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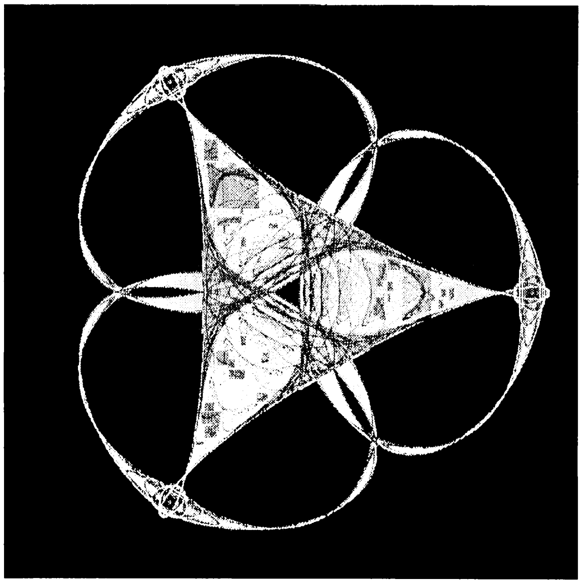

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A NEW GENERATING FUNCTION FROM THE VIEW POINT OF CHANGE IN THE NATURE OF RANDOM VARIABLE IN HYPERGEOMETRIC DISTRBUTIONS

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Contrivance of a new generating function, to cope up with the variation in the nature of random variable in hypergeometric distributions, has been the subject of study in the present paper. Bunch differentiation and integral formulae have been also acquired for the function.

Key words:- Hypergeometric distributions / random variable / transcedental functions / bunch formula

1. INTRODUCTION

Statisticians dealing with the theory of exceedances or problems of drawing balls from an urn come across hypergeometric distributions discussed in the book of Johnson and Kotz[2]. In some cases such situations may arise where the random variable assumes values, different from the traditional ones, may be multiples of prime numbers or may involve different powers. To overcome such situations a new type of generating function is required and it has been the reason of genesis of the present paper.

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2. GENERAL FORMULATION

Let us consider a generating function defined in the following manner

$${}_{K}A^{p\ \ r}_{q\ \ s} = A^{p\ \ r}_{q\ \ s}\left[\begin{matrix}|(a_p), p_1| & ; & |(c_r), r_1| & ; & (z,k)\\ |(b_q), q_1| & & |(d_s), s_1| & & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{(a_p)_{p_1 n}\ (c_r)_{-r_1 n}}{(b_q)_{q_1 n}\ (d_s)_{-s_1 n}}\right)\frac{z^{kn}}{(kn)!} \qquad ....(2.1)$$

Which can be expressed in the form

$$\sum_{n=0}^{\infty}\frac{p_1^{p_1 n}\ (a_p/p_1)_n\ (|(a_p)+1|/p_1)_n \text{ ----------- } (|(a_p)+p_1-1|/p_1)_n}{q_1^{q_1 n}\ (b_q/q_1)_n\ (|(b_q)+1|/q_1)_n \text{ ----------- } (|(b_q)+q_1-1|/q_1)_n}$$

$$\cdot\ \frac{(r_1)^{-r_1 n}\ (c_r/r_1)_{-n}\ (|(c_r)+1|/r_1)_{-n}\text{-------}\ (|(c_r)+r_1-1|/r_1)_{-n}\ \ z^{kn}}{(s_1)^{-s_1 n}\ (d_s/s_1)_{-n}\ (|(d_s)+1|/s_1)_{-n}\text{---------}(|(d_s)+s_1-1|/s_1)_{-n}(1)_{kn}} \qquad ....(2.2)$$

$$= \sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p_1-1}(|(a_p)+i|/p_1)_n\ \prod_{i=0}^{r_1-1}(|(c_r)+i|/r_1)_{-n}\ \left[\frac{p_1^{p_1}\ \ s_1^{s_1}\ \ z^k}{q_1^{q_1}\ \ r_1^{r_1}\ \ k^k}\right]^n}{\prod_{i=0}^{q_1-1}(|(b_q)+i|/q_1)_n\ \prod_{i=0}^{s_1-1}(|(d_s)+i|/s_1)_{-n}\ \prod_{i=0}^{k-2}(|1+i|/k)_n\ \ n!} \qquad ....(2.3)$$

Whose value in terms of B - function, generated in our earlier study[1], shall be

$$B^{pp_1\ \ \ rr_1}_{qq_1+k-1\ \ ss_1}\left[\begin{matrix}|(a_p)+i|/p_1 & & |(c_r)+i_2|/r_1 & & \\ & ; & & ; & \frac{p_1^{p_1}\ \ s_1^{s_1}\ \ (z/k)^k}{q_1^{q_1}\ \ r_1^{r_1}}\\ |(b_q)+j)/q_1\ ,\ |1+i_1|/k & & |(d_s)+j_1|/s_1 & & \end{matrix}\right]$$

...........(2.4)

3. SPECIAL CASES

The function defined by (2.1) possesses, as usual, the imporant property of special functions as it can be expressed in terms of other transcedental functions. It is evidentfrom the following special cases.

(1). Taking

$$r = 0 = s, \quad p_1 = 1 = q_1 \;, \quad r_1 = 0 = s_1 \;, \quad k = 1$$

We get

$$A^{p\;\;0}_{q\;\;0}\left[\begin{matrix}(a_p, 1) \\ (b_q, 1)\end{matrix} \;;\; \begin{matrix}- \\ -\end{matrix} \quad (z, 1)\right] = \sum_{n=0}^{\infty} \frac{(a_p)_n}{(b_q)_n} \frac{z^n}{n!}$$

$$= {}_pF_q\left[\begin{matrix}(a_p) \\ (b_q)\end{matrix} \;;\; z\right] \text{...............(3.1)}$$

(II) When

$$r = r_1 = s = s_1 = 0\,, \qquad p_1 = 2,\; q_1 = 1,\; k = 1, \qquad p = q = 1$$

The functions gets converted into the form of Gauss hypergeometric function as depicted below

$$A^{1\;\;0}_{1\;\;0}\left[\begin{matrix}(a_1, 2) \\ (b_1, 1)\end{matrix} \;;\; \begin{matrix}- \\ -\end{matrix} \;;\; (z, 1)\right] = \sum_{n=0}^{\infty} \frac{(a_1)_{2n}}{(b_1)_n} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{2^{2n} \quad (a_1/2)_n \quad ([a_1+1]/2)_n}{(b_1)_n}\right] \frac{z^n}{n!}$$

$$= {}_2F_1 \; [a_1/2, (a_1+1)/2; b_1 \;;\; 4z] \text{ ....(3.2)}$$

(iii) When $p_1 = 2, \; p = q_1 = q = 1, \; k = 2, \; r = r_1 = s = s_1 = 0$

$$A^{1\;\;0}_{1\;\;0}\left[\begin{matrix}(a_1, 2) \\ (b_1, 1)\end{matrix} \;;\; \begin{matrix}- \\ -\end{matrix} \;;\; (z, 2)\right] = {}_2F_2 \quad [a_1/2, (a_1+1)/2; b_1, 1/2; z^2] \text{ ....(3.3)}$$

(vi)
In case if

$$p = 1,\ p_1 = 3 \quad ; \quad q = q_1 = 1;\ k = 3$$

and the rest parameters are zero, then

$$A\ {}^{1\ \ 0}_{1\ \ 0}\left[\begin{matrix}(a_1, 3) & - \\ ; & \\ (b_1, 1) & - \end{matrix} ;\ (z, 3)\right] = {}_3F_3|a_1/3, (a_1+1)/3, (a_1+2)/3; 1/2, 1/3, 2/3; z^3| \qquad \text{..........................(3.4)}$$

Result |(3.2) to (3.4) | delineate the different type of generalized hypergeometric functions involving multiples and powers of z.

4. BUNCH DIFFERENTIATION OF THE FUNCTION

Let $\theta = d^k/dz^k$ so that

$$\theta\, z^{kn} = kn\ (kn-1) \text{--------} (kn - k + 1)\, z^{kn-k}$$

and for the function given by (2.1)

$$\theta A = \sum_{n=0}^{\infty} \left(\frac{(a_p)_{p_1(n+1)}\ (c_r)_{-r_1(n+1)}}{(b_q)_{q_1(n+1)}\ (d_s)_{-s_1(n+1)}} \right) \frac{z^{kn}}{(kn)!} \qquad \text{..............(4.1)}$$

Which can be represented in the form

$$\sum_{n=0}^{\infty} \left(\frac{(a_p)_{p_1}\ (a_p + p_1)_{n\,p_1}\ (c_r)_{-r_1}\ (c_r - r_1)_{-r_1 n}}{(b_q)_{q_1}\ (b_q + q_1)_{n\,q_1}\ (d_s)_{-s_1}\ (d_s - s_1)_{-s_1 n}} \right) \frac{z^{kn}}{kn!} \qquad \text{......(4.2)}$$

or

$$\frac{\prod_{i=1}^{p} (a_i)_{p_1} \prod_{i=1}^{r} (c_i)_{-r_1}}{\prod_{i=1}^{q} (b_i)_{q_1} \prod_{i=1}^{s} (d_i)_{-s_1}}\ A\ {}^{p\ \ r}_{q\ \ s}\left[\begin{matrix}(a_p + p_1, p_1) & (c_r - r_1, r_1) \\ ; & \\ (b_q + q_1, q_1) & (d_s - s_1, s_1)\end{matrix} ;\ (z, k)\right]$$

......................(4.3)

So that the m^{th} successive differential coefficient involving bunch differential coefficients is obtained as

$$\frac{d^{mk}A}{dz^{mk}} = \prod_{i=0}^{m-1} \frac{((a_p)+ip_1)_{p_1} \quad ((c_r)-ir_1)_{r_1}}{((b_q)+iq_1)_{q_1} \quad ((d_s)-is_1)_{s_1}}$$

$$\cdot \quad A \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} (a_p)+mp_1, p_1) & (c_r)-mr_1, r_1) \\ & ; & ;(z,k) \\ (b_q)+mq_1, q_1) & (d_s)-ms_1, s_1) \end{matrix} \right] \quad ....(4.4)$$

5. BUNCH INTEGRAL FORMULA

$$\text{Let} \quad \int^{k} = \int_0^z \int_0^z \; ...... \int_0^z \quad k \text{ times}$$

We observe that

$$\int^{k} {}_{k}A \begin{matrix} p & r \\ q & s \end{matrix} (dz)^k = \sum_{n=0}^{\infty} \left(\frac{(a_p)_{p_1 n} \quad (c_r)_{-r_1 n}}{(b_q)_{q_1 n} \quad (d_s)_{-s_1 n}} \right) \frac{z^{kn+k}}{(kn+k)!} \quad ....(5.1)$$

Which can be further simplified in order to yield meaningful value, as

$$\sum_{n=1}^{\infty} \left(\frac{(a_p)_{-p_1} \quad (a_p - p_1)_{p_1 n} \quad (c_r)_{r_1} \quad (c_r + r_1)_{-r_1 n}}{(b_q)_{-q_1} \quad (b_q - q_1)_{q_1 n} \quad (d_s)_{-s_1} \quad (d_s + s_1)_{-s_1 n}} \right) \frac{z^{kn}}{(kn)!}$$

$$= \frac{\pi(a_i)_{-p_1} \quad \pi(c_i)_{r_1}}{\pi(b_i)_{-q_1} \quad \pi(d_i)_{s_1}} \left({}_{k}A \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} (a_p - p_1, p_1) & (c_r + r_1, r_1) \\ & ; & ;(z,k) \\ (b_q - q_1, q_1) & (d_s + s_1, s_1) \end{matrix} \right] - 1 \right)$$

$$...................(5.2)$$

Following result of double integration

$$\int^{2k} {}_{k}A \begin{matrix} p & r \\ q & s \end{matrix} (dz)^{2k} = \frac{(a_p)_{-2p_1} \quad (c_r)_{2r_1}}{(b_q)_{-2q_1} \quad (d_s)_{2s_1}} \left({}_{k}A \begin{matrix} p & r \\ q & s \end{matrix} \left[\begin{matrix} (a_p - 2p_1, p_1) & (c_r + 2r_1, r_1) \\ & ; & ;(z,k) \\ (b_q - 2q_1, q_1) & (d_s + 2s_1, s_1) \end{matrix} \right] - 1 \right)$$

$$- \left(\frac{(a_p)_{-p_1} \quad (c_r)_{r_1}}{(d_s)_{s_1} \quad (b_q)_{-q_1}} \right) \frac{z^k}{k!} \quad ....(5.3)$$

gives inference about the m^{th} bunch integral, having the value

$$\int^{mk} {}_kA\begin{matrix} p & r \\ q & s \end{matrix} (dz)^{mk}$$

$$= \frac{(a_p)_{-mp_1} \quad (c_r)_{mr_1}}{(s_q)_{-mq_1} \quad (d_s)_{ms_1}} \; {}_kA\begin{matrix} p & r \\ q & s \end{matrix}\left[\begin{matrix}(a_p - mp_1,p_1) \\ (b_q - mq_1,q_1)\end{matrix} ; \begin{matrix}(c_r + mr_1,r_1) \\ (d_s + ms_1,s_1)\end{matrix} ; (z, k)\right]$$

$$\sum_{j=1}^{m} \left(\frac{(a_p)_{-jp_1} \quad (c_r)_{jr_1}}{(s_q)_{-jq_1} \quad (d_s)_{js_1}}\right) \frac{z^{k(m-j)}}{|\,k(m-j)|\,!} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5.4)$$

We propose to deal with application part of the present paper in statistical distribution in our subsequant studies.

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