ANALYSIS OF A MATHEMATICAL MODEL OF THE GROWTH OF NECROTIC TUMORS

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Abstract. In this paper we study a model of necrotic tumor growth. The tumor comprises of necrotic cells which occupy a radially symmetric core, and life proliferating cells which occupy a radially symmetric shell adjacent to the core. The proliferating cells receive nutrients both through diffusion from the outer boundary as well as by means of blood flow through a network of capillary vessels. The mathematical model describes the evolution of the nutrient concentration σ between the boundary of the necrotic core $r = \rho(t)$ and the outer boundary of the tumor $r = R(t)$; within the core itself the concentration is a constant $\sigma = \sigma_{\text{nec}}$, a level under which life cells cannot be sustained. Both surfaces $r = \rho(t)$ and $r = R(t)$ are free boundaries, which are unknown in advance. Under some assumptions on the parameters, we prove that (i) there exists a stationary solution with radii $r = \rho_s$, $r = R_s$; (ii) for any initial data near the stationary solution, the time dependent model has a unique solution $\sigma(r,t)$ with free boundaries $r = \rho(t)$, $r = R(t)$, and (iii) $\rho(t) \to \rho_s$ and $R(t) \to R_s$ as $t \to \infty$.

Key words. tumor, necrotic, parabolic equations, free boundary problems.

1. The model. In this paper we study a model recently proposed by Byrne and Chaplain [7] for the growth of necrotic tumor in the absence of inhibitors. The tumor consists of a spherical core of dead cells (necrotic core) and a spherical shell of life cells surrounding the core (nonnecrotic shell). The nonnecrotic region receives blood supply through a developed network of capillary vessels (vascularized tumor). The blood supply provides the nonnecrotic region with nutrients. On the other hand there is no blood supply within the necrotic region, and the concentration of nutrients within it is at constant level, the threshold at which life cells cannot be sustained. The boundaries of the necrotic core $r = \rho(t)$ (where $r = |x|$, $x = (x_1, x_2, x_3)$), and of the outer nonnecrotic shell $r = R(t)$ are free boundaries, unknown in advance. An earlier model was developed by Greenspan [13].

The present paper develops mathematical techniques for rigorous analysis of transient and stationary solutions to such models with two free boundaries. In the particular model under study they allow us to confirm, but also significantly extend, the results obtained in [7] through numerical studies.

As in other models developed in the last few decades (see, e.g., [1–6, 13] and the references cited there), Byrne and Chaplain represent the tumor's evolution by means of the levels of diffusing nutrient concentration. Their model also takes into account the possible blood-tissue nutrient transfer that occurs *in vivo* through angiogenesis as described and modelled in [3, Chap. 5], [4]. (Angiogenesis is a process by which tumors induce blood vessels to sprout capillary tips which migrate toward, and penetrate into, the tumor thus providing it with circulating blood supply.) Further, their model includes a well-motivated cell loss mechanism, *apoptosis* or *shrinkage necrosis* in the nonnecrotic region, and cell loss in the necrotic region. However, in contrast with previous models the present model includes two free boundaries: $r = \rho(t)$, the boundary of the necrotic core, and, $r = R(t)$, the (outer) boundary of the tumor.

As in [7], after nondimensionalization, the nutrient concentration, $\hat\sigma(r,t)$, will satisfy the reaction-diffusion equation

$$c\,\frac{\partial\hat\sigma}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\hat\sigma}{\partial r}\right) + H(r-\rho(t))\Big[\Gamma(\sigma_B-\hat\sigma)-\lambda_0\hat\sigma\Big] \tag{1.1}$$

where $H(s)$ is the Heaviside function: $H(s)=0$ if $s\le 0$ and $H(s)=1$ if $s>0$. Here the constants σ_B and Γ denote the (dimensionless) nutrient concentration in the vasculature and the rate of nutrient-in-blood tissue transfer per unit length, respectively. Thus $\Gamma(\sigma_B-\hat\sigma)$ accounts for the transfer of nutrient by means of the vasculature (in the nonnecrotic shell), whose presence stems from angiogenesis. The term $\lambda_0\hat\sigma$ is the nutrient consumption rate, and $c = T_{\text{diffusion}}/T_{\text{growth}}$ is the ratio of nutrient diffusion time scale to the tumor growth (e.g., tumor doubling) time scale. Note that, typically, $T_{\text{diffusion}}\approx 1$ minute (see [3, pp. 194–195]) while $T_{\text{growth}}\approx 1$ day, so that $c\ll 1$.

In (1.1)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right)$$

is meant to denote the radial part of the Laplace operator in 3-dimensions, and therefore one must include the condition

$$\frac{\partial\hat\sigma}{\partial r}(0,t)=0\ . \tag{1.2}$$

In the sequel we take

$$\hat\sigma(r,t)=\text{const.}=\sigma_{\text{nec}}>0 \qquad \text{if} \quad r=\rho(t)$$

and assume that initially

$$\hat\sigma(r,0)=\sigma_{\text{nec}} \qquad \text{if} \quad 0\le r<\rho(0).$$

Then, by the maximum principle,

$$\hat\sigma(r,t)=\sigma_{\text{nec}} \qquad \text{if} \quad 0\le r<\rho(t),\ t>0. \tag{1.3}$$

We shall also prove that $\hat\sigma_r(r,t)>0$ if $\rho(t)<r<R(t)$, so that

$$\hat\sigma(r,t)>\sigma_{\text{nec}} \qquad \text{if} \quad \rho(t)<r\le R(t),\ t>0. \tag{1.4}$$

Assuming that the mass density of cells is constant in the nonnecrotic region, the principle of conservation of mass coincides with the principle of conservation of volume. A reasonable simplified approach to this principle, developed in [13], gives the relation

$$\frac{d}{dt}\Big(\frac{4}{3}\,\pi R^3(t)\Big) = \int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_{\rho(t)}^{R(t)} S(\hat\sigma)\,r^2\sin\theta\,dr\,d\theta\,d\varphi - M(\sigma_{\text{nec}})\,\rho^3(t) \tag{1.5}$$

where $S(\hat\sigma)$ denotes the cell proliferation rate within the nonnecrotic region of the tumor, and $M(\sigma_{\text{nec}})$ represents the cell loss due to necrosis.

We should clarify (cf. [7]) that whereas apoptosis refers to natural cell death caused for example, by aging, necrosis represents cell death caused by the microenvironment which occurs, for example, when the level of nutrient concentration is below a critical value necessary to sustain the cell.

For simplicity we restrict ourselves to the proliferation rate [6, 7]

$$S(\sigma) = \mu_0(\hat{\sigma} - \stackrel{\approx}{\sigma}) \tag{1.6}$$

where μ_0 and $\stackrel{\approx}{\sigma}$ are positive constants. This means that the cell birth-rate is $\mu_0\hat{\sigma}$ while the cell death-rate (apoptosis) is given by $\mu_0 \stackrel{\approx}{\sigma}$. We also set

$$4\pi\mu = M(\hat{\sigma}_{\text{nec}}) \ ; \qquad \mu > 0 \ . \tag{1.7}$$

Finally, the external nutrient concentration is assumed to be a constant $\bar{\bar{\sigma}}$, so that

$$\hat{\sigma} = \bar{\bar{\sigma}} \qquad \text{at} \qquad r = R(t) \ . \tag{1.8}$$

The above model in the nonnecrotic case was studied, by rigorous mathematical methods, by Friedman and Reitich [12]; the case where inhibitors are present was studied, by rigorous mathematical analysis, by Cui and Friedman [8]. In both of these papers the existence of stationary solutions was established and their asymptotic stability was analyzed. The present paper is written in the same spirit. However the mathematical analysis turns out to be far more dedicate due to the presence of *two* free boundaries.

It will be convenient to work with the function

$$\sigma = \hat{\sigma} - \frac{\Gamma\sigma_B}{\Gamma + \lambda_0}$$

and the quantities

$$\begin{aligned} \bar{\sigma} \ &= \bar{\bar{\sigma}} - \frac{\Gamma\sigma_B}{\Gamma + \lambda_0} \ , \qquad & \tilde{\sigma} \ &= \stackrel{\approx}{\sigma} - \frac{\Gamma\sigma_B}{\Gamma + \lambda_0} \ , \\ \sigma_0 \ &= \ \sigma_{\text{nec}} - \frac{\Gamma\sigma_B}{\Gamma + \lambda_0} \ , \qquad & \lambda \ &= \ \Gamma + \lambda_0 \ . \end{aligned}$$

Choosing also, for definiteness, $\mu_0 = 1$, we can rewrite the model in the form:

$$c\,\frac{\partial\sigma}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\sigma}{\partial r}\right) + \lambda\sigma\, H(\sigma - \sigma_0) = 0 \ , \qquad \text{a.e. for} \quad 0 < r < R(t), \quad t > 0 \ , \tag{1.9}$$

$$\frac{\partial\sigma}{\partial r}(0,t) \ = \ 0 \qquad \text{if} \qquad t > 0 \ , \tag{1.10}$$

$$\sigma(R(t),t) \ = \ \bar{\sigma} \qquad \text{if} \qquad t > 0 \ , \tag{1.11}$$

$$\sigma(r,t) \ = \ \sigma_0 \qquad \text{if} \qquad 0 \le r \le \rho(t) \ , \ t > 0 \ , \tag{1.12}$$

$$\frac{\partial\sigma}{\partial r}(r,t) \ = \ 0 \qquad \text{if} \qquad r = \rho(t) \ , \tag{1.13}$$

$$R^2(t)\,\frac{dR(t)}{dt} = \int_{\rho(t)}^{R(t)} (\sigma(r,t) - \tilde{\sigma})\, r^2\, dr - \mu\rho^3(t) \ , \qquad t > 0 \tag{1.14}$$

with initial data

$$\begin{gathered} 0 < \rho(0) < R(0) \ , \\ \sigma(r,0) = \sigma_0 \quad \text{if} \quad 0 \le r \le \rho(0) \ , \ \sigma(R(0),0) = \bar{\sigma} \ , \ \sigma_0 < \sigma(r,0) < \bar{\sigma} \quad \text{if} \quad \rho(0) < r < R(0) \ . \end{gathered} \tag{1.15}$$

Of course as long as $\rho(t)$ remains positive, the condition (1.10) is automatically satisfied.

Note that (1.13) means that $\partial\sigma/\partial r$ is continuous from both sides of $r = \rho(t)$ and its value at $r = \rho(t)$ is zero.

We shall be proving later on as that $\sigma_r(r,t) \geq 0$, so that $\rho(t)$ is such that $\sigma(r,t) = \sigma_0$ if $0 \leq r \leq \rho(t)$ and $\sigma(r,t) > \sigma_0$ if $\rho(t) < r \leq R(t)$; cf. also (1.4). As a consequence we could replace (1.9) by

$$c\frac{\partial\sigma}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma}{\partial r}\right) + \lambda\sigma = 0 \quad \text{if} \quad \rho(t) < r < R(t)\ ,\ t > 0\ . \tag{1.16}$$

Since on the free boundary $r = \rho(t)$ we have the two boundary conditions

$$\sigma(\rho(t),t) = \sigma_0\ , \qquad \sigma_r(\rho(t),t) = 0\ , \qquad t > 0\ , \tag{1.17}$$

σ is also a solution of the parabolic variational inequality

$$\begin{aligned} c\frac{\partial\sigma}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma}{\partial r}\right) + \lambda\sigma \ &\geq\ 0\ , \qquad \sigma \geq \sigma_0\ , \\ \left[c\frac{\partial\sigma}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma}{\partial r}\right) + \lambda\sigma\right](\sigma - \sigma_0) \ &=\ 0 \\ &\text{a.e. for} \quad 0 < r < R(t),\ t > 0\ . \end{aligned} \tag{1.18}$$

For a given $R(t)$, this variational inequality has a unique solution (satisfying given initial data and (1.11)) (see [11, Chap. 1]); however, in our case the curve $r = R(t)$ is not given, and, in fact, it is a free boundary. This complicates substantially the proof of existence of a solution to the time dependent problem.

The structure of the paper is as follows: In §2 we establish the existence and uniqueness of a stationary solution to (1.9)–(1.14), under certain assumptions on the parameters. The remaining part of the paper is devoted to proving that if the initial data ($\sigma(r,0)$, $R(0)$, $\rho(0)$) are "close enough" to a stationary solution ($\sigma_s(r)$, R_s, ρ_s) then a global solution to (1.9)–(1.15) exists, and any such solution converges to the stationary solution as $t \to \infty$. Below we elaborate in more detail on the outline of the proof.

In §§3,4 we consider the free boundary problem (1.9)–(1.13) with fixed boundary $r = R(t)$ ("Problem (R_T)") and prove that it has a unique solution, and that the free boundary $r = \rho(t)$ is in C^∞. It is assumed that the initial data $\sigma^0(r) = \sigma(r,0)$ and $R(t)$ satisfy a set of conditions, (3.5) and (3.7), respectively.

In §5 we prove that if c is small enough then the free boundary problem (1.9)–(1.14) has a unique solution provided $\sigma^0(r)$ satisfies the condition in (3.5), and the solution exists as long as $R(t)$ remains in an interval (R_0, R_1) which contains the point R_s; here $R_s - R_0$ is assumed to be small (but independent of c). Both free boundaries $r = R(t)$ and $r = \rho(t)$ are proved to be in C^∞.

Making the additional assumption (6.1), we derive in §6 the priori bound $|d\rho(t)/dt| \leq N$ where N is a constant independent of c. This bound allows us, in §7, to prove that for the solution established in §5, $R(t)$ never leaves the interval (R_0, R_1); consequently the solution exists for all $t > 0$. Finally, in §7 it is shown that the solution converges to the stationary solution as $t \to \infty$, and the rate of convergence is $e^{-K\sqrt{t}}$, where K is a positive constant.

2. Stationary solutions. As easily seen [7] if $(\sigma_s(r), \rho_s, R_s)$ is a stationary solution of (1.9)–(1.14) then

$$\begin{aligned}\sigma_s(r) &= \frac{A}{r}\left[\sinh(\sqrt{\lambda}(r-\rho_s)) + \sqrt{\lambda}\rho_s\cosh(\sqrt{\lambda}(r-\rho_s))\right] \qquad \text{for} \qquad \rho_s < r < R_s\ ,\\ &= \sigma_0 \qquad \text{for} \qquad 0 \le r \le \rho_s\ ,\end{aligned} \tag{2.1}$$

where $A\sqrt{\lambda} = \sigma_0$, so that the condition $\sigma_s(R_s) = \bar{\sigma}$ becomes

$$\frac{\bar{\sigma}}{\sigma_0}\sqrt{\lambda}R_s = \sinh(\sqrt{\lambda}(R_s-\rho_s)) + \sqrt{\lambda}\rho_s\cosh(\sqrt{\lambda}(R_s-\rho_s))\ . \tag{2.2}$$

Finally, the condition (1.14) yields the relation

$$\frac{\sigma_0}{\lambda^{3/2}}\Big[\sqrt{\lambda}(R_s-\rho_s)\cosh(\sqrt{\lambda}(R_s-\rho_s)) + (\lambda R_s\rho_s - 1)\sinh(\sqrt{\lambda}(R_s-\rho_s))\Big] - \frac{\tilde{\sigma}}{3}(R_s^3-\rho_s^3) - \mu\rho_s^3 = 0\,. \tag{2.3}$$

Setting

$$\eta = \sqrt{\lambda}R_s\ , \qquad \zeta = \sqrt{\lambda}\rho_s\ , \tag{2.4}$$

equations (2.2), (2.3) become

$$\sinh(\eta-\zeta) + \zeta\cosh(\eta-\zeta) = \frac{\bar{\sigma}}{\sigma_0}\eta\ , \tag{2.5}$$

$$(\eta-\zeta)\cosh(\eta-\zeta) + (\eta\zeta-1)\sinh(\eta-\zeta) = \frac{\tilde{\sigma}}{3\sigma_0}(\eta^3-\zeta^3) + \frac{\mu}{\sigma_0}\zeta^3\ . \tag{2.6}$$

We summarize:

LEMMA 2.1. *$(\sigma_s(r), R_s, P_s)$ is a stationary solution of (1.9)–(1.14) if and only if $\sigma_s(r)$ is given by (2.1) where R_s and ρ_s satisfy (with the notation (2.4)) the equations (2.5), (2.6).*

We now proceed to determine the solutions of (2.5), (2.6), setting

$$a = \frac{\tilde{\sigma}}{\sigma_0}\ , \qquad b = \frac{\mu}{\sigma_0}\ , \qquad \gamma = \frac{\bar{\sigma}}{\sigma_0}\ . \tag{2.7}$$

We shall henceforth make the natural assumption that $\gamma > 1$.

If we define

$$\xi = \eta - \zeta \qquad \text{i.e.,} \quad \eta = \xi + \zeta$$

then (2.5), (2.6) reduce to

$$\sinh\xi + \zeta\cosh\xi = \gamma\xi + \gamma\zeta\ , \tag{2.8}$$

$$\xi\cosh\xi + (\zeta^2 + \xi\zeta - 1)\sinh\xi - \frac{a}{3}\xi(3\zeta^2 + 3\xi\zeta + \xi^2) - b\zeta^3 = 0\ . \tag{2.9}$$

From (2.8) we get

$$\zeta = \frac{\gamma\xi - \sinh\xi}{\cosh\xi - \gamma}\ . \tag{2.10}$$

Substituting this into (2.9), we obtain

$$f(\xi) = 0 \tag{2.11}$$

where

$$f(\xi) = \xi \cosh \xi + \left[\left(\frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} \right)^2 + \xi \left(\frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} \right) - 1 \right] \sinh \xi$$
$$- \frac{a}{3} \xi \left[3 \left(\frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} \right)^2 + 3\xi \left(\frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} \right) + \xi^2 \right] - b \left(\frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} \right)^3 . \tag{2.12}$$

We have thus proved:

LEMMA 2.2. *The stationary solutions (η, ζ) are given by $\eta = \xi + \zeta$ and (2.10) where ξ is the solution of (2.11), with f defined by (2.12).*

Of course we are looking only for such solution for which $\eta > \zeta > 0$; the corresponding ξ will be called *admissible*; note that $\xi > 0$.

LEMMA 2.3. *If ξ is an admissible solution of (2.11) then*

$$\cosh \xi > \gamma > \frac{\sinh \xi}{\xi} . \tag{2.13}$$

Proof. Since $\zeta > 0$, (2.10) implies that either

$$\gamma\xi - \sinh \xi > 0 \qquad \text{and} \qquad \cosh \xi - \gamma > 0 , \tag{2.14}$$

or $\gamma\xi - \sinh \xi < 0$ and $\cosh \xi - \gamma < 0$. But since, as easily verified,

$$\cosh \xi > \frac{\sinh \xi}{\xi} , \tag{2.15}$$

the second case cannot take place; thus (2.14) must be satisfied, and this yields (2.13).

Since $\gamma > 1$, there exist unique solutions ξ_1, ξ_2 of

$$\cosh \xi_1 = \gamma , \qquad \frac{\sinh \xi_2}{\xi_2} = \gamma ; \tag{2.16}$$

by (2.15), $0 < \xi_1 < \xi_2$. Lemma 2.3 says that the admissible solutions of (2.11) must lie in the interval (ξ_1, ξ_2); for such a solution ξ,

$$\begin{aligned} \eta &= \xi + \frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} = \frac{\xi \cosh \xi - \sinh \xi}{\cosh \xi - \gamma} , \\ \zeta &= \frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} \end{aligned} \tag{2.17}$$

form a solution of (2.5), (2.6) with $\eta > \zeta > 0$. Note that

$$\eta\zeta = \left(\frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} \right)^2 + \xi \frac{\gamma\xi - \sinh \xi}{\cosh \xi - \gamma} . \tag{2.18}$$

□

LEMMA 2.4. *The functions $\eta = \eta(\xi)$, $\zeta = \zeta(\xi)$ defined in (2.17) are strictly monotone decreasing in ξ, $\xi_1 < \xi < \xi_2$; in fact $\eta'(\xi) < 0$, $\zeta'(\xi) < 0$ if $\xi_1 < \xi < \xi_2$, and*

$$\lim_{\xi \to \xi_1 +} \eta(\xi) = \infty , \qquad \lim_{\xi \to \xi_2 -} \eta(\xi) = \xi_2 , \tag{2.19}$$

$$\lim_{\xi \to \xi_1 +} \zeta(\xi) = \infty , \qquad \lim_{\xi \to \xi_2 -} \zeta(\xi) = 0 . \tag{2.20}$$

Proof. Computing $\eta'(\xi)$ and using the inequality $\gamma\xi - \sinh\xi > 0$, we find that $\eta'(\xi) < 0$. Differentiating $\zeta(\xi)$ we get a fraction with positive denominator and with numerator $g(\xi) = 2\gamma\cosh\xi - \gamma^2 - 1 - \gamma\xi\sinh\xi$. But $g(\xi)$ is negative since $g(0) < 0$ and $g'(\xi) = \gamma\sinh\xi - \gamma\xi\cosh\xi < 0$ by (2.15), so that $\zeta'(\xi) < 0$. Finally, the verification of (2.19) and (2.20) is immediate. □

COROLLARY 2.5. *If (η, ζ) is a solution of (2.5), (2.6) then $\eta > \xi_2$.*

We note that

$$\lim_{\xi\to\xi_1+} f(\xi) = -\infty\ , \qquad f(\xi_2) = \xi_2^3\left(\frac{\xi_2\cosh\xi_2 - \sinh\xi_2}{\xi_2^3} - \frac{a}{3}\right)\ , \tag{2.21}$$

and set

$$m(\eta) \equiv \frac{\eta\cosh\eta - \sinh\eta}{\eta^3}\ , \tag{2.22}$$

so that

$$f(\xi_2) = \xi_2^3\left[m(\xi_2) - \frac{a}{3}\right]\ . \tag{2.23}$$

LEMMA 2.6. *(i) $m'(\eta) > 0$ for all $\eta > 0$;*

(ii) $\lim_{\eta\to 0} m(\eta) = \dfrac{1}{3}\ , \qquad \lim_{\eta\to\infty} m(\eta) = \infty\ .$

Proof. We compute that $m'(\eta) = h(\eta)/\eta^6$ where $h(\eta) = \eta^2\sinh\eta - 3\eta\cosh\eta + 3\sinh\eta$. Since $h(0) = 0$ and

$$h'(\eta) = -\eta\sinh\eta + \eta^2\cosh\eta > 0\ ,$$

by (2.15), the assertion (i) follows. The assertions in (ii) are rather immediate. □

From Lemma 2.6 it follows that $m(\eta) > \frac{1}{3}$, and, for each $a > 1$, there exists a unique $\eta_a^* > 0$ such that $m(\eta_a^*) = \frac{a}{3}$. Combining this with (2.23) we conclude:

COROLLARY 2.7. *(i) If $a \le 1$ then*

$$f(\xi_2) > \frac{(1-a)\xi_2^3}{3} \ge 0\ ;$$

(ii) If $a > 1$ then

$$f(\xi_2) > 0 \quad \text{for} \quad \xi_2 > \eta_a^*\ ,$$
$$f(\xi_2) < 0 \quad \text{for} \quad \xi_2 < \eta_a^*\ .$$

THEOREM 2.8. *(i) If $a \le 1$ then (2.11) has at least one solution in (ξ_1, ξ_2);*

(ii) If $a > 1$ and $\gamma > (\sinh\eta_a^*)/\eta_a^*$ then (2.11) has at least one solution in (ξ_1, ξ_2).

Indeed, (i) follows from Corollary 2.7 (i) and the first part of (2.21). To prove (ii) note that by the definition of ξ_2 in (2.16), $\xi_2 > \eta_a^*$ so that, by Corollary 2.7 (ii), $f(\xi_2) > 0$. Now use the first part of (2.21) to complete the proof.

We next prove uniqueness:

THEOREM 2.9. *If either $a \le 1$ or $a > 1$ and $\gamma > \cosh\eta_a^*$ then the solution established in Theorem 2.8 is unique.*

Proof. Using (2.18) and the last equation in (2.17), we can rewrite (2.12) in the form

$$\begin{aligned} \frac{f(\xi)}{\zeta^3} &= \frac{1}{\zeta^3}\left[\left(\xi\cosh\xi - \sinh\xi - \frac{a}{3}\xi^3\right) + \eta\zeta(\sinh\xi - a\xi)\right] - b \\ &= \frac{\xi}{\zeta}\frac{\eta}{\zeta}\left[\frac{\xi^2}{\eta\zeta}\left(\frac{\xi\cosh\xi - \sinh\xi}{\xi^3} - \frac{a}{3}\right) + \left(\frac{\sinh\xi}{\xi} - a\right)\right] - b\ . \end{aligned} \tag{2.24}$$

We shall use the last expression to prove that $f(\xi)/\zeta^3$ is strictly monotone increasing in ξ, for $\xi_1 < \xi < \xi_2$.

First note that the functions

$$\frac{\xi}{\zeta}\,, \quad \frac{\eta}{\zeta}\Big(= 1 + \frac{\xi}{\zeta}\Big) \quad \text{and} \quad \frac{\xi^2}{\eta\zeta}\Big(= \frac{\xi}{\eta}\frac{\xi}{\zeta}\Big)$$

are strictly monotone increasing, by Lemma 2.4, and the function

$$\frac{\xi\cosh\xi - \sinh\xi}{\xi^3} - \frac{a}{3}$$

is strictly monotone increasing, by Lemma 2.6. Furthermore $(\sinh\xi)/\xi$ is clearly also strictly monotone increasing. We can also easily verify that

$$\frac{\sinh\xi}{\xi} > 3\,\frac{\xi\cosh\xi - \sinh\xi}{\xi^3} \qquad \text{for all} \quad \xi > 0\,. \tag{2.25}$$

Consider the case $a \le 1$. Then

$$\frac{\sinh\xi}{\xi} - a > 3\Big[\frac{\xi\cosh\xi - \sinh\xi}{\xi^3} - \frac{a}{3}\Big] = 3\Big(m(\xi) - \frac{a}{3}\Big) > 0$$

so that the expression in brackets in (2.24) is positive and strictly monotone increasing, and the same then holds for $f(\xi)/\zeta^3$.

Consider next the case $a > 1$ with $\gamma > \cosh\eta_a^*$. Then $\xi_1 > \eta_a^*$ by definition of ξ_1 in (2.16), and so, by (2.25) and Lemma 2.6(i),

$$\frac{\sinh\xi}{\xi} - a > 3[m(\xi) - m(\eta_a^*)] > 3[m(\xi_1) - m(\eta_a^*)] > 0\,.$$

Thus $f(\xi)/\zeta^3$ is again strictly monotone increasing in ξ. Consequently, in both cases $f(\xi)$ can have just one zero in the interval $\xi_1 < \xi < \xi_2$. □

3. Auxiliary problem. Consider the function

$$F(R,\rho) = \frac{1}{\sqrt{\lambda}R}\Big[\sinh(\sqrt{\lambda}(R-\rho)) + \sqrt{\lambda}\rho\cosh(\sqrt{\lambda}(R-\rho))\Big]\,. \tag{3.1}$$

It is easy to see that

$$F_R(R,\rho) > 0\,, \qquad F_\rho(R,\rho) < 0 \qquad \text{for} \qquad 0 < \rho \le R < \infty\,.$$

By (2.2), if (σ_s, R_s, ρ_s) is a stationary solution then

$$F(R_s,\rho_s) = \gamma\,. \tag{3.2}$$

Consequently, for any $0 < \rho_0 < \rho_s$ with $\rho_s - \rho_0$ small, there exists an R_0, $\rho_s < R_0 < R_s$, such that

$$F(R_0,\rho_0) = \gamma\,. \tag{3.3}$$

In this section we shall consider problem (1.9)–(1.13) with $R(t)$ *fixed* and prove existence and uniqueness of a solution, with Lipschitz continuous $\rho(t)$, as long as

$$R(t) > R_0 \; . \tag{3.4}$$

In the sequel we assume that the initial nutrient concentration $\sigma(r,0) \equiv \sigma^0(r)$ satisfies the following condition:

$$\begin{aligned}
&\text{(a)} && \sigma^0 \in C^1[0,R(0)] \cap C^3[\rho(0),R(0)], \qquad R(0) > R_0 \, , \\
&\text{(b)} && \sigma^0(r) = \sigma_0 \quad \text{if} \quad 0 \le r \le \rho(0) \, , \quad \rho(0) > \rho_0 \, , \quad \rho(0) < R_0 \, , \\
&\text{(c)} && \sigma^0(r) \le \bar\sigma \, \frac{F(r,\rho_0)}{F(R_0,\rho_0)} \quad \text{if} \quad \rho_0 < r < R_0 \, , \qquad \sigma^0(r) \le \bar\sigma \quad \text{if} \quad R_0 \le r \le R(0) \, , \\
&\text{(d)} && \frac{\partial}{\partial r}\sigma^0(r) > 0 \quad \text{if} \quad \rho(0) < r \le R(0) \, , \\
&\text{(e)} && \sigma^0(R(0)) = \bar\sigma \, , \\
&\text{(f)} && A\,\frac{\partial}{\partial r}\sigma^0(r) \ge \left| \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial \sigma^0(r)}{\partial r}\right) - \lambda\sigma^0(r) \right| \quad \text{if} \quad \rho(0) < r < R(0)
\end{aligned} \tag{3.5}$$

for some positive constant A.

Note that, by our assumptions, $\rho_0 < \rho(0) < R_0 < R(0)$. Note also that (c) is consistent with (b) and (e) since

$$\sigma_0 \le \bar\sigma \, \frac{F(r,\rho_0)}{F(R_0,\rho_0)} \qquad \text{if} \qquad \rho_0 < r \le \rho(0)$$

as $F(r,\rho_0) > F(\rho_0,\rho_0) = 1$ whereas $F(R_0,\rho_0) = \gamma = \bar\sigma/\sigma_0$ by (3.3); the inequality (c) in fact holds if $|\sigma^0 - \sigma_s|_{L^\infty(\rho(0),R(0))}$ is sufficiently small. We finally observe that the condition (3.5)(f) is satisfied if and only if

$$\frac{\partial^2}{\partial r^2}\sigma^0(\rho(0)) - \lambda\sigma^0(\rho(0)) = 0 \; ; \tag{3.6}$$

this condition holds if $\sigma^0(r)$ is a stationary solution; see also Remark 3.2.

In this section and in the following one we assume that $R(t)$ is a given function satisfying the following conditions:

$$\begin{aligned}
&\text{(a)} && R \in C^{1+\alpha/2}[0,T] \, , \qquad 0 < \alpha < 1 \, , \\
&\text{(b)} && R_0 < R(t) < R_1 \quad \text{for} \quad 0 \le t \le T \quad (R_1 \text{ is given}) \, , \\
&\text{(c)} && \left[\dot R(0)\,\frac{\partial \sigma^0}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial \sigma^0}{\partial r}\right)\right]_{r=R(0)} = \lambda\bar\sigma \, .
\end{aligned} \tag{3.7}$$

The last condition (on $\dot R(0)$) arises from

$$0 = \frac{d}{dt}\sigma(R(t),t) \quad \text{as} \quad t \to 0$$

and is a natural consistency condition between the initial data $\sigma^0(r)$ and the boundary data $\sigma(R(t),t) = \bar\sigma$; it will allow us to use the Schauder estimates up to the boundary at $(R(0),0)$.

DEFINITION 3.1. *Given $\sigma^0(r)$ as in (3.5) and $R(t)$ as in (3.7), the problem of solving (1.9)–(1.13) for (σ,ρ) for $0 \le t \le T$ will be called Problem (R_T).*

THEOREM 3.1. *There exists a unique solution (σ, ρ) to problem (R_T), provided T is sufficiently small, and*

$$\rho_0 \le \rho(t) \ , \qquad 0 < t < T \ , \tag{3.8}$$

$$0 \le \sigma_r(r,t) \le K_0 \qquad \text{if} \quad \rho(t) < r \le R(t) \ , \quad 0 < t \le T \ , \tag{3.9}$$

$$|\sigma_t(r,t)| \le K\sigma_r(r,t) \le C \qquad \text{if} \quad \rho(t) < r \le R(t) \ , \quad 0 < t \le T \ , \tag{3.10}$$

$$|\dot\rho(t)| \le C_1 \qquad \text{a.e.} \quad \text{for} \quad 0 < t < T \ , \tag{3.11}$$

where K_0 is a constant independent of T, c, and K, C, C_1 are constants which depend on T, c.

Proof. Let $H_\varepsilon(s)$ be $C^\infty(-\infty,\infty)$ functions for $0 < \varepsilon < 1$, such that

$$\begin{aligned} H_\varepsilon{}'(s) &\ge 0 \ , \\ H_\varepsilon(s) &= 1 \qquad \text{if} \qquad s \ge \varepsilon \ , \\ H_\varepsilon(s) &= 0 \qquad \text{if} \qquad s \le 0 \end{aligned}$$

and let

$$Q_T = \{0 < r < R(t) \ , \qquad 0 < t < T\} \ .$$

Consider the equation

$$c\sigma_t - \Delta\sigma + \lambda\sigma H_\varepsilon(\sigma - \sigma_0) = 0 \qquad \text{in} \quad Q_T \tag{3.12}$$

with boundary and initial conditions

$$\begin{aligned} \sigma &= \bar\sigma \qquad && \text{if} \qquad r = R(t) \quad 0 < t < T \ , \\ \sigma(r,0) &= \sigma^0(r) \qquad && \text{if} \qquad 0 < r < R(0) \ . \end{aligned} \tag{3.13}$$

The system (3.12), (3.13) has a unique solution σ_ε, and, by L^p estimates [14],

$$||\sigma_\varepsilon||_{L^p(Q_T)} + ||D_x\sigma_\varepsilon||_{L^p(Q_T)} + ||D_x^2\sigma_\varepsilon||_{L^p(Q_T)} + ||D_t\,\sigma_\varepsilon||_{L^p(Q_T)} \le C \tag{3.14}$$

for any $1 < p < \infty$, where C is a constant independent of ε. By the maximum principle we also have

$$\sigma_0 \le \sigma_\varepsilon < \bar\sigma \qquad \text{in} \qquad Q_T \ , \tag{3.15}$$

$$\frac{\partial\sigma_\varepsilon}{\partial r} > 0 \qquad \text{at} \qquad r = R(t) \ , \quad 0 < t \le T \ . \tag{3.16}$$

We can then take a sequence $\varepsilon = \varepsilon' \to 0$ such that $\sigma_{\varepsilon'} \to \sigma$ uniformly in Q_T, and $D_t\sigma_{\varepsilon'} \to D_t\sigma$, $D_x\sigma_{\varepsilon'} \to D_x\sigma$, $D_x^2\sigma_{\varepsilon'} \to D_x^2\sigma$, $H_\varepsilon(\sigma_{\varepsilon'} - \sigma_0) \to \kappa$ weakly in $L^p(Q_T)$, where κ is a bounded measurable function, so that

$$||\sigma||_{L^p(Q_T)} + ||D_x\sigma||_{L^p(Q_T)} + ||D_x^2\sigma||_{L^p(Q_T)} + ||D_t\,\sigma||_{L^p(Q_T)} \le C \ , \tag{3.17}$$

and

$$c\sigma_t - \Delta\sigma + \lambda\sigma\kappa = 0 \qquad \text{a.e.} \quad \text{in} \quad Q_T \ .$$

From the last equation it follows that $\kappa = 0$ a.e. on the set $\{\sigma = \sigma_0\}$ (since $D_t\sigma = 0$, $D_x\sigma = 0$, $D_x^2\sigma = 0$ a.e. on the set $\{\sigma = \sigma_0\}$). Also, $\kappa = 1$ on the set $\{\sigma > \sigma_0\}$ (since $H_\varepsilon(\sigma_\varepsilon - \sigma_0) = 1$

on any set $\{\sigma \geq \sigma_0 + \delta\}$ ($\delta > 0$) if ε is sufficiently small). Consequently, $\kappa = H(\sigma - \sigma_0)$ a.e. (recall that $H(0) = 0$), and thus

$$c\sigma_t - \Delta\sigma + \lambda\sigma H(\sigma - \sigma_0) = 0 \qquad \text{a.e. in} \quad Q_T \; . \tag{3.18}$$

Clearly $\sigma_0 \leq \sigma \leq \bar{\sigma}$ and, by the maximum principle

$$\sigma_0 \leq \sigma < \bar{\sigma} \qquad \text{in} \quad Q_T \; , \tag{3.19}$$

$$\frac{\partial \sigma}{\partial r} > 0 \qquad \text{at} \qquad r = R(t) \; , \quad 0 < t \leq T \; . \tag{3.20}$$

Next we apply the maximum principle to $v = \partial\sigma_\varepsilon/\partial r$. Since

$$cv_t - \Delta v + \frac{2}{r^2}\, v + \lambda G v = 0$$

where $G = \frac{d}{ds}(sH_\varepsilon(s - s_0))|_{s=\sigma_\varepsilon} \geq 0$, and

$$\begin{aligned} v > 0 & \qquad \text{on} \quad r = R(t) \qquad \text{(by (3.16))} \; , \\ v(r, 0) \geq 0 & \qquad \text{by (3.5)(d)} \; , \end{aligned}$$

we conclude that

$$\frac{\partial \sigma_\varepsilon}{\partial r} \geq 0 \qquad \text{in} \quad Q_T \; ,$$

so that also

$$\frac{\partial \sigma}{\partial r} \geq 0 \qquad \text{in} \quad Q_T \tag{3.21}$$

and, by strong maximum principle,

$$\frac{\partial \sigma}{\partial r} > 0 \qquad \text{in} \quad Q_T \cap \{\sigma > \sigma_0\} \; . \tag{3.22}$$

Defining

$$\rho(t) = \inf\{r \, ; \; \sigma(r, t) > \sigma_0\} \tag{3.23}$$

we conclude that

$$\sigma(r, t) = \sigma_0 \qquad \text{if and only if} \qquad 0 \leq r \leq \rho(t) \; .$$

Later on we shall prove that $\rho(t)$ is continuous, and then (3.18) will hold everywhere for $r \neq \rho(t)$.

Uniqueness follows from the fact that the function $\sigma H(\sigma - \sigma_0)$ is monotone increasing in σ.

We next prove (3.8). Consider the function

$$\begin{aligned} \bar{u}(r, t) = \bar{u}(r) \quad & = \frac{\bar{\sigma} F(r, \rho_0)}{F(R_0, \rho_0)} \qquad & \text{for} \qquad \rho_0 \leq r \leq R_0 \\ & = \; \sigma_0 & \text{for} \qquad 0 < r < \rho_0 \; . \end{aligned}$$

It is easily seen that $\bar{u}(r)$ is continuously differentiable at $r = \rho_0$ and

$$\begin{aligned} \bar{u}_t - \Delta\bar{u} + \lambda\bar{u}\, H(\bar{u} - \sigma_0) \;&=\; 0 \qquad\qquad \text{if} \qquad 0 < r < R_0 \; , \\ \bar{u}(R_0, t) \;&=\; \bar{\sigma} \; , \\ \bar{u}(r, 0) \;&\geq\; \sigma^0(r) \qquad\qquad (\text{by(3.5)(c) for } \; \sigma^0(r)) \; . \end{aligned}$$

Hence, by comparison,

$$\sigma(r,t) \le \bar{u}(r) \qquad \text{for} \qquad 0 \le r \le R_0 \ , \quad t \ge 0$$

and, in particular, $\sigma(r,t) = \sigma_0$ if $r < \rho_0$, so that (3.8) holds.

To prove the upper bound on σ_r, in (3.9), consider the function

$$\omega_\varepsilon(r,t) = \sigma_\varepsilon(r,ct) \ .$$

It satisfies

$$\omega_t - \Delta\omega + \lambda\omega H_\varepsilon(\omega - \sigma_0) = 0 \ , \qquad 0 \le r < \tilde{R}(t) \ , \quad 0 < t < \frac{T}{c}$$

where $\tilde{R}(t) = R(ct)$. The L^p parabolic estimates hold for ω_ε in any region

$$0 \le r < \tilde{R}(t) \qquad a \le t \le a+1 \ , \qquad \text{for any} \quad a < \frac{T}{c} - 1$$

with a bound independent of a and c. By interpolation (see [14, Lemma 3.3]) we then have

$$\left\| \frac{\partial \omega_\varepsilon}{\partial r} \right\|_{C^{\alpha_0}} \le K_0 \ , \qquad r < \tilde{R}(t) \ , \quad 0 < t < \frac{T}{c} \tag{3.24}$$

for some $\alpha_0 > 0$, where K_0 is a constant independent of c. Taking $\varepsilon \to 0$, we obtain the bound $\sigma_r \le K$.

For any $0 < \nu < \rho_0$ consider the system (1.9)–(1.13) (with obvious modification) in the domain

$$Q_{T,\nu} = \{\nu < r < R(t) \ , \quad 0 < t < T\} \ .$$

All the results established above hold also for this new problem. Denoting the solution by $\sigma_\nu(r,t)$ and extending it by σ_0 to $r \le \nu$, it is clear that the extended function is a solution to problem (R_T), so that by uniqueness,

$$\sigma_\nu(r,t) \equiv \sigma(r,t) \ . \tag{3.25}$$

We shall next use (3.25) in order to prove (3.10). Consider the function

$$\omega = K_* e^{\frac{Mt}{c}} \frac{\partial \sigma_{\nu,\varepsilon}}{\partial r} \pm \frac{\partial \sigma_{\nu,\varepsilon}}{\partial t} \qquad (K_* > 0 \ , \ M > 0) \ ,$$

where $\sigma_{\nu,\varepsilon}$ is the solution σ_ε in $Q_{T,\nu}$ with $\sigma_{\nu,\varepsilon} = \sigma_0$ on $r = \nu$. Then

$$c\omega_t - \Delta\omega + G\omega = \Big(M - \frac{2}{r^2} \Big) \frac{\partial \sigma_{\nu,\varepsilon}}{\partial r} > 0 \qquad \text{in} \quad Q_{T,\nu}$$

if $M > 2/\nu^2$, where $G \ge 0$ since the function $s \to s\,H_\varepsilon(s - \sigma_0)$ is monotone increasing. We assert that

$$\omega(r,0) \ge 0 \qquad \text{if} \qquad K_* \ge A/c \ , \qquad \text{for} \quad r < \rho(0) \ \text{ and } \ r > \rho(0) + C_0\sqrt{\varepsilon}$$

where C_0 is a constant independent of ε. Indeed, by (3.6) and Taylor's formula,

$$\sigma^0(r) \ge \sigma_0 + \frac{1}{2}\lambda\sigma_0(r - \rho(0))^2 + O((r - \rho(0))^3)$$

for $r > \rho(0)$, so that

$$\sigma^0(r) > \sigma^0(r) + \varepsilon \qquad \text{and} \qquad H_\varepsilon(\sigma^0(r) - \sigma_0) = 1$$

if $r \geq \rho(0) + C_0\sqrt{\varepsilon}$. Hence, $|\partial\sigma_{\nu,\varepsilon}(r,0)/\partial t|$, for $r > \rho(0) + C_0\sqrt{\varepsilon}$, is given by the right-hand side of (3.5)(f), so that $\omega(r,0) \geq 0$ if $K_* \geq A/C$. For $r < \rho(0)$, $\omega(r,0)$ is clearly equal to zero. Next, by (3.20),

$$\frac{\partial\sigma_{\nu,\varepsilon}}{\partial r}(R(t),t) \geq C_0 > 0$$

so that

$$\omega(R(t),t) \geq 0$$

if K_* is sufficiently large. Finally, since $\sigma_{\nu,\varepsilon} \equiv \sigma_0$ on $r = \nu$ which implies $\dfrac{\partial\sigma_{\nu,\varepsilon}}{\partial t} \equiv 0$ on $r = \nu$, we also have $\omega(\nu,t) \geq 0$. Let ζ_ε be the solution of

$$c\zeta_t - \Delta\zeta + G\zeta = 0 \qquad \text{in} \quad Q_{T,\nu}$$

with zero values on the parabolic boundary of $Q_{T,\nu}$, except for

$$\zeta(r,0) = 1 \qquad \text{for} \qquad \rho(0) < r < \rho(0) + C_0\sqrt{\varepsilon}\ .$$

We can apply the maximum principle to $\omega + A\zeta_\varepsilon$ in $Q_{T,\nu}$ where A is a sufficiently large constant (independent of ε) and conclude that $\omega + A\zeta_\varepsilon \geq 0$ in $Q_{T,\nu}$.

As $\varepsilon \to 0$, $\zeta_\varepsilon \to 0$ and we obtain the inequality

$$|\sigma_t(r,t)| \leq K\,\sigma_r(r,t) \qquad \text{a.e. in} \quad \Omega_T\ , \tag{3.26}$$

where $K = K_* e^{\frac{MT}{c}}$.

Having proved (3.10) we now take any level curve $r = \rho_\eta(t)$ of $\sigma(r,t) = \sigma_0 + \eta$, $\eta > 0$. By (3.26)

$$\left|\frac{d}{dt}\rho_\eta(t)\right| = \left|\frac{\sigma_t}{\sigma_r}\right|_{r=\rho_\eta(t)} \leq K$$

so that

$$|\rho_\eta(t_2) - \rho_\eta(t_1)| \leq K|t_2 - t_1|\ .$$

Since $\rho_\eta(t) \to \rho(t)$ as $\eta \to 0$, it follows that $|\rho(t_2) - \rho(t_1)| \leq K|t_2 - t_1|$, and the proof of (3.11) is complete. ◻

Remark 3.1. Existence and uniqueness for problem (R_T) follow also from the variational inequality formulation of the problem, by standard results for parabolic variational inequalities [11].

Remark 3.2. For the variational inequality corresponding to the Stefan problem one can show (see [11, Chap. 1]) that σ_t is continuous across the free boundary. It follows from (3.26) and the continuity of σ_r (so that $\sigma_r(\rho(t)+0,t) = 0$) that in our case σ_t is also continuous across the free boundary $r = \rho(t)$, i.e. $\sigma_t(\rho(t)+0,t) = 0$, so that, by (3.18), $\sigma_{rr}(\rho(t)+0,t) = \lambda\sigma_0$. This equation for $t = 0$ coincides with the assumption (3.6).

4. $\rho(t)$ is in C^∞.

THEOREM 4.1. *Under the assumptions of Theorem 3.1, the function $\rho(t)$ is in $C^\infty(0,T]$.*

A similar result for the Stefan problem was proved by Schaeffer [15].

Proof. Let

$$\begin{aligned} y &= r - \rho(t) \ , \qquad & U(y,t) &= \sigma(y+\rho(t),t) - \sigma_0 \\ R_1(t) &= R(t) - \rho(t) \ , & b &= \bar{\sigma} - \sigma_0 \end{aligned} \tag{4.1}$$

and introduce the domains

$$Q_T^0 = \{0 < y < R_1(t) \ , \quad 0 < t < T\} \ ,$$

$$Q_T^\delta = \{0 < y < R_1(t) - \delta \ , \quad 0 < t < T\}$$

for any small $\delta > 0$. Then

$$cU_t - U_{yy} - \left[\frac{2}{y+\rho(t)} + c\dot{\rho}(t)\right] U_y + \lambda U + \lambda\sigma_0 = 0 \qquad \text{in} \quad Q_T^0 \ , \tag{4.2}$$

$$U(R_1(t),t) = \bar{\sigma} \qquad \text{if} \quad 0 < t < T \ , \tag{4.3}$$

$$U(0,t) = 0 \ , \qquad \frac{\partial U}{\partial y}(0,t) = 0 \qquad \text{if} \quad 0 < t < T \ , \tag{4.4}$$

$$U(y,0) = U_0(y) \equiv \sigma^0(y+\rho(0)) \qquad \text{for} \quad 0 < y < R_1(0) \ . \tag{4.5}$$

Since the leading coefficients in (4.2) are smooth and the lower order coefficients are bounded (recall (3.8) and (3.11)), we have, by L^p boundary estimates,

$$U,\ U_t,\ U_y,\ U_{yy},\ U_t \in L^p(Q_T^\delta) \qquad \text{for any} \qquad 1 < p < \infty$$

and, by embedding [14, Chap. 2, Lemma 3.3], $U_y \in C_{y,t}^{\alpha,\alpha/2}$ for any $0 < \alpha < 1$.

Consider next the function $w = U_y$. It satisfies

$$cw_t - w_{yy} - \left[\frac{2}{y+\rho(t)} + c\dot{\rho}(t)\right] w_y + \left[\lambda + \frac{2}{(y+\rho(t))^2}\right] w = 0 \qquad \text{in} \quad Q_T^0 \ . \tag{4.6}$$

Since $w(0,t) = 0$ we can again apply L^p boundary estimates to deduce that

$$w,\ w_t,\ w_y,\ w_{yy} \in L^p(Q_T^{\delta,\varepsilon}) \qquad \text{for any} \qquad 1 < p < \infty$$

where $Q_T^{\delta,\varepsilon} = Q_T^\delta \cap \{t > \varepsilon\}$. Thus

$$U_y,\ U_{ty},\ U_{yy},\ U_{yyy} \in L^p(Q_T^{\delta,\varepsilon}) \tag{4.7}$$

and again, by embedding, also

$$U_{yy} \in C_{y,t}^{\alpha,\alpha/2} \ . \tag{4.8}$$

For any $0 < t_1 < t_2 < T$ and sufficiently small y, we integrate (4.2) with respect to t, $t_1 < t < t_2$, and divide by $t_2 - t_1$, to get

$$\begin{aligned} \frac{1}{t_2-t_1}\int_{t_1}^{t_2} U_{yy}(y,t)dt &= c\,\frac{U(y,t_2)-U(y,t_1)}{t_2-t_1} - \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[\frac{2}{y+\rho(t)} + c\dot{\rho}(t)\right] U_y(y,t)dt \\ &\quad + \frac{\lambda}{t_2-t_1}\int_{t_1}^{t_2} U(y,t)dt + \lambda\sigma_0 \ . \end{aligned}$$

Taking $y \to 0$ and using (4.4) and (4.8), we obtain

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} U_{yy}(0,t)dt = \lambda\sigma_0 \ .$$

Letting $t_2 - t_1 \to 0$, we deduce that

$$U_{yy}(0,t) = \lambda\sigma_0 \ , \qquad 0 < t < T \tag{4.9}$$

and, by (4.2), (4.4), also

$$U_t(0,t) = 0, \qquad 0 < t < T \ . \tag{4.10}$$

Consider next the function $z = U_{yy}$. It satisfies a parabolic equation obtained from (4.6) by differentiation with respect to y. Since, by (4.9), $z(0,t) = \lambda\sigma_0$, we can apply L^p estimates to z, and as before deduce that (cf. (4.8))

$$U_{yyy} \in C_{y,t}^{\alpha,\alpha/2} \ . \tag{4.11}$$

Next we integrate (4.6) with respect to t, $t_1 < t < t_2$, and let $y \to 0$. We get, after using (4.4), (4.9) and (4.11),

$$-\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} U_{yyy}(0,t)\,dt - \frac{\lambda\sigma_0}{t_2 - t_1} \int_{t_1}^{t_2} \left[\frac{2}{y + \rho(t)} + c\dot{\rho}(t)\right] dt = 0 \ .$$

Taking $t_2 - t_1 \to 0$ we conclude that

$$U_{yyy}(0,t) + \lambda\sigma_0 \left[\frac{2}{\rho(t)} + c\dot{\rho}(t)\right] = 0 \ . \tag{4.12}$$

Recalling (4.11), we deduce that $\rho \in C^{1+\alpha/2}(0,T]$.

This allows us to repeat the above process with $C^{2+\alpha}$ Schauder estimates instead of L^p estimates. Hence the functions in (4.7) and their y-derivatives are in $C_{y,t}^{\alpha,\alpha/2}$. In particular, U_{tyyy} is in $C_{y,t}^{\alpha,\alpha/2}$ so that, by (4.12), $\dot{\rho} \in C^{1+\alpha/2}(0,T]$, or $\rho \in C^{2+\alpha/2}(0,T]$.

We can now repeat the above process with $C^{3+\alpha}$ Schauder estimates (instead of $C^{2+\alpha}$), to deduce (from (4.12)) that $\rho \in C^{4+\alpha}(0,T]$. Proceeding step-by-step, we find that $\rho(t)$ is in $C^\infty(0,T]$. □

THEOREM 4.2. *Under the assumptions of Theorem 3.1, $\rho(t)$ is continuously differentiable up to $t = 0$.*

Proof. The proof will be based on an integral equation representation for $\rho(t)$ or, rather, $\sigma_{rt}(\rho(t),t)$.

To derive the integral equation we shall use fact (proved in Sections 3, 4) that $\rho \in C^{0,1}[0,T] \cap C^\infty(0,T]$. By differentiating the relation $\sigma(\rho(t),t) = \sigma_0$ we get (cf. (4.10))

$$\sigma_t = 0 \qquad \text{on} \qquad r = \rho(t) \tag{4.13}$$

Next, differentiating the relation $\sigma_r(\rho(t),t) = 0$ we get

$$\sigma_{rr}\dot{\rho} + \sigma_{rt} = 0$$

and, since

$$\sigma_{rr} + \frac{2}{r}\,\sigma_r - \lambda\sigma = c\sigma_t = 0 \qquad \text{on} \qquad r = \rho(t) \ ,$$

we obtain the relation

$$(\sigma_t)_r + (\lambda\sigma_0)\dot{\rho} = 0 \qquad \text{on} \qquad r = \rho(t) \ . \tag{4.14}$$

The function

$$\omega = r\sigma_t \tag{4.15}$$

is easily seen to satisfy the equation

$$c\omega_t - \omega_{rr} + \lambda\omega = 0 \qquad \text{in} \quad \tilde{\Omega}_{R,T} \tag{4.16}$$

where

$$\tilde{\Omega}_{R,T} = \{(r,t) \ ; \quad \rho(t) < r < R(t) \ , \quad 0 < t < T\}$$

and, by (4.13), (4.14),

$$\omega = 0 \ , \quad \omega_r + B\rho\dot{\rho} = 0 \qquad \text{on} \qquad r = \rho(t) \ , \quad 0 < t < T \quad (B = \lambda\sigma_0) \ . \tag{4.17}$$

Furthermore,

$$\omega(r,0) = r[\Delta\sigma^0(r) - \lambda\sigma^0(r)] \equiv \omega^0(r) \ , \qquad \rho(0) < r < R(0) \ . \tag{4.18}$$

We introduce the fundamental solutions

$$G^{\pm}(x,t;\,\xi,\tau) = \sqrt{c}\left\{\frac{1}{[4\pi(t-\tau)]^{1/2}}\,e^{-\frac{c(x-\xi)^2}{4(t-\tau)}} \pm \frac{1}{[4\pi(t-\tau)]^{1/2}}\,e^{-\frac{c(x+\xi)^2}{4(t-\tau)}}\right\}e^{-\frac{\lambda(t-\tau)}{c}} \ .$$

Next we let

$$\beta = \rho(0) + \frac{R(0)-\rho(0)}{2}$$

so that

$$\rho(t) + \frac{R(0)-\rho(0)}{4} \le \beta \le R(t) - \frac{R(0)-\rho(0)}{4} \ , \qquad 0 < t < T$$

if T is small enough, and introduce a change of variables:

$$x = \beta - r \ , \qquad W(x,t) = \omega(r,t) \ , \qquad s(t) = \beta - \rho(t) \ . \tag{4.19}$$

The pair (W,s) forms a solution of the Stefan problem

$$\begin{aligned}
W_t - W_{xx} + \lambda W = 0 \ , \qquad & 0 < x < s(t) \ , \quad 0 < t < T \ , \\
W = 0 \ , \qquad W_x = -B(\beta - s)\dot{s} \qquad & \text{if} \quad x = s(t) \ , \quad 0 < t < T \ , \\
W(x,0) = w^0(\beta - x) \ , \qquad & 0 < x < s(0) \ .
\end{aligned}$$

As in [9, 10], we can represent $V(t) \equiv W_x(s(t),t)$ in the form

$$\begin{aligned}
V(t) \ = \ & 2\int_0^t V(\tau)\ G_x^-\Big(s(t),t;\,s(\tau),\tau\Big)\,d\tau \\
& + 2\int_0^t W(0,\tau)\ G_{x\xi}^-\Big(s(t),t;\,0,\tau\Big)\,d\tau \\
& - 2\int_0^{s(0)} G^+\Big((s(t),t;\,\xi,0\Big)\ W_\xi(\xi,0)\,d\xi + 2G^+\Big(s(t),t;\,0,0\Big)W(0,0) \ .
\end{aligned} \tag{4.20}$$

Here we used the smoothness of W up to $x = s(t)$, $t > 0$, the Lipschitz continuity of $s(t)$ up to $t = 0$ and the fact that $W(s(0), 0) = 0$. We also have

$$V(t) = -B(\beta - s)\dot{s} \ . \tag{4.21}$$

This integral equation with $c = 1$, $\lambda = 0$ and $\beta - s$ replaced by 1 in (4.21)) was considered in [9]. However the estimates and results established in [9] extend with minor differences to the present case. Thus existence and uniqueness for a bounded solution $V(t)$ of (4.20), (4.21) can be established by a fixed point argument. But the analysis in [9] also shows that the solution $V(t)$ is continuous for $0 \le t \le T$; hence the same is true for $\dot{\rho}(t)$. □

5. Solution to (1.9)–(1.14). In this section we prove:

THEOREM 5.1. *Assume that (3.5) holds and that $R_0 < R(0) < R_1$, where R_1 is any given positive number larger than R_s. Then there exists a unique solution to problem (1.9)–(1.14) for $0 \le t < T$ with $R(t)$ in $C^{1+\alpha}[0,T) \cap C^\infty(0,T)$ and $\rho(t)$ in $C^1[0,T) \cap C^\infty(0,T)$ such that $R_0 < R(t) < R_1$ for all $0 \le t < T$; furthermore, either $T = \infty$, or $T < \infty$ in which case $R \in C^\infty(0,T]$ and either $R(T) = R_0$ or $R(T) = R_1$.*

Proof. Define $\dot{R}(0)$ by (3.7)(c) and set

$$M_0 = |\dot{R}(0)| + 1 \ .$$

For any positive number M introduce the class of functions

$$\mathcal{A}_{M,T} = \Big\{ \tilde{R}(t) \ , \quad 0 \le t \le T \ ; \qquad \tilde{R}(0) = R(0) \ , \qquad \frac{d\tilde{R}}{dt}(0) = \dot{R}(0) \ , \\ \left| \frac{d}{dt}\tilde{R}(t) \right| \le M_0 \ , \qquad \left| \frac{d}{dt}\tilde{R}(t) \right|_\alpha \le M \Big\} \tag{5.1}$$

and take T small enough so that, in particular,

$$R_0 \ < \ \tilde{R}(t) \ < \ R_1 \qquad \text{if} \qquad 0 \le t < T \ , \tag{5.2}$$

$$T M_0 \ < \ \frac{1}{4}\Big(R(0) - \rho(0)\Big) \ . \tag{5.3}$$

For any $\tilde{R} \in \mathcal{A}_{M,T}$ consider the solution (σ, ρ) established in Theorem 3.1 (with $R(t) = \tilde{R}(t)$) and define a function $\hat{R}(t)$ by (cf. (1.14))

$$\tilde{R}^2 \frac{d\hat{R}}{dt} = \int_{\rho(t)}^{\tilde{R}(t)} (\sigma(r,t) - \tilde{\sigma})\, r^2 dr - \mu\rho^3(t) \ . \tag{5.4}$$

This determines a mapping S: $\hat{R} = S\tilde{R}$, and we shall prove:

$$S\mathcal{A}_{M,T} \subset \mathcal{A}_{M,T} \ , \tag{5.5}$$

$$S \ \text{ is continuous} \ , \tag{5.6}$$

provided T is small enough.

To prove (5.5) note, by (5.4), that

$$\left| \frac{d\hat{R}}{dt} \right| \le C_0 \ , \qquad C_0 \text{ is independent of } M \ .$$

Also, since σ_t is uniformly bounded (by (3.10)), we also have, by (5.4) and (3.11),

$$\left|\frac{d^2\hat{R}}{dt^2}\right| \le C_1(M,T)$$

where $C_1(M,T)$ depends on M and T in such a way that $\lim\limits_{T\to 0+} C_1(M,T) = C_1(M,0) < \infty$. Therefore

$$\left|\frac{d\hat{R}}{dt}\right|_\alpha \le C_1(M,T)\,T^{1-\alpha} < M$$

if T is small enough and

$$\left|\frac{d\hat{R}(t)}{dt}\right| \le |\dot{R}(0)| + C_1(M,T)\,t < |\dot{R}(0)| + 1 = M_0$$

if T is small enough. This completes the proof of (5.5).

To prove (5.6) suppose $R_m \in \mathcal{A}_{M,T}$, $R_m \to R$ in $C^{1+\alpha}[0,T]$. Then the corresponding (σ_m, ρ_m) and (σ, ρ) satisfy

$$\sup_{0<t\le T} |\dot{\rho}_m(t)| \le C\ , \qquad \sup_{0<t\le T} |\dot{\rho}(t)| \le C$$

and, by Theorem 3.1,

$$\left|\frac{\partial}{\partial r}\,\sigma_m\right| \le C\ , \qquad \left|\frac{\partial}{\partial t}\,\sigma_m\right| \le C\ .$$

It follows that, for a subsequence,

$$\sigma_m \to \sigma\ , \qquad \rho_m \to \rho$$

uniformly, and

$$\int_{\rho_m}^{R_m} (\sigma_m - \tilde{\sigma})\, r^2\, dr \to \int_\rho^R (\sigma - \tilde{\sigma})\, r^2\, dr$$

in $C^\beta[0,T]$, for any $0 < \beta < 1$; clearly also

$$\mu\rho_m^3 \to \mu\rho^3 \qquad \text{in} \qquad C^\beta[0,T]\ .$$

Hence $SR_m \to SR$ in $C^{1+\beta}[0,T]$, and the proof of (5.6) follows.

This proof also shows that the set $S\mathcal{A}_{M,T}$ lies in $C^{1+\beta}[0,T]$, so that upon taking $\alpha < \beta < 1$ we conclude that S maps $\mathcal{A}_{M,T}$ into a compact subset. We can then apply the Schauder fixed point theorem to deduce that S has a fixed point in $\mathcal{A}_{M,T}$, which is of course a solution of (1.9)–(1.14).

If $R_0 < R(T) < R_1$ then we can continue the solution to a larger interval $0 \le t \le T + T_1$ by working with the class of $C^{1+\alpha}[T,\, T+T_1]$ functions $\tilde{R}(t)$ satisfying:

$$\begin{aligned}
&\tilde{R}(T) = R(T)\ , \qquad \frac{d\tilde{R}}{dt}(T) = \frac{dR}{dt}(T)\ ,\\
&\left|\frac{d}{dt}\,\tilde{R}(t)\right| \le |\dot{R}(T)| + 1 \qquad \text{for} \qquad T \le t \le T + T_1\ ,\\
&\left|\frac{d}{dt}\,\tilde{R}(t)\right|_\alpha \le M \qquad \text{for} \qquad T \le t \le T + T_1
\end{aligned}$$

and choosing T_1 small enough so that $R_0 < R(t) < R_1$ if $T \le t < T + T_1$. The initial values are taken, at $t = T$, to coincide with the values at $t = T$ of the solution already constructed.

Proceeding in this way step-by-step, we obtain a global solution unless $R(t)$ happens to decrease to R_0 or to increase to R_1 at a finite time. Here, it should be pointed out that as long as $R(t)$ remains in the interval (R_0, R_1), the estimates established in Theorem 3.1 do not degenerate. Although the constant K in (3.10) grows with T like e^{CT}, for any given time interval, say $0 < t < T_0$, K remains a priori bounded as the solution is extended to this interval, and we can therefore extend the solution in a finite number of steps (provided $R(t)$ it remains in the interval (R_0, R_1)).

We have already proved the asserted regularly for $\rho(t)$. To prove that $R(t)$ is in $C^\infty(0,T)$ note that as $R(t)$ is in $C^{1+\alpha}(0,T)$, σ is in $C^{2+\alpha,\,1+\alpha/2}_{r,t}$ by the Schauder estimates, so that, by (1.14), $R(t)$ is in $C^{2+\alpha/2}$. This enables us to assert that σ is in $C^{3+\alpha/2,\,(3+\alpha/2)/2}_{r,t}$ and thus, by (1.14), $R(t)$ is in $C^{5/2+\alpha/4}$. Proceeding in this way step-by-step, we deduce that $R(t)$ is indeed in $C^\infty(0,T)$. □

It remains to prove uniqueness. It suffices to prove it for $0 \le t \le T_0$ where T_0 is small enough, for then we can proceed step-by-step to prove global uniqueness.

Suppose (σ_1, R_1, ρ_1) and (σ_2, R_2, ρ_2) are two solutions and let

$$R(t) = \min(R_1(t), R_2(t)) \ .$$

Consider the function $\sigma = \sigma_1 - \sigma_2$ in $0 \le r \le R(t)$, $0 \le t \le T_0$. If

$$\delta = \sup_{0 \le t \le T} |R_1(t) - R_2(t)|$$

then

(5.7) $$|\sigma(R(t),t)| \le C_0 \delta \qquad \text{for} \qquad 0 \le t \le T_0 \ ,$$

where C_j will denote constants independent of T_0 and δ. Since the function $s \to s\,H(s - \sigma_0)$ is monotone increasing, we deduce by comparison and (5.7) that

(5.8) $$|\sigma(r,t)| \le C_0 \delta \ , \qquad 0 \le r \le R(t) \ , \qquad 0 \le t \le T_0 \ .$$

Setting

$$\beta = \frac{R(0) + \rho(0)}{2} \ ,$$

and using (5.8) and parabolic estimates for σ in $\beta - \varepsilon_0 < r < \beta + \varepsilon_0$, $0 \le t \le T_0$ for some small positive ε_0, we derive the estimate

(5.9) $$|\sigma_t(\beta,t)| + |\nabla \sigma_t(\beta,t)| + |\nabla^2 \sigma_t(\beta,t)| \le C_1 \delta$$

for $0 \le t \le T_0$.

Consider now the integral equation (4.20). As proved in [9], the right-hand side, which we shall denote by SV, is a contraction mapping; more precisely,

(5.10) $$|SV_1 - SV_2|_{L^\infty(0,T_0)} \le C_2 \sqrt{T_0}\, |V_1 - V_2|_{L^\infty(0,T_0)} \ .$$

Using the estimate (5.9) we can deduce by the same argument used to prove (5.10) that a similar bound hold for the solutions of (4.20) corresponding to σ_1 and σ_2. Consequently, by means of the transformation (4.19),

$$|\dot\rho_1 - \dot\rho_2|_{L^\infty(0,T_0)} \le C_3 \sqrt{T_0}\, \delta \ .$$

If we use this inequality and (5.8) in the equation (1.14) for R_1, R_2, we find that

$$|\dot{R}_1 - \dot{R}_2|_{L^\infty(0,T_0)} \le C_4 \delta \ .$$

By integration

$$\delta = |R_1 - R_2|_{L^\infty(0,T_0)} \le C_4 T_0 \delta \ ,$$

so that $\delta = 0$ if $C_4 T_0 < 1$, and then the two solutions coincide.

REMARK 5.1. The proof of uniqueness requires only $C^{1+\alpha}[0,T]$ regularity of $R(t)$ and $\rho(t)$.

REMARK 5.2. Since, by (3.9),

$$\bar{\sigma} - \sigma_0 = \sigma(R(t),t) - \sigma(\rho(t),t) \le K_0(R(t) - \rho(t)) \ ,$$

the solution established in Theorem 5.1 has the property that

$$R(t) - \rho(t) \ge C_0 > 0 \qquad (0 < t < T) \tag{5.11}$$

where C_0 is a positive constant independent of T, c.

6. A priori bound on $\dot{\rho}(t)$. In this section we shall improve the bound (3.11) by showing that the constant C_1 can be chosen to be independent of T and c. This fact will be needed in order to prove that a global solution exists, i.e., that $T = \infty$ in Theorem 5.1.

We shall assume, in addition to (3.5), that

$$\left| \sigma^0_{rr}(r) + \frac{2}{r}\,\sigma^0_r(r) - \lambda \sigma^0(r) \right| \le C_*\, c(r - \rho(0)) \ , \qquad \rho(0) \le r < R(0) \tag{6.1}$$

where C_* is a positive constant independent of c. This condition strengthens the assumption (3.5)(f).

REMARK 6.1. The assumptions (3.5), (6.1) hold if $|\sigma^0 - \sigma_s|_{C^1[\rho(0),R(0)]}$ is sufficiently small and $\sigma_t|_{t=0}$ is bounded by $C_* c \sigma^0_r$, for some positive constant C_*.

THEOREM 6.1. *Under the additional condition (6.1), there exist positive constants c_0, N such that the solution of Theorem 5.1 satisfies:*

$$|\dot{\rho}(t)| < N \qquad \textit{for} \qquad 0 < t < T \tag{6.2}$$

provided $0 < c \le c_0$.

Proof. As in the proof of Theorem 4.2, the function $v = \sigma_t$ satisfies:

$$\begin{aligned}
c v_t - \Delta v + \lambda v = 0 \ , &\qquad \rho(t) < r < R(t) \ , \qquad 0 < t < T \ , \\
v(\rho(t),t) = 0 \ , &\qquad 0 < t < T \ , \\
v(R(t),t) = g(t) \ , &\qquad 0 < t < T \ , \\
v(r,0) = h(r) \ , &\qquad \rho(0) \le r \le R(0)
\end{aligned}$$

and

$$\dot{\rho}(t) = -\frac{1}{\lambda \sigma_0}\, v_r(\rho(t),t) \ , \qquad 0 < t < T \tag{6.3}$$

where

$$|\dot{R}(t)| \le M_0 \ , \tag{6.4}$$

$$g(t) = -\sigma_r(R(t),t)\, \dot{R}(t) \ , \qquad |g(t)| \le M_1 \ , \tag{6.5}$$

and

$$h(r) = \frac{1}{c}\left[\sigma^0_{rr}(r) + \frac{2}{r}\,\sigma^0_r(r) - \lambda\sigma^0(r)\right], \qquad |h| \le M_2\ ; \tag{6.6}$$

the constants M_i are independent of T, c. The estimate (6.4) can be derived from (1.14) and the boundedness of $\sigma(r,t)$, $R(t)$ and $\rho(t)$; the estimate in (6.5) follows from (6.4) and the boundedness of $\sigma_r(r,t)$ (c.f. (3.9)); and the last estimate follows from (6.1).

Let $V = V(r;\, a,b)$ be the solution of

$$-V'' - \frac{2}{r}V' + \lambda V = 1 \qquad \text{for} \qquad a < r < b$$
$$V|_{r=a} = 0\ , \qquad V|_{r=b} = 1\ .$$

It is given by

$$\begin{aligned} V(r;\, a,b) \;=\; & \frac{1}{\lambda}\left(1 - \frac{a\cosh(\sqrt{\lambda}(r-a))}{r}\right) \\ & + \left[1 - \frac{1}{\lambda}\left(1 - \frac{a\cosh(\sqrt{\lambda}(b-a))}{b}\right)\right]\frac{b\sinh(\sqrt{\lambda}(r-a))}{r\sinh(\sqrt{\lambda}(b-a))} \end{aligned} \tag{6.7}$$

and, as easily computed,

$$0 \le \frac{\partial V}{\partial r}(r;\, a,b) \le C_0 \tag{6.8}$$

$$\left|\frac{\partial V}{\partial a}(r;\, a,b)\right| \le C_0\ , \qquad \left|\frac{\partial V}{\partial b}(r;\, a,b)\right| \le C_0\ , \qquad \text{for} \quad a \le r \le b\ . \tag{6.9}$$

Here C_0 is a constant which depends only on λ, lower bounds of a and $b-a$, and an upper bound of b.

Since the function

$$\tilde V(r;\, a,b) = \frac{b\sinh(\sqrt{\lambda}(r-a))}{r\sinh(\sqrt{\lambda}(b-a))}$$

satisfies

$$-\tilde V'' - \frac{2}{r}\tilde V' + \lambda\tilde V = 0 \qquad \text{for} \qquad a < r < b\ ,$$
$$\tilde V|_{r=a} = 0\ , \qquad \tilde V|_{r=b} = 1\ ,$$

we obtain, by comparison,

$$V(r;\, a,b) \ge \tilde V(r;\, a,b) \ge \frac{b}{a}\,\frac{\sqrt{\lambda}(r-a)}{\sinh(\sqrt{\lambda}(b-a))} \tag{6.10}$$

and, consequently,

$$\frac{\partial V(a;\, a,b)}{\partial r} \ge \frac{b}{a}\,\frac{\sqrt{\lambda}}{\sinh(\sqrt{\lambda}(b-a))}\ . \tag{6.11}$$

By (6.3)

$$\dot\rho(0) = -\frac{1}{\lambda\sigma_0}\,h'(\rho(0))$$

and by (6.1)

$$|h'(\rho(0))| \le C_* \ .$$

Hence if $N > C_1 = \dfrac{C_*}{\lambda\sigma_0}$ then $|\dot\rho(0)| < N$ and, by continuity,

$$|\dot\rho(t)| < N \tag{6.12}$$

if $0 \le t \le t_0$ for some $t_0 > 0$. Denote by t^* the supremum of all $\tilde t$ such that (6.12) holds for all $0 \le t < \tilde t$. We shall prove that for a certain choice of N, independent of c and T, t^* is equal to T, which means that (6.12) holds for all $0 \le t < T$.

Suppose the assertion is not true. Then $t^* < T$ and either $\dot\rho(t^*) = -N$ or $\dot\rho(t^*) = N$; we take for definiteness

$$\dot\rho(t^*) = -N \ ; \tag{6.13}$$

the case $\dot\rho(t^*) = N$ can be handled in a similar way.

Consider the function

$$\bar V(r,t) = MV(r;\ \rho(t), R(t)) \ .$$

It satisfies

$$\begin{aligned} &\bar V = 0 = v \qquad \text{on} \qquad r = \rho(t) \ , \\ &\bar V = M \ge v \qquad \text{on} \qquad r = R(t) \qquad \text{if} \qquad M > M_1 \qquad (\text{by } (6.5)) \ , \end{aligned}$$

and

$$\bar V(r,0) \ge M\,\frac{R(0)}{\rho(0)}\,\frac{\sqrt\lambda(r-\rho(0))}{\sinh\left(\sqrt\lambda(R(0)-\rho(0))\right)} > h(r) = v(r,0)$$

for $\rho(0) < r < R(0)$, by (6.1), (6.10), provided

$$M\,\frac{R(0)}{\rho(0)}\,\frac{\sqrt\lambda}{\sinh\left(\sqrt\lambda(R(0)-\rho(0))\right)}\,\frac{1}{\lambda\sigma_0} > C_* \ . \tag{6.14}$$

Further, with C_0 as in (6.9),

$$c\bar V_t - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\bar V}{\partial r}\right) + \lambda\bar V = cM\left(\frac{\partial V}{\partial a}\,\dot\rho + \frac{\partial V}{\partial b}\,\dot R\right) + M \ge -cMC_0(N+M_0) + M > 0$$

if

$$0 < c \le c_0 \le \frac{1}{C_0(N+M_0)} \ .$$

Thus, if $M > M_1$ and M satisfies (6.14) then $\bar V$ is a supersolution to v and, by comparison,

$$\bar V(r,t) \ge v(r,t) \qquad \text{if} \qquad \rho(t) \le r < R(t) \ , \qquad 0 \le t \le t^* \ .$$

Since $\bar V = v$ on $r = \rho(t)$ it follows that

$$\frac{\partial\bar V}{\partial r}(\rho(t),t) \ge \frac{\partial v}{\partial r}(\rho(t),t) \qquad \text{for} \qquad 0 < t \le t^* \ .$$

Recalling (6.3) we conclude that

$$\dot\rho(t) \ge -\frac{1}{\lambda\sigma_0}\,M\,\frac{\partial V}{\partial r}(\rho(t);\ \rho(t), R(t)) \ge -\frac{1}{\lambda\sigma_0}\,MC_0 > -N$$

by (6.8) (here we used (3.7), (3.8) and (5.7)), provided N/M is sufficiently large. But for $t = t^*$ this is a contradiction to (6.13). ∎

7. Global existence. In this section we shall prove global existence of a solution to (4.9)–(4.11), namely, that $T = \infty$ in Theorem 5.1.

Introduce the functions

$$A(t) = \frac{\bar\sigma\, R(t)}{\left[\sinh\left(\sqrt\lambda(R(t)-\rho(t))\right) + \sqrt\lambda\rho(t)\cosh\left(\sqrt\lambda(R(t)-\rho(t))\right)\right]}\,, \tag{7.1}$$

$$u(r,t) = \frac{A(t)}{r}\left[\sinh\left(\sqrt\lambda(r-\rho(t))\right) + \sqrt\lambda\rho(t)\cosh\left(\sqrt\lambda(r-\rho(t))\right)\right]\,, \tag{7.2}$$

LEMMA 7.1. *Assume that (3.5), (6.1) hold. There exists a constant C independent of T and c such that*

$$|\sigma(r,t) - u(r,t)| \le C\left(c + e^{-\lambda t/c}\right) \qquad \textit{for} \qquad \rho(t) < r < R(t)\,, \qquad 0 < t < T\,, \tag{7.3}$$

and

$$|\sqrt\lambda\, A(t) - \sigma_0| \le C\left(c + e^{-\lambda t/c}\right) \qquad \textit{for} \qquad 0 < t < T\,. \tag{7.4}$$

Proof. One can easily verify that

$$\begin{aligned} &\Delta u - \lambda u = 0 \qquad \text{if} \qquad \rho(t) < r < R(t) \qquad 0 < t < T\,,\\ &\frac{\partial u}{\partial r}(\rho(t),t) = 0\,, \qquad u(R(t),t) = \bar\sigma\,, \qquad 0 < t < T\,. \end{aligned}$$

Also

$$\begin{aligned} u_t = {} & \frac{\dot R(t)}{R(t)}\,u - \frac{\lambda A(t)}{r}\,\rho(t)\,\dot\rho(t)\,\sinh(\sqrt\lambda(r-\rho(t)))\\ & - \frac{u}{\bar\sigma R(t)}\,A(t)\Big\{\Big[\sqrt\lambda\cosh(\sqrt\lambda(R(t)-\rho(t))) + \lambda\rho(t)\sinh(\sqrt\lambda(R(t)-\rho(t)))\Big]\\ & \qquad \cdot(\dot R - \dot\rho) + \sqrt\lambda\cosh(\sqrt\lambda(R(t)-\rho(t)))\,\dot\rho(t)\Big\}\,. \end{aligned} \tag{7.5}$$

Since the function

$$\frac{1}{r}\left[\sinh(\sqrt\lambda(r-\rho(t))) + \sqrt\lambda\rho(t)\,\cosh(\sqrt\lambda(r-\rho(t)))\right]$$

is monotone increasing in r,

$$\sinh(\sqrt\lambda(R(t)-\rho(t))) + \sqrt\lambda\rho(t)\,\cosh(\sqrt\lambda(R(t)-\rho(t))) \ge \frac{R(t)}{\rho(t)}\sqrt\lambda\rho(t)$$

so that

$$0 < A(t) < \frac{\bar\sigma}{\sqrt\lambda}\,.$$

Then, after cancelling out the terms $\sqrt\lambda\cosh(\sqrt\lambda(R(t)-\rho(t)))\,\dot\rho$ in $\{\cdots\}$ in (7.5), we get

$$\begin{aligned} |u_t| \le {} & \frac{|\dot R(t)|}{R(t)}\,\bar\sigma + \sqrt\lambda\bar\sigma|\dot\rho(t)|\rho(t)\sinh(\sqrt\lambda(R(t))\\ & + \frac{\bar\sigma}{\sqrt\lambda R(t)}\Big[\sqrt\lambda\cosh(\sqrt\lambda R(t)) + \lambda R(t)\sinh(\sqrt\lambda R(t))\Big]|\dot R(t)|\\ & + \frac{\bar\sigma}{\sqrt\lambda R(t)}\,\lambda R(t)\sinh(\sqrt\lambda R(t))\,|\dot\rho(t)| \le C \end{aligned}$$

by Theorem 6.1. Consequently

$$|cu_t - \Delta u + \lambda u| \le Cc \qquad \text{if} \qquad \rho(t) < r < R(t)\ , \qquad 0 < t < T\ .$$

Applying the maximum principle to

$$C(c + e^{-\lambda t/c}) \pm (\sigma - u)$$

the assertion (7.3) follows. Taking $r = \rho(t)$ in (7.3), we get the estimate (7.4). □

We recall some facts from §2 which we shall state in terms of the functions

$$\begin{aligned} G(\eta,\zeta) =&\ \sinh(\eta-\zeta) + \zeta\cosh(\eta-\zeta) - \gamma\eta\ , \\ F(\eta,\zeta) =&\ (\eta-\zeta)\cosh(\eta-\zeta) + (\eta\zeta - 1)\sinh(\eta-\zeta) - \frac{a}{3}(\eta^3-\zeta^3) - \frac{\mu}{3}\zeta^3\ . \end{aligned}$$

Setting $\xi = \eta - \zeta$ we can solve for $G = 0$ to get (see (2.8))

$$\begin{aligned} \zeta &= \zeta(\xi) = \frac{\gamma\xi - \sinh\xi}{\cosh\xi - \gamma}\ , \\ \eta &= \eta(\xi) = \frac{\xi\cosh\xi - \sinh\xi}{\cosh\xi - \gamma}\ , \end{aligned} \tag{7.6}$$

and $\zeta' < 0\,,\ \eta' < 0$. A stationary solution is determined by the equation $f(\xi) = 0$ where $f(\xi)$ is defined in (2.12), and

$$f(\xi) = F(\eta(\xi), \zeta(\xi))\ . \tag{7.7}$$

From the proof of Theorem 2.9 we have

$$\frac{d}{d\xi}\frac{f(\xi)}{(\zeta(\xi))^3} > 0\ , \qquad \xi_1 < \xi < \xi_2$$

and upon writing $\xi = \xi(\eta)$, $\zeta = \zeta(\eta)$, we then also have

$$\frac{d}{d\eta}\frac{F(\eta,\zeta(\eta))}{(\zeta(\eta))^3} < 0\ . \tag{7.8}$$

THEOREM 7.2. *If (3.5), (6.1) hold then the solution established in Theorem 5.1 exists for all $t > 0$ provided c is sufficiently small.*

Proof. If the assertion is not true then $T < \infty$ and either

$$R(T) = R_1 \tag{7.9}$$

or

$$R(T) = R_0 \tag{7.10}$$

Consider the case (7.9). Since $R(T) < R$ for all $t < T$, we have

$$\dot{R}(T) \ge 0\ . \tag{7.11}$$

Using (7.3) in (1.14) we get

$$
\begin{aligned}
R^2(t)\,\dot R(t) \le & \int_{\rho(t)}^{R(t)} u(r,t)\,r^2\,dr - \frac{1}{3}\,\tilde\sigma(R^3(t)-\rho^3(t)) - \mu\rho^3(t) \\
& \qquad + C(c+e^{-\lambda t/c})(R^3(t)-\rho^3(t)) \\
\le & \frac{A(t)}{\lambda}\Big[\sqrt{\lambda}(R(t)-\rho(t))\cosh(\sqrt{\lambda}(R(t)-\rho(t))) \\
& \qquad + (\lambda R(t)\rho(t)-1)\sinh(\sqrt{\lambda}(R(t)-\rho(t)))\Big] \\
& \qquad - \frac{1}{3}\,\tilde\sigma(R^3(t)-\rho^3(t)) - \mu\rho^3(t) + C(c+e^{-\lambda t/c})
\end{aligned}
$$

Recalling (7.4) and setting

$$
\eta(t) = \sqrt{\lambda}R(t)\ , \qquad \zeta(t) = \sqrt{\lambda}\rho(t)\ ,
$$

we obtain the inequality

$$
\begin{aligned}
\eta^2(t)\,\dot\eta(t) \ \le\ & \Big[\sigma_0 + C(c+e^{-\lambda t/c})\Big]\ \Big[(\eta(t)-\zeta(t))\cosh(\eta(t)-\zeta(t)) \\
& + (\eta(t)\zeta(t)-1)\sinh(\eta(t)-\zeta(t))\Big] \\
& - \frac{1}{3}\,\tilde\sigma[(\eta^3(t)-\zeta^3(t))] - \mu\zeta^3(t) + C(c+e^{-\lambda t/c})\ ,
\end{aligned}
$$

or

$$
\eta^2(t)\,\dot\eta(t) \le \sigma_0\,F(\eta(t),\zeta(t)) + C(c+e^{-\lambda t/c}). \tag{7.12}
$$

The estimate (7.4) also yields

$$
|\sinh(\eta(t)-\zeta(t)) + \zeta(t)\cosh(\eta(t)-\zeta(t)) - \gamma\eta(t)| \le C(c+e^{-\lambda t/c})\ ,
$$

i.e.,

$$
|G(\eta(t),\zeta(t))| \le C(c+e^{-\lambda t/c})\ . \tag{7.13}
$$

Now set

$$
\eta_0 = \eta(T)\ , \qquad \zeta_0 = \zeta(T)\ , \qquad \beta = G(\eta_0,\zeta_0)\ .
$$

It follows easily from (6.4), (7.9),

$$
T \ge \frac{R_1 - R(0)}{M_0}\ .
$$

Hence, by (7.12), (7.13),

$$
\eta_0^2\,\dot\eta(T) \le \sigma_0\,F(\eta_0,\zeta_0) + Cc\ , \qquad |\beta| \le Cc\ . \tag{7.14}
$$

From the equation $G(\eta_s,\zeta_s)=0$ and the fact that $G_\zeta \ne 0$ it follows, by the implicit function theorem, that if c is sufficiently small, then we can solve, for any β', $|\beta'| \le Cc$, the equation

$$
G(\eta,\zeta) = \beta'
$$

uniquely for η near η_s, in the form $\zeta = \phi(\eta,\beta')$ where ϕ is continuously differentiable, and $\phi(\eta_s,0) = \zeta_s$. In particular, $\phi(\eta_0,\beta) = \zeta_0$.

Setting $\eta_s = \sqrt{\lambda} R_s$ we have

$$\frac{F(\eta_s, \phi(\eta_s, 0))}{(\phi(\eta_s, 0))^3} = \frac{f(\xi_s)}{\zeta_s^3} = 0$$

where $\xi_s = \eta^{-1}(\eta_s)$ and $\zeta_s = \zeta(\xi_s)$ (see (7.6)). Consequently, by (7.8) and the fact that $\eta_0 > \eta_s$ (since $R_1 > R_s$), we get

$$F(\eta_0, \phi(\eta_0, 0)) \leq -\delta_0 < 0 \ , \qquad \delta_0 \text{ is independent of } c.$$

By continuity it then follows that, for sufficiently small c,

$$\sigma_0 \, F(\eta_0, \phi(\eta_0, \beta)) + Cc < 0 \ , \qquad \text{since} \qquad |\beta| \leq Cc \ ,$$

i.e.,

$$\sigma_0 \, F(\eta_0, \zeta_0) + Cc < 0 \ .$$

Using this in (7.14) we conclude that $\dot{\eta}(T) < 0$, a contradiction to (7.11).

Similarly, if (7.10) holds then $\dot{R}(T) \leq 0$, but using Lemma 7.1 and (1.14) as before, we can deduce, by the previous arguments, that $\dot{R}(T) > 0$, which is a contradiction. ☐

8. Asymptotic behavior. We extend $\sigma_s(r)$ by (2.1) also to $r > R_s$.

In this section we prove that the global solution established in Theorem 7.2 converges to the corresponding stationary solution.

LEMMA 8.1. *Under the assumptions of Theorem 7.2, for any $\alpha_0 > 0$ and $0 < \alpha < \alpha_0$, if*

$$|R(t) - R_s| \leq \alpha \ , \qquad |\rho(t) - \rho_s| < \alpha \ , \tag{8.1}$$

$$|\dot{R}(t)| \leq \alpha \ , \qquad |\dot{\rho}(t)| \leq \alpha \ , \tag{8.2}$$

$$|\sigma(r,t) - \sigma_s(r)| \leq \alpha \tag{8.3}$$

for all $\rho(t) \leq r \leq R(t)$, $t > 0$, and if $0 < c \leq c_0$ for some sufficiently small positive number c_0 independent of α, then there exist positive constants C, K such that

$$|R(t) - R_s| \leq Cc\alpha \ , \qquad |\rho(t) - \rho_s| < Cc\alpha \ , \tag{8.4}$$

$$|\dot{R}(t)| \leq Cc\alpha \ , \qquad |\dot{\rho}(t)| \leq Cc\alpha \ , \tag{8.5}$$

$$|\sigma(r,t) - \sigma_s(r)| \leq Cc\alpha \tag{8.6}$$

for $\rho(t) \leq r \leq R(t)$, $t \geq 1 + Kc \log \dfrac{1}{c\alpha}$; C and K are independent of α, c.

Proof. Proceeding as in the proof of Lemma 7.1 we can establish that

$$|\sigma(r,t) - u(r,t)| \leq C\alpha(c + e^{-\lambda t/c}) \ ,$$
$$|\sqrt{\lambda}\, A(t) - \sigma_0| \leq C\alpha(c + e^{-\lambda t/c}) \ ,$$

for $\rho(t) \leq r < R(t)$, $t \geq 0$, so that

$$\begin{aligned} |\sigma(r,t) - u(r,t)| &\leq Cc\alpha \ , \\ |\sqrt{\lambda}\, A(t) - \sigma_0| &\leq Cc\alpha \end{aligned} \tag{8.7}$$

if $\rho(t) \le r \le R(t),\ \ t \ge 1$ with another constant C. Using this in (1.14) we get

$$|\eta(t)^2\,\dot\eta(t) - \sigma_0\,F(\eta(t),\zeta(t))| \le Cc\alpha \tag{8.8}$$

$$|G(\eta(t),\zeta(t))| \le Cc\alpha \tag{8.9}$$

for $t \ge 1$. Set

$$\beta(t) = G(\eta(t),\zeta(t))\ . \tag{8.10}$$

As in the proof of Theorem 7.2, we can express, from (8.10), $\zeta(t)$ in terms of $\eta(t)$ and $\beta(t)$,

$$\zeta(t) = \phi(\eta(t),\beta(t))\ ; \tag{8.11}$$

moreover,

$$\frac{d}{d\eta}\,\frac{F(\eta,\phi(\eta,\beta(t)))}{(\phi(\eta,\beta(t)))^3} < 0\ , \tag{8.12}$$

and $F(\eta,\phi(\eta,\beta(t)))$ has a unique zero $\eta = \eta^0_{\beta(t)}$ sufficiently near η_s, provided c is small enough.

Substituting (8.11) into (8.8) we get

$$|\eta(t)^2\,\dot\eta(t) - \sigma_0\,F(\eta(t),\phi(\eta(t),\beta(t)))| \le Cc\alpha \qquad \text{if} \qquad t \ge 1$$

and then (using (8.12)) we deduce (cf. [8, Lemma 5.4]) that

$$|\eta(t) - \eta_s| \le Cc\alpha \qquad \text{if} \qquad t \ge 1 \tag{8.13}$$

or, equivalently

$$|R(t) - R_s| \le Cc\alpha \qquad \text{if} \qquad t \ge 1 \tag{8.14}$$

Next, since $\zeta_s = \phi(\eta_s, 0)$,

$$|\zeta(t) - \zeta_s| = |\phi(\eta(t),\beta(t)) - \phi(\eta_s,0)| \le C\Big[|\eta(t)-\eta_s| + |\beta(t)|\Big]$$

and using (8.13) and (8.10), (8.9), we get

$$|\zeta(t) - \zeta_s| \le Cc\alpha\ , \tag{8.15}$$

or

$$|\rho(t) - \rho_s| \le Cc\alpha \qquad \text{if} \qquad t \ge 1\ . \tag{8.16}$$

If we substitute (8.13), (8.15) into (8.8), we get

$$|\dot\eta(t)| \le Cc\alpha\ ,$$

or

$$|\dot R(t)| \le Cc\alpha \qquad \text{if} \qquad t \ge 1 \tag{8.17}$$

We next claim that

$$|\dot\rho(t)| \le Cc\alpha \qquad \text{if} \qquad t > 1 + Kc\,\log\frac{1}{c\alpha} \tag{8.18}$$

where K is a positive constant independent of c, α.

To prove this estimate note that, by (6.3),

$$\dot\rho(t) = -\frac{1}{\lambda\sigma_0} v_r(\rho(t),t) \ . \tag{8.19}$$

where $v = \sigma_t$. Also

$$v(\rho(t),t) = 0 \ ,$$

and, by (8.17),

$$|v(R(t),t)| \le |\sigma_r(R(t),t)\,\dot R(t)| \le Cc\alpha \ .$$

Hence, by the maximum principle,

$$Cc\alpha + Ce^{-\lambda t/c} \pm v \ge 0 \qquad \text{if} \qquad \rho(t) \le r \le R(t) \ , \quad t \ge 1 \ .$$

It follows that

$$|v| \le 2Cc\alpha \qquad \text{if} \qquad \rho(t) < r < R(t) \ , \quad t > t_1 = \frac{c}{\lambda}\log\frac{1}{c\alpha} \ .$$

Now, by Theorem 6.1 and the proof of Theorem 4.1,

$$|\rho^{(j)}(t)| \le C \qquad \text{if} \qquad t \ge 1 \quad (j=1,2) \ .$$

We can therefore apply the interior-boundary Schauder estimate [10, 14] to $v/c\alpha$ to deduce that

$$\left|\frac{\partial}{\partial r}\frac{v}{c\alpha}\right| \le C \qquad \text{at} \qquad r = \rho(t) \ , \quad t > t_1 + 1 \ .$$

Substituting this into (8.19), the proof of (8.18) is complete.

Finally (8.6) follows from (8.7) and (8.4). □

Having proved Lemma 8.1 we can now apply it with some $\alpha = C_0$. If we then choose c small enough so that $Cc < \frac{1}{2}$ (Cc is as in (8.4)–(8.6)) we find that the assumption (8.1)–(8.3) hold with $\alpha/2$ and $t \ge t_1$ where

$$t_1 = Kc\,\log\frac{1}{c\,\alpha/2} \ .$$

Repeating this process with $\alpha/2$ and initial time $t = t_1$, then with $\alpha/2^2$ and $t = t_2$, etc., we arrive after n steps at the estimates

$$\begin{aligned} |R(t) - R_s| &\le \frac{\alpha}{2^n} \ , &\qquad |\rho(t) - \rho_s| &\le \frac{\alpha}{2^n} \ , \\ |\dot R(t)| &\le \frac{\alpha}{2^n} \ , &\qquad |\dot\rho(t)| &\le \frac{\alpha}{2^n} \ , \\ |\sigma(r,t) - \sigma_s(r)| &\le \frac{\alpha}{2^n} \end{aligned}$$

for

$$t \ge t_1 + t_2 + \cdots + t_n \equiv T_n$$

But for $T_n < t < T_n + t_{n+1}$ we have $K_0 n^2 c\,\log\frac{1}{c} \ge t$ for some positive constant K_0, so that

$$\frac{\alpha}{2^n} \le Ce^{-\delta(t/(c\,\log\frac{1}{c}))^{1/2}}$$

for some $\delta > 0$. We thus conclude:

THEOREM 8.2. *If (3.5), (6.1) hold and if c is sufficiently small, the (σ, R, ρ) converges to the stationary solution as $t \to \infty$; more precisely:*

$$|R(t) - R_s| \le Ce^{-\delta_0\sqrt{t}}\ , \qquad |\rho(t) - \rho_s| \le Ce^{-\delta_0\sqrt{t}}\ ,$$
$$|\dot{R}(t)| \le Ce^{-\delta_0\sqrt{t}}\ , \qquad |\dot{\rho}(t)| \le Ce^{-\delta_0\sqrt{t}}\ ,$$
$$|\sigma(r,t) - \sigma_s(r)| \le Ce^{-\delta_0\sqrt{t}}$$

for $\rho(t) \le r \le R(t),\ t > 0$, where $\delta_0 = \delta(c \log \frac{1}{c})^{-1}$ and C, δ are positive constants independent of c.

Conclusion. In this paper we considered a model of necrotic tumor with spherical necrotic core of radius $r = \rho(t)$ surrounded by a shell $\rho(t) < r < R(t)$ of life cells. The nutrient concentration satisfies a diffusion equation which incorporates vascularized structure as well as apoptosis. The free boundaries $r = \rho(t)$ and $r = R(t)$ satisfy a conservation of mass law which includes the effect of cell proliferation in the shell $\rho(t) < r < R(t)$ and shrinkage due to necrosis. We have proved by rigorous mathematical methods that, for a certain range of parameters, there exist stationary solutions (dormant tumors), with radii, say, ρ_s and R_s. Further, any such dormant tumor is stable in the following sense: for any initial data "near" the stationary solution, there exists a unique global solution of the time evolution problem, and its radii $\rho(t)$ and $R(t)$ converge to the radii ρ_s and R_s, respectively, as $t \to \infty$.

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