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MULTIPLE HYPERGEOMETRIC FUNCTIONS WHEN PARAMETERS  
AS WELL AS THE VARIABLES ARE INCREASED**

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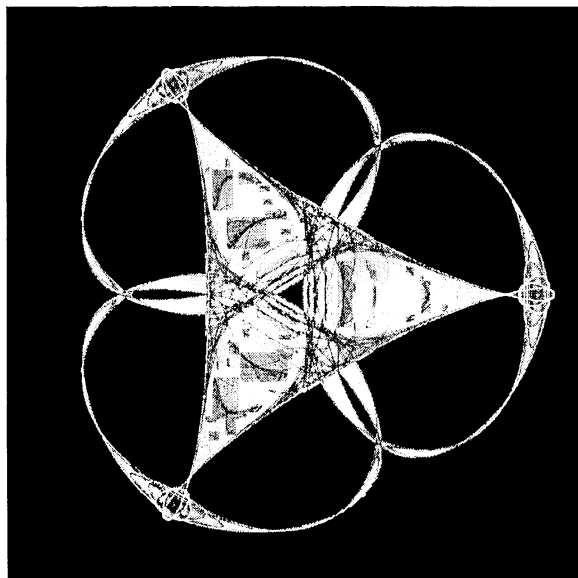
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**MULTIPLICATIVE GROUPS AND SUBGROUPS FOR LAURICEELLA TYPE  
MULTIPLE HYPERGEOMETRIC FUNCTIONS WHEN PARAMETERS AS WELL  
AS THE VARIABLES ARE INCREASED**

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The present paper is in continuation of our earlier work dealing with the construction of multiplicative groups for the cases, where on one hand the variables are increased and on the other hand the parameters are augmented. A subgroup exhibiting congruence with number theory has been considered.

**Key words :** Lauricella function / identity element / relatively prime / order of group.

**1. INTRODUCTION**

Multiple hypergeometric functions are capable of dealing with many problems of applied nature and their application potential has increased immensely with the development of new generation computers. In this context we can cite the works of Exton<sup>4,5</sup> and Agarwal<sup>6</sup>. The usefulness of group representation theory for the solution of a variety of physical problems makes it natural that representation matrix elements are important special functions from many problems in mathematical physics. The books of Srivastava and Rassias<sup>2,3</sup> have opened new vistas of interest for workers of the field. In our earlier work<sup>1</sup>, finite multiplicative group has been constructed for multiple hypergeometric functions involving two and three variables. Here an attempt has been made to conform them for more generalized cases involving more than three variables and generalized parameters.

**2. GENERAL FORMULATION**

B-function defined by equation (3.1) of our earlier work<sup>1</sup> can assume following form, if we take two elements in the upper half

$$B \begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix} \left[ \begin{matrix} a, b & d \\ & ; & ; x, y \\ & c & e \end{matrix} \right] \dots\dots\dots(2.1)$$

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for which Lauricella pattern functions can be obtained in the form

$$\begin{bmatrix} a & b \\ & c \end{bmatrix}$$

Where

- 0 for normal function, no change
  - 1 for change in the first element in upper.
  - 1' for change in the second element in upper.
  - 2 for change in lower and first element of upper
  - 2' for change in lower and second element of upper
  - 3 for change in the lower
  - 4 for change in lower and both upper element
  - 5 for change in all the three
- } .....(2.2)

Set for the different functions generated according to the rule, discussed above shall be

$$S = \{ B^{pq} \} \dots\dots\dots(2.3)$$

Where the ordered pair (pq) shall take one of the values from the following set

$$\{0,1,1',2',3,4,5\} \times \{0,1,2,3\} - \{0,0\}$$

and total elements of S shall be  $2^5 - 1$

One element of the set can be defined as

$$B = \begin{bmatrix} (a_1, a_2)(b_1, b_2) & (d_2, d_1) \\ (c_2, c_1) & (e_2, e_1) \end{bmatrix} = \begin{bmatrix} (1,0)(1,0) & (1,0) \\ (1,0) & (1,0) \end{bmatrix}$$

whose inverse element shall be  $B^{40}$

Identity element for the set S shall be  $B^{52}$ .

### 3. MULTIPLICATIVE GROUP FOR B-FUNCTION INVOLVING GENERALIZED PARAMETERS (Variables being two only)

Let the function be defined as

$$B \begin{bmatrix} a_p & c_r \\ b_q & d_s \end{bmatrix} ; x, y = \sum_{m_1, m_2=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_{m_1} \prod_{i'=1}^r (c_{i'})_{m_2} x^{m_1} y^{m_2}}{\prod_{j=1}^q (b_j)_{m_1} \prod_{j'=1}^s (d_{j'})_{m_2} (m_1)! (m_2)!} \dots\dots\dots(3.1)$$

Where  $m = m_1 + m_2$

For conversion in Lauricella pattern function for first half

$$\begin{bmatrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{bmatrix}$$

We shall use following notations

0 for no change

$i_1, i_2, \dots, i_p$   
1 for change in  $i_1, i_2, \dots, i_p$  elements of upper.

$i_1, i_2, \dots, i_p$  for change in  $i_1, i_2, \dots, i_p$  elements of upper  
2 and  $j_1, j_2, \dots, j_q$  elements of lower

$j_1, j_2, \dots, j_q$

3 for change in  $j_1, j_2, \dots, j_q$  elements of lower

$j_1, j_2, \dots, j_q$

(3.2)

Same criterion can be adopted for change in second half.

Total distinct elements of the multiplicative group of set

$$S = \{ B^{i j} \} \text{ shall be}$$

$$\frac{p+q+r+s}{2} - 1$$

Identity element of the set S shall be

$$B = \begin{matrix} i_1, i_2, \dots, i_p & i'_1, i'_2, \dots, i'_r \\ j_1, j_2, \dots, j_q & j'_1, j'_2, \dots, j'_s \end{matrix}$$

One element of the set may be defined as

$$B = \begin{bmatrix} (a^1_1, a^1_2) (a^2_1, a^2_2) \dots (a^p_1, a^p_2) & (c^1_1, c^1_2) (c^2_1, c^2_2) \dots (c^r_1, c^r_2) \\ (b^1_1, b^1_2) (b^2_1, b^2_2) \dots (b^q_1, b^q_2) & (d^1_1, d^1_2) (d^2_1, d^2_2) \dots (d^s_1, d^s_2) \end{bmatrix}$$

$$= \begin{bmatrix} (1,0) (1,0) \dots p \text{ times} & (1,0) (1,0) \dots r \text{ times} \\ (1,0) (1,0) \dots q \text{ times} & (1,0) (1,0) \dots s \text{ times} \end{bmatrix}$$

**4. MULTIPLICATIVE GROUP FOR B FUNCTION INVOLVING MORE THEN TWO VARIABLE (Parameters being four only)**

Taking the B function

$$B \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \left[ \begin{matrix} a & c \\ ; & ; \\ b & d \end{matrix} ; x_1, x_2, x_3 \right] = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a)_m (c)_{-m} x_1^{m_1} x_2^{m_2} x_3^{m_3}}{(b)_m (d)_{-m} m_1! m_2! m_3!} \dots(4.1)$$

Where  $m = m_1 + m_2 + m_3$

Which can be easily converted to Lauricella type B - function by using the explanation given in previous section

As an example we may cite

$$B \begin{matrix} 1 & 0 \\ 3 & \end{matrix} = \left[ \begin{matrix} (a_3, a_1, a_2) & (c_1, c_2, c_3) \\ ; & \\ (b_1, b_2, b_3) & (d_1, d_2, d_3) \end{matrix} \right] = \left[ \begin{matrix} (1,0,0) & (1,0,0) \\ ; & \\ (1,0,0) & (1,0,0) \end{matrix} \right] \dots\dots\dots(4.2)$$

The fact that total number of distinct elements of the group shall be 80 is evident from the matrix form given by (4.2) whose each element acquires three different functional values and it implies that total elements shall be  $3^4$  out of which one, being ordinary B -function, shall be taken out.

Here if we take n variable for  $B_{11}^{11}$  then the set S can be constructed for different functions like  $B_{(r,s)}^{pq}$  defined where pq shall assume fifteen values as discussed in section (3) of our earlier study.

We may cite one particular case

$$B \begin{matrix} 1 & 1 \\ (r,s) & \end{matrix} = \left[ \begin{matrix} (a_r, a_1, a_2, \dots, a_{r-1}, a_{r+1}, \dots, a_n) & (c_s, c_1, c_2, \dots, c_{s-1}, c_{s+1}, \dots, c_n) \\ ; & \\ (b_1, b_2, \dots, b_n) & (d_1, d_2, \dots, d_n) \end{matrix} \right]$$

$$= \left[ \begin{matrix} (1,0,0, \dots, 0) & (1,0,0, \dots, 0) \\ ; & \\ (1,0,0, \dots, 0) & (1,0,0, \dots, 0) \end{matrix} \right]$$

Total number of such function shall be  $n^4 - 1$ .

## 5. MULTIPLICATIVE GROUP

(When both parameters and variables assume generalized form)

Let the function be

$$B_{r s}^{p q} \left[ \begin{array}{cc} a_p & c_r \\ b_q & d_s \end{array} ; x_1, x_2, \dots, x_n \right] \dots\dots\dots(5.1)$$

Lauricella type function for which shall form a set

$$S = \{ B^{k \ell} \} \dots\dots\dots(5.2)$$

Whose identity element shall be

$$B_{i j} = \left[ \begin{array}{l} (a^1_1, a^1_2, \dots, a^1_n) (a^2_2, a^2_1, \dots, a^2_n) \dots (a^p_p, a^p_1, \dots, a^p_n) \\ (b^1_1, b^1_2, \dots, b^1_n) (b^2_2, b^2_1, \dots, b^2_n) \dots (b^q_q, b^q_1, \dots, b^q_n) \\ (c^1_1, c^1_2, \dots, c^1_n) (c^2_2, c^2_1, \dots, c^2_n) \dots (c^r_r, c^r_1, \dots, c^r_n) \\ (d^1_1, d^1_2, \dots, d^1_n) (d^2_2, d^2_1, \dots, d^2_n) \dots (d^s_s, d^s_1, \dots, d^s_n) \end{array} \right] ;$$

$$= \left[ \begin{array}{ll} (1, 0, 0, \dots, 0) \dots \dots \dots p \text{ times} & (1, 0, 0, \dots, 0) \text{ } r \text{ times} \\ (1, 0, 0, \dots, 0) \dots \dots \dots q \text{ times} & (1, 0, 0, \dots, 0) \text{ } s \text{ times} \end{array} \right]$$

Total numbers of distinct element in the set S shall be  $n^{p+q+r+s-1}$ .

## 6. SUBGROUP EXHIBITING CONGRUENCE WITH NUMBER THEORY

We know that order of a generalized group involving n variables and generalized parameters is

$$n^{p+q+r+s-1}$$

Here if g. c. d. (n, p + q + r + s + 1) = 1 \dots\dots\dots(6.1)

and p + q + r + s + 1 = P (say) is a prime number then generalized group shall have a subgroup of order P.

P.T.O.[5]

Which can be used in the verification of Euler's theorem stating that if  $\text{g.c.d.}(a, p) = 1$  ( $p$  being a prime number)

$$\text{then } a^{p-1} \equiv 1 \pmod{p} \dots\dots(6.2)$$

Here if we consider the following subgroup of  $(s, \cdot)$  obtained in our earlier work<sup>1</sup>,

$$S_1 = \{B^{12}, B^{21}, B^{23}, B^{32}, B^{22}\} \dots\dots(6.3)$$

then above statement is verified

$$\text{as } 2^4 - 1 \equiv 0 \pmod{5}$$

Lagrange's theorem can be verified in a similar fashion.

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