Well-posedness of a Structural Acoustics Control Model with Point Observation of the Pressure

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Abstract

We consider a controlled and observed partial differential equation (PDE) which describes a structural acoustics interaction. Physically, this PDE describes an acoustic chamber with a flexible chamber wall. The control is applied to this flexible wall, and the class of controls under consideration includes those generated by piezoceramic patches. The observation we consider is point measurements of acoustic pressure inside the cavity. Mathematically, the model consists of a wave equation coupled, through boundary trace terms, to a structurally damped plate (or beam) equation, and the point controls and observations for this system are modeled by highly unbounded operators. We analyze the map from the control to the observation, since the properties of this map are central to any control design which is based upon this observation. We also show that for an appropriate state space $\mathcal{X}$, if the initial state is in $\mathcal{X}$ and the control is in L^2, then the state evolves continuously in $\mathcal{X}$ and the observation is in L^2. The analysis of this system entails a microlocal analysis of the wave component of the system, and the use of pseudodifferential machinery.

1 Introduction

1.1 Motivation

In this paper we consider a controlled and observed PDE associated with the mathematical modeling of certain structural acoustic interactions. The spatial domain for the PDE system is a bounded region $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, with boundary Γ. We refer to a subset Γ_0 of Γ as the "active" part of this boundary. The motivating physical example is sound waves in a cavity Ω which has a flexible wall at Γ_0. The control is applied to Γ_0 and observations are taken of acoustic pressure inside the cavity. In the motivating example the control is implemented via piezoceramic patches on Γ_0.

Mathematically, the system is comprised in part of a wave equation satisfied by the function $z(t, x)$ at time t and position $x \in \Omega$. The acoustic pressure inside of the cavity Ω is proportional to $z_t(t, x)$. Moreover, the wave equation is coupled through boundary "trace" terms to a parabolic–like equation which is satisfied on Γ_0. Specifically, when $n = 2$ this parabolic–like equation is a damped beam equation of Kelvin–Voigt type, and when $n = 3$ the parabolic component is a plate equation, likewise with Kelvin–Voigt damping. The displacement of this beam or plate is given by the function $v(t, \xi)$, at time t and position $\xi \in \Gamma_0$. We consider general control input terms of the form $Bu(t)$ in the parabolic component of the system for some operator B. In the motivating example, the patches generate bending moments on the beam or plate, and the control $u(t) \in L^2(0, T; \mathbb{R}^k)$ contains in its ith component $(i = 1, ..., k)$ the amount of voltage to be conducted through the ith patch at time t. The PDE model will be explicitly given in the next subsection; for details about the physical motivation for the model see [5, 6, 7, 8] and the references therein.

We are interested in the following physically motivated observations $y(t)$ for this system:

(A) The acoustic pressure at points inside the cavity Ω;

(B) The displacement of the beam (or plate) at points on Γ_0;

(C) The velocity of the beam (or plate) at points on Γ_0.

In order to suppress noise in the cavity, a control $u(t)$ is typically synthesized by a feedback (dynamic or static) of the observation $y(t)$; see [5, 7] for examples of this practice. Not surprisingly then, a proper understanding of the properties of the *input-output* map $u \to y$ (with zero initial data) is crucial for the purposes of feedback control design and analysis. For instance, it is the boundedness properties of this map (or lack thereof) which determine whether or not a given feedback stabilization is robust with respect to a large class of perturbations, see [17, 30]. To make precise this notion of boundedness, we introduce the following:

Definition 1.0.1 *Given a control space U and an observation space Y, an input–output map L_T, defined on $L^2(0, T; U)$ by $L_T u = y$, is said to be well–posed if*

$$L_T \in \mathcal{L}\left(L^2(0, T; U), L^2(0, T; Y)\right).$$

In Avalos, Lasiecka, Rebarber [3], it is shown that when Ω is a rectangle in $\mathbb{R}^2$, then L_T is a well-posed input–output map for all three of the observations listed above. If Ω is a general region with a smooth boundary, it is not hard to show that L_T is also well-posed if the observation is limited to (B) and (C) only; this is carried out in [3] for $n = 2$, and for $n = 3$ it is a much simplified version of the work in this paper. Therefore, for the remainder of this paper we will assume that $y(t)$ is a single observation of the form (A). It is much more difficult to prove the well-posedness of L_T when the observation is taken to be (A) because techniques which attempt to invoke the Sobolev Embedding Theorem at some point are insufficient for this case. One of the main results in this paper is a demonstration of this well-posedness of L_T, to be accomplished by a microlocal analysis of the dynamics.

A regular input–output map is a well-posed map, which satisfies an additional property; see [39] for details about regular systems, and [3] (Definitions 2.6 and 2.7) for the definition of a regular input–output map. In Section 4 of [3] it is shown that this additional property holds for any of the three observations listed above when the input–output map is already known to be well-posed. The arguments for this proof do not depend upon the geometry of Ω, or whether $n = 2$ or 3. Therefore if we can prove well-posedness of the input–output map, we immediately obtain regularity of the map as well. This concept of regularity is quite useful for both control design and analysis; see [25, 26, 30, 31, 32, 40] for a sampling of papers using regularity. For instance, in [3] we proved that in the case that $\Omega \subset \mathbb{R}^2$ is a rectangular domain, and that the structural acoustics model is input-output stabilized by dynamic (or static) feedback using linear combinations of any and all of the three observations listed above, then that stabilization is not robust with respect to arbitrarily small delays in the feedback loop. With the results in this paper, the same non-robustness results are immediately obtained for the structural acoustics model when Ω has a smooth boundary and $n = 2$ or 3; see section 5 in [3] for precise statements of these notions.

The other main result in this paper is in the choosing of an appropriate state space $\mathcal{X}$ for the controlled, observed structural acoustics model. In particular, we wish to find a Hilbert space $\mathcal{X}$ so that, for any $T > 0$, if the initial state is in $\mathcal{X}$ and the control is in $L^2(0,T;\mathbb{R}^k)$, then $y \in L^2(0,T)$, and the state evolves continuously in $\mathcal{X}$. A natural candidate would be the finite energy space $\mathbf{H}$ given in (1.4). Unfortunately however, this space is "too rough" to accomodate pointwise measurements of the acoustic pressure. Appealing to the general theory in Salamon [34], we know that once a system is shown to have a well-posed input-output map, there exists a state $\mathcal{X}$ which has these properties. However, the construction of this space does not result in a physically motivated Sobolev space. Roughly speaking, the $\mathcal{X}$ which we choose has one-half (spatial) derivative more in the wave component than that for the basic space of well-posedness $\mathbf{H}$. The analysis undertaken to validate this choice of state space also has a large microlocal component.

1.2 The Model and the Main Results

In this subsection we describe in detail the controlled, observed PDE model describing structural acoustic interactions. This model originates from [9, 28] and more recently has been analyzed in papers by several authors, including [5, 6, 8, 10, 11, 12, 13, 24]. We shall use here a slightly abstracted version of this model, which retains all basic characteristics of the original structural acoustic problem.

Let Ω be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a smooth boundary Γ. Furthermore, let Γ_0 be a smooth segment of Γ, with its boundary denoted by $\partial\Gamma_0$. Let $z = z(t,x)$ for $t \in [0,T]$ and $x \in \Omega$, and let $v = v(t,\xi)$ for $t \in [0,T]$ and $\xi \in \Gamma_0$. Denote the outward normal derivative to Γ by $\partial/\partial\nu$, and the outward normal derivative to $\partial\Gamma_0$ by $\partial/\partial n$. Let the control space be $U = \mathbb{R}^k$, and suppose

$$B \in \mathcal{L}\left(U, H^{-\frac{5}{3}}(\Gamma_0)\right). \tag{1.1}$$

With this notation, we will refer to the following system as the *structural acoustics model:*

$$\begin{cases} z_{tt} = \Delta z \text{ on } (0,T)\times\Omega; \\ \dfrac{\partial z}{\partial \nu} = \begin{cases} v_t & \text{on } (0,T)\times\Gamma_0; \\ 0 & \text{on } (0,T)\times\Gamma\setminus\Gamma_0; \end{cases} \\ v_{tt} = -\Delta^2 v - \Delta^2 v_t - z_t + Bu \text{ on } (0,T)\times\Gamma_0; \\ v|_{\partial\Gamma_0} = \dfrac{\partial v}{\partial n}|_{\partial\Gamma_0} = 0 \quad \text{on } (0,T)\times\Gamma_0; \\ [z(t=0), z_t(t=0), v(t=0), v_t(t=0)] = X_0 := [z_0, z_1, v_0, v_1]. \end{cases} \tag{1.2}$$

The particular structure of the control operator B is not at all important for the subsequent analysis; its specified degree of unboundedness in (1.1) comes from our wish to consider those input operators B which are associated with control by smart materials. In the motivating example, in which B represents the action of piezoceramic patches bonded to the flexible wall Γ_0, B will take the form of a linear combination of derivatives of delta functions, and as such is an element of $\mathcal{L}(U, H^{-\frac{5}{3}}(\Gamma_0))$ for $n = 2$. To see this, let ξ_0 be a given point in Γ_0. By the continuity of the Sobolev embeddings we have $H^{\frac{5}{3}}(\Gamma_0) \subset H^{3/2+\epsilon}(\Gamma_0) \subset C^1(\Gamma_0)$, from which we easily deduce that $\delta'(\xi_0) \in H^{-\frac{5}{3}}(\Gamma_0)$.

As indicated in the previous subsection, there are three natural observations for the PDE model (1.2), and in this paper we are interested in the most "unbounded" of these, the observation of acoustic pressure at a given point in the cavity, since this is the observation for which well-posedness of the control-to-observation map is difficult to prove. Let $\mathbf{x_0}$ be a specified point in $\bar{\Omega}$. If $[z, v]$ is a solution of (1.2), we then define

$$y(t) = z_t(t, \mathbf{x_0}), \tag{1.3}$$

so that, in this paper, the observation space is $Y = \mathbb{R}$. This observation is proportional to the acoustic pressure at $\mathbf{x_0}$, a physical quantity which in practice could be measured by means of a microphone. Initially, (1.3) is only a formal expression, inasmuch as there is no guarantee that $z_t(t, \cdot)$ can be evaluated pointwise in Ω, see Remark 1.5 below. More generally, one can consider any finite collection of points $\{\mathbf{x_i}\}_{i=0}^m \in \bar{\Omega}$, and with it take our observation $y(t)$ to be a linear combination of $\{z_t(t, \mathbf{x}_i)\}_{i=0}^m$. This would clearly not affect the analysis below.

One may represent (1.2), (1.3) formally by a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C}_{x_0})$ (defined below) on the "natural" state space $\mathbf{H}$, where

$$\mathbf{H} = H^1(\Omega) \times L^2(\Omega) \times H_0^2(\Gamma_0) \times L^2(\Gamma_0). \tag{1.4}$$

For $X_0 = [z_0, z_1, v_0, v_1]^T \in \mathbf{H}$, we define $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ by

$$\mathcal{A}X_0 = \begin{bmatrix} z_1 \\ \Delta z_0 \\ v_1 \\ -\Delta^2 v_0 - \Delta^2 v_1 - z_1 \end{bmatrix},$$

$$\text{with } D(\mathcal{A}) = \Big\{ [z_0, z_1, v_0, v_1]^T \in \left[H^1(\Omega)\right]^2 \times \left[H_0^2(\Gamma_0)\right]^2 : \Delta z_0 \in L^2(\Omega),$$
$$\frac{\partial z_0}{\partial \nu} = v_1 \text{ on } \Gamma_0, \ \frac{\partial z_0}{\partial \nu} = 0 \text{ on } \Gamma \setminus \Gamma_0; \ \Delta^2 v_0 + \Delta^2 v_1 \in L^2(\Gamma_0) \Big\}. \tag{1.5}$$

One can readily show that $\{e^{\mathcal{A}t}\}_{t \geq 0}$ generates a C_0–semigroup on $\mathbf{H}$; see for instance [2]. Letting $X(t) = [z(t), z_t(t), v(t), v_t(t)]^T$ and $\mathcal{B} = [0, 0, 0, B]^T$, (1.2) is equivalent to

$$\frac{d}{dt} X(t) = \mathcal{A}X(t) + \mathcal{B}u, \qquad X(0) = X_0.$$

Therefore the solution of (1.2) may be given by the variation of parameters formula

$$X(t) = e^{\mathcal{A}t} X_0 + + \mathcal{L}_T u(t),$$

where

$$\mathcal{L}_T u(t) \equiv \int_0^t e^{\mathcal{A}(t-s)} \mathcal{B}u(s) ds, \quad \text{for all } t \in [0, T].$$

It is shown in [2] that if $u \in L^2(0, T; U)$, then $X(t) \in \mathbf{H}$ for every $X_0 \in \mathbf{H}$. In systems theoretic language then, $\mathcal{B}$ is an "admissible control operator" for $e^{\mathcal{A}t}$ in $\mathbf{H}$, see Weiss [39].

For arbitrary $X_0 = [z_0, z_1, v_0, v_1]^T \in \mathbf{H}$, we formally define the observation map $\mathcal{C}_{x_0}$ by

$$\mathcal{C}_{x_0} X_0 = z_1(\mathbf{x_0}), \text{ where } \mathbf{x_0} \in \overline{\Omega}, \tag{1.6}$$

so the observation map $y(t)$ may be formally expressed as

$$y(t) = \mathcal{C}_{x_0} e^{\mathcal{A}t} X_0 + \mathcal{C}_{x_0} \mathcal{L}_T u(t). \tag{1.7}$$

The input–output map L_T may then be formally represented by $L_T u(t) = \mathcal{C}_{x_0} \mathcal{L}_T u(t)$ for $u \in L^2(0, T; U)$. The fact that both $\mathcal{B}$ and $\mathcal{C}_{x_0}$ are unbounded operators makes the analysis of this map especially challenging. Our main results, Theorem 1.1 and 1.4, give a proper meaning to the observed state (1.7).

Theorem 1.1 *For every $0 < T < \infty$ and all $\mathbf{x_0} \in \overline{\Omega}$, L_T is well–posed (in the sense of Definition 1.0.1), with the norm bound of L_T generally depending on T.*

Remark 1.2 *In the special case that Ω is a rectangle, this was proved in Section 3 of [3] by a Fourier/Harmonic analysis. These techniques are wholly inapplicable in the present case that Ω is an arbitrary domain with smooth boundary.*

Note that it is *not* true that $\mathcal{C}_{x_0} e^{\mathcal{A}t} X_0 \in L^2(0,T)$ for all $X_0 \in \mathbf{H}$—that is to say, in the language of systems theory, $\mathcal{C}_{x_0}$ is not an "admissible" observation operator for $e^{\mathcal{A}t}$ on the space $\mathbf{H}$, see [39]. We wish to identify a state space $\mathcal{X}$ such that $\mathcal{B}$ is an admissible control operator and $\mathcal{C}_{x_0}$ an admissible observation operator. To this end, we define the elliptic operator $A_N : \frac{L^2(\Omega)}{\mathbb{R}} \supset D(A_N) \to \frac{L^2(\Omega)}{\mathbb{R}}$ by

$$\begin{aligned} A_N &= -\Delta \text{ with} \\ D(A_N) &= \left\{ f \in \frac{H^2(\Omega)}{\mathbb{R}} : \frac{\partial f}{\partial \nu} = 0 \text{ on } \Gamma \right\}. \end{aligned} \tag{1.8}$$

A_N is positive definite, self–adjoint on $\frac{L^2(\Omega)}{\mathbb{R}}$, and so by the characterization of the fractional powers in [15], we have

$$\begin{aligned} D(A_N^{\alpha}) &= \frac{H^{2\alpha}(\Omega)}{\mathbb{R}}, \quad 0 \le \alpha < \frac{3}{4}; \\ D(A_N^{\frac{3}{4}}) &= \left\{ f \in \frac{H^{\frac{3}{2}}(\Omega)}{\mathbb{R}} \ni \nabla f \in L^2_{-\frac{1}{2}}(\Omega) \right\}, \end{aligned} \tag{1.9}$$

where, as in [15], $L^2_{-\frac{1}{2}}(\Omega)$ denotes the space of functions $h(x)$ such that $\varrho(x)^{-\frac{1}{2}} h(x) \in L^2(\Omega)$, with $\varrho(x)$ being the distance from x to boundary Γ. Furthermore, we define the Neumann map $N : L^2(\Gamma) \to \frac{L^2(\Omega)}{\mathbb{R}}$ by $Ng = h$ if

$$\begin{aligned} \Delta h &= 0 \text{ on } \Omega\,; \\ \frac{\partial h}{\partial \nu} &= g \text{ on } \Gamma. \end{aligned} \tag{1.10}$$

By standard elliptic theory (see e.g., [20]), we have that for all real s,

$$N \in \mathcal{L}\left(H^s(\Gamma), \frac{H^{s+\frac{3}{2}}(\Omega)}{\mathbb{R}} \right). \tag{1.11}$$

With these operator definitions, we now present our choice of state space $\mathcal{X}$ which is principally motivated by the following "trace" result of R. Triggiani, which gives meaning to pointwise spatial evaluations of solutions of the wave equation.

Lemma 1.3 *For $n = 2$, let $z(t,x)$ be the solution of the wave equation*

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } (0,T) \times \Omega; \\ \frac{\partial z}{\partial \nu} = 0 \text{ on } (0,T) \times \Gamma; \\ z(0,\cdot) = z_0 \in D(A_N^{\frac{3}{4}}), \;\; z_t(0,\cdot) = z_1 \in \frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}}; \end{cases} \tag{1.12}$$

Then for any $\mathbf{x_0} \in Int(\Omega)$, there exists $C_T > 0$ such that

$$\|z_t(\cdot, \mathbf{x_0})\|_{L^2(0,T)} \le C_T \left\| [z_0, z_1]^T \right\|_{D(A_N^{\frac{3}{4}}) \times \frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}}}.$$

Proof: Note that the wave equation above is well posed with $[z, z_t]^T \in C([0,T]; D(A_N^{\frac{3}{4}}) \times H^{\frac{1}{2}}(\Omega)/\mathbb{R})$. Thus for all $0 \le t \le T$, $z_{tt}(t) = A_N z(t) \in \left[H^{\frac{1}{2}}(\Omega)/\mathbb{R}\right]'$, after using the characterization in (1.9). If we make the change of variable $p = z_t$, then p solves

$$\begin{cases} p_{tt} = \Delta p \quad \text{on } (0,T) \times \Omega; \\ \frac{\partial p}{\partial \nu} = 0 \text{ on } (0,T) \times \Gamma; \\ p(0,\cdot) = z_1 \in \frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}}, \;\; p_t(0,\cdot) = A_N z_0 \in \left[\frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}}\right]'. \end{cases}$$

By Theorem 3.2 of [38], we then have for arbitrary $\mathbf{x_0}$ in the interior of Ω,

$$\begin{aligned} \|p(\cdot, \mathbf{x_0})\|_{L^2(0,T)} &\le C_T \left\| [z_1, A_N z_0]^T \right\|_{H^{\frac{1}{2}}(\Omega)/\mathbb{R} \times [H^{\frac{1}{2}}(\Omega)/\mathbb{R}]'} \\ &\le C_T \left\| [z_0, z_1]^T \right\|_{D(A_N^{\frac{3}{4}}) \times \frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}}}. \end{aligned} \tag{1.13}$$

Transforming back to variable z_t now gives the result. □

With this result in mind, we define our prospective state space to be functions in the following product of Hilbert spaces, which moreover satisfy a certain compatibility condition:

$$\begin{aligned} \mathcal{X} := \Big\{ &[z_0, z_1, v_0, v_1] \in \frac{H^{\frac{3}{2}}(\Omega)}{\mathbb{R}} \times \frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}} \times H_0^2(\Gamma_0) \times L^2(\Gamma_0), \\ &\text{such that } z_0 - N v_1 \in D(A_N^{\frac{3}{4}}) \Big\}. \end{aligned} \tag{1.14}$$

As given, then $\mathcal{X}$ has the same regularity as $\mathbf{H}$ in the beam component, but has one-half more derivative regularity in the wave component. $\mathcal{X}$ is easily seen to be a Hilbert space with the norm

$$\begin{aligned} &\|[z_0, z_1, v_0, v_1]\|_{\mathcal{X}} \\ &= \left(\|z_0\|^2_{H^{\frac{3}{2}}(\Omega)} + \|z_1\|^2_{H^{\frac{1}{2}}(\Omega)} + \|v_0\|^2_{H_0^2(\Gamma_0)} + \|v_1\|^2_{L^2(\Gamma_0)} + \left\| A_N^{\frac{3}{4}}(z_0 - N v_1) \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \end{aligned} \tag{1.15}$$

We are now in a position to state our second main result, which shows that the given $\mathcal{X}$ is a state space for the structural acoustics model (1.2), and moreover allows pointwise observations of the acoustic pressure.

Theorem 1.4 *(i) Let $n = 2, 3$. Then $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{X}$.*

(ii) Let $n = 2, 3$. Then $\mathcal{L}_T \in \mathcal{L}(L^2(0,T;U), C([0,T];\mathcal{X}))$.

(iii) Let $n = 2$. Then for $\mathbf{x_0} \in Int(\Omega)$, $\mathcal{C}_{x_0} e^{\mathcal{A}(\cdot)} \in \mathcal{L}\left(\mathcal{X}, L^2(0,T)\right)$.

In (ii) and (iii), the respective norm bounds will generally depend upon T.

Remark 1.5 *For $X_0 \in \mathcal{X}$ and $u \in L^2(0,T;U)$, the known interior regularity for this problem (see [2]) is $z_t \in C([0,T]; L_2(\Omega))$. This is not sufficient to allow a well-defined pointwise evaluation (in Ω) of z_t via the Sobolev Embedding Theorem. Thus, Theorems 1.1 and 1.4 provide an additional trace regularity for the hyperbolic component of (1.2). In systems theoretic language, the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C}_{x_0})$ is a well-posed system (see [39]); using the techniques in Theorem 4.8 of [3] we can see that $(\mathcal{A}, \mathcal{B}, \mathcal{C}_{x_0})$ is in fact a regular system with "feedthrough" 0. We should also note that for the admissibility of the observation operator $\mathcal{C}_{x_0}$ (part (iii) in Theorem 1.4), we need to restrict ourselves to $n = 2$ and $\mathbf{x_0} \in \Omega$, rather than the more general situation considered in parts (i) and (ii); this is owing to our critical use of Lemma 1.3 in proving (iii).*

2 Proof of Theorem 1.1

2.1 Analysis of the wave component

For the remainder of this paper the letter C will denote a generic constant, which will vary according to context. If C depends on a variable, we put that variable as a subscript on C; e.g., C_T.

We begin by listing some properties of the structural acoustic system which were established in [1, 2], and which will be used throughout this paper. In addition to the regularity property for z_t noted in Remark 1.5, note that we also have the improved regularity of the velocity of the displacement v_t.

Proposition 2.1 *(See [1],[2]). For all $u \in L^2(0,T;U)$ and $X_0 \in \mathbf{H}$, the solution of (1.2) satisfies the following estimate:*

$$\left\| [z, z_t, v, v_t]^T \right\|_{C([0,T];\mathbf{H})} + \left\| \left[z_t|_{\Gamma_0}, v_t \right]^T \right\|_{L^2(0,T;H^{-\frac{1}{3}}(\Gamma_0)\times H_0^2(\Gamma_0))} \\ \leq C_T \left[\|u\|_{L^2(0,T;U)} + \|X_0\|_{\mathbf{H}} \right].$$

Remark 2.2 *In [1, 2] this result was only established for $n=2$, but the same proof holds for $n=3$. It should be noted that the proof of Proposition 2.1 relies in part upon estimates derived from a rather technical microlocal analysis of the wave component of (1.2). In particular, the "sharp" regularity of traces of solutions to the Neumann problem, which was established recently in [1, 21, 22], plays a critical role.*

The extra regularity for v_t cited in Proposition 2.1 leads us to a study of the following wave equation:

$$\begin{aligned} &z_{tt} = \Delta z \text{ in } (0,T)\times\Omega; \\ &\frac{\partial z}{\partial \nu} = g \text{ on } (0,T)\times\Gamma; \\ &z(t=0) = z_t(t=0) = 0; \text{ in } \Omega, \end{aligned} \tag{2.1}$$

where the boundary data g is taken to be better than L^2 in space. Indeed, the bulk of our effort in this section will be devoted to establishing the following Lemma:

Lemma 2.3 *Let z solve (2.1) with $g \in L^2(0,T;H^2(\Gamma))$ and fixed $\mathbf{x_0} \in \overline{\Omega} \subset \mathbb{R}^n$, $n \leq 3$. Then,*

$$\|z_t(\cdot,\mathbf{x_0})\|_{L^2(0,T)} \leq C_T \|g\|_{L^2(0,T;H^2(\Gamma))}. \tag{2.2}$$

Assuming for the time being the validity of Lemma 2.3, the proof of Theorem 1.1 is now straightforward. In fact, let

$$g := \begin{cases} v_t \text{ on } \Gamma_0 \\ 0 \text{ on } \Gamma\setminus\Gamma_0; \end{cases} \tag{2.3}$$

so by Proposition 2.1 and the fact that $v_t|_{\partial\Gamma_0} = \left.\frac{\partial v_t}{\partial n}\right|_{\partial\Gamma_0} = 0$, we see that $g \in L^2(0,T;H^2(\Gamma))$. Theorem 1.1 follows immediately then from Lemma 2.3.

Remark 2.4 *We note that the result stated in Lemma 2.3 does not follow from the previously known regularity for the Neumann problem. Indeed, the classical hyperbolic results in [20, 29], and even the "sharp" hyperbolic results in [21] require much more regularity on g in the time variable, a regularity which is not available in our present problem.*

Our proof of Lemma 2.3 is to be done in the forthcoming sections. In Section 2.2 we first prove the Lemma for the case where Ω is a half-space. Our reason for singling out this canonical geometry is that the corresponding computations are much simpler than in the general case, yet an analysis on the half space reveals key features of the problem, and illuminates the proof in the general case. The proof of Lemma 2.3 for a smooth, bounded domain will require the use of microlocal analysis, and will be undertaken in Section 2.3.

2.2 The Proof of Lemma 2.3 for the Half-Space Problem

In this subsection we assume that

$$\Omega = \{(x, \mathbf{y});\ x > 0,\ \mathbf{y} \in \mathbb{R}^{n-1}\};\quad \Gamma = \{0\} \times \mathbb{R}^{n-1},$$

where again, $n \leq 3$. With this geometry we consider the equation

$$\begin{aligned} &z_{tt}(t, x, \mathbf{y}) = z_{xx}(t, x, \mathbf{y}) + \Delta_y z(t, x, \mathbf{y}) \text{ on } (0, T) \times \Omega; \\ &z_x(t, 0, \mathbf{y}) = g(t, \mathbf{y}) \text{ on } (0, T) \times \Gamma; \\ &z(0, x, \mathbf{y}) = z_t(0, x, \mathbf{y}) = 0, \end{aligned} \tag{2.4}$$

where $0 < T < \infty$, and $\Delta_y = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial y_i^2}$.

In this special case, we are in a position to prove a somewhat stronger result than the one in Lemma 2.3. In fact, we shall show the following:

Lemma 2.5 *Let $\mathbf{x_0} = (x_0, \mathbf{y_0}) \in \bar{\Omega}$, and $\epsilon > 0$ be arbitrary. Then if z solves (2.4) with $g \in L^2(0, T; H^{3/2+\epsilon}(\Gamma))$, we have the estimate:*

$$\|z_t(\cdot, x_0, \mathbf{y_0})\|_{L^2(0,T)} \leq C_T \|g\|_{L^2(0,T;H^{3/2+\epsilon}(\Gamma))}.$$

Proof: The key component in the proof of this Lemma is to show the following estimate for fixed $x_0 \in \mathbb{R}$:

$$\int_0^T \|z_t(t, x_0, \cdot)\|^2_{H^{1+\epsilon}(\Gamma)} dt \leq C_T \|g\|^2_{L^2(0,T;H^{3/2+\epsilon}(\Gamma))}. \tag{2.5}$$

We extend $g(t, \cdot)$ to all of $\mathbb{R}$ by zero outside the interval $t \in [0, T]$ (given that the problem (2.4) has zero initial data) and apply the Fourier–Laplace transform. For this purpose we will denote the Fourier variable corresponding to $\mathbf{y}$ by $\eta \in \mathbb{R}^{n-1}$, and the Laplace variable corresponding to t by $s = \gamma + i\sigma \in \mathbb{C}$, where $\gamma > 0$ is fixed.

With $\tilde{z}$ denoting the Fourier–Laplace Transform of z, we obtain, after using the fact that (2.4) has zero initial data,

$$\begin{aligned} s^2 \tilde{z}(s, x, \eta) &= \tilde{z}_{xx}(s, x, \eta) - |\eta|^2 \tilde{z}(s, x, \eta) \\ \tilde{z}_x(s, 0, \eta) &= \tilde{g}(s, \eta). \end{aligned} \tag{2.6}$$

Solving this initial value problem explicitly gives the formulae

$$\tilde{z}(s, x, \eta) = -\frac{\tilde{g}(s, \eta)}{\sqrt{s^2 + |\eta|^2}} e^{-x\sqrt{s^2+|\eta|^2}};$$

$$\tilde{z}_t(s,x,\eta) = -\frac{s\tilde{g}(s,\eta)}{\sqrt{s^2+|\eta|^2}} e^{-x\sqrt{s^2+|\eta|^2}}. \tag{2.7}$$

The behaviour of the symbol $s^2+|\eta|^2$ is critical here. Since

$$s^2+|\eta|^2 = |\eta|^2+\gamma^2-\sigma^2+2i\gamma\sigma,$$

we have asymptotically as $|\eta|\to\infty$ and $|\sigma|\to\infty$,

$$\left|s^2+|\eta|^2\right| \sim \left(\left||\eta|^2+\gamma^2-\sigma^2\right|+|\sigma|\right). \tag{2.8}$$

We shall now localize the Fourier variable according to the following partition of $\mathbb{R}^n$: let

$$R_0 \equiv \{(\eta,\sigma) : |\sigma|^2+|\eta|^2 \le 1\};\ R_1 \equiv \{(\eta,\sigma) : |\sigma| \ge 2|\eta|\};\quad R_2 \equiv \mathbb{R}^n \backslash R^1.$$

Step 1: Analysis in $R_1 \setminus R_0$.

In $R_1 \setminus R_0$ we have from (2.8) that asymptotically

$$|s^2+|\eta|^2| \ge C\sigma^2. \tag{2.9}$$

Hence in $R_1 \setminus R_0$, we have

$$\left| s\frac{\tilde{g}(\eta,s)}{\sqrt{s^2+|\eta|^2}} \right| \le C\frac{|s\tilde{g}(\eta,s)|}{|\sigma|} \le C|\tilde{g}(\eta,s)|. \tag{2.10}$$

This estimate and (2.7) thus yield

$$|\tilde{z}_t(x_0,\eta,s)||\eta|^{1+\epsilon} \le C|\eta|^{1+\epsilon}||\tilde{g}(\eta,s)|. \tag{2.11}$$

Step 2: Analysis in $R_2 \setminus R_0$.

In this region $|\sigma| < 2|\eta|$, and so as $\left|s^2+|\eta|^2\right| \ge C\,|\sigma|$, we then have

$$\left| s\frac{\tilde{g}(\eta,s)}{\sqrt{s^2+|\eta|^2}} \right| \le C|\sigma|^{1/2}|\tilde{g}(\eta,s)| \le C|\eta|^{1/2}|\tilde{g}(\eta,s)|. \tag{2.12}$$

Therefore (2.7) and (2.12) show that in $R_2 \setminus R_0$

$$|\tilde{z}_t(x_0,\eta,s)||\eta|^{1+\epsilon} \le C|\eta|^{3/2+\epsilon}|\tilde{g}(\eta,s)|. \tag{2.13}$$

Step 3 : Analysis in $\mathbb{R}^n$.

Combining (2.11) and (2.13), we obtain

$$|\tilde{z}_t(x_0,\eta,s)||\eta|^{1+\epsilon} \le C_\gamma|\eta|^{3/2+\epsilon}|\tilde{g}(\eta,s)| \ \text{ for all } (\eta,\sigma)\in\mathbb{R}^n\backslash R_0. \tag{2.14}$$

On the other hand, since R_0 is bounded then easily we deduce from (2.7) and (2.14) that for all $(\eta,\sigma)\in\mathbb{R}^n$

$$|\tilde{z}_t(s,x_0,\eta)||\eta|^{1+\epsilon} \le C[|\eta|^{3/2+\epsilon}+1]|\tilde{g}(\eta,s)|.$$

Using this inequality and the generalized Parseval's relation (see p. 212 of [14]) we obtain

$$
\begin{aligned}
& 2\pi e^{-2\gamma T}\int_0^T \|z_t(t,x_0,\cdot)\|^2_{H^{1+\epsilon}(\Gamma)}\,dt \le 2\pi\int_0^\infty e^{-2\gamma t}\|z_t(t,x_0,\cdot)\|^2_{H^{1+\epsilon}(\Gamma)}\,dt \\
= \;& \int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}} |\tilde{z}_t(\gamma+i\sigma,x_0,\eta)|^2\left(1+|\eta|^2\right)^{1+\epsilon} d\eta d\sigma \\
= \;& \le C\int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}} \left[(|\eta|^{3/2+\epsilon}+1)^2|\tilde{g}(\eta,s)|^2\right] d\eta d\sigma \\
\le \;& 2\pi C\int_0^T e^{-2\gamma t}\|g(t)\|^2_{H^{\frac{3}{2}+\epsilon}(\Gamma)}\,dt,
\end{aligned}
$$

from which the estimate (2.5) follows.

Finally, by the Sobolev embedding theorem $H^{1+\epsilon}(\Gamma)\subset C(\Gamma)$ (recall that the dimension of $\Gamma\le 2$), and this combined with (2.5) yields,

$$
\|z_t(\cdot,x_0,\mathbf{y_0})\|_{L_2(0,T)} \le C_T\|z_t(\cdot,x_0,\cdot)\|_{L^2(0,T;H^{1+\epsilon}(\Gamma))} \le C_T\|g\|_{L^2(0,T;H^{3/2+\epsilon}(\Gamma))},
$$

which is the desired result of Lemma 2.5. □

Remark 2.6 *The above computations reveal that the trace regularity of z_t is better in the sector R_1, where the Lopatinski condition holds true. The deterioration occurs in R_2, which contains the characteristic sector $|\eta| = |\sigma|$. This structure is typically seen in hyperbolic problems. In fact, we shall observe the same phenomenon in the general case of smooth domains. The difference, however, will be that the regularity of traces established for the general case requires more tangential differentiability of g than when Ω is a half plane. This is due to the appearance of commutator terms.*

Remark 2.7 *Note that if $\dim(\Omega)\le 2$, then the result of Lemma 2.5 holds with $g\in L^2(0,T;H^{1+\epsilon}(\Omega))$ only.*

2.3 Proof of Lemma 2.3 (The General Case)

2.3.1 Space Localization

Since the initial conditions are zero, we can extend the boundary data g by zero for $t<0$ and consider the system (2.1) for all $t\in\mathbb{R}$. To make things easier we also multiply g by $e^{-\gamma t}$ with $\gamma>0$. This does not influence the regularity over a finite time horizon, but it does allow the application of the transform over an infinite interval. We adopt the following notation:

$$
Q_t := \Omega_t\times(0,t); \qquad Q := Q_T;
$$

$$
\Sigma_t := \Gamma\times(0,t); \qquad \Sigma := \Sigma_T;
$$

$$
|u|_{s,\mathcal{D}} := \|u\|_{H^s(\mathcal{D})},
$$

for any $s\in\mathbb{R}$ and any appropriate smooth domain $\mathcal{D}$. Letting, as usual, $OPS^s(\Omega)$, $OPS^s(\Gamma)$, $OPS^s(Q)$, and $OPS^s(\Sigma)$ denote the spaces of pseudodifferential operators (ΨDO's) with homogenous symbol of order s (see [36]), we now proceed to prove a trace regularity result for wave equations from which Lemma 2.3 immediately follows. Namely, our main result in this section is the following:

Lemma 2.8 *With* $\Omega \subset \mathbb{R}^n$, $n \leq 3$, *let* z *solve the wave equation*

$$\begin{aligned} z_{tt} &= \Delta z + Lz \ \text{ in } (0,T) \times \Omega; \\ \frac{\partial z}{\partial \nu} &= g + Mz \ \text{ on } (0,T) \times \Gamma; \\ z(t=0) &= z_t(t=0) = 0; \ \text{ in } \Omega, \end{aligned} \tag{2.15}$$

where

$$g \in L^2(0,T;H^2(\Gamma)), \text{ and } \Psi DO\text{'s } L \in OPS^1(\Omega);\ M \in OPS^0(\Gamma). \tag{2.16}$$

Then for all fixed $\mathbf{x_0} \in \overline{\Omega}$, *we have the estimate*

$$\|z_t(\cdot, \mathbf{x_0})\|_{L^2(0,T)} \leq C_T \|g\|_{L^2(0,T;H^2(\Gamma))}. \tag{2.17}$$

Remark 2.9 *In the case that* $\dim(\Omega) = 2$, *then Lemma 2.8 holds true with* $g \in L^2(0,T;H^{\frac{3}{2}-\epsilon}(\Gamma))$ *only.*

Proof of Lemma 2.8: As before, we extend g by zero for $T < t < 0$, so that this extension of g is in $L^2(0,T;H^2(\Gamma))$. We shall first consider the case when the point $\mathbf{x_0}$ lies on the boundary. The general case of $\mathbf{x_0} \in \Omega$ can readily be reduced to the former (see Remark 2.15 below). For each $\xi_0 \in \Gamma$, we choose a neighborhood N_{ξ_0} of ξ_0. Subsequently, we introduce, in N_{ξ_0}, a local coordinate coordinate system (ν, τ), where ν is the normal vector to Γ at ξ_0 and τ the tangent vector to Γ at ξ_0. This can be done due to the smoothness requirements imposed upon the boundary Γ. D_ν will denote the directional derivative in the direction of ν. It is known (see e.g., [16, 27]) that for N_{ξ_0} sufficiently small,

$$\Delta = D_\nu^2 + R\left(\nu(x), \tau(x), \frac{\partial}{\partial \tau}\right), \tag{2.18}$$

where $\nu(x)$ and $\tau(x)$ are the components of x in the directions of ν and τ; and where in N_{ξ_0}, $R(\nu(x), \tau(x), \frac{\partial}{\partial \tau})$ is a second order, strongly elliptic operator in the tangential direction τ. In the sequel, we shall frequently denote this local normal and tangential representation of the Laplacian as simply Δ.

Let $[A, B]$ denote the commutator between operators A and B. Furthermore, let ϕ denote a $C^\infty(\Omega)$–function whose support is contained in N_{ξ_0}; we will also denote the associated (multiplication) ΨDO by ϕ. We will use this ϕ to localize the wave equation (2.1); to wit, the function ϕz satisfies

$$\begin{aligned} (\phi z)_{tt} &= \Delta(\phi z) - [\Delta, \phi] z + \phi L z \text{ on } Q; \\ D_\nu(\phi z) &= \phi g + (\nabla \phi \cdot \nu) z + \phi M z \text{ on } \Sigma. \end{aligned} \tag{2.19}$$

In addition, we introduce the Fourier transform variables $\eta \in R^{n-1}$ (corresponding to tangential τ) and $\sigma - i\gamma$ (corresponding to t), where $\sigma \in R^1$ and $\gamma > 0$ is fixed. Given the spatial (time) dual variable, we let D_τ^s (D_σ^s) denote the ΨDO with symbol $\sqrt{|\eta|^2 + 1}^s$ $(\sqrt{|\sigma|^2 + 1}^s$see e.g. [36]). With this ΨDO, we make the change of variable

$$w := {D_\tau}^{3/2}(\phi z), \tag{2.20}$$

where z solves (2.15). Applying $D_\tau^{3/2}$ to the PDE (2.19), we have that w, supported in a neighborhood of the boundary Γ, satisfies the following equation:

$$\begin{aligned} w_{tt} &= \Delta w + [{D_\tau}^{3/2}, \Delta]\phi z - {D_\tau}^{3/2}[\Delta, \phi] z + {D_\tau}^{3/2}(\phi L z) \ \text{ on } \mathbb{R} \times \Omega; \\ D_\nu w &= D_\tau^{3/2}(\phi g) + {D_\tau}^{3/2}(\nabla \phi \cdot \nu) z + {D_\tau}^{3/2}(\phi M z) \text{ on } \mathbb{R} \times \Gamma. \end{aligned} \tag{2.21}$$

We now evaluate the commutator terms which appear in (2.21). We begin with the commutators in the PDE. First, note that by the algebra for ΨDO's we have

$$[D_\tau{}^{3/2}, \Delta]\phi z = [D_\tau{}^{3/2}, \Delta] D_\tau{}^{-3/2} w;$$
$$D_\tau{}^{3/2}[\Delta, \phi] z = D_\tau{}^{3/2}[\Delta, \phi] D_\tau{}^{-3/2} D_\tau{}^{3/2} z. \tag{2.22}$$

By (2.18) and the commutator and product rules (see e.g., Theorems 4.3 and 4.4 of [36]),

$$\begin{aligned} [D_\tau{}^{3/2},\ \Delta] D_\tau{}^{-3/2} &\in OPS^1(\Omega); \\ [\Delta, \phi] &\in OPS^1(\Omega). \end{aligned} \tag{2.23}$$

As for the commutators in the boundary functions of (2.21), we similarily have

$$D_\tau{}^{3/2} \left(\nabla\phi \cdot \nu\right) z = D_\tau{}^{3/2}(\nabla\phi \cdot \nu) D_\tau{}^{-3/2} D_\tau{}^{3/2} z. \tag{2.24}$$

Setting

$$\begin{aligned} f_0(z) &:= \quad [D_\tau{}^{3/2}, \Delta]\phi z - D_\tau{}^{3/2}[\Delta, \phi] z + D_\tau^{\frac{3}{2}} \left(\phi L z\right); \\ g_0(z) &:= \quad D_\tau{}^{3/2}(\nabla\phi \cdot \nu) z + D_\tau^{\frac{3}{2}} \left(\phi M z\right), \end{aligned}$$

we rewrite (2.21) as

$$\begin{aligned} w_{tt} &= \Delta w + f_0(z) \text{ in } Q \\ D_\nu w &= g_0(z) + D_\tau^{3/2}(\phi g) \text{ on } \Sigma. \end{aligned} \tag{2.25}$$

Using (2.22)–(2.24), (2.16), and the Sobolev continuity of ΨDO operators (see [16, 36]), we obtain the following (pointwise in time) estimate:

$$\begin{aligned} |f_0(z)(t)|_{0,\Omega} &\le C\left[|w(t)|_{1,\Omega} + |D_\tau{}^{3/2} z(t)|_{1,\Omega}\right]; \\ |g_0(z)(t)|_{\frac{1}{2},\Gamma} &\le C|D_\tau{}^{3/2} z(t)|_{\frac{1}{2},\Gamma}. \end{aligned} \tag{2.26}$$

Now applying Theorem 3 in [29] (with $k = 0$ therein) to equation (2.25), the regularity estimates in (2.26), classical trace theory (see [20]), and the fact that w has zero initial data, we obtain for all $0 < t \le T$:

$$\begin{aligned} |w|^2_{1,Q_t} + |w(t)|^2_{1,\Omega} + |D_\tau{}^{-1/2} w|_\Gamma|^2_{1,\Sigma_t} &\le C_T\left[|f_0(z)|^2_{0,Q_t} + \left\|g_0(z) + D_\tau^{3/2}\phi g\right\|^2_{L^2(0,t;H^{1/2}(\Gamma))}\right] \\ \le C_T\left[\|w\|^2_{L^2(0,t;H^1(\Omega))} + \left\|D_\tau{}^{3/2} z\right\|^2_{L^2(0,t;H^1(\Omega))} \right. &+ \left.\left\|D_\tau{}^{3/2}\phi g\right\|^2_{L^2(0,t;H^{1/2}(\Gamma))}\right]. \end{aligned} \tag{2.27}$$

To refine the right hand side, we have

$$w = D_\tau^{\frac{3}{2}}\phi z = \phi D_\tau^{\frac{3}{2}} z + \left[D_\tau^{\frac{3}{2}}, \phi\right] D_\tau^{-\frac{3}{2}} D_\tau^{\frac{3}{2}} z,$$

and with this and the Sobolev continuity of ΨDO's, the above inequality becomes

$$|w|^2_{1,Q_t} + |w(t)|^2_{1,\Omega} + |D_\tau{}^{-1/2} w|_\Gamma|^2_{1,\Sigma_t} \le C_T\left[\left\|D_\tau{}^{3/2} z\right\|^2_{L^2(0,t;H^1(\Omega))} + \|g\|^2_{L^2(0,t;H^2(\Gamma))}\right]. \tag{2.28}$$

It should be noted that in (2.28), the constant C_T depends upon the choice of N_{ξ_0}.

Let us now say that $\phi = \phi_i$ is the ith component of the resolution of the identity over a relatively compact set containing Ω, corresponding to a partition of unity, and with the support of each ϕ_i being small enough so that in the normal and tangential coordinates, $\Delta|_{\text{supp}(\phi_i)}$ has the general elliptic representation in (2.18). Using $w = D_\tau{}^{3/2}(\phi z)$, we can apply the same argument to each ϕ_i (noting that analysis for those ϕ_i supported in the interior of Ω is much simpler) and sum up the estimates in (2.28) with respect to i. This gives for all $0 < t \le T$:

$$|D_\tau{}^{3/2} z|^2_{1,Q_t} + |D_\tau{}^{3/2} z(t)|^2_{1,\Omega} + |D_\tau z|_\Gamma|^2_{1,\Sigma_t} \le C_T \left[\left\| D_\tau{}^{3/2} z \right\|^2_{L^2(0,t;H^1(\Omega))} + \|g\|^2_{L^2(0,T;H^2(\Gamma))} \right], \quad (2.29)$$

(where here, we are implicitly using $z = \sum \phi_i z$). Applying Gronwall's inequality to (2.29), we obtain

$$\|D_\tau^{3/2} z\|^2_{L^2(0,t;H^1(\Omega))} \le C_T \|g\|^2_{L^2(0,T;H^2(\Gamma))}.$$

In turn, this inequality applied to (2.29) and (2.28) gives

$$|D_\tau{}^{3/2} z|^2_{1,Q} + |D_\tau z|_\Gamma|^2_{1,\Sigma} + |w|^2_{1,Q} + |D_\tau{}^{-1/2} w|_\Gamma|^2_{1,\Sigma} \le C_T \|g\|^2_{L^2(\mathbb{R};H^2(\Gamma))}. \quad (2.30)$$

(Alternatively, we could have obtained (2.30) by taking the parameter γ "large enough" in the formulation of Theorem 1 in [29].) Applying this estimate to (2.26) yields

$$|f_0(z)|_{0,Q} + \|g_0(z)\|_{L^2(\mathbb{R};H^{1/2}(\Gamma))} \le C_T \|g\|_{L^2(\mathbb{R};H^2(\Gamma))}. \quad (2.31)$$

Using the notation

$$f_1 := f_0(z); \qquad g_1 := D_\tau{}^{3/2}(\phi g) + g_0(z), \quad (2.32)$$

we rewrite equation (2.25) as

$$\begin{aligned} w_{tt} &= \Delta w + f_1 \quad \text{on } Q, \\ D_\nu w &= g_1 \ \text{ on } \Sigma, \end{aligned} \quad (2.33)$$

whereby (2.31), the "forcing term" f_1 and boundary term g_1 satisfy

$$|f_1|_{0,Q} + \|g_1\|_{L^2(\mathbb{R};H^{1/2}(\Gamma))} \le C_T \|g\|_{L^2(\mathbb{R};H^2(\Gamma))}. \quad (2.34)$$

It is the equation (2.33) which now constitutes a basis for further analysis. In this analysis, we will frequently invoke the following estimate, which follows immediately from (2.30):

$$|w|_{1,Q} + \left| D_\tau{}^{-1/2} w_t|_\Gamma \right|_{0,\Sigma} \le C_T \|g\|_{L^2(0,T;H^2(\Gamma))}. \quad (2.35)$$

We first analyze (2.33) with a generic choice of ϕ (recall that w is associated with ϕ by (2.20)), and after obtaining the appropriate bounds for w, we will again use the particular partition of unity $\{\phi_i\}$ chosen above. Our next step is microlocalization.

2.3.2 Microlocalization (Continuation of the Proof of Lemma 2.8)

Having introduced the transform variables η, σ, we shall microlocalize the problem to a conic neighborhood $(\tau, t, \eta, \sigma) \in T^*(\Sigma)$. Let $c > 1$ and

$$R_1 := \{(\eta,\sigma) : |\sigma| > 2c|\eta|\}; \qquad R_3 := \{(\eta,\sigma) : |\sigma| < c|\eta|\}; \qquad R_2 = \mathbb{R} \setminus \{R_1 \cup R_3\}.$$

The microlocalization is done with a help of a "localizer", by which we mean here the zero order ΨDO $\Lambda \in OPS^0(\Sigma)$ (see [16]), with associated symbol $\lambda \in S^0(\Sigma)$ given by

$$\lambda(\eta,\sigma) := \begin{cases} 1 \ \text{ in } R_1 \\ \{\text{a } C^\infty \text{ function with range } [0,1]\} \ \text{ in } R_2 \\ 0 \quad \text{in } R_3. \end{cases} \tag{2.36}$$

We shall study the behaviour of the solution w to (2.33), particularly on Γ, in each sector R_i. To this end we consider the decomposition

$$w = \Lambda w + (I - \Lambda)w.$$

The idea here is that in the sector R_1 the Lopatinski condition is satisfied, so the trace of w will accordingly have "better" regularity in R_1. Lacking the Lopatinski condition in the characteristic sector $R_2 \cup R_3$, we instead take advantage of extra spatial regularity prescribed on Γ (that is, the regularity of g_1), and the fact that in this sector "space dominates time", so as to compensate for the loss of ellipticity.

We formalize the above discussion in the proofs of the following Propositions:

Proposition 2.10 *If w is the solution of the wave equation*

$$\begin{cases} w_{tt} = \Delta w + f_1 \quad \text{on } Q \\ \dfrac{\partial w}{\partial \nu} = g_1 \quad \text{on } \Sigma, \end{cases} \tag{2.37}$$

where arbitrary $f_1 \in L^2(Q)$ and $g_1 \in L^2(0,T;H^{\frac{1}{2}}(\Gamma))$, then

$$|\Lambda\, w_t|_\Gamma|_{0,\Sigma} \leq C\left[|w|_{1,Q} + |f_1|_{0,Q}\right].$$

Proposition 2.11 *If w is the solution of (2.37), then*

$$|(I-\Lambda)\, w|_\Gamma|_{\frac{3}{5},\Sigma} \leq C\left[|w|_{1,Q} + |f_1|_{0,Q} + \|g_1\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}\right].$$

The proofs of both Propositions are given in the next subsection. Taking for granted the validity of Propositions 2.10 and 2.11, we shall continue with the proof of Lemma 2.8 (and thus Lemma 2.3).

The estimate in Proposition 2.10, in combination with (2.34) and (2.35), yields

$$|\Lambda w_t|_\Gamma|_{0,\Sigma} \leq C_T \|g\|_{L^2(\mathbb{R};H^2(\Gamma))}. \tag{2.38}$$

With this estimate in mind, we recall the definition of w to write

$$\begin{aligned} & D_\tau{}^{3/2}\Lambda(\phi z_t|_\Gamma) = D_\tau{}^{3/2}\Lambda D_\tau{}^{-3/2} D_\tau{}^{3/2}(\phi z_t|_\Gamma) \\ = \ & D_\tau{}^{3/2}\left[\Lambda, D_\tau{}^{-3/2}\right] w_t|_\Gamma + \Lambda w_t|_\Gamma \\ = \ & \left(D_\tau{}^{3/2}\left[\Lambda, D_\tau{}^{-3/2}\right] D_\tau{}^{1/2}\right) D_\tau{}^{-1/2}(w_t|_\Gamma) + \Lambda w_t|_\Gamma. \end{aligned}$$

Norming and majorizing both sides of this expression with the commutator rule, (2.35) and (2.38), we obtain

$$\|\Lambda\phi z_t|_\Gamma\|_{L^2(\mathbb{R};H^{3/2}(\Gamma))} \leq C_T \|g\|_{L_2(\mathbb{R};H^2(\Gamma))}.$$

Since the dimension of Γ is 1 or 2, the Sobolev Embedding theorem gives $H^{3/2}(\Gamma) \subset C(\Gamma)$, and so

$$\|\Lambda\phi z_t|_\Gamma(\mathbf{x_0})\|_{L^2(\mathbb{R})} \leq C_T \|g\|_{L^2(\mathbb{R};H^2(\Gamma))} \quad \text{for fixed } \mathbf{x_0} \in \Gamma. \tag{2.39}$$

To handle the quantity $(I-\Lambda)\phi z_t$, we first write out the equality

$$\begin{aligned} D_\tau^{\frac{3}{5}}(I-\Lambda)D_\tau^{\frac{1}{2}}\phi\, z_t|_\Gamma &= D_\tau^{\frac{3}{5}}(I-\Lambda)\frac{d}{dt}D_\tau^{\frac{1}{2}}\phi\, z|_\Gamma \\ = \quad & D_\tau^{\frac{3}{5}}(I-\Lambda)\frac{d}{dt}D_\tau^{\frac{1}{2}}D_\tau^{-\frac{3}{2}}D_\tau^{\frac{3}{2}}\phi\, z|_\Gamma \\ = \quad & \left[D_\tau^{\frac{3}{5}}(I-\Lambda), \frac{d}{dt}D_\tau^{-1}\right] w|_\Gamma + \frac{d}{dt}D_\tau^{-1}D_\tau^{\frac{3}{5}}(I-\Lambda)\, w|_\Gamma\,. \end{aligned} \tag{2.40}$$

To majorize this last expression, we note initially that Proposition 2.11, (2.34) and (2.35) give collectively

$$|(I-\Lambda)w|_\Gamma|_{3/5,\Sigma} \le C_T\, \|g\|_{L^2(\mathbb{R};H^2(\Gamma))}\,. \tag{2.41}$$

We next notice that on the supp$(1-\lambda)$ we have the dominance of the tangential transform variable η over that of time; i.e.,

$$|\sigma| \le 2c|\eta| \quad \text{on supp}\,(I-\lambda)\,.$$

We thus conclude that

$$\frac{d}{dt}D_\tau{}^{-1} \text{ is bounded on supp}\,(I-\Lambda)\,. \tag{2.42}$$

Hence, taking the L^2-norm in (2.40), and majorizing the resulting expression by means of (2.42), (2.41), trace theory, (2.35), and the Sobolev continuity of ΨDO's, we obtain

$$\left| D_\tau^{3/5}(I-\Lambda)D_\tau{}^{1/2}\phi z_t|_\Gamma \right|_{0,\Sigma} \le C_T\, \|g\|_{L^2(\mathbb{R};H^2(\Gamma))}\,. \tag{2.43}$$

Moreover,

$$D_\tau^{3/5}\left[D_\tau^{1/2},(I-\Lambda)\right]\phi z_t|_\Gamma = D_\tau^{3/5}\left[D_\tau^{1/2},(I-\Lambda)\right]D_\tau^{-1}\left(D_\tau^{-1/2}w_t|_\Gamma\right).$$

Using (2.35) and the commutator rule, we then have

$$\left| D_\tau^{3/5}[D_\tau^{1/2},(I-\Lambda)]\phi z_t|_\Gamma \right|_{0,\Sigma} \le C_T\|g\|_{L^2(0,T;H^2(\Gamma))}. \tag{2.44}$$

Combining (2.43) and (2.44) with another application of the commutator rule now gives

$$|(I-\Lambda)\phi z_t|_\Gamma|_{L^2(\mathbb{R};H^{11/10}(\Gamma))} \le C_T\, \|g\|_{L^2(\mathbb{R};H^2(\Gamma))}\,. \tag{2.45}$$

By the Sobolev's Embedding $H^{11/10}(\Gamma)\subset C(\Gamma)$ (recall dim $\Gamma\le 2$), so we conclude that

$$\|(I-\Lambda)\phi z_t|_\Gamma(\mathbf{x_0})\|_{L^2(\mathbb{R})} \le C_T\, \|g\|_{L^2(\mathbb{R};H^2(\Gamma))} \quad \text{for all } \mathbf{x_0}\in\Gamma. \tag{2.46}$$

Combining (2.39) and (2.46) gives the pointwise (in space) estimate

$$\|\phi z_t|_\Gamma(\mathbf{x_0})\|_{L^2(\mathbb{R})} \le \|\Lambda\phi z_t|_\Gamma(\mathbf{x_0})\|_{L^2(\mathbb{R})} + \|(I-\Lambda)\phi z_t|_\Gamma(\mathbf{x_0})\|_{L^2(\mathbb{R})} \le C\,\|g\|_{L^2(\mathbb{R};H^2(\Gamma))}\,.$$

This inequality holds with $\phi=\phi_i$, for every ϕ_i in our partition of unity. Summing over i leads to the final estimate

$$\|z_t|_\Gamma(\mathbf{x_0})\|_{L^2(0,T)} \le C\,\|g\|_{L^2(\mathbb{R};H^2(\Gamma))} \quad \text{for all } \mathbf{x_0}\in\Gamma.$$

This will complete the proof of Lemma 2.8 (and thus Lemma 2.3), as soon as we prove Propositions 2.10 and 2.11. □.

2.3.3 The Proof of Proposition 2.10

We start by applying the localizing operator Λ to both sides of the first equation in (2.37). This leads to the PDE

$$\Lambda w_{tt} = \Delta\Lambda w + \Lambda f_1 + [\Lambda, \Delta]w \tag{2.47}$$

By the commutator rule and the Sobolev regularity of ΨDO's we have

$$|[\Lambda, \Delta]w|_{0,Q} \leq C|w|_{1,Q}.$$

Therefore, denoting

$$\widehat{f} := \Lambda f_1 + [\Lambda, \Delta]w,$$

we have (since $\Lambda \in \mathcal{L}(H^s(Q))$)

$$|\widehat{f}|_{0,Q} \leq C[|w|_{1,Q} + |f_1|_{0,Q}]. \tag{2.48}$$

With this new forcing function $\widehat{f}$, we rewrite (2.47) as

$$\Lambda w_{tt} = \Delta\Lambda w + \widehat{f}. \tag{2.49}$$

To this equation, we next apply a multiplier method in very much the same way as in [18, 21, 22]. We merely sketch the method here; for a full proof, see Appendix A of Triggiani [37]. We set the vector $\nu = [\nu_1, \ldots \nu_n]^T$, and moreover, for $x \in \mathbb{R}^n$ let H be the $n \times n$ matrix defined by

$$H(x) = \left[\frac{\partial \nu_i(x)}{\partial x_j}\right]_{i,,j}.$$

We multiply both sides of the equation (2.49) by $\nu \cdot \nabla\Lambda w = D_\nu \Lambda w$ and subsequently integrate by parts. Manipulations of this equation which involve the use of Green's Formula and the Divergence Theorem eventually give:

$$\frac{1}{2}\int_\Sigma \left[|\Lambda w_t|^2 + |D_\nu\Lambda w|^2 - \left|\frac{\partial}{\partial\tau}\Lambda w\right|^2\right] d\Sigma = \int_Q H\nabla\Lambda w \cdot \nabla\Lambda w dQ$$
$$+\frac{1}{2}\int_Q \left(\Lambda w_t^2 - |\nabla\Lambda w|^2\right) div(\nu) dQ - \int_Q \hat{f} D_\nu \Lambda w dQ$$

(here we have also used the fact that w has zero initial data).

Estimating the right hand side of this expression, using Cauchy-Schwartz, the estimate (2.48), and fact that $\Lambda \in \mathcal{L}(H^s(Q))$, we obtain

$$\frac{1}{2}\int_\Sigma \left[|\Lambda w_t|^2 + |D_\nu\Lambda w|^2 - \left|\frac{\partial}{\partial\tau}\Lambda w\right|^2\right] d\Sigma \leq C\left[|w|_{1,Q}^2 + |f_1|_{0,Q}^2\right].$$

Rewriting the inequality above in terms of the transform variables (σ, η) gives

$$\int_{\mathbb{R}^n} [\sigma^2 - |\eta|^2||\lambda(\sigma,\eta)\tilde{w}(\sigma,\eta)|^2] d\eta d\sigma + |D_\nu\Lambda w|_{0,\Sigma}^2 \leq C\left[|w|_{1,Q}^2 + |f_1|_{0,Q}^2\right], \tag{2.50}$$

where $\tilde{w}$ is the Fourier (in the tangential and time variable) transform of $w|_\Gamma$. Note that by (2.36), the symbol $\sigma^2 - |\eta|^2$ is elliptic in σ on supp(λ). Thus (2.50) yields

$$\int_{\mathbb{R}^n} [\sigma^2|\lambda(\sigma,\eta)\tilde{w}(\sigma,\eta)|^2] d\eta d\sigma \leq C\left[|w|_{1,Q}^2 + |f_1|_{0,Q}^2\right],$$

or

$$\int_{\Sigma} |\Lambda w_t|^2 d\Sigma \leq C \left[|w|^2_{1,Q} + |f_1|^2_{0,Q} \right].$$

This completes Proposition 2.10. $\square$.

Remark 2.12 *We notice that the proof of Proposition 2.10 did not use any boundary conditions satisfied by Λw. Accordingly, the same argument will apply if we replace Σ in Proposition 2.10 by any other noncharacteristic time-like surface.*

2.3.4 Proof of Proposition 2.11

The proof of Proposition 2.11 is based on sharp trace regularity results established for solutions to the Neumann problem in [21, 22, 23] . These results improve the classical results [20, 29] by "1/6 derivative". As we shall see, this improvement is critical to the analysis below.

To begin, we apply the operator $I - \Lambda$ to both sides of the first equation in (2.37), leading to

$$(I - \Lambda) w_{tt} = \Delta (I - \Lambda) w + (I - \Lambda) f_1 + [I - \Lambda, \Delta] w. \tag{2.51}$$

Since the commutator term $[I - \Lambda, \Delta] \in OPS^1(Q)$, we then have, by the Sobolev continuity for ΨDO's,

$$|[I - \Lambda, \Delta] w|_{0,Q} \leq C |w|_{1,Q}.$$

Therefore, denoting

$$\widehat{f} := (I - \Lambda) f_1 + [I - \Lambda, \Delta] w,$$

we have

$$|\widehat{f}|_{0,Q} \leq C[|w|_{1,Q} + |f_1|_{0,Q}]. \tag{2.52}$$

We thus have from (2.37) the wave equation

$$\begin{aligned} (I - \Lambda) w_{tt} &= \Delta (I - \Lambda) w + \widehat{f} \\ D_\nu (I - \Lambda) w &= (I - \Lambda) g_1. \end{aligned} \tag{2.53}$$

To analyze (2.53), we recall the sharp regularity results for the Neumann-to-Dirichlet map defined on smooth domains:

Theorem 2.13 *(see [23]) Let $\mathcal{A}(x, \partial)$ be an arbitrary second order elliptic operator with smooth coefficients, and let $\frac{\partial}{\partial \nu_{\mathcal{A}}}$ denote the associated conormal derivative. For the system*

$$\begin{aligned} p_{tt} + \mathcal{A}(x, \partial) p = f \quad & in\ \Omega \times (0, T); \\ \frac{\partial p}{\partial \nu_{\mathcal{A}}} = h \quad & on\ \Gamma \times (0, T); \\ p(t=0) = p_t(t=0) = 0 \quad & 0n\ \Omega, \end{aligned} \tag{2.54}$$

the following regularity holds:

(i) The mapping $[f, h = 0] \to [p, p|_\Sigma]$ is bounded from $L^2(Q)$ into $\left[H^1(Q), H^{\frac{3}{5}}(\Sigma) \right]$.

(ii) The mapping $[f=0,h] \to [p, p|_\Sigma]$ *is bounded from* $L^2(\Sigma)$ *into* $\left[H^{\frac{3}{5}-\epsilon}(Q), H^{\frac{1}{5}-\epsilon}(\Sigma)\right]$.

(iii) The mapping $[f=0,h] \to p|_\Sigma$ *is bounded from* $H^{1/2}(\Sigma)$ *into* $H^{\frac{7}{10}-\epsilon}(\Sigma)$.

Parts (i)–(ii) of this theorem is a direct statement from Theorems 3.3 and 3.1 in [23]. The result (iii) comes from interpolating between Theorems 3.1 and 3.2(b).

Remark 2.14 *We note that in comparison to the Dirichlet problem, for which the Lopatinski condition holds, the traces of the solutions lose some measure of differentiability. Indeed, for the Dirichlet problem, a boundary trace statement equivalent to that for (2.54) would be that* $p|_\Sigma \in H^1(\Sigma)$. *Thus, the Neumann solutions lose "*$3/10+\epsilon$*" derivative. The fact that this is unavoidable (when the dimension is higher than one) follows from elementary computations performed on special domains like spheres or parallelepipeds (see [23]).*

We now apply these trace results directly to the analysis of (2.53). Since on the support of $I-\Lambda$ the space tangential variable dominates that of time (i.e., $|\sigma| \le 2c|\eta|$ on $\text{supp}(1-\lambda)$),

$$g_1 \in L^2(\mathbb{R}; H^{1/2}(\Gamma)) \Rightarrow (I-\Lambda)g_1 \in H^{1/2}(\Sigma).$$

Hence, using Theorem 2.13(iii) we obtain that

$$[\hat{f}=0, g_1] \to (I-\Lambda)w|_\Sigma \in H^{\frac{7}{10}-\epsilon}(\Sigma). \tag{2.55}$$

In addition, as $\widehat{f} \in L^2(Q)$, we can use (2.55) and Theorem 2.13(i) to obtain

$$|(I-\Lambda)w|_\Gamma|_{3/5,\Sigma} \le C[|\widehat{f}|_{0,Q} + |(I-\Lambda)g_1|_{1/2,\Sigma}] \le C[|\widehat{f}|_{0,Q} + \|g_1\|_{L^2(\mathbb{R},H^{1/2}(\Gamma))}\cdot \tag{2.56}$$

Proposition 2.11 now follows from estimating the right hand side of (2.56) with (2.52) . □

Remark 2.15 *In order to obtain the same result valid for an arbitrary point* $\mathbf{x}_0 \in \Omega$ *(rather than just* $\mathbf{x}_0 \in \Gamma$*), we proceed as follows: For a given* $\mathbf{x}_0$ *we choose a noncharacteristic time-like surface, say* Σ_1*, such that* $\mathbf{x}_0 \in \Sigma_1$. *As noted in Remark 2.12, the result of Proposition 2.10 holds for any such surface (i.e. replace* Σ *with* Σ_1*). The reason for this is that the boundary conditions are not used in the proof thereof. As for proving the counterpart of the result of Proposition 2.11 for* $\mathbf{x}_0 \in \Sigma_1$*: instead of invoking Theorem 2.13, one can use Theorem 2 and 3 in [35] which gives the same trace regularity for any* $H^1(Q)$ *solutions, and any time-like noncharacteristic surface without any a priori knowledge of the boundary conditions. With these modifications, the rest of the argument is the same.*

3 Proof of Theorem 1.4

3.1 Two Supporting Results

To begin, we give a regularity result for the velocity of the beam component, which essentially follows from Proposition 2.1.

Proposition 3.1 *For* $u \in L^2(0,T;U)$ *and* $X_0 = [z_0, z_1, v_0, v_1]^T \in \mathbf{H}$*, the component* v_t *of the solution* $[z, z_t, v, v_t]^T$ *to (1.2) satisfies*

$$\|v_t\|_{H^{\frac{1}{2}}(0,T;L^2(\Gamma_0))} \le C_T\left[\|u\|_{L^2(0,T;U)} + \|X_0\|_{\mathbf{H}}\right].$$

Proof: Taking norms in the third equation of (1.2), we have first

$$\|v_{tt}\|_{L^2(0,T;H^{-2}(\Gamma_0))} = \left\|-\Delta^2 v - \Delta^2 v_t - z_t|_{\Gamma_0} + Bu\right\|_{L^2(0,T;H^{-2}(\Gamma_0))}. \tag{3.1}$$

To majorize this relation, we note from Proposition 2.1 that

$$\left[\|[v,v_t]\|_{[L^2(0,T;H_0^2(\Gamma_0))]^2} + \|z_t|_\Gamma\|_{L^2(0,T;H^{-2}(\Gamma_0))}\right] \le C_T\left[\|u\|_{L^2(0,T;U)} + \|X_0\|_{\mathbf{H}}\right]. \tag{3.2}$$

Majorizing (3.1) with (3.2) and (1.1) we obtain

$$\|v_{tt}\|_{L^2(0,T;H^{-2}(\Gamma_0))} \le C_T\left[\|u\|_{L^2(0,T;U)} + \|X_0\|_{\mathbf{H}}\right];$$

and so

$$\|v_t\|_{H^1(0,T;H^{-2}(\Gamma_0))} \le C_T\left[\|u\|_{L^2(0,T;U)} + \|X_0\|_{\mathbf{H}}\right]. \tag{3.3}$$

Interpolation between (3.3) and (3.2) now gives the asserted result.□

In the sequel we will make frequent use of the parametrized Sobolev norms introduced in [29]. To wit, for positive integer m and function f defined on $\mathbb{R}\times(0,\infty)\times\mathbb{R}^{n-1}\equiv Q_\infty$ (time and space), we set

$$|f|_{s,\gamma}^2 := \sum_{i+j+k+|\alpha|=m}\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}^{n-1}}\left|e^{-\gamma t}\gamma^i D_t^j D_x^k D_y^{|\alpha|} f(t,x,y)\right|^2 dydxdt. \tag{3.4}$$

We also define the boundary norms $\langle f\rangle_{s,\gamma}$, by having for arbitrary real s and function f defined on $\mathbb{R}\times\mathbb{R}^{n-1}\equiv\Sigma_\infty$,

$$\langle f\rangle_{s,\gamma}^2 := \sum_{i+j+|\alpha|=m}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left|e^{-\gamma t}\gamma^i D_t^j D_y^{|\alpha|} f(t,x,y)\right|^2 dydt. \tag{3.5}$$

For all real s, the norms $|\cdot|_{s,\gamma}$ and $\langle\cdot\rangle_{s,\gamma}$ can subsequently be defined by interpolation.

With these norms, we next give a nontrivial regularity result for the wave equation with prescribed Neumann data and forcing term, which is due to Miyatake.

Lemma 3.2 *([29], Theorem 1) Let $z(t,x)$ satisfy the wave equation*

$$\begin{cases} z_{tt} = \Delta z + f \quad on\ Q_\infty \\ \dfrac{\partial z}{\partial\nu} = g \quad on\ \Sigma_\infty \end{cases}$$

where $f\in H^{\frac12}(Q_\infty)$, and $g\in H^{\frac12}(\mathbb{R};H^{\frac12}(\mathbb{R}^{n-1}))\cap L^2(\mathbb{R};H^1(\mathbb{R}^{n-1}))$. Then there exists a $\gamma_0>0$, such that for $\gamma>\gamma_0$, z satisfies the estimate

$$\gamma\,|z|_{\frac32,\gamma}^2 \le \frac{C}{\gamma}\left(|f|_{\frac12,\gamma}^2 + \left\langle D_\tau^{\frac12} g\right\rangle_{\frac12,\gamma}^2\right).$$

3.2 Sharp Regularity Results for Wave Equations

In this section, we prove two regularity result for wave equations under the action of boundary data of prescribed smoothness, in time and space. This result is in support of the proof of Theorem 1.4.

Lemma 3.3 *For Ω a smooth, bounded subset of $\mathbb{R}^n$, $n \leq 3$, Let $z(t,x)$ satisfy the following wave equation*

$$\begin{cases} z_{tt} = \Delta z \quad on\ (0,T)\times\Omega \\ \dfrac{\partial z}{\partial \nu} = g \quad on\ (0,T)\times\Gamma \\ z(0,\cdot) = 0, \ \ z_{t,}(0,\cdot) = 0. \end{cases} \tag{3.6}$$

Here the boundary data $g = g_1 + g_2$, where $g_1 \in G_\eta \equiv \left[L^2(0,T;H^\eta(\Gamma)) \cap H^{\frac{1}{2}}(\Sigma) \cap C([0,T];L^2(\Gamma))\right]$, and $g_2 \in H^\eta(0,T;L^2(\Gamma))$, where (to be specified) $\eta > \frac{1}{2}$. Moreover, we assume $g(t=0)=0$.

(i) Let dimension $n = 2$ or 3, and parameter $\eta = 1$. Then the corresponding solution $[z, z_t] \in C\left([0,T]; \frac{H^{\frac{3}{2}}(\Omega)}{\mathbb{R}} \times \frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}}\right)$, with the accompanying estimate

$$\|[z,z_t]\|_{C([0,T];H^{\frac{3}{2}}(\Omega)\times H^{\frac{1}{2}}(\Omega))} \leq C_T \left[\|g_1\|_{G_1} + \|g_2\|_{H^1(0,T;L^2(\Gamma))}\right]. \tag{3.7}$$

(ii) Let dimension $n=2$ and parameter $\eta = \frac{3}{2} - \epsilon$. Then for fixed $\mathbf{x}_0 \in \overline{\Omega}$, we have that the velocity z_t satisfies the pointwise (in space) estimate

$$\|z_t(\cdot,\mathbf{x}_0)\|_{L^2(0,T)} \leq C_T \left(\|g_1\|_{G_{\frac{3}{2}-\epsilon}} + \|g_2\|_{H^{\frac{3}{2}-\epsilon}(0,T;L^2(\Gamma))}\right). \tag{3.8}$$

Remark 3.4 *If we define the operator*

$$A_1 := \begin{bmatrix} 0 & I \\ -A_N & 0 \end{bmatrix}; \qquad D(A_1) = D(A_N) \times \frac{H^1(\Omega)}{\mathbb{R}},$$

where A_N is the elliptic operator given in (1.8), then $A_1 : D(A_N) \subset \frac{H^1(\Omega)}{\mathbb{R}} \times \frac{L^2(\Omega)}{\mathbb{R}} \to \frac{H^1(\Omega)}{\mathbb{R}} \times \frac{L^2(\Omega)}{\mathbb{R}}$ generates a C_0 unitary group $\left\{e^{A_1 t}\right\}_{t\geq 0}$ on $\frac{H^1(\Omega)}{\mathbb{R}} \times \frac{L^2(\Omega)}{\mathbb{R}}$, and by extension, on $D(A_N^{\frac{3}{4}}) \times \frac{H^{\frac{1}{2}}(\Omega)}{\mathbb{R}}$. With these dynamics, the solution to (3.6) can be written as

$$\begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} = e^{A_1 t} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ A_N N g(s) \end{bmatrix} ds. \tag{3.9}$$

Proof of (i): To start, we again extend the function g for $t < 0$; and so given the compatibility condition $g(t=0)=0$, we have that the Sobolev regularity of g with this extension is retained at the microlocal level. That is to say, with the partition of unity $\{\phi_i\}$ introduced in Subsection 2.3.1, and the space defined in (3.5) we have that $\phi_i g = \phi_i g_1 + \phi_i g_2$, where

$$\phi_i g_1 \in H_{\frac{1}{2},\gamma}(\Sigma_\infty); \ D_\tau^{\frac{1}{2}} \phi_i g_1, \ D_\sigma \phi_i g_2 \in H_{0,\gamma}(\Sigma_\infty). \tag{3.10}$$

We next recall the localizing function $\lambda(\eta,\sigma)$, and its corresponding ΨDO Λ which were introduced in Subsection 2.3.2 (see (2.36)). With this localizer, we will now analyze each component of the decomposition

$$\phi_i z = \Lambda \phi_i z + (I - \Lambda)\phi_i z. \tag{3.11}$$

Handling the component $\Lambda\phi_i z$: Applying the ΨDO $\Lambda\phi_i$ to (3..6) we obtain

$$\left\{ \begin{array}{l} (\Lambda\phi_i z)_{tt} = \Delta\left(\Lambda\phi_i z\right) + [\Lambda\phi_i, \Delta]\, z \quad \text{on } Q_\infty \\ \dfrac{\partial}{\partial\nu}\left(\Lambda\phi_i z\right) = \Lambda\phi_i g + \Lambda\left(\nabla\phi_i \cdot \nu\right) z \quad \text{on } \Sigma_\infty. \end{array} \right. \tag{3.12}$$

We decompose $\Lambda\phi_i z$ into $\Lambda\phi_i z \equiv w^{(a)} + w^{(b)}$, where

$$\left\{ \begin{array}{l} w_{tt}^{(a)} = \Delta w^{(a)} + [\Lambda\phi_i, \Delta]\, z \quad \text{on } Q_\infty \\ \dfrac{\partial}{\partial\nu} w^{(a)} = \Lambda\left(\nabla\phi_i \cdot \nu\right) z \text{ on } \Sigma_\infty; \end{array} \right. \tag{3.13}$$

$$\left\{ \begin{array}{l} w_{tt}^{(b)} = \Delta w^{(b)} \quad \text{on } Q_\infty \\ \dfrac{\partial}{\partial\nu} w^{(b)} = \Lambda\phi_i g \quad \text{on } \Sigma_\infty. \end{array} \right. \tag{3.14}$$

We note first, by interpolation, that if $z \in H^{\frac{3}{2}}(Q)$, then $z \in H^{\frac{1}{2}}(0,T;H^1(\Omega))$. Coupling this with trace theory, and using the spaces defined in (3.4) and (3.5), we then have that

$$z \in H^{\frac{3}{2}}(Q) \Rightarrow D_\tau^{\frac{1}{2}}\left(\nabla\phi_i \cdot \nu\right) z\Big|_{\Sigma_\infty} \in H_{\frac{1}{2},\gamma}(\Sigma_\infty). \tag{3.15}$$

To handle the wave equation in (3.13), one can then directly appeal to Lemma 3.2, followed by a use of the commutator rule and (3.15) to obtain the following estimate for $\gamma > 0$ large enough:

$$\gamma \left| w^{(a)} \right|_{\frac{3}{2},\gamma}^2 \leq \frac{C}{\gamma}\left[\left|[\Lambda\phi_i, \Delta]\, z\right|_{\frac{1}{2},\gamma}^2 + \left\langle D_\tau^{\frac{1}{2}} \Lambda\left(\nabla\phi_i \cdot \nu\right) z \right\rangle_{\frac{1}{2},\gamma}^2 \right] \leq \frac{C}{\gamma}\left|z\right|_{\frac{3}{2},\gamma}^2. \tag{3.16}$$

To handle the wave equation in (3.14), we note that in supp(λ), "time dominates space", and so we deduce from from (3.10) that

$$[g_1, g_2]^T \in G_1 \times H^1(0,T;L^2(\Gamma)) \Rightarrow \Lambda\phi_i g \in H_{\frac{1}{2},\gamma}(\Sigma_\infty).$$

Recalling now from Section 2.3.3 that the wave operator satisfies the Lopatinski condition in supp(λ), we can apply the results of [21] (see also [33]) to show that $w^{(b)} \in H_{\frac{3}{2},\gamma}(\Sigma_\infty)$, with the estimate

$$\left| w^{(b)} \right|_{\frac{3}{2},\gamma}^2 \leq C_0 \left\|\Lambda\phi_i g\right\|_{\frac{1}{2},\gamma}^2 \leq C_\gamma \left(\|g_1\|_{G_1}^2 + \|g_2\|_{H^1(0,T;L^2(\Gamma))}^2 \right). \tag{3.17}$$

Combining (3.16) and (3.17) then gives

$$\gamma \left|\Lambda\phi_i z\right|_{\frac{3}{2},\gamma}^2 \leq \frac{C}{\gamma}\left|z\right|_{\frac{3}{2},\gamma}^2 + C_\gamma \left(\|g_1\|_{G_1}^2 + \|g_2\|_{H^1(0,T;L^2(\Gamma))}^2 \right). \tag{3.18}$$

Handling the component $(I - \Lambda)\phi_i z$: Here, we will need a further decomposition. To this end, we define the sectors

$$R_e := \{(\eta,\sigma) : |\sigma| < c_0|\eta|\}; \qquad R_{ne} := \{(\eta,\sigma) : |\eta| < |\sigma|\}; \qquad R_{tr} = \mathbb{R} \setminus \{R_e \cup R_{ne}\},$$

where the constant c_0 is chosen small enough so that on R_e, the symbol corresponding to $\Delta - \partial_{tt}$ is elliptic in η. Therewith we define the (elliptic) localizer $\lambda_e(\eta,\sigma)$ by

$$\lambda_e(\eta,\sigma) = \left\{ \begin{array}{l} 1 \text{ in } R_e \\ \{\text{a } C^\infty \text{ function with range } [0,1]\} \text{ in } R_{tr} \\ 0 \quad \text{in } R_{ne}, \end{array} \right. \tag{3.19}$$

with $\Lambda_e \in OPS^0(\Omega)$ being its corresponding ΨDO. We now write $(I-\Lambda)\,\phi_i z = \left(I-\widehat{\Lambda}\right)\phi_i z + \Lambda_e \phi_i z$, where

$$\widehat{\Lambda} \equiv \Lambda + \Lambda_e. \tag{3.20}$$

The symbol $\widehat{\lambda}$ being so defined, we then have from (2.36) and (3.19)

$$\mathrm{supp}\left(1-\widehat{\lambda}\right) \subset \{(\eta,\sigma) : c_1|\eta| \le |\sigma| \le c_2|\eta|\}, \tag{3.21}$$

for appropriately chosen constants c_1 and c_2.

(a) Handling the subcomponent $\left(I-\widehat{\Lambda}\right)\phi_i z$: Applying $(I-\widehat{\Lambda})\phi_i$ to (3.6), we see that $(I-\widehat{\Lambda})\phi_i z$ solves the PDE

$$\left\{\begin{array}{ll} (I-\widehat{\Lambda})\phi_i z_{tt} = \Delta(I-\widehat{\Lambda})\phi_i z + \left[(I-\widehat{\Lambda})\phi_i, \Delta\right] z & \text{on } Q_\infty \\ \dfrac{\partial}{\partial\nu}\left((I-\widehat{\Lambda})\phi_i z\right) = (I-\widehat{\Lambda})\phi_i g + (I-\widehat{\Lambda})\,(\nabla\phi_i\cdot\nu)\,z & \text{on } \Sigma_\infty. \end{array}\right. \tag{3.22}$$

From (3.21), we see that on $\mathrm{supp}(1-\widehat{\lambda})$, the tangential spatial variable "is comparable" to that of time. In consequence,

$$[g_1, g_2]^T \in G_1 \times H^1(0,T;L^2(\Gamma)) \Rightarrow D_\tau^{\frac{1}{2}}(I-\widehat{\Lambda})\phi_i g \in H_{\frac{1}{2},\gamma}(\Sigma_\infty). \tag{3.23}$$

Using this information and (3.15), we can again appeal to Lemma 3.2 to have for large enough $\gamma > 0$,

$$\begin{aligned} \gamma\left|(I-\widehat{\Lambda})\phi_i z\right|^2_{\frac{3}{2},\gamma} &\le \frac{C}{\gamma}\left(\left|\left[(I-\widehat{\Lambda})\phi_i,\Delta\right] z\right|^2_{\frac{1}{2},\gamma} + \left\langle D_\tau^{\frac{1}{2}}\left[(I-\widehat{\Lambda})\,(\nabla\phi_i\cdot\nu)\,z + (I-\widehat{\Lambda})\phi_i g\right]\right\rangle^2_{\frac{1}{2},\gamma}\right) \\ &\le \frac{C}{\gamma}\,|z|^2_{\frac{3}{2},\gamma} + C_\gamma\left(\|g_1\|^2_{G_1} + \|g_2\|^2_{H^1(0,T;L^2(\Gamma))}\right). \end{aligned} \tag{3.24}$$

(b) Handling the component $\Lambda_e\phi_i z$: Applying $\Lambda_e\phi_i$ to (3.6), we have the PDE

$$\left\{\begin{array}{ll} (\Lambda_e\phi_i z)_{tt} = \Delta\,(\Lambda_e\phi_i z) + [\Lambda_e\phi_i,\Delta]\,z & \text{on } Q_\infty \\ \dfrac{\partial}{\partial\nu}\,(\Lambda_e\phi_i z) = \Lambda_e\phi_i g + \Lambda_e\,(\nabla\phi_i\cdot\nu)\,z & \text{on } \Sigma_\infty. \end{array}\right.$$

Here we use the fact that on $\mathrm{supp}(\Lambda_e)$, the operator $\Delta - \partial_{tt}$ is elliptic, and so by classical elliptic theory we obtain the estimate

$$\begin{aligned} |\Lambda_e\phi_i z|^2_{\frac{3}{2},\gamma} &\le C\left[\langle\Lambda_e\phi_i g + \Lambda_e\,(\nabla\phi_i\cdot\nu)\,z\ \rangle^2_{0,\gamma} + |[\Lambda_e\phi_i,\Delta]\,z\,|^2_{-\frac{1}{2},\gamma}\right] \\ &\le C\left[|g|^2_{0,\Sigma} + |\,z\,|^2_{\frac{1}{2},\gamma}\right] \\ &\le C\,|g|^2_{0,\Sigma} \le C\left(\|g_1\|^2_{G_1} + \|g_2\|^2_{H^1(0,T;L^2(\Gamma))}\right), \end{aligned} \tag{3.25}$$

where above, we have also used the *a priori* regularity given in Theorem 2.13(ii). Combining the decomposition (3.20) with (3.24) and (3.25) thus yields

$$\begin{aligned} \gamma\,|(I-\Lambda)\phi_i z|^2_{\frac{3}{2},\gamma} &\le \gamma\left|(I-\widehat{\Lambda})\phi_i z\right|^2_{\frac{3}{2},\gamma} + \gamma\,|\Lambda_e\phi_i z|^2_{\frac{3}{2},\gamma} \\ &\le \frac{C}{\gamma}\,|z|^2_{\frac{3}{2},\gamma} + C_\gamma\left(\|g_1\|^2_{G_1} + \|g_2\|^2_{H^1(0,T;L^2(\Gamma))}\right). \end{aligned} \tag{3.26}$$

With (3.18) and (3.26), we hence get the local estimate

$$\gamma\left|\phi_i z\right|^2_{\frac{3}{2},\gamma} \leq \frac{C}{\gamma}\left|z\right|^2_{\frac{3}{2},\gamma} + C_\gamma\left(\|g_1\|^2_{G_1} + \|g_2\|^2_{H^1(0,T;L^2(\Gamma))}\right). \tag{3.27}$$

Now using the decomposition $z = \sum \phi_i z$ and taking γ large enough in this inequality, we obtain

$$\left|z\right|^2_{\frac{3}{2},Q} \leq C_{T,\gamma}\left(\|g_1\|^2_{G_1} + \|g_2\|^2_{H^1(0,T;L^2(\Gamma))}\right). \tag{3.28}$$

Interpolation with this inequality further gives that $[z, z_t]^T \in L^2(0,T; H^{\frac{3}{2}}(\Omega) \times H^{\frac{1}{2}}(\Omega))$, with continuous dependence on the data. The improvement to continuity in time now follows from the "lifting" argument invoked in [19]. This completes the proof of **(i)**.

Proof of (ii): Again, owing to the compatibility condition $g(t=0) = 0$, we extend $g = g_1 + g_2$ by zero to all of time and space via the partition of unity $\{\phi_i\}$, thereby preserving the Sobolev regularity of this data. Now invoking the localizer Λ_e of (3.19), we apply the decomposition $\phi_i z = \Lambda_e \phi_i z + (I - \Lambda_e)\phi_i z$ to (3.6) to obtain

$$\begin{cases} (\Lambda_e \phi_i z)_{tt} = \Delta\left(\Lambda_e \phi_i z\right) + \left[\Lambda_e \phi_i, \Delta\right] z \quad \text{on } Q_\infty \\ \dfrac{\partial}{\partial\nu}\left(\Lambda_e \phi_i z\right) = \Lambda_e \phi_i g + \Lambda_e\left(\nabla\phi_i \cdot \nu\right) z \quad \text{on } \Sigma_\infty; \end{cases} \tag{3.29}$$

$$\begin{cases} (I - \Lambda_e)\phi_i z_{tt} = \Delta(I - \Lambda_e)\phi_i z + \left[(I - \Lambda_e)\phi_i, \Delta\right] z \quad \text{on } Q_\infty \\ \dfrac{\partial}{\partial\nu}\left((I - \Lambda_e)\phi_i z\right) = (I - \Lambda_e)\phi_i g + (I - \Lambda_e)\left(\nabla\phi_i \cdot \nu\right) z \text{ on } \Sigma_\infty. \end{cases} \tag{3.30}$$

Handling the wave equation (3.29): With $p \equiv \left(\Lambda_e \phi_i z\right)_t$, then p solves the wave equation

$$\begin{cases} p_{tt} = \Delta p + \left[\Lambda_e \phi_i, \Delta\right] z_t \quad \text{on } Q_\infty \\ \dfrac{\partial p}{\partial\nu} = \Lambda_e \phi_i g_t + \Lambda_e\left(\nabla\phi_i \cdot \nu\right) z_t \quad \text{on } \Sigma_\infty. \end{cases} \tag{3.31}$$

In analyzing the data of this equation, we have by the regularity posted in part **(i)** of the theorem that

$$[g_1, g_2]^T \in G_1 \times H^1(0,T; L^2(\Gamma)) \Rightarrow z \in H^{\frac{3}{2}}(Q). \tag{3.32}$$

This, together with the algebra for ΨDO's gives

$$\left|\left[\Lambda_e \phi_i, \Delta\right] z_t\right|_{H_{-\frac{1}{2},\gamma}(Q_\infty)} \leq C_T\left(\|g_1\|_{G_1} + \|g_2\|_{H^1(0,T;L^2(\Gamma))}\right). \tag{3.33}$$

Moreover, by (3.32), the fact that space dominates time on supp(Λ_e) (see (3.19)) and trace theory, we have

$$\begin{aligned} [g_1, g_2]^T &\in G_1 \times H^1(0,T; L^2(\Gamma)) \\ &\Rightarrow \Lambda_e \phi_i g_t + \Lambda_e\left(\nabla\phi_i \cdot \nu z_t\right) \in H_{0,\gamma}(\Sigma_\infty). \end{aligned} \tag{3.34}$$

To solve the problem (3.31), we can now use the fact that supp(Λ_e) was chosen so that $\Delta - \partial_{tt}$ is elliptic therein; so as to have by classical elliptic theory that $\left(\Lambda_e \phi_i z\right)_t = p \in H_{\frac{3}{2},\gamma}(Q_\infty)$. This, combined with (3.33) and (3.34), gives

$$\left|\Lambda_e \phi_i z_t\right|_{H_{\frac{3}{2},\gamma}(Q_\infty)} \leq C_T\left(\|g_1\|_{G_1} + \|g_2\|_{H^1(0,T;L^2(\Gamma))}\right). \tag{3.35}$$

Combining the Sobolev Embedding Theorem (for $\dim(\Omega) \leq 3$) with the estimate above now gives for arbitrary $\mathbf{x}_0 \in \overline{\Omega}$,

$$\|\Lambda_e \phi_i z_t(\mathbf{x}_0)\|_{L^2(0,T)} \leq C_T\left(\|g_1\|_{G_1} + \|g_2\|_{H^1(0,T;L^2(\Gamma))}\right). \tag{3.36}$$

Handling the wave equation (3.30): Here we use the fact that on $\mathrm{supp}(1-\lambda_e)$, time dominates space, so that

$$\begin{array}{rl} [g_1, g_2]^T \in & G_{\frac{3}{2}-\epsilon} \times H^{\frac{3}{2}-\epsilon}(0,T;L^2(\Gamma)) \\ \Rightarrow & (I-\Lambda_e)\phi_i g \in L^2(0,T;H^{\frac{3}{2}-\epsilon}(\Gamma)). \end{array}$$

Moreover, $[(I-\Lambda_e)\phi_i, \Delta] \in OPS^1(\Omega)$, and $(I-\Lambda_e)(\nabla\phi_i \cdot \nu)\, z \in OPS^1(\Gamma)$. Appealing to Lemma 2.8 and Remark 2.9 thus yields

$$\begin{aligned} \|(I-\Lambda_e)\phi_i z_t(\cdot, \mathbf{x_0})\|_{L^2(0,T)} &\leq C_T \|(I-\Lambda_e)\phi_i g\|_{L^2(0,T;H^{\frac{3}{2}-\epsilon}(\Gamma))} \\ &\leq C_T \left(\|g_1\|_{G_{\frac{3}{2}-\epsilon}} + \|g_2\|_{H^{\frac{3}{2}-\epsilon}(0,T;L^2(\Gamma))} \right). \end{aligned} \tag{3.37}$$

Finally, combining (3.36) and (3.37) with the partition of unity $\{\phi_i\}$ gives the asserted point evaluation, thereby completing the proof of **(ii)**. ■

3.3 Proof of Theorem 1.4(i)-(ii)

The proof of Theorem 1.4(i)-(ii) will readily follow as a deduction from the following Lemma.

Lemma 3.5 *For initial data $X_0 \in \mathcal{X}$ and control $u \in L^2(0,T;L^2(\Gamma_0))$, one has that the corresponding solution $[z, z_t, v, v_t]$ of (1.2) is an element of $C([0,T];\mathcal{X})$, with the estimate*

$$\|[z, z_t, v, v_t]\|_{C([0,T];\mathcal{X})} \leq C \left(\|X_0\|_{\mathcal{X}} + \|u\|_{L^2(0,T;U)} \right). \tag{3.38}$$

Proof: By virtue of the definition of $\mathcal{X}$ in (1.14), and the estimate in Propostion 2.1, we can restrict our attention to the wave component of (1.2).

Step 1 (Demonstration of L^2–time regularity).

(1.A). Taking initial data $X_0 = [z_0, z_1, v_0, v_1]$ from $\mathcal{X}$ and $u \in L^2(0,T;U)$, we can write the wave component $[z, z_t]$ via the representation (see (3.9))

$$\begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} = e^{A_1 t} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ A_N N v_t(s) \end{bmatrix} ds. \tag{3.39}$$

Integrating by parts, we have then

$$\begin{aligned} &\begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} \\ = \;& e^{A_1 t} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \int_0^t e^{A_1(t-s)} A_1 A_1^{-1} \begin{bmatrix} 0 \\ A_N N v_1 \end{bmatrix} ds + \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ A_N N(v_t(s) - v_1) \end{bmatrix} ds \\ = \;& e^{A_1 t} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} - \int_0^t \frac{d}{ds} e^{A_1(t-s)} A_1^{-1} \begin{bmatrix} 0 \\ A_N N v_1 \end{bmatrix} ds + \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ A_N N(v_t(s) - v_1) \end{bmatrix} ds \\ = \;& e^{A_1 t} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \int_0^t \frac{d}{ds} e^{A_1(t-s)} \begin{bmatrix} N v_1 \\ 0 \end{bmatrix} ds + \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ A_N N(v_t(s) - v_1) \end{bmatrix} ds \\ = \;& e^{A_1 t} \begin{bmatrix} z_0 - N v_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} N v_1 \\ 0 \end{bmatrix} + \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ A_N N(v_t(s) - v_1) \end{bmatrix} ds \\ = \;& \begin{bmatrix} z^{(0)}(t) \\ z_t^{(0)}(t) \end{bmatrix} + \begin{bmatrix} z^{(1)} \\ z_t^{(1)} \end{bmatrix} + \begin{bmatrix} z^{(2)}(t) \\ z_t^{(2)}(t) \end{bmatrix}, \end{aligned} \tag{3.40}$$

where

$$\begin{bmatrix} z^{(0)}(t) \\ z_t^{(0)}(t) \end{bmatrix} = e^{A_1 t} \begin{bmatrix} z_0 - Nv_1 \\ z_1 \end{bmatrix}; \quad \begin{bmatrix} z^{(1)} \\ z_t^{(1)} \end{bmatrix} = \begin{bmatrix} Nv_1 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} z^{(2)}(t) \\ z_t^{(2)}(t) \end{bmatrix} = \int_0^t e^{A_1(t-s)} \begin{bmatrix} 0 \\ A_N N\left(v_t(s) - v_1\right) \end{bmatrix} ds. \tag{3.41}$$

To estimate $\left[z^{(0)}, z_t^{(0)}\right]$: As the initial data was taken from $\mathcal{X}$, then using the invariance of the group $\left\{e^{A_1 t}\right\}_{t\geq 0}$, we have

$$\left\| \begin{bmatrix} z^{(0)} \\ z_t^{(0)} \end{bmatrix} \right\|_{C([0,T]; D(A_N^{\frac{3}{4}}) \times D(A_N^{\frac{1}{4}}))} \leq \left\| \begin{bmatrix} z_0 - Nv_1 \\ z_1 \end{bmatrix} \right\|_{D(A_N^{\frac{3}{4}}) \times D(A_N^{\frac{1}{4}})} \leq \|X_0\|_{\mathcal{X}}. \tag{3.42}$$

To estimate $\left[z^{(1)}, z_t^{(1)}\right]$: By (1.11), we have

$$\left\| \begin{bmatrix} z^{(1)} \\ z_t^{(1)} \end{bmatrix} \right\|_{H^{\frac{3}{2}}(\Omega) \times H^{\frac{1}{2}}(\Omega)} \leq C \|v_1\|_{L^2(\Gamma_0)} \leq C \|X_0\|_{\mathbf{H}}. \tag{3.43}$$

To estimate $\left[z^{(2)}, z_t^{(2)}\right]$: We first note that $v_t \in L^2(0,T; H^1(\Gamma_0)) \cap H^{\frac{1}{2}}(0,T; L^2(\Gamma_0)) \cap C([0,T]; L^2(\Gamma_0))$, with continuous dependence on the data and control, by Propositions 2.1 and 3.1. Moreover, initial datum $v_1 \in L^2(\Gamma_0)$. Given then that $\left[z^{(2)}, z_t^{(2)}\right]$, as it appears in (3.41), solves the wave equation with Neumann data $v_t - v_1$, with $v_t(t=0) - v_1 = 0$, we can use Lemma 3.3 (with $g = v_t - v_1$ therein) to obtain

$$\begin{aligned} &\left\| \begin{bmatrix} z^{(2)} \\ z_t^{(2)} \end{bmatrix} \right\|_{C([0,T]; H^{\frac{3}{2}}(\Omega) \times H^{\frac{1}{2}}(\Omega))} \\ &\leq C \left\| [v_t, v_1] \right\|_{\left[L^2(0,T;H^1(\Gamma_0)) \cap H^{\frac{1}{2}}(0,T;L^2(\Gamma_0)) \cap C([0,T];L^2(\Gamma_0))\right] \times [H^1(0,T;L^2(\Gamma_0))]} \\ &\leq C \left(\|X_0\|_{\mathbf{H}} + \|u\|_{L^2(0,T;U)} \right). \end{aligned} \tag{3.44}$$

Together, the estimates (3.42)–(3.44) give

$$\left\| \begin{bmatrix} z \\ z_t \end{bmatrix} \right\|_{C([0,T]; H^{\frac{3}{2}}(\Omega) \times H^{\frac{1}{2}}(\Omega))} \leq C \left(\|X_0\|_{\mathcal{X}} + \|u\|_{L^2(0,T;U)} \right). \tag{3.45}$$

(1.B) We have by means of the representation (3.40),

$$z(t) - Nv_t(t) = z^{(0)}(t) + z^{(1)}(t) + z^{(2)}(t) - Nv_t(t). \tag{3.46}$$

In analyzing the right hand side of this expression, we note by (3.41) that

$$\frac{\partial}{\partial \nu} \left(z^{(1)}(t) + z^{(2)}(t) - Nv_t(t) \right) \Big|_{\Gamma_0} = 0;$$

and this, combined with (1.11), (3.43), (3.44) and the characterization (1.9) gives the deduction that

$$\left\| z^{(1)} + z^{(2)} - Nv_t \right\|_{C([0,T]; D(A_N^{\frac{3}{4}}))} \leq C \left(\|X_0\|_{\mathbf{H}} + \|u\|_{L^2(0,T;U)} \right). \tag{3.47}$$

Combining (3.46), (3.42), (3.47) and Proposition 2.1 yields then

$$\|[z, z_t, v, v_t]\|_{C([0,T];\mathcal{X})} \leq C \left(\|X_0\|_{\mathcal{X}} + \|u\|_{L^2(0,T;U)} \right), \tag{3.48}$$

the desired result.

3.4 Proof of Theorem 1.4(iii)

Again we consider here the solution $[z, z_t, v, v_t]^T$ of (1.2) with initial data $X_0 := [z_0, z_1, v_0, v_1]^T \in \mathcal{X}$ and control $u = 0$. As we have done before (see (3.40)), the wave component $[z, z_t]^T$ of the solution of can be written explicitly as

$$\begin{aligned} \begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} &= e^{A_1 t}\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \int_0^t e^{A_1(t-s)}\begin{bmatrix} 0 \\ A_N N v_t(s) \end{bmatrix} ds \\ &= e^{A_1 t}\begin{bmatrix} z_0 - Nv_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} Nv_1 \\ 0 \end{bmatrix} + \int_0^t e^{A_1(t-s)}\begin{bmatrix} 0 \\ A_N N\left(v_t(s) - v_1\right) \end{bmatrix} ds. \end{aligned} \tag{3.49}$$

For $x_0 \in \mathrm{Int}(\Omega)$, let $\Pi_{x_0} : D(\Pi_{x_0}) \subset H^1(\Omega) \times L^2(\Omega)$ be defined by

$$\Pi_{x_0}[z_0, z_1]^T = z_1(\mathbf{x_0}).$$

Note that from Propositions 2.1 and 3.1 (with $u = 0$), we have

$$\|v_t\|_{L^2(0,T;H^2(\Gamma))\cap H^{1/2}(\Sigma)\cap C([0,T];L^2(\Gamma))} \le \|X_0\|_{\mathbf{H}}.$$

We can hence invoke Lemma 3.3(ii) to have

$$\begin{aligned} &\left\| \Pi_{x_0}\int_0^{(\cdot)} e^{A_1(\cdot - s)}\begin{bmatrix} 0 \\ A_N N\left(v_t(s) - v_1\right) \end{bmatrix} ds \right\|_{L^2(0,T)} \\ &\quad \le C_T\left(\|v_t\|_{L^2(0,T;H^2(\Gamma))\cap H^{1/2}(\Sigma)\cap C([0,T];L^2(\Gamma))} + \|v_1\|_{L^2(\Gamma)} \right) \le C_T \|X_0\|_{\mathbf{H}}. \end{aligned} \tag{3.50}$$

Moreover, as $X_0 \in \mathcal{X}$, then $[z_0 - Nv_1, z_1]^T \in D(A_N^{\frac{3}{4}}) \times \frac{H^1(\Omega)}{\mathbb{R}}$, and so we have from Triggiani's Lemma 1.3,

$$\left\| \Pi_{x_0} e^{A_1(\cdot)}\begin{bmatrix} z_0 - Nv_1 \\ z_1 \end{bmatrix} \right\|_{L^2(0,T)} \le C_T \left\| [z_0 - Nv_1, z_1] \right\|_{D(A_N^{\frac{3}{4}})\times \frac{H^1(\Omega)}{\mathbb{R}}} \le C_T \|X_0\|_{\mathcal{X}}. \tag{3.51}$$

Combining (3.50) and (3.51) with the expression in (3.49) and the definition of the observation in (1.6) now leads to the desired result.

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