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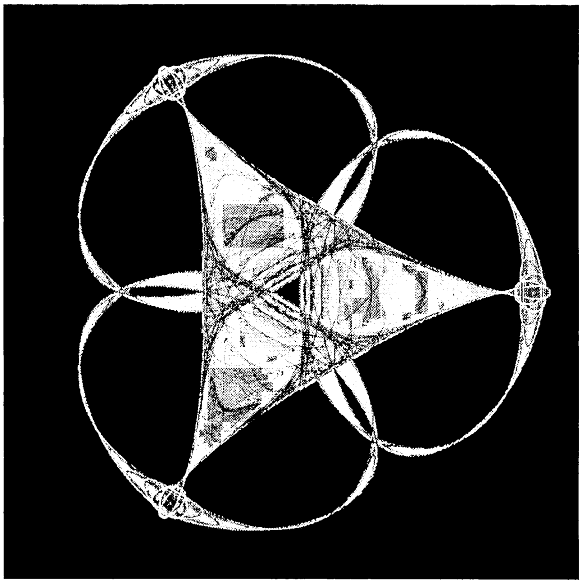

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GENERATION OF BASIC, BILATERAL AND BASIC BILATERAL SERIES FOR B- FUNCTION

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The present paper deals with the formation of basic, bilateral and basic bilateral series representations of our newly generated B- function and derivation of some of their properties.

Key words **:-** Complex number / Appell series / Lauricella function / Bilateral series.

1. INTRODUCTION

Interpreting a basic number as

$$a_q = \frac{1-q^a}{1-q} \qquad \text{............ (1.1)}$$

where q and a are real or complex numbers, so that as

$$\text{as} \qquad q \to 1, \qquad [\,(1-q^a)\ /\ (1-q)\,] \to a\,;$$

the basic analogue of the Gauss series is defined as

$$1- \frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)}\, x + \frac{(1-q^a)\,(1-q^{a+1})\,(1-q^b)\,(1-q^{b+1})}{(1-q^c)\,(1-q^{c+1})\,(1-q)\,(1-q^2)}\, x^2 + \qquad \text{............ (1.2)}$$

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where $|q| < 1$.

Such type of extension to the basic number field, so that we have basic exponential, trigonometric and hyperbolic functions, basic analogues of Bessel, Weber and Airy functions, and basic Legendre, Laguerre, Hermite and Gegenbaur polymonials ; has applications in the field of pure mathematics, particularly in number theory, modular equations and elliptic integrals. The advent of electronic computers has opened the field of their utility in applied mathematics branches also.

For Latest contributions in the field of basic functions we can cite the references of Devendra Kandu[2] for establishing certain expansions involving basic hypergeometric functions of two variables and certain symbolic relations and expansions associated with basic hypergeometric functions of three variables. Saxena & Gupta[4] have derived certain q-expansions of multivariate basic hypergeometric functions and their transformations. A most general known analytic auxiliary functional generalization has been given by Singh[5] which can be used to give combinatorial interpretations of generalized q - identities of the Rogers- Ramanujan type.

2. GENERAL FORMULATION

In addition of established expansions we shall employ following notations

$$(a,q)_{-n} = q \Big/ (1-a/q^2)(1-a/q^3) \ldots\ldots\ldots (1-a/q^{n+1}) \qquad \ldots\ldots\ldots (2.1)$$

whose special case shall be

$$(q,q)_{-n} = q \Big/ (1-1/q)(1-1/q^2) \ldots\ldots\ldots (1-1/q^n), \quad q \neq 1 \qquad \ldots\ldots (2.2)$$

and

$$(a, q)_{N-n} = (aq^{-n}, q)_N \Big/ (a, q)_{-n} \qquad \text{.......... (2.3)}$$

under these expansions & notations the q - analogue of the B- function defined by our earlier study[1] relation (2.3) shall be

$$ {}_{Pr}Y_{qs} = \begin{matrix} P & r \\ & Y \\ q & s \end{matrix} \left[\begin{matrix} (a_p) & & (c_r) & & \\ & ; & & ; & q, z \\ (b_q) & & (d_s) & & \end{matrix} \right]$$

$$= \sum_{m=0}^{\infty} \frac{(a_p, q)_m \quad (c_r, q)_{-m} \quad z^m}{(b_q, q)_m \quad (d_s, q)_{-m} \quad (q, q)_m} \qquad \text{....... (2.4)}$$

which shall be basic B- function

The bilateral and basic bilateral B- function shall be

$$ {}_{AC}P_{BD} \left[\begin{matrix} (a_A) ; (c_C) & \\ & ; x \\ (b_B) ; (d_D) & \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_A)_n (c_C)_{-n}}{(b_B)_n (d_D)_{-n}} x^n \qquad \text{...... (2.5)}$$

$$ {}_{AC}X_{BD} \left[\begin{matrix} (a_A) ; (c_C) & \\ & ; q, x \\ (b_B) ; (d_D) & \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_A, q)_n (c_C, q)_{-n}}{(b_B, q)_n (d_D, q)_{-n}} x^n \qquad \text{..... (2.6)}$$

which is symmetric in itself so far as parameters are concerned however the variables assume the inverse form, as it is evident from the following result

$$ {}_{AC}X_{BD} \left[\begin{matrix} (a_A) ; (c_C) & \\ & ; q, x \\ (b_B) ; (d_D) & \end{matrix} \right] = {}_{CA}X_{DB} \left[\begin{matrix} (c_C) ; (a_A) & \\ & ; q, x^{-1} \\ (d_D) ; (b_B) & \end{matrix} \right] \qquad \text{... (2.7)}$$

3. REPRESENTATIONS IN THE FORM OF ORDINARY BASIC, BILATERAL & BASIC BILATERAL FUNCTIONS

The basic B- function defined by (2.4) can be expressed as

$$ {}_{Pr}Y_{qr}\left[\begin{matrix}(a_p)\ ;\ (c_r) \\ \\ (b_q)\ ;\ (d_r)\end{matrix}\ ;\ q,\ x\right] = {}_{P+r}\Phi_{q+r}\left[\begin{matrix}(a_p)\ ;\ (q/d_r) \\ \\ (b_q)\ ;\ (q/c_r)\end{matrix}\ ;\ q,\ (d_1 \ldots\ldots d_r\ /\ c_1\, c_2 \ldots\ldots c_r)\, x\right] \quad \ldots.\ (3.1)$$

If $d_i = c_i$ then it will become ordinary basic hypergeomitric function

$$ {}_P\Phi_q\ [\ a_p\ ;\ b_q\ ;\ q,\ x\ ] \quad \ldots\ldots\ldots\ (3.2)$$

expression (2.5) can be converted as

$$ {}_{AC}P_{BD}\left[\begin{matrix}(a_A)\ ;\ (c_C) \\ \\ (b_B)\ ;\ (d_D)\end{matrix}\ ,\ x\right] = {}_{A+D}H_{B+C}\left[\begin{matrix}(a_A)\ ,\ (1-(d_D)) \\ \\ (b_B)\ ,\ (1-(c_C))\end{matrix}\ ;\ (-1)^{C-D}\, x\right] \quad \ldots\ldots.(3.3)$$

which shall become ordinary bilateral series in case if $c_i = d_i$.

Result (2.6) can be transmogrified if $D = C$

$$ {}_{A+D}\Psi_{B+C}\left[\begin{matrix}(a_A)\ ,\ (q/d_D) \\ \\ (b_B)\ ,\ (q\ /\ c_C)\end{matrix}\ ;\ (d_D\ /\ c_C)\, x\right] \quad \ldots\ldots\ (3.4)$$

which shall assume the form of ordinary basic bilateral series under the same condition as applied for (3.2).

4. APPLICATION OF GAUSS THEOREM

Result of Gauss theorem, given in the book of Slater[3] can be applied to (2.5) for the case $A = B = C = D = 1,$ so as to produce

$$ {}_{11}P_{11}\begin{bmatrix} a\ ;\ c & \\ & ;\ 1 \\ b\ ;\ d & \end{bmatrix} = {}_2H_2\begin{bmatrix} a,\ 1-d & \\ & ;\ 1 \\ b,\ 1-c & \end{bmatrix} \qquad \ldots\ldots (4.1)$$

which can also be expressed in the from of generalized hypergeometric function as

$$ {}_3F_2\begin{bmatrix} 1,\ a,\ 1-d & \\ & ;\ 1 \\ b,\ 1-c & \end{bmatrix} + \frac{(1-b)\ c}{(1-a)\ d}\ {}_3F_2\begin{bmatrix} 1,\ 2-b,\ 1+c & \\ & ;\ 1 \\ 2-a,\ 1+d & \end{bmatrix} \qquad \ldots\ldots\ldots (4.2)$$

for general values of $A,\ B\ \&\ C$ we shall have

$$ P^{AC}_{BC}\begin{bmatrix} (a_A)\ \ (c_C) & \\ ; & ;\ q,\ x \\ (b_B)\ \ (d_C) & \end{bmatrix} - P^{AC}_{BC}\begin{bmatrix} (a_A)\ \ (c_C) & \\ ; & ;\ q,\ qx \\ (b_B)\ \ (d_C) & \end{bmatrix} $$

$$ = \frac{\prod_{i=1}^{A}(1-a_i)}{\prod_{i=1}^{B}(1-b_i)}\ \prod_{i=1}^{r}\frac{d_r - q}{c_r - q}\ P^{AC}_{BC}\begin{bmatrix} (a_A q),\ (c_C/q) & \\ & ;\ q,\ x \\ (b_B q),\ (d_C/q) & \end{bmatrix} \qquad \ldots\ldots (4.3)$$

and

$$ P^{AC}_{BC}\begin{bmatrix} (a_A)\ \ (c_C) & \\ ; & ;\ q,\ x \\ (b_B)\ \ (d_C) & \end{bmatrix} - a_k\ P^{AC}_{BC}\begin{bmatrix} (a_A)\ \ (c_C) & \\ ; & ;\ q,\ qx \\ (b_B)\ \ (d_C) & \end{bmatrix} $$

$$= (1-a_k)\ P^{AC}_{BC}\left[\begin{array}{ccc} a_1, a_2, \ldots\ldots a_{k-1}, a_{k+1}, \ldots\ldots, a_A, a_k q & & c_C \\ & ; & ; q, x \\ (b_B) & (d_C) & \end{array}\right] \qquad \ldots(4.4)$$

Result (4.1) can assume following forms also.

$${}_{11}P_{11}\left[\begin{array}{ccc} a & c & \\ ; & & ; 1 \\ b & d & \end{array}\right] = {}_{11}P_{11}\left[\begin{array}{cccc} 1-d & c & & \\ & ; & ; & 1 \\ b & 1-a & & \end{array}\right] \qquad \ldots\ldots (4.5)$$

$$= {}_{11}P_{11}\left[\begin{array}{cccc} 1-d & 1-b & & \\ & ; & ; & 1 \\ 1-c & 1-a & & \end{array}\right] \qquad \ldots. (4.6)$$

$$= {}_{02}P_{02}\left[\begin{array}{cccc} - & c\ , & 1-b & \\ ; & & & ;\ 1 \\ - & d\ , & 1-a & \end{array}\right] \qquad \ldots\ldots (4.7)$$

$$= {}_{20}P_{20}\left[\begin{array}{cccc} a, & 1-d & - & \\ & & ; & ;\ 1 \\ b\ , & 1-c & - & \end{array}\right] \qquad \ldots. (4.8)$$

Above explanation suggests that the number of total results for the function (2.5) shall be equal to the P(A+D) . P (B+C)

where P (A) stands for partition of A in which

we shall exclude all results of the form ${}_{AC}P_{BD}$, $C \neq D$.

5. BASIC APPELL SERIES FOR B- FUNCTION

Result (2.5) can generate basic Appell series of following type

$$P^{(1)}\left[\begin{matrix} a\,, & (b,\, b') & d & \\ & ; & ; & q,\ x,\ y \\ c & & e & \end{matrix}\right]$$

$$= \sum_{m\,n=0}^{\infty} \frac{(a,\ q)_{m+n}\ (b,q)_m\ (b',\ q)_n\ (d,\ q)_{-(m+n)}\ x^m\ y^n}{(c,\ q)_{m+n}\ \ (e)_{-(m+n)}\ \ (q,\ q)_m\ \ (q,\ q)_n} \qquad \text{....... (5.1)}$$

whose general form shall be

$$\begin{matrix}(p_1,\ p_2)\ (r_1,\ r_2) \\ P \\ (q_1,\ q_2)\ (s_1,\ s_2)\end{matrix}\left[\begin{matrix} (a_A^{\ k},\ \ell_k),\ (a_\alpha) & (c_C^{\ k},\ \ell_k),\ (c_\chi) & \\ ; & ; & q,\ x_1,\ x_2,\ \ldots\ldots,\ x_n \\ (a_B^{\ k},\ \ell_k),\ (b_B) & (d_D^{\ k},\ \ell_k)\ (d_\delta) & \end{matrix}\right]$$

$$= \sum_{m_i=0}^{\infty}\left[\prod_{i=1}^{k} \frac{(a_A^{\ i},\ q)_{\sum m_{\ell_i}}\ \ (c_C^{\ i},\ q)_{-\sum m_{\ell_i}}}{(b_B i,\ q)_{\sum m_{\ell_i}}\ \ (d_D^{\ i},\ q)_{-\sum m_{\ell_i}}}\right.$$

$$\left. \cdot\ \frac{(a_\alpha,\ q)_m\ \ (c_\chi,\ q)_{-m}\ \ x_1^{\ m_1}\ \ x_2^{\ m_2}\ \ldots\ldots\ldots\ x_n^{\ m_n}}{(b_\beta,\ q)_m\ \ (d_\delta\,,\ q)_{-m}\ (q,\ q)_{m_1}\ (q,\ q)_{m_2}\ \ldots\ldots\ (q,\ q)_{m_n}}\right] \qquad \text{....... (5.2)}$$

Lauricella form for result (5.1) shall be

$$P\left[\begin{matrix} a\,,\ (b_1,\ b_2,\ \ldots\ldots\ldots,\ b_n) & d & \\ ; & ; & q,\ \ x_1,\ \ x_2,\ \ldots\ldots\ldots,\ x_n \\ c & e & \end{matrix}\right]$$

$$= \sum_{m_i = 0}^{\infty} \frac{(a, q)_{m_i} (b_1, q)_{m_1} (b_2, q)_{m_2} \cdots\cdots (b_n, q)_{m_n} (d)_{-m_i} x_1^{m_1} \cdots\cdots x_n^{m_n}}{(c, q)_{m_i} \; (e)_{-m_i} \; (q, q)_{m_1} \; (q, q)_{m_2} \cdots\cdots (q, q)_{m_n}} \qquad \text{.....(5.3)}$$

6. APPLICATION OF JACKSON'S THEOREM TO BILATERAL SERIES

Jakson's theorem usable in the deduction of simple basic summation theorems can be applied to (2.5) so as to produce.

$${}_{43}^{53}P \left[\begin{matrix} a, \; b, \; c, \; d, \; e & ; & b/a, \; c/a, \; d/a \\ \sqrt{a}, \; -\sqrt{a}, \; aq^{N+1}, \; aq/c & & 1/\sqrt{a}, \; -1/\sqrt{a}, \; q^{N+1} \end{matrix} ; q, \; \frac{-q^{N+2} a^2}{bcd} \right]$$

$$= \left[\begin{matrix} aq, \; aq/cd, \; aq/bd, \; aq/bc \\ aq/b, \; aq/c, \; aq/d, \; aq/bcd \end{matrix} \right] (q, N) \qquad \text{.... (6.1)}$$

Here if we replace d by aq / d and e by adq^n / bc , then we shall have

$${}_{34}^{44}P \left[\begin{matrix} a, \; b, \; c, \; adq^N / bc & ; & b/a, \; c/a, \; q^N / bc, \; q^{-N} / a \\ \sqrt{a}, \; -\sqrt{a}, \; d & & 1/\sqrt{a}, \; -1/\sqrt{a}, \; d/a, \; q^{N+1} \end{matrix} ; q, \; -daq^{N+1} \right]$$

$$= \frac{(aq, q)_N \; (d/c, q)_N \; (d/b, q)_N \; (aq/bc, q)_N}{(aq/b, q)_N \; (aq/c, q)_N \; (d, q)_N \; (d/bc, q)_N} \qquad \text{....... (6.2)}$$

which shall take the form of Saalschutz's in case if $a \to \infty$, whose result shall be

$${}_{11}^{21}P \left[\begin{matrix} b, \; c \quad dq^N / bc & ; & \\ d & & q^{N+1} \end{matrix} ; q, \; bcq^2 / d \right]$$

$$= \frac{(d/c, q)_N \; (d/b, q)_N}{(d, q)_N \; (d/bc, q)_N} \qquad \text{.... (6.3)}$$

which is a basic analogue of saalschutz's theorem.

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