Superintegrability on the two dimensional hyperboloid II

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Abstract

This work is devoted to the investigation of the quantum mechanical systems on the two dimensional hyperboloid which admit separation of variables in at least two coordinate systems. Here we consider two potentials introduced in a paper of C.P.Boyer, E.G.Kalnins and P.Winternitz, which haven't yet been studied. We give an example of an interbasis expansion and work out the structure of the quadratic algebra generated by the integrals of motion.

1 Introduction

Superintegrable systems on the two-dimensional hyperboloid were introduced and developed in the papers [1, 2, 3]. In distinction to the cases of two-dimensional Euclidean space and the two-sphere, the classification of superintegrable systems on the hyperboloid is difficult. To date only the four potentials studied in [3] and two more listed in [1] are known. In the present paper two potentials are considered, which were constructed in the work [1] but have not previously been investigated. These potentials both have only a finite number of bound states. At this point we have treated all the potentials that arise by restriction from hermitean hyperbolic space. We follow the approach of [3], which contains an introduction and motivation.

The two dimensional hyperboloid is characterized via the cartesian coordinates $\omega_0, \omega_1, \omega_2$ where $\omega_0^2 - \omega_1^2 - \omega_2^2 = 1, \omega_0 > 1$. The requirement $\omega_0 > 1$ means that we consider only

upper sheet of the double-sheet hyperboloid. Throughout this paper we will consider the Schrödinger equation on the hyperboloid in the form ($\hbar = m = 1$)

$$H\Psi \equiv \left(-\frac{1}{2}\Delta_{LB} + V\right)\Psi = E\Psi \tag{1}$$

where V is a potential function and the Laplace-Beltrami operator Δ_{LB} is written as

$$\Delta_{LB} = K_3^2 + K_2^2 - M_1^2. \tag{2}$$

Here K_3, K_2, M_1 generate the Lie algebra $so(2,1)$ [4, 5]

$$K_3 = \omega_0\partial_{\omega_1} + \omega_1\partial_{\omega_0}, \quad K_2 = \omega_0\partial_{\omega_2} + \omega_2\partial_{\omega_0}, \quad M_1 = \omega_1\partial_{\omega_2} - \omega_2\partial_{\omega_1}. \tag{3}$$

and

$$[K_3, K_2] = M_1, \quad [K_2, M_1] = -K_3, \quad [K_3, M_1] = K_2 \tag{4}$$

The Schrödinger equation (1) for $V = 0$ separates in nine coordinate systems [6]. Introduction of a potential breaks the symmetry and, in general, reduces the number of coordinate systems permitting separability, usually to zero. We consider the following two potentials (see Table), constructed in [1], for which (1) is superintegrable.

Table

Potential $V(\omega)$	Coordinate system
$V_1 = \frac{\alpha^2}{\omega_2^2} - \frac{\gamma^2}{(\omega_0-\omega_1)^2} + \beta^2\frac{\omega_0+\omega_1}{(\omega_0-\omega_1)^3}$	Equidistant Elliptic-parabolic Hyperbolic-parabolic Horicyclic
$V_2 = \frac{\alpha^2}{\omega_2^2} + \gamma^2\frac{\omega_0\omega_1}{(\omega_0^2+\omega_1^2)^2}$ $+(\alpha^2-\beta^2)\frac{\omega_0^2-\omega_1^2}{(\omega_0^2+\omega_1^2)^2}$	Equidistant Semi-Hyperbolic

Recall that (1) is *superintegrable* for a given potential V if it is separable simultaneously in at least two coordinate systems.

2 First Potential

The first considered potential is

$$V_1 = \frac{\alpha^2}{\omega_2^2} - \frac{\gamma^2}{(\omega_0-\omega_1)^2} + \beta^2\frac{\omega_0+\omega_1}{(\omega_0-\omega_1)^3} \tag{5}$$

where α, β, γ are positive constants . The corresponding Schrödinger equation admits separable solutions in four coordinate systems: equidistant, elliptic–parabolic, hyperbolic–parabolic and horicyclic.

2.1 Solutions of the Schrödinger equation

1.1 Equidistant coordinates. In this coordinate system

$$\omega_0 = \cosh\tau_1 \cosh\tau_2, \quad \omega_1 = \cosh\tau_1 \sinh\tau_2, \quad \omega_2 = \sinh\tau_1$$

$[\tau_1, \tau_2 \in (-\infty, \infty)]$ the potential V_1 has the form

$$V_1(\tau_1, \tau_2) = \frac{\alpha^2}{\sinh^2\tau_1} + \frac{1}{\cosh^2\tau_1}\frac{\beta^2 - \gamma^2(\cosh\tau_2 - \sinh\tau_2)^2}{(\cosh\tau_2 - \sinh\tau_2)^4} \tag{6}$$

After putting

$$\Psi(\tau_1, \tau_2) = (\cosh\tau_1)^{-1/2} S_1(\tau_1) S_2(\tau_2) \tag{7}$$

we come to the system of equations:

$$\frac{d^2S_2}{d\tau_2^2} + \left[-\mu^2 - 2\beta^2 e^{4\tau_2} + 2\gamma^2 e^{2\tau_2}\right] S_2 = 0 \tag{8}$$

$$\frac{d^2S_1}{d\tau_1^2} + \left[(2E - \frac{1}{4}) + \frac{\mu^2 - \frac{1}{4}}{\cosh^2\tau_1} - \frac{2\alpha^2}{\sinh^2\tau_1}\right] S_1 = 0 \tag{9}$$

where μ is the equidistant separation constant. The first equation (8) could be considered as a one dimensional Shrödinger equation for the Morse potential [9] and the orthonormalised solution is given by the expression:

$$\begin{aligned} S_2(\tau_2) \equiv S_m^{(\beta,\mu)}(z) &= \sqrt{\frac{2\mu\Gamma(m+\mu+1)}{m!\Gamma^2(\mu+1)}} e^{-z/2} z^{\mu/2}\, {}_2F_1(-m, \mu+1; z) \\ &= \sqrt{\frac{2\mu m!}{\Gamma(m+\mu+1)}} e^{-z/2} z^{\mu/2} L_m^{\mu}(z), \quad z = \sqrt{2}\beta e^{2\tau_2} \end{aligned} \tag{10}$$

where $L_m^{\mu}(z)$ are the Laguerre polynomials [7]. The separation constant is quantized as

$$\mu = -2m - 1 + \frac{\gamma^2}{\sqrt{2}\beta}, \quad 0 \le m \le \left[\frac{1}{2}\left(\frac{\gamma^2}{\sqrt{2}\beta} - 1\right)\right] \tag{11}$$

The second equation (9) represents the modified Pöschl–Teller equation [3, 8]. The orthonormalised wave function is given by:

$$\begin{aligned} S_1(\tau_1) \equiv S_n^{(\alpha,\mu)}(\tau_1) &= \sqrt{\frac{2(\mu - \sqrt{2\alpha^2 + 1/4} - 2n - 1)\Gamma(\mu - n) n!}{\Gamma(\mu - \sqrt{2\alpha^2 + 1/4} - n)\Gamma(1 + n + \sqrt{2\alpha^2 + 1/4})}} \\ &\times (\sinh\tau_1)^{\frac{1}{2} + \sqrt{2\alpha^2 + 1/4}} (\cosh\tau_1)^{\frac{1}{2} - \mu} P_n^{(\sqrt{2\alpha^2 + 1/4}, -\mu)}(\cosh 2\tau_1), \end{aligned} \tag{12}$$

with $n = 0, 1, \ldots \left[\frac{1}{2}\left(\mu - 1 - \sqrt{2\alpha^2 + \frac{1}{4}}\right)\right]$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial [7]. The quantized energy is

$$\begin{aligned} E_N &= -\frac{1}{2}(\mu - \sqrt{2\alpha^2 + 1/4} - 2n - 1)^2 + \frac{1}{8} \\ &= -\frac{1}{2}\left(2N + 2 + \sqrt{2\alpha^2 + 1/4} - \frac{\gamma^2}{\sqrt{2}\beta}\right)^2 + \frac{1}{8} \end{aligned} \tag{13}$$

where $N = m + n$ is the principal quantum number and the bound states occur for

$$0 \leq N \leq \left[\frac{1}{2}\left(\frac{\gamma^2}{\sqrt{2}\beta} - \sqrt{2\alpha^2 + 1/4} - 2\right)\right] \tag{14}$$

The orthonormalized total wave function $\Psi_{nm}(\tau_1, \tau_2)$ is given by (7), (10) and (12).

The symmetry operator describing this coordinate system is

$$\begin{aligned} L_1\Psi_{nm}(\tau_1, \tau_2) &\equiv \left[K_3^2 - 2\beta^2\left(\frac{\omega_0 + \omega_1}{\omega_0 - \omega_1}\right)^2 + 2\gamma^2\frac{\omega_0 + \omega_1}{\omega_0 - \omega_1}\right]\Psi_{nm}(\tau_1, \tau_2) \\ &= \left(-2m - 1 + \frac{\gamma^2}{\sqrt{2}\beta}\right)^2 \Psi_{nm}(\tau_1, \tau_2) \end{aligned} \tag{15}$$

1.2 Horicyclic coordinates. In the horicyclic coordinates

$$\omega_0 = \frac{x^2 + y^2 + 1}{2y}, \quad \omega_1 = \frac{x^2 + y^2 - 1}{2y}, \quad \omega_2 = \frac{x}{y}, \tag{16}$$

$[y > 0, x \in (-\infty, \infty)]$ the potential V_1 is

$$V_1(x, y) = y^2\left[\frac{\alpha^2}{x^2} + \beta^2(x^2 + y^2) - \gamma^2\right] \tag{17}$$

and the Schrödinger equation has the following form

$$-\frac{1}{2}y^2\left[\frac{\partial^2}{\partial x^2} - \frac{2\alpha^2}{x^2} - 2\beta^2x^2 + \frac{\partial^2}{\partial y^2} - 2\beta^2y^2 + 2\gamma^2\right]\Psi(x, y) = E\Psi(x, y) \tag{18}$$

Via putting

$$\Psi(x, y) = \psi_1(x)\psi_2(y) \tag{19}$$

it admits a separation

$$\frac{d^2\psi_1}{dx^2} + 2\left[\gamma^2(\lambda_1 + 1) - \beta^2x^2 - \frac{\alpha^2}{x^2}\right]\psi_1 = 0 \tag{20}$$

$$\frac{d^2\psi_2}{dy^2} + 2\left[\gamma^2(\lambda_2 - 1) - \beta^2y^2 + \frac{E}{y^2}\right]\psi_2 = 0 \tag{21}$$

where λ_1 and λ_2 are the horicyclic separation constants with the relation: $\lambda_1 + \lambda_2 = 1$.

The orthonormalized solutions of the equations (20), (21) for $(-2E + 1/4) > 0$ are

$$\begin{aligned}\psi_1(x) &\equiv \psi_{n_1}^{(\alpha,\beta)}(x) = \sqrt{\frac{n_1!(\sqrt{2}\beta)^{1/2}}{\Gamma(n_1 + \sqrt{2\alpha^2 + 1/4} + 1)}} \\ &\times e^{-\frac{\beta x^2}{\sqrt{2}}}(\sqrt{\sqrt{2}\beta x^2})^{\frac{1}{2}+\sqrt{2\alpha^2+1/4}} L_{n_1}^{\sqrt{2\alpha^2+1/4}}(\sqrt{2}\beta x^2) \qquad (22)\end{aligned}$$

$$\begin{aligned}\psi_2(y) &\equiv \psi_{n_2}^{(\gamma,\beta)}(y) = \sqrt{\frac{n_2!(\sqrt{2}\beta)^{1/2}}{\Gamma(n_2 + \sqrt{-2E + 1/4} + 1)}} \\ &\times e^{-\frac{\beta y^2}{\sqrt{2}}}(\sqrt{\sqrt{2}\beta y^2})^{\frac{1}{2}+\sqrt{-2E+1/4}} L_{n_2}^{\sqrt{-2E+1/4}}(\sqrt{2}\beta y^2). \qquad (23)\end{aligned}$$

The separation constants λ_1, λ_2 are quantized as:

$$\lambda_1 = \frac{\sqrt{2}\beta}{\gamma^2}(2n_1 + \sqrt{2\alpha^2 + 1/4} + 1) - 1; \quad \lambda_2 = \frac{\sqrt{2}\beta}{\gamma^2}(2n_2 + \sqrt{-2E + 1/4} + 1) + 1 \qquad (24)$$

and according to the relation $\lambda_1 + \lambda_2 = 1$, we come to the energy spectrum as in (13). The operator characterizing the separation in horicyclic coordinates is:

$$\begin{aligned}L_2\Psi_{n_1n_2}(x,y) &\equiv \left[(K_2 - M_1)^2 - \frac{2\beta^2\omega_2^2}{(\omega_0 - \omega_1)^2} - \frac{2\alpha^2(\omega_0 - \omega_1)^2}{\omega_2^2} + 2\gamma^2\right]\Psi_{n_1n_2}(x,y) \\ &= -\left[2\sqrt{2}\beta(2n_1 + \sqrt{2\alpha^2 + 1/4} + 1) + 2\gamma^2\right]\Psi_{n_1n_2}(x,y) \qquad (25)\end{aligned}$$

1.3 Elliptic–parabolic coordinates. In this coordinate system

$$\omega_0 = \frac{\cosh^2 a + \cos^2\theta}{2\cosh a\cos\theta}, \quad \omega_1 = \frac{\sinh^2 a - \sin^2\theta}{2\cosh a\cos\theta}, \quad \omega_2 = \tanh a\tan\theta, \qquad (26)$$

$[a > 0,\ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})]$ the potential V_1 has the form:

$$\begin{aligned}V_1(a,\theta) &= \frac{\cosh^2 a\cos^2\theta}{\cosh^2 a - \cos^2\theta}\Big[\beta^2(\cosh^2 a\sinh^2 a + \cos^2\theta\sin^2\theta) \\ &- \gamma^2(\cosh^2 a - \cos^2\theta) + \alpha^2\left(\frac{1}{\sinh^2 a} + \frac{1}{\sin^2\theta}\right)\Big] \qquad (27)\end{aligned}$$

The Shrödinger equation is:

$$\begin{aligned}&-\frac{1}{2}\frac{\cosh^2 a\cos^2\theta}{\cosh^2 a - \cos^2\theta}\Big[\frac{\partial^2}{\partial a^2} - 2\beta^2\cosh^2 a\sinh^2 a + 2\gamma^2\cosh^2 a - \frac{2\alpha^2}{\sinh^2 a} \\ &+ \frac{\partial^2}{\partial\theta^2} - 2\beta^2\cos^2\theta\sin^2\theta - 2\gamma^2\cos^2\theta - \frac{2\alpha^2}{\sin^2\theta}\Big]\Psi(a,\theta) = E\Psi(a,\theta) \qquad (28)\end{aligned}$$

Putting for the wave function $\Psi(a,\theta) = S(a)S(\theta)$, after separation of variables we get two identical equations:

$$\frac{d^2S(\rho)}{d\rho^2} + \left[\lambda - 2\beta^2\cosh^2\rho\sinh^2\rho + 2\gamma^2\cosh^2\rho - \frac{2\alpha^2}{\sinh^2\rho} - \frac{2E}{\cosh^2\rho}\right]S(\rho) = 0 \qquad (29)$$

where λ is the elliptic–parabolic separation constant and $\rho \equiv a, i\theta$. After changing the variables $x = \cosh^2\rho$ in eq. (29), we obtain

$$4x(x-1)\frac{d^2S}{dx^2} + 2(2x-1)\frac{dS}{dx} + \left[\lambda - 2\beta^2 x(x-1) + 2\gamma^2 x - \frac{2\alpha^2}{x-1} - \frac{2E}{x}\right]S = 0 \qquad (30)$$

Thus the region $x \in [1,\infty]$ in eq. (30) belongs to the wave function $S(a)$ and $x \in [0,1]$ to the wave function $S(\theta)$. Putting

$$S(x) = (x-1)^s x^t e^{-\beta x/\sqrt{2}} G(x), \qquad (31)$$

where

$$s = \frac{1}{4} + \frac{1}{\sqrt{2}}\sqrt{\alpha^2 + \frac{1}{8}}, \quad t = \frac{1}{4} + \frac{1}{\sqrt{2}}\sqrt{-E + \frac{1}{8}} \qquad (32)$$

we get

$$\begin{aligned} \frac{d^2G}{dx^2} &+ \frac{1}{2}\left[\frac{1+4t}{x} + \frac{1+4s}{x-1} - \frac{4\beta}{\sqrt{2}}\right]\frac{dG}{dx} \qquad (33) \\ &+ \frac{1}{4}\left\{\frac{[2\gamma^2 - 4\beta(1+2(t+s))/\sqrt{2}]x + \nu + \sqrt{2}\beta(1+4t) + 4(t+s)^2}{x(x-1)}\right\}G = 0. \end{aligned}$$

If we now substitute

$$G(x) = \prod_{i=1}^{N}(x - \theta_i) \qquad (34)$$

and take into account (32), we find that θ_i satisfies the equation:

$$\begin{aligned} 2\theta_i(1-\theta_i)\left(\sum_{\substack{k=1\\k\neq i}}^{N}\frac{1}{\theta_k - \theta_i} + \frac{\beta}{\sqrt{2}}\right) + 2(1-\theta_i)N + \frac{\sqrt{2}\gamma^2}{4\beta}\theta_i \\ + \frac{\gamma^2}{\sqrt{2}\beta} + \sqrt{2\alpha^2 + \frac{1}{4}} + 1 \quad = \quad 0. \qquad (35) \end{aligned}$$

The quantization for the energy is given via:

$$\sqrt{-2E + \frac{1}{4}} + \sqrt{2\alpha^2 + \frac{1}{4}} + 2N + 2 - \frac{\gamma^2}{\sqrt{2}\beta} = 0 \qquad (36)$$

and we obtain the expression (13). The separation constant λ is:

$$\lambda = \frac{8\beta}{\sqrt{2}}\sum_{i=1}^{N}\theta_i - \left(\frac{\gamma^2}{\sqrt{2}\beta} - 1\right)^2 + \frac{4\beta}{\sqrt{2}}\left(1 + \sqrt{2\alpha^2 + \frac{1}{4}}\right) - 2\gamma^2. \tag{37}$$

Thus the total solution $\Psi(a,\theta)$ is represented as:

$$\begin{aligned}\Psi_{Npq}(a,\theta) &= S_{Np}(a)S_{Nq}(\theta) = (\sinh a \sin\theta)^{\frac{1}{2}+\sqrt{2\alpha^2+\frac{1}{4}}}(\cosh a\cos\theta)^{\frac{\gamma^2}{\sqrt{2}\beta}-\sqrt{2\alpha^2+\frac{1}{4}}-2N-\frac{3}{2}} \\ &\cdot \exp\left\{-\frac{\beta}{\sqrt{2}}(\cosh^2 a + cos^2\theta)\right\}\prod_{i=1}^{N}(\cosh^2 a - \theta_i)(\cos^2\theta - \theta_i)\end{aligned} \tag{38}$$

where p and q is the number of zeroes for the wave functions $S(a)$ and $S(\theta)$ in the regions $[0,1]$, $[1,\infty]$ correspondingly; and the total number of zeroes is $N = p + q$.

Eliminating the energy E from equation (30), we see that the additional integral of motion here is

$$\begin{aligned}L_3\Psi_{Npq}(a,\theta) &= \frac{1}{\cos^2\theta - \cosh^2 a}\Big\{\cosh^2 a\frac{\partial^2}{\partial a^2} + \cos^2\theta\frac{\partial^2}{\partial\theta^2} - 2\beta^2(\cosh^4 a\sinh^2 a \\ &+ \cos^4\theta\sin^2\theta) + 2\gamma^2(\cosh^4 a - \cos^4\theta) - 2\alpha^2(\coth^2 a - \cot^2\theta)\Big\}\Psi_{Npq}(a,\theta) \\ &= \Big\{-(K_2 - M_1)^2 - K_3^2 + 2\beta^2\frac{(w_0+w_1)^2 + w_2^2}{(w_0-w_1)^2} + 2\alpha^2\left(\frac{w_0-w_1}{w_2}\right)^2 \\ &- 4\gamma^2\frac{w_0}{w_0-w_1}\Big\}\Psi_{Npq}(a,\theta) = \lambda\Psi_{Npq}(a,\theta).\end{aligned} \tag{39}$$

1.4 Hyperbolic–parabolic coordinates. In this coordinate system

$$\omega_0 = \frac{\cosh^2 b + \cos^2\theta}{2\sinh b\sin\theta}, \quad \omega_1 = \frac{\sinh^2 b - \sin^2\theta}{2\sinh b\sin\theta}, \quad \omega_2 = \coth b\cot\theta, \tag{40}$$

$[b > 0,\ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})]$ the potential V_1 has the form:

$$\begin{aligned}V_1(b,\theta) &= \frac{\sinh^2 b\sin^2\theta}{\sinh^2 b + \sin^2\theta}\Big[\beta^2(\sinh^2 b\cosh^2 b + \sin^2\theta\cos^2\theta) \\ &- \gamma^2(\sinh^2 b + \sin^2\theta) + \alpha^2\left(\frac{1}{\cos^2\theta} - \frac{1}{\cosh^2 b}\right).\Big]\end{aligned} \tag{41}$$

The Shrödinger equation is

$$\begin{aligned}&-\frac{1}{2}\frac{\sinh^2 b\sin^2\theta}{\sinh^2 b + \sin^2\theta}\Big[\frac{\partial^2}{\partial b^2} - 2\beta^2\sinh^2 b\cosh^2 b + 2\gamma^2\sinh^2 b + \frac{2\alpha^2}{\cosh^2 b} \\ &+ \frac{\partial^2}{\partial\theta^2} - 2\beta^2\sin^2\theta\cos^2\theta + 2\gamma^2\sin^2\theta - \frac{2\alpha^2}{\cos^2\theta}\Big]\Psi(b,\theta) = E\Psi(b,\theta).\end{aligned} \tag{42}$$

Putting for the wave function $\Psi(b,\theta) = S(b)S(\theta)$, after separation of variables we get two identical equations:

$$\frac{d^2S(\rho)}{d\rho^2} + 2\Big[\frac{\tau}{2} - \beta^2 \sinh^2\rho\cosh^2\rho + \gamma^2\sinh^2\rho + \frac{\alpha^2}{\cosh^2\rho} + \frac{E}{\sinh^2\rho}\Big]S(\rho) = 0 \qquad (43)$$

where τ is the hyperbolic–parabolic separation constant and $\rho \equiv b, i\theta$. After changing the variables $x = \sinh^2\rho$ in eq. (43), we come to the equation

$$4x(x+1)\frac{d^2S}{dx^2} + 2(2x+1)\frac{dS}{dx} + \left[\tau - 2\beta^2 x(x+1) + 2\gamma^2 x + \frac{2\alpha^2}{x+1} + \frac{2E}{x}\right]S = 0. \qquad (44)$$

Choosing

$$P(x) = (1+x)^s x^t e^{-\beta x/\sqrt{2}} \prod_{i=1}^{N}(x-\theta_i) \qquad (45)$$

where t and s are given by the formulas (32), we obtain the energy spectrum (36). Here θ_i satisfies the equations

$$2\theta_i(1+\theta_i)\left(\sum_{\substack{k=1\\k\neq i}}^{N}\frac{1}{\theta_i-\theta_k} - \frac{\beta}{\sqrt{2}}\right) - 2(1+\theta_i)N + \frac{\sqrt{2}\gamma^2}{4\beta}\theta_i \quad +$$

$$\frac{\gamma^2}{\sqrt{2}\beta} - \sqrt{2\alpha^2+\frac{1}{4}} - 1 \quad = \quad 0. \qquad (46)$$

The separation constant τ is:

$$\tau = \frac{8\beta}{\sqrt{2}}\sum_{i=1}^{N}\theta_i - \left(\frac{\gamma^2}{\sqrt{2}\beta} - 1\right)^2 - \frac{4\beta}{\sqrt{2}}\left(1 + \sqrt{2\alpha^2+\frac{1}{4}}\right) + 2\gamma^2, \qquad (47)$$

so the total solution $\Psi(b,\theta)$ is represented as:

$$\Psi_{Nlk}(b,\theta) \quad = \quad S_{Nl}(b)S_{Nk}(\theta) = (\cosh b\cos\theta)^{\frac{1}{2}+\sqrt{2\alpha^2+\frac{1}{4}}}(\sinh b\sin\theta)^{\frac{\gamma^2}{\sqrt{2}\beta}-\sqrt{2\alpha^2+\frac{1}{4}}-2N-\frac{3}{2}}$$

$$\cdot \quad \exp\left\{-\frac{\beta}{\sqrt{2}}(\sinh^2 b - \sin^2\theta)\right\}\prod_{i=1}^{N}(\sinh^2 b - \theta_i)(\sin^2\theta + \theta_i). \qquad (48)$$

The total number of zeros is N, and k of them are located in the interval $[-1,0]$ and l are in $[0,\infty]$.

Each solution $\Psi_{Nlk}(b,\theta)$ satisfies the eigenvalue equation

$$L_4\Psi_{Nlk}(b,\theta) \quad = \quad -\frac{1}{\sin^2 b + \sin^2\theta}\Big\{\sinh^2 b\frac{\partial^2}{\partial b^2} - \sin^2\theta\frac{\partial^2}{\partial\theta^2} - 2\beta^2(\cosh^2 b\sinh^4 b$$

$$- \quad \cos^2\theta\sin^4\theta) + 2\gamma^2(\sinh^4 b - \sin^4\theta) + 2\alpha^2(\tanh^2 b + \tan^2\theta)\Big\}\Psi_{Nlk}(b,\theta)$$

$$= \quad \Big\{(K_2 - M_1)^2 - K_3^2 + 2\beta^2\frac{(w_0+w_1)^2 - w_2^2}{(w_0-w_1)^2} - 2\alpha^2\left(\frac{w_0-w_1}{w_2}\right)^2$$

$$- \quad 4\gamma^2\frac{w_1}{w_0-w_1}\Big\}\Psi_{Nlk}(b,\theta) = \tau\,\Psi_{Nlk}(b,\theta). \qquad (49)$$

2.2 Algebra

Among the operators $\{L_1, L_2, L_3, L_4\}$, corresponding to the four separable coordinate systems, only two are independent, as

$$L_3 = -L_2 - L_1, \quad L_4 = L_2 - L_1. \tag{50}$$

Consider the operators N_1, N_2 and R where

$$\begin{aligned}
N_1 &= \tilde{L}_2 = L_1, \quad N_2 = \tilde{L}_1 = L_2 - 2\gamma^2, \\
R &\equiv [N_1, N_2] = 2\{K_3, \{K_2, M_1\}\} - 2\{K_3, K_2^2\} - 2\{K_3, M_1^2\} + 8\Big[\alpha^2\Big(\frac{\omega_0 - \omega_1}{\omega_2}\Big)^2 + \\
&+ \beta^2\Big(\frac{\omega_2}{\omega_0 - \omega_1}\Big)^2\Big]K_3 + \frac{16\beta^2\omega_2}{(\omega_0 - \omega_1)^2}(\omega_0 K_2 - \omega_1 M_1) + \frac{8\gamma^2\omega_2}{\omega_0 - \omega_1}(M_1 - K_2) \\
&- 4\left[\gamma^2 + 2\alpha^2\Big(\frac{\omega_0 - \omega_1}{\omega_2}\Big)^2 - 2\beta^2\frac{1 + 2\omega_2^2}{(\omega_0 - \omega_1)^2}\right]
\end{aligned} \tag{51}$$

We have

$$[R, N_2] = -8N_2^2 - 64\beta^2 H - 16\gamma^2 N_2 - 32\beta^2 N_1 + 16\beta^2(4\alpha^2 - 1) \tag{52}$$

$$[R, N_1] = 4\{N_1, N_2\} + 32\gamma^2 H - 16N_2 + 16\gamma^2 N_1 + 16\gamma^2(2\alpha - 1) \tag{53}$$

$$\begin{aligned}
R^2 &= \frac{8}{3}\{N_2, N_2, L_1\} - \frac{176}{3}N_2^2 + 32\beta^2 N_1^2 + 128\beta^2 H^2 + 64\gamma^2 H N_2 + 128\beta^2 H N_1 \\
&+ 16\gamma^2\{N_1, N_2\} + \left(\frac{128}{3} + 256\alpha^2\beta^2\right)H + \left(64\alpha^2\gamma^2 - \frac{352}{3}\gamma^2\right)N_2 \\
&+ \left(\frac{352}{3} - 128\alpha^2\beta^2\right)N_1 + (128\alpha^4\beta^2 + 128\gamma^4\alpha^2 - \frac{128}{3}\alpha^2\beta^2 - \frac{64}{3}\beta^2 - 48\gamma^2)
\end{aligned}$$

where $\{A, B\} = AB + BA$ and

$$\{A, B, C\} = ABC + ACB + BCA + BAC + CAB + CBA.$$

The integrals of motion N_1, N_2 and H generate a quadratic algebra.

2.3 Interbasis expansion

For a fixed value of energy, we can write the equidistant wave function (7) in terms of the horicyclic ones (19) as

$$\Psi_{n_1 n_2}(x, y) = \sum_{m=0}^{n_1+n_2} W_{n_1 n_2}^{nm}(\alpha, \beta, \gamma)\Psi_{nm}(a, b) \tag{54}$$

where $n_1 + n_2 = n + m$. The connection between the equidistant (a, b) and horicyclic (x, y) coordinates is

$$x = e^b \tanh a, \quad y = e^b \frac{1}{\cosh a}. \tag{55}$$

Going over to the horicyclic coordinates in the left side of expansion (54), then considering the limit $b \to \infty$ and using the asymptotic formula for Laguerre polynomials [7]

$$\lim_{x\to\infty} L_n^{\alpha}(x) \to (-1)^n \frac{x^n}{n!} \tag{56}$$

we see that dependence on b cancels on both sides of (54). Now using the orthogonality condition for the angular wave functions (12) we find the following expression for the interbasis coefficients $W_{n_1 n_2}^{nm}$:

$$W_{n_1 n_2}^{nm} = (-1)^n \sqrt{\frac{m!n!\sqrt{2}\beta(\mu-d-2n-1)\Gamma(\mu+m+1)\Gamma(\mu-n)}{n_1!n_2!\mu\Gamma(n_1+d+1)\Gamma(n_2+d+1)\Gamma(n+d+1)\Gamma(\mu-d-n)}} B_{n_1 n_2}^{nm} \tag{57}$$

where

$$B_{n_1 n_2}^{nm} = \int_{-\infty}^{+\infty} (\sinh a)^{1+2d+2n_1} (\cosh a)^{1-2\mu-2m} P_n^{(d,-\mu)}(\cosh 2a)\ da \tag{58}$$

and $d = \sqrt{2\alpha^2 + 1/4}$. The integral B_{n_1,n_2}^{nm} can be evaluated by expressing the Jacobi polynomial through the hypergeometric function ${}_2F_1$ [7]:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)n!} {}_2F_1\left(\begin{matrix} -n,\ n+\alpha+\beta+1 \\ \beta+1 \end{matrix} \middle| \frac{1+x}{2} \right). \tag{59}$$

Representing the function ${}_2F_1$ as a series we come to a sum of integrals, each of which can be calculated by using the formula [7]:

$$\int_0^{+\infty} (\sinh\tau)^{\alpha} (\cosh\tau)^{-\beta}\ d\tau = \frac{1}{2} B\left(\frac{1+\alpha}{2}, \frac{\beta-\alpha}{2}\right), \quad [Re\alpha > -1, Re(\alpha-\beta) < 0]. \tag{60}$$

We thus obtain

$$W_{n_1 n_2}^{nm} = \frac{(-1)^n}{2} \sqrt{\frac{m!\sqrt{2}\beta(\mu-d-2n-1)(\mu+m)\Gamma(n_1+d+1)}{n!n_1!n_2!\mu\Gamma(n_2+d+1)\Gamma(n+d+1)\Gamma(\mu-n-d)}} \tag{61}$$

$$\frac{\Gamma(\mu)\Gamma(\mu+m-d-n_1-1)}{\sqrt{\Gamma(\mu-n)\Gamma(\mu+m)}} {}_3F_2\left(\begin{matrix} -n,\ n+d-\mu+1,\ 1-\mu-m \\ 1-\mu,\ 2+n_1+d-\mu-m \end{matrix} \middle| 1 \right).$$

Alternatively, by using the formula [10] for the Hahn polynomials $h_n^{(\alpha,\beta)}(x,N)$.

$$h_n^{(\alpha,\beta)}(x,N) = \frac{(-1)^n \Gamma(N)\Gamma(\beta+n+1)}{n!\Gamma(N-n)\Gamma(\beta+1)} {}_3F_2\left(\begin{matrix} -n\ ;\ \alpha+\beta+n+1\ ;\ -x \\ \beta+1\ ;\ 1-N \end{matrix} \middle| 1 \right) \tag{62}$$

we obtain the following expression for the expansion coefficients

$$\begin{aligned} W_{n_1 n_2}^{nm} &= \frac{(-1)^n}{2} \sqrt{\frac{m!n!\sqrt{2}\beta(\mu-d-2n-1)(\mu+m)}{n_1!n_2!\mu\Gamma(n+d+1)\Gamma(\mu-n-d)}} \\ &\cdot \sqrt{\frac{\Gamma(n_1+d+1)\Gamma(\mu-n)}{\Gamma(n_2+d+1)\Gamma(\mu+m)}} \Gamma(\mu+m-d-n_1-n-1) \\ &\cdot h_n^{(d,-\mu)}(\mu+m+1, \mu+m-d-n_1-1), \end{aligned} \tag{63}$$

in terms of Hahn polynomials.

3 Second Potential.

The second considered potential is

$$V_2 = \frac{\alpha^2}{\omega_2^2} + \gamma^2 \frac{\omega_0 \omega_1}{(\omega_0^2+\omega_1^2)^2} + (\alpha^2-\beta^2)\frac{\omega_0^2-\omega_1^2}{(\omega_0^2+\omega_1^2)^2} \tag{64}$$

where α, β and γ are positive constants. The corresponding Schrödinger equation admits separable solutions in two coordinate systems: equidistant and semi–hyperbolic.

3.1 Solutions of the Schrödinger equation

2.1 Equidistant coordinates. In this coordinate system

$$\omega_0 = \cosh\tau_1 \cosh\tau_2, \quad \omega_1 = \cosh\tau_1 \sinh\tau_2, \quad \omega_2 = \sinh\tau_1, \tag{65}$$

$[\tau_1, \tau_2 \in (-\infty, \infty)]$, the potential V_2 has the form

$$V_2(\tau_1,\tau_2) = -\frac{\alpha^2}{\sinh^2\tau_1} + \frac{1}{\cosh^2\tau_1}\frac{\alpha^2-\beta^2+\gamma^2\cosh\tau_2\sinh\tau_2}{(\cosh^2\tau_2+\sinh^2\tau_2)^2}. \tag{66}$$

After putting

$$\Psi(\tau_1,\tau_2) = (\cosh\tau_1)^{-1/2} S(\tau_1) Z(\tau_2) \tag{67}$$

we arrive at two equations:

$$\frac{d^2S}{d\tau_2^2} + \left[-\mu^2 - \frac{2(\alpha^2-\beta^2)+\gamma^2\sinh(2\tau_2)}{\cosh^2(2\tau_2)}\right] S = 0 \tag{68}$$

$$\frac{d^2S}{d\tau_1^2} + \left[2E - \frac{1}{4} + \frac{\mu^2-\frac{1}{4}}{\cosh^2\tau_1} - \frac{2\alpha^2}{\sinh^2\tau_1}\right] S = 0 \tag{69}$$

where μ is the equidistant separation constant.

Let us consider the first equation (68). The substituon $x = \sinh 2\tau_2$ transforms this equation to

$$4(1+x^2)\frac{d^2S}{dx^2} + 4x\frac{dS}{dx} + \left[-\mu^2 + \frac{2(\beta^2-\alpha^2)-\gamma^2 x}{(1+x^2)}\right] S = 0 \tag{70}$$

where the physical region is $x \in (-\infty, \infty)$. The equation (68) has three regular singularities in the point $x = -i, i, \infty$ and may be solved in term of hypergeometric functions. The solution of the equation (68) for a large $|x|$ can be written as:

$$S(x) = A_1 (x-i)^{-\frac{b+\mu}{2}-\frac{1}{4}}(x+i)^{\frac{b}{2}+\frac{1}{4}}\, {}_2F_1\left(\frac{a+b+1+\mu}{2}, \frac{b-a+1+\mu}{2}; 1+\mu; \frac{2i}{i-x}\right)$$

$$+ A_2 (x-i)^{-\frac{b-\mu}{2}-\frac{1}{4}}(x+i)^{\frac{b}{2}+\frac{1}{4}}\, {}_2F_1\left(\frac{a+b+1-\mu}{2}, \frac{b-a+1-\mu}{2}; 1-\mu; \frac{2i}{i-x}\right) \tag{71}$$

with

$$a^2 = (b^2)^* = \frac{2\beta^2 - 2\alpha^2 + 1 - i\gamma^2}{4}. \tag{72}$$

Let the separation constant μ be a positive number (the equation (71) is symmetric with respect to the replacement $\mu \to -\mu$). Then the second term in formula (71) behaves like $|x|^{\frac{\mu}{2}}$ at ∞ and must be omitted. Thus for $S(x)$ we obtain

$$S(x) = A\,(x-i)^{-\frac{b+\mu}{2}-\frac{1}{4}}(x+i)^{\frac{b}{2}+\frac{1}{4}}\,{}_2F_1\left(\frac{a+b+1+\mu}{2}, \frac{b-a+1+\mu}{2}; \mu+1; \frac{2i}{i-x}\right) \tag{73}$$

The hypergeometric function in equation (73) converges if x lies out of the circle C on Fig.1,

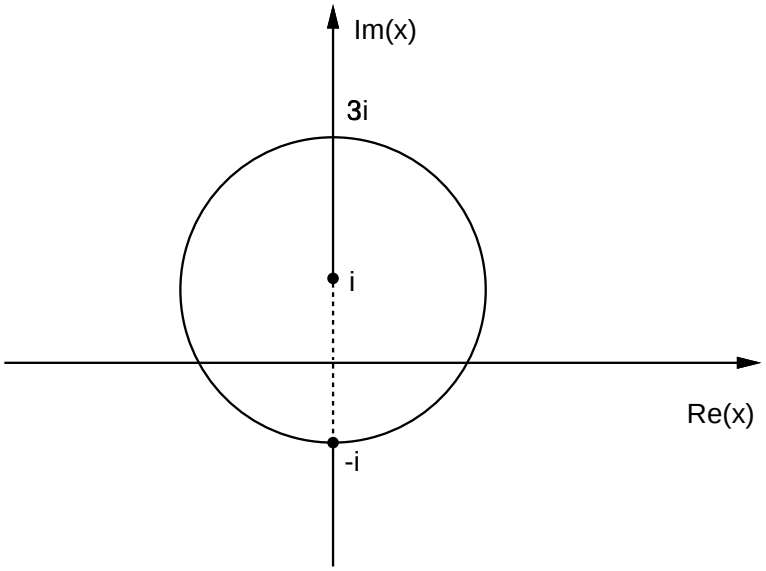

Figure 1: Domain of Convergence

defined by $|i - x| = 2$, and converges on the circle C with the condition $Re(b) < 0$. The function $S(x)$ exists everywhere inside C except the interval $x \in [-i, i]$ (see 1), since the hypergeometric function in (73) has a cut along the argument $\frac{2i}{i-x} \in [1, \infty)$). That means that the solution (73) along the real axes inside C in general is not a continuous function and may have a jump at the point $x = 0$. Let us now consider the analytic continuation of (73) inside the circle C

$$\begin{aligned} S(x) &= A\Big\{(x-i)^{\frac{a}{2}+\frac{1}{4}}(x+i)^{\frac{b}{2}+\frac{1}{4}}\frac{\Gamma(\mu+1)\Gamma(-a)}{\Gamma(\frac{b-a+1+\mu}{2})\Gamma(\frac{-b-a+1+\mu}{2})(2i)^{\frac{a+b+1+\mu}{2}}} \\ &\cdot\ {}_2F_1\left(\frac{a+b+1+\mu}{2}, \frac{a+b+1-\mu}{2}; a+1; \frac{i-x}{2i}\right) \\ &+\ (x-i)^{-\frac{a}{2}+\frac{1}{4}}(x+i)^{\frac{b}{2}+\frac{1}{4}}\frac{\Gamma(\mu+1)\Gamma(a)}{\Gamma(\frac{b+a+1+\mu}{2})\Gamma(\frac{-b+a+1+\mu}{2})(2i)^{\frac{-a+b+1+\mu}{2}}} \end{aligned}$$

$$\cdot \quad {}_2F_1\left(\frac{-a+b+1+\mu}{2}, \frac{-a+b+1-\mu}{2}; -a+1; \frac{i-x}{2i}\right)\Big\}. \tag{74}$$

¿From equation (72) follow two possibilities

$$a = b^*, \qquad a = -b^*. \tag{75}$$

Putting the $a = b^*$ ($Re(a) = Re(b) < 0$) we find that the first term in (74) represents an analytic function, while the second term is discontinuous at $x = 0$. [Note since the both terms in equation (74) transform to each other with replacement $a \to -a$ the choice $a = -b^*$ means that the first term in (74) is discontinuous while the second term is continous at $x = 0$.] Thus the *sufficient* condition for the existence of the continuous solution requires the relation

$$\mu + a + a^* + 1 = -2m, \qquad m = 0, 1, 2..., \left[-\frac{a+a^*+1}{2}\right], \tag{76}$$

so from (72) we have

$$\mu = -2m - 1 + \frac{1}{\sqrt{2}}\sqrt{2\beta^2 - 2\alpha^2 + 1 + \sqrt{(2\beta^2 - 2\alpha^2 + 1)^2 + \gamma^4}}. \tag{77}$$

Finally, the orthonormalized eigenfunction of equation (68) may be written in the form

$$\begin{aligned} S(\tau_2) &= (-1)^{\frac{3m}{2}}\Gamma(-a)\sqrt{\frac{(-2m-a-a^*-1)\Gamma(-m-a^*)}{\pi m! 2^{a+a^*+1}\Gamma(-m-a)\Gamma(-m-a-a^*)}} \\ &\cdot \quad (1 + i\sinh 2\tau_2)^{\frac{a}{2}+\frac{1}{4}}(1 - i\sinh 2\tau_2)^{\frac{a^*}{2}+\frac{1}{4}} \\ &\cdot \quad {}_2F_1\left(-m, m+a+a^*+1; a+1; \frac{1+i\sinh 2\tau_2}{2}\right) \\ &= (-1)^{\frac{m}{2}}\sqrt{\frac{(-2m-a-a^*-1)m!\Gamma(-m-a)\Gamma(-m-a^*)}{\pi 2^{a+a^*+1}\Gamma(-m-a-a^*)}} \\ &\cdot \quad (1 + i\sinh 2\tau_2)^{\frac{a}{2}+\frac{1}{4}}(1 - i\sinh 2\tau_2)^{\frac{a^*}{2}+\frac{1}{4}} P_m^{(a,a^*)}(-i\sinh 2\tau_2) \end{aligned} \tag{78}$$

where

$$\begin{aligned} a = \frac{1}{2^{\frac{3}{2}}}\Big\{ &- \sqrt{\sqrt{(2\beta^2 - 2\alpha^2 + 1)^2 + \gamma^4} + 2\beta^2 - 2\alpha^2 + 1} \\ &+ i\sqrt{\sqrt{(2\beta^2 - 2\alpha^2 + 1)^2 + \gamma^4} - (2\beta^2 - 2\alpha^2 + 1)}\Big\}. \end{aligned}$$

The second equation (69) is quite like (9) and has a solution:

$$\begin{aligned} Z(\tau_1) &\equiv S_n^{(\alpha,\mu)}(\tau_1) = \sqrt{\frac{2(\mu - \sqrt{2\alpha^2 + 1/4} - 2n - 1)\Gamma(\mu - n)n!}{\Gamma(\mu - \sqrt{2\alpha^2 + 1/4} - n)\Gamma(1 + n + \sqrt{2\alpha^2 + 1/4})}} \\ &\times \quad (\sinh\tau_1)^{\frac{1}{2}+\sqrt{2\alpha^2+1/4}}(\cosh\tau_1)^{\frac{1}{2}-\mu} P_n^{(\alpha,-\mu)}(\cosh 2\tau_1) \end{aligned} \tag{79}$$

with $n = 0, 1, ...$

The quantized energy is

$$E = -\frac{1}{2}(\mu - \sqrt{2\alpha^2 + 1/4} - 2n - 1)^2 + \frac{1}{8} = -\frac{1}{2}\Bigg\{2N + 2 + \sqrt{2\alpha^2 + 1/4} - \frac{1}{\sqrt{2}}\sqrt{2\beta^2 - 2\alpha^2 + 1 + \sqrt{(2\beta^2 - 2\alpha^2 + 1)^2 + \gamma^4}}\Bigg\}^2 + \frac{1}{8} \tag{80}$$

where $N = n + m$ is the principal quantum number and the bound state occurs for

$$0 \leq N \leq \left[\frac{1}{\sqrt{8}}\sqrt{2\beta^2 - 2\alpha^2 + 1 + \sqrt{(2\beta^2 - 2\alpha^2 + 1)^2 + \gamma^4}} - \frac{1}{2}\sqrt{2\alpha^2 + 1/4} - 1\right]. \tag{81}$$

The additional operator describing this coordinate system is

$$L_1\Psi_{nm}(\tau_1, \tau_2) \equiv \left[K_3^2 - 2(\alpha^2 - \beta^2)\left(\frac{\omega_0^2 - \omega_1^2}{\omega_0^2 + \omega_1^2}\right)^2 - 2\gamma^2\frac{\omega_0\omega_1(\omega_0^2 - \omega_1^2)}{(\omega_0^2 + \omega_1^2)^2}\right]\Psi_{nm}(\tau_1, \tau_2)$$

$$= \left\{2m + 1 - \frac{1}{\sqrt{2}}\sqrt{2\beta^2 - 2\alpha^2 + 1 + \sqrt{(2\beta^2 - 2\alpha^2 + 1)^2 + \gamma^4}}\right\}^2\Psi_{nm}(\tau_1, \tau_2). \tag{82}$$

2.2 Semi–hyperbolic coordinates. Here

$$\begin{aligned}
\omega_0^2 &= -\frac{(\mu - e_3)(\nu - e_3)}{2[(e_3 - a)^2 + b^2]} + \frac{1}{2} - \frac{1}{2b}\left[\frac{[(\mu - a)^2 + b^2][(\nu - a)^2 + b^2]}{(e_3 - a)^2 + b^2}\right]^{1/2} \\
\omega_1^2 &= \frac{(\mu - e_3)(\nu - e_3)}{2[(e_3 - a)^2 + b^2]} - \frac{1}{2} - \frac{1}{2b}\left[\frac{[(\mu - a)^2 + b^2][(\nu - a)^2 + b^2]}{(e_3 - a)^2 + b^2}\right]^{1/2} \\
\omega_2^2 &= -\frac{(\mu - e_3)(\nu - e_3)}{(e_3 - a)^2 + b^2}
\end{aligned} \tag{83}$$

$[\nu < e_3 < \mu]$, where $\sinh 2f = (e_3 - a)/b$ and $2f$ is the distance between the focii of the semi-hyperbolas and the bases of their equidistants [6].

If we change variables according to

$$\omega_0 = (s_1 + s_2)/\sqrt{2}, \quad \omega_1 = -i(s_1 - s_2)/\sqrt{2}, \quad \omega_2 = -is_3, \tag{84}$$

the Schrödinger equation becomes

$$\frac{1}{2}\left[\left(s_1\frac{\partial}{\partial s_2} - s_2\frac{\partial}{\partial s_1}\right)^2 + \left(s_1\frac{\partial}{\partial s_3} - s_3\frac{\partial}{\partial s_1}\right)^2 + \left(s_3\frac{\partial}{\partial s_2} - s_2\frac{\partial}{\partial s_3}\right)^2\right]\Psi - \left[E + \frac{1}{2}\left(\frac{k_1^2 - \frac{1}{4}}{s_1^2} + \frac{k_2^2 - \frac{1}{4}}{s_2^2} + \frac{k_3^2 - \frac{1}{4}}{s_3^2}\right)\right]\Psi = 0 \tag{85}$$

with

$$\frac{1}{2}(k_1^2 - \frac{1}{4}) = \frac{1}{4}(\beta^2 - \alpha^2) - \frac{i}{8}\gamma^2, \quad \frac{1}{2}(k_2^2 - \frac{1}{4}) = \frac{1}{4}(\beta^2 - \alpha^2) + \frac{i}{8}\gamma^2, \quad \frac{1}{2}(k_3^2 - \frac{1}{4}) = \alpha^2.$$

Noting

$$\omega_0^2 - \omega_1^2 - \omega_2^2 = s_1^2 + s_2^2 + s_3^2 = 1$$

and considering eq.(85) we see that the problem we wish to solve using the real coordinates ω_0, ω_1 and ω_2 is a real case of the corresponding problem on the sphere with coordinates s_1, s_2, s_3 and energy $\varepsilon = -E$.

Inverting the relations (84) we have

$$s_1 = (\omega_0 + i\omega_1)/\sqrt{2}, \quad s_2 = (\omega_0 - i\omega_1)/\sqrt{2}, \quad s_3 = i\omega_2.$$

Now choose elliptic coordinates on the complex sphere according to

$$s_1^2 = \frac{(\mu - e_1)(\nu - e_1)}{(e_1 - e_2)(e_1 - e_3)}, \quad s_2^2 = \frac{(\mu - e_2)(\nu - e_2)}{(e_2 - e_1)(e_2 - e_3)}, \quad s_3^2 = \frac{(\mu - e_3)(\nu - e_3)}{(e_3 - e_2)(e_3 - e_1)}.$$

This choice of real coordinates μ, ν will work for the real coordinates $\omega_k, k = 0, 1, 2$ if we take $e_1 = e_2^* = a + ib, a, b$ real and $\nu < e_3 < \mu$.

In terms of the coordinates μ and ν the Schrödinger equation has the form:

$$\begin{aligned}
\frac{4}{(\mu-\nu)}\Big\{&(\mu - e_2^*)(\mu - e_2)(\mu - e_3)\Big[\frac{\partial^2\Psi}{\partial\mu^2} + \frac{1}{2}\Big(\frac{1}{\mu - e_2^*} + \frac{1}{\mu - e_2} + \frac{1}{\mu - e_3}\Big)\frac{\partial\Psi}{\partial\mu}\Big] \\
&-(\nu - e_2^*)(\nu - e_2)(\nu - e_3)\Big[\frac{\partial^2\Psi}{\partial\nu^2} - \frac{1}{2}\Big(\frac{1}{\nu - e_2^*} + \frac{1}{\nu - e_2} + \frac{1}{\nu - e_3}\Big)\frac{\partial\Psi}{\partial\nu}\Big]\Big\} \\
&+\Big[(k_1^2 - \frac{1}{4})\frac{(e_2^* - e_2)(e_2^* - e_3)}{(\mu - e_2^*)(\nu - e_2^*)} + (k_2^2 - \frac{1}{4})\frac{(e_2 - e_2^*)(e_2 - e_3)}{(\mu - e_2)(\nu - e_2)} \\
&+(k_3^2 - \frac{1}{4})\frac{(e_3 - e_2)(e_3 - e_2^*)}{(\mu - e_3)(\nu - e_3)}\Big]\Psi = -2E\Psi. \qquad (86)
\end{aligned}$$

The separation equations are:

$$\begin{aligned}
&(\rho - e_2^*)(\rho - e_2)(\rho - e_3)\Big[\frac{d^2\Psi}{d\rho^2} + \frac{1}{2}\Big(\frac{1}{\rho - e_2^*} + \frac{1}{\rho - e_2} + \frac{1}{\rho - e_3}\Big)\frac{d\Psi}{d\rho}\Big] \\
&-\frac{1}{4}\Big[(k_1^2 - \frac{1}{4})\frac{(e_2^* - e_2)(e_2^* - e_3)}{(\rho - e_2^*)} + (k_2^2 - \frac{1}{4})\frac{(e_2 - e_2^*)(e_2 - e_3)}{(\rho - e_2)} \\
&+ (k_3^2 - \frac{1}{4})\frac{(e_3 - e_2)(e_3 - e_2^*)}{(\rho - e_3)} - 2E\rho + \lambda\Big]\psi(\rho) = 0 \qquad (87)
\end{aligned}$$

where $\rho = \mu, \nu$. The operator L_2 with eigenvalue λ is

$$\begin{aligned}
L_2\Psi = \frac{-4}{(\mu-\nu)}\Big\{&\nu(\mu - e_1)(\mu - e_2)(\mu - e_3)\Big[\frac{\partial^2\Psi}{\partial\mu^2} + \frac{1}{2}\Big(\frac{1}{\mu - e_2^*} + \frac{1}{\mu - e_2} + \frac{1}{\mu - e_3}\Big)\frac{\partial\Psi}{\partial\mu}\Big] \\
&-\mu[(\nu - e_1)(\nu - e_2)(\nu - e_3)\Big[\frac{\partial^2\Psi}{\partial\nu^2} + \frac{1}{2}\Big(\frac{1}{\nu - e_2^*} + \frac{1}{\nu - e_2} + \frac{1}{\nu - e_3}\Big)\frac{\partial\Psi}{\partial\nu}\Big]\Big\} \\
-\Big[&(k_1^2 - \frac{1}{4})\frac{(e_2^* - e_2)(e_2^* - e_3)}{(\mu - e_2^*)(\nu - e_2^*)}(\mu + \nu - e_2^*) + (k_2^2 - \frac{1}{4})\frac{(e_2 - e_2^*)(e_2 - e_3)}{(\mu - e_2)(\nu - e_2)}(\mu + \nu - e_2) \\
&+ (k_3^2 - \frac{1}{4})\frac{(e_3 - e_2)(e_3 - e_2^*)}{(\mu - e_3)(\nu - e_3)}(\mu + \nu - e_3)\Big]\Psi. \qquad (88)
\end{aligned}$$

In order to find the bound state solutions of this system in semi–hyperbolic systems we first observe the identity

$$\begin{aligned}\frac{s_1^2}{\theta_j-e_2^*}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3} &= \frac{(\omega_0^2-\omega_1^2)(\theta_j-a)-2\omega_0\omega_1 b}{(\theta_j-a)^2+b^2}-\frac{\omega_2^2}{\theta_j-e_3}\\ &= \frac{(\mu-\theta_j)(\nu-\theta_j)}{(\theta_j-e_2^*)(\theta_j-e_2)(\theta_j-e_3)}.\end{aligned} \tag{89}$$

If we then look for solutions of the form

$$\Psi=\prod_{\ell=1}^{3}s_\ell^{k_\ell+\frac{1}{2}}\prod_{j=1}^{N}\Big(\frac{s_1^2}{\theta_j-e_2^*}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}\Big), \tag{90}$$

we see that the corresponding zeros satisfy the equations

$$\frac{k_1+1}{\theta_m-e_2^*}+\frac{k_2+1}{\theta_m-e_2}+\frac{k_3+1}{\theta_m-e_3}+\sum_{j\neq m}^{N}\frac{2}{(\theta_m-\theta_j)}=0. \tag{91}$$

For the energy E we have

$$E=-\frac{1}{2}(2N+2+k_1+k_2+k_3)^2+\frac{1}{8}, \tag{92}$$

which coincides with the formula (73), note (86). For the separation constant λ we obtain:

$$\begin{aligned}\lambda &= -2[k_1(e_2+e_3)+k_2(e_2^*+e_3)+k_3(e_2+e_2^*)]-2[e_3k_1k_2+e_2k_1k_3+e_2^*k_2k_3]\\ &- \frac{3}{2}(e_2^*+e_2+e_3)-4e_2e_3(k_1+1)\sum_{m=1}^{q}\frac{1}{(\theta_m-e_2^*)}-e_2^*e_3(k_2+1)\sum_{m=1}^{q}\frac{1}{(\theta_m-e_2)}\\ &- 4e_2e_2^*(k_3+1)\sum_{m=1}^{q}\frac{1}{(\theta_m-e_3)}.\end{aligned} \tag{93}$$

In terms of variables w_i the total wave function is written:

$$\Psi=\Big(\frac{\omega_0+i\omega_1}{\sqrt{2}}\Big)^{k_1+\frac{1}{2}}\Big(\frac{\omega_0-i\omega_1}{\sqrt{2}}\Big)^{k_2+\frac{1}{2}}(i\omega_2)^{k_3+\frac{1}{2}}\prod_{j=1}^{N}\Big[\frac{(\omega_0^2-\omega_1^2)(\theta_j-a)-2\omega_0\omega_1 b}{(\theta_j-a)^2+b^2}-\frac{\omega_2^2}{\theta_j-e_3}\Big].$$

The algebra of second order symmetries for this potential is generated by the operators

$$L_{jk}=(s_j\partial_{s_k}-s_k\partial_{s_j})^2+(\frac{1}{4}-k_j^2)\frac{s_k^2}{s_j^2}+(\frac{1}{4}-k_k^2)\frac{s_j^2}{s_k^2} \tag{94}$$

for $j,k=1,2,3$ and $j\neq k$. The Hamiltonian of the system is expressed in terms of L_{jk} as:

$$H=\frac{1}{2}(L_{12}+L_{13}+L_{23})-\frac{1}{2}\sum_{i=1}^{3}k_i^2+\frac{3}{4}. \tag{95}$$

The relevant generators in the real case we are considering are then

$$L_{12} = -K_3^2 + \left(\frac{1}{4} - k_1^2\right)\left(\frac{\omega_0 - i\omega_1}{\omega_0 + i\omega_1}\right)^2 + \left(\frac{1}{4} - k_2^2\right)\left(\frac{\omega_0 + i\omega_1}{\omega_0 - i\omega_1}\right)^2 \quad (96)$$

$$L_{13} = \frac{1}{2}(M_1 - iK_2)^2 + \left(\beta^2 - \alpha^2 - \frac{i}{2}\gamma^2\right)\frac{\omega_2^2}{(\omega_0 + i\omega_1)^2} + \alpha^2\frac{(\omega_0 + i\omega_1)^2}{w_2^2} \quad (97)$$

$$L_{23} = \frac{1}{2}(M_1 + iK_2)^2 + \left(\beta^2 - \alpha^2 + \frac{i}{2}\gamma^2\right)\frac{\omega_2^2}{(\omega_0 - i\omega_1)^2} + \alpha^2\frac{(\omega_0 - i\omega_1)^2}{w_2^2}. \quad (98)$$

The commutation relations and resulting quadratic algebra can then be deduced from the relations for the complex forms in terms of the L_{ij}. It is easy to show that the additional integrals of motion, corresponding to the separation in equidistant and semi–hyperbolic coordinates can be written as

$$L_1 = -L_{12} + \beta^2 - \alpha^2 \quad (99)$$

and

$$L_2 = e_3L_{12} + e_2L_{13} + e_1L_{32} - k_1^2(e_2 + e_3 - e_1) - k_2^2(e_1 + e_3 - e_2) - k_3^2(e_1 + e_2 - e_3) + \frac{1}{4}(e_1 + e_2 + e_3). \quad (100)$$

The algebra for the operators (99), (100) is found in the work [11].

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