B_2 = B\begin{bmatrix} [A_p]+[A'_p] & [C_r]+[c'_r] & \\ & ; & ;\ x \\ [B_q]+[B'_q] & [D_s]+[D'_s] & \end{bmatrix} \quad ......(2.5)$$

and F = { field of real no .}

such that $\alpha \in F$ and $B \in S$

$$\alpha \odot B = B\begin{bmatrix} [\alpha A_p] & [\alpha C_r] & \\ & ; & ;\ x \\ [\alpha B_q] & [\alpha D_s] & \end{bmatrix} \quad ....(2.6)$$

then S (F) shall be a vector space

The algebraic spaces shall be linear, if for the space

$$S(F) = \{B_1, B_2, ..., B_n\} \quad .....(2.7)$$

the condition

$$\alpha_1 B_1 + \alpha_2 B_2 + ... + \alpha_n B_n = I \quad ....(2.8)$$

gets converted as

$$B\begin{bmatrix} \alpha_1 A_1 + \alpha_2 A_2 + ... + \alpha_n A_n & \alpha_1 C_1 + \alpha_2 C_2 + ... + \alpha_n C_n & \\ & ; & ;\ x \\ \alpha_1 B_1 + \alpha_2 B_2 + ... + \alpha_n B_n & \alpha_1 D_1 + \alpha_2 D_2 + ... + \alpha_n D_n & \end{bmatrix} = I \quad ......(2.9)$$

that is

$$\left.\begin{aligned} \Sigma \alpha_i \; A_i &= 0 \\ \Sigma \alpha_i \; B_i &= 0 \\ \Sigma \alpha_i \; C_i &= 0 \\ \Sigma \alpha_i \; D_i &= 0 \end{aligned}\right] \qquad \text{..... (2.10)}$$

3. SPECIAL CASES

(a) when $A = B = I$ and $C = D = 0$ (Null matrices)

then

$$B^{p\,q}_{r\,s}\left[\begin{matrix} I & - \\ & ; & ; & x \\ I & - \end{matrix}\right] = e^x \; I \qquad \text{...... (3.1)}$$

where I is a unit matrix of order (n×n)

(b) when $A_p = B_q = I$, $C_r = D_s = 0$

and $p = q + 1$ then

$$B^{p\,r}_{q\,s}\left[\begin{matrix} I, I, \ldots\ldots\ldots, I & - \\ & ; & ; & x \\ I, I, \ldots.., I & - \end{matrix}\right] = I\;(1 + x\,e^{x}) \qquad \text{..... (3.2)}$$

4. DIFFERENTIALS, INTEGRALS AND RECURRENCE RELATIONS

From result (1.2) we can have following result as first differential coefficient

$$\frac{d}{dx}\left(B^{p\,r}_{q\,s}\left[\begin{matrix} A_p & C_r \\ & ; & ; & x \\ B_q & D_s \end{matrix}\right]\right)$$

$$= \frac{\prod_{i=1}^{p} [A_i] \quad \prod_{i=1}^{r} [C_i]_{-1}}{\prod_{i=1}^{q} [B_i] \quad \prod_{i=1}^{s} [D_i]_{-1}} \; B^{p\,r}_{q\,s}\left[\begin{matrix} A_p + I & C_r - I \\ & ; & ; & x \\ B_q + I & D_r - I \end{matrix}\right] \qquad \text{... (4.1)}$$

and

$$\frac{d^m}{dx^m}\left\{ B^{p\ r}_{q\ s}\left[\begin{matrix} A_p & C_r & \\ & ; & ; \quad x \\ B_q & D_s & \end{matrix}\right]\right\}$$

$$= \frac{\prod\limits_{i=1}^{p} [A_i]_m \quad \prod\limits_{i=1}^{r} [C_i]_{-m}}{\prod\limits_{i=1}^{q} [B_i]_m \quad \prod\limits_{i=1}^{s} [D_i]_{-m}} \; B^{p\ r}_{q\ s}\left[\begin{matrix} A_p+mI & C_r-mI & \\ & ; & ; \quad x \\ B_q+mI & D_s-mI & \end{matrix}\right] \qquad \text{.... (4.2)}$$

Results on account of integration process shall be

$$\int_0^x \left\{ B^{p\ r}_{q\ s}\left[\begin{matrix} A_p & C_r & \\ & ; & ; \quad x \\ B_q & D_s & \end{matrix}\right]\right\} dx$$

$$= \frac{\prod\limits_{i=1}^{p} [A_i]_{-1} \quad \prod\limits_{i=1}^{r} [C_i]}{\prod\limits_{i=1}^{q} [B_i]_{-1} \quad \prod\limits_{i=1}^{s} [D_i]} \left\{ B^{p\ r}_{q\ s}\begin{matrix} A_p-I & C_r+I & \\ & ; & ; \quad x \quad - I \\ B_q-I & D_s+I & \end{matrix}\right\} \qquad \text{..... (4.3)}$$

and

$$\int_0^x \int_0^x \cdots \int_0^x B^{p\ r}_{q\ s}\left[\begin{matrix} A_p & C_r \\ & ; & ; x \\ B_q & D_s \end{matrix}\right] (dx)^m$$

$$= \frac{\prod_{i=1}^{p} [A_i]_{-m} \prod_{i=1}^{r} [C_i]_m}{\prod_{i=1}^{q} [C_i]_{-m} \prod_{i=1}^{s} [D_i]_m} \left\{ B^{p\ r}_{q\ s} \left[\begin{matrix} A_p - mI & C_r + mI \\ & ; & ; x & - I \\ B_q - mI & D_s + mI \end{matrix}\right] \right\}$$

$$- \sum_{k=0}^{m} \left[\frac{\prod_{i=1}^{p} [A_i]_{-k} \prod_{i=1}^{r} [C_i]_k}{\prod_{i=1}^{q} [B_i]_k \prod_{i=1}^{s} [D_i]_k} \right] \frac{x^{n-k}}{(n-k)!} \qquad \text{..... (4.4)}$$

Recurrence relations for the function (1.2) shall be of following eight different types when parameters A_p, B_q, C_r, D_s assume single values.

$$B(A^+) = B(A + I), \qquad B(A^-) = B(A - I) \qquad \text{.... (4.5)}$$

$$B(B^+) = B(B + I) \qquad B(B^-) = B(B - I) \qquad \text{.... (4.6)}$$

$$B(C^+) = B(C + I) \qquad B(C^-) = B(C - I) \qquad \text{.... (4.7)}$$

$$B(D^+) = B(D + I) \qquad B(D^-) = B(D - I) \qquad \text{.... (4.8)}$$

Total such results for $B^{p\ r}_{q\ s}$ shall be

$2^{p+q+r+s}-1.$

If $$ {}_{r\,s}^{p\,q}B\left[\begin{matrix} A_p & C_r & \\ ; & ; & x \\ B_q & D_s & \end{matrix}\right] = \sum_{n=0}^{\infty} \delta_n \frac{x^n}{n!} \qquad \text{...... (4.9)}$$

the following results can be seen

$$B[A^+] = \sum_{n=0}^{\infty} (I + nA^{-1}) \frac{\delta_n x^n}{n!} \qquad \text{.... (4.10)}$$

$$B[A^-] = \sum_{n=0}^{\infty} \frac{(I - A^{-1})}{I - (n-1)A^{-1}} \frac{\delta_n x^n}{n!} \qquad \text{..... (4.11)}$$

$$B[B^+] = \sum_{n=0}^{\infty} (I + nB^{-1})^{-1} \frac{\delta_n x^n}{n!} \qquad \text{..... (4.12)}$$

$$B[B^-] = \sum_{n=0}^{\infty} \frac{(I+(n-1)B^{-1})}{(I-B^{-1})} \frac{\delta_n x^n}{n!} \qquad \text{...... (4.13)}$$

$$B[C^+] = \sum_{n=0}^{\infty} (I-nC^{-1}) \frac{\delta_n x^n}{n!} \qquad \text{..... (4.14)}$$

$$B[C^-] = \sum_{n=0}^{\infty} \frac{(I-C^{-1})}{I-(n-1)C^{-1}} \frac{\delta_n x^n}{n!} \qquad \text{..... (4.15)}$$

$$B(D^+) = \sum_{n=0}^{\infty} \frac{I}{(I - nD^{-1})} \frac{\delta_n x^n}{n!} \qquad \text{...... (4.16)}$$

$$B(D^-) = \sum_{n=0}^{\infty} \frac{[I - (n-1)D^{-1}]}{[I-D^{-1}]} \frac{\delta_n x^n}{n!} \qquad \text{....... (4.17)}$$

5. APPLICATIONS IN TENSOR ALGEBRA

Assuming result (1.2) in the form

$$T^{p\quad r}_{q\quad s} = B^{p\quad r}_{q\quad s}\begin{bmatrix} A_p & & C_r & & \\ & : & & ; & x \\ B_q & & D_s & & \end{bmatrix} \qquad \ldots (5.1)$$

where T is multiscript tensor for which dimensions of contravariants of first category shall vary from 1 to p while dimensions of its second category shall vary from 1 to r. Same is true for two types of covariants. It finally justifies the outer product of tensors as

$$T^{p\quad r}_{q\quad s} = T^{p}_{q} \;.\; T^{s}_{r} \qquad \ldots (5.2)$$

if all matrices are unequal

above result follows from the properties of B- function given by result, (2.2) & (2.7) of our earlier studies[5].

If p_1 matrices of first pair of contra and covariant are equal then result (5.2) shall be applicable to inner products in the form

$$T^{(p,\,p_1)\quad r}_{(q,\,p_1)\quad s} = T^{p-p_1\quad r}_{q-p_1,\quad s}$$

$$= T^{p\,-\,p_1}_{q\,-\,p_1} \;.\; T^{s}_{r} \qquad \ldots (5.3)$$

Same criterion shall hold true for second kind contra and convariant pairs. The tensors described here represent those curves for which n dimensional point shall be

(1,1,1, ..., 1), (2,2,2, ..., 2) ..... (n, n, n, ..., n).

B- function defined by relation (1.2) shall generate the norm of tensor in case if the matrices are replaced by their respective determinants so as to obtain

$$T^{p\,r}_{q\,s} \; . \; T^{r\,p}_{s\,q} = \| T \|^2 \qquad \text{... (5.4)}$$

Here if $\| T \| = 1$

then the two tensors

$$T^{p\,r}_{q\,s} \quad \text{and} \quad T^{r\,p}_{s\,q}$$

shall automatically be reciprocal.

with the help of result (2.5) of our ealier study[4] we shall also have

$$T_1{}^{p\,r}_{q\,s} + T_2{}^{p\,r}_{q\,s} = (T_1+T_2)^{p\,r}_{q\,s} \qquad \text{... (5.5)}$$

which depicts the fact that two same type of tensors can be added.

Scalar multiplication follows from result (2.6), in the form

$$(\lambda T)^{p\,r}_{q\,s} = \lambda \; T^{p\,r}_{q\,s} \qquad \text{.... (5.6)}$$

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