

SPATIAL HIDDEN SYMMETRIES IN PATTERN FORMATION

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Abstract.

Partial differential equations that are invariant under euclidean transformations are traditionally used as models in pattern formation. These models are often posed on finite domains (in particular, multidimensional rectangles). Defining the appropriate boundary conditions is often a very subtle problem. On the other hand, there is pressure to choose the boundary conditions which are most attractive to mathematical treatment. Geometrical shapes and mathematically friendly boundary conditions usually imply spatial symmetry. The presence of symmetry effects which are 'hidden' in the boundary conditions was noticed and carefully investigated by several researchers during the past 15 years. Here we review developments in this subject and introduce a unifying formalism to uncover spatial *hidden symmetries* (hidden translations and hidden rotations) in multidimensional rectangular domains with Neumann boundary conditions.

1. Introduction. The diversity of natural patterns is fascinating. The design of laboratory experiments which reproduce such patterns has attracted many researchers from various fields of science and engineering. It has become traditional to model pattern forming systems with partial differential equations. For example, reaction-diffusion equations are well established models in chemistry [42] and biology [46, 36, 38], the Boussinesq equations [37] model thermal convection in a thin layer of fluid, the Kuramoto-Sivashinsky [45] equations model instability in flames. These models are often posed on finite domains and need a specification for the boundary conditions. Geometrical shapes are favourite to mathematical analysis and computations, and some boundary conditions are more attractive than others. Geometrical domains and mathematically friendly boundary conditions usually imply *symmetry*. Symmetry is a very important ingredient in a pattern formation model. It is essential that the model exhibits the 'right' symmetries. Here we address the problem of spatial symmetries which may be 'hidden' in the boundary conditions. They may be wanted. Or maybe not. In either case it is important to understand their effects. Problems of unwanted spatial *hidden symmetries* in models may be resolved by choosing domains and boundary conditions which do not support them.

Typically a pattern forms as a bifurcation parameter is varied across a critical value. A technique generally used in this situation is to reduce the dynamics to a simpler set of equations whose general form depends on symmetry considerations. There are essentially two reduction formalisms, which address different issues in pattern formation:

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1. The first formalism extracts the universal properties of regular patterns. A small number of Fourier modes provides a good characterization of such patterns [20]. In this case, the dynamics reduce to a finite set of ordinary differential equations (center manifold reduction [26]). The variables in the equations are the amplitudes of the relevant modes. A Liapunov-Schmidt reduction [20] is an alternative method, leading to a set of algebraic equations that retain information about the equilibria of the system, but not the dynamics.
2. The second formalism is a natural scheme which additionally extracts pattern irregularities by considering that the spectrum of growth rates is continuous. In this case, the amplitudes introduced are the envelopes of the various mode shapes. The envelopes depend on space and time and so are governed by a set of coupled partial differential equations (Newell-Whitehead-Segel [41, 44]). Further considerations about the symmetries of this type of reduced equations can be found in [27].

This article is directed towards the first formalism. In particular, we expose a group theoretical approach which provides information about the general form of the Liapunov-Schmidt reduced equations.

The partial differential equations modelling pattern forming systems are often invariant under some group Γ of euclidean transformations (translations, rotations and reflections of the physical space). Any PDE that is posed on a domain \mathcal{D} and is invariant under the group Γ will inherit those symmetries in Γ that preserve the domain and the boundary conditions. The same invariance will hold for any reduced bifurcation equations derived from the PDE. Naively, one would not expect a system modelled by a PDE on a bounded domain to inherit any symmetries that do not leave the domain invariant (in particular, translations and certain rotations). However, when only the symmetries of the domain are taken into account, it is not unusual to find unexpected conjugacies and unexpected highly developed patterns.

It is now well known that many of these degeneracies can be created by *hidden symmetries* that leave invariant the PDE and the boundary conditions, but not the domain. Given a PDE on a bounded domain \mathcal{D} with certain boundary conditions (e.g. an n -dimensional rectangle with Neumann boundary conditions), the obvious conjugates to a solution u are obtained by acting on u with the euclidean transformations that leave invariant the PDE, the domain, and the boundary conditions. In order to investigate the presence of more symmetries, an abstract extension to a larger domain without boundaries (e.g. the whole euclidean space \mathbb{R}^n) is constructed. The original problem is embedded in the larger abstract problem. The abstract problem in the unbounded domain allows more freedom for symmetry transformations to act. Therefore, the extended problem will typically have a larger class of solutions, possibly with finer structure. From

this large class we select the solutions which satisfy the original boundary conditions. The interesting observation is that this method may lead to solutions that coincide neither with u nor with its image under the euclidean transformations that leave the domain invariant.

This article reviews the effects of spatial hidden symmetries in rectangular domains of arbitrary dimension. It contains a brief description of hidden translational symmetries [11, 21, 22, 23, 24, 6], hidden rotational symmetries [9, 25], and forced symmetry breaking of the boundary conditions [3, 4, 8, 10, 29]. A generalization to a larger class of PDEs and more complicated domains was established in [16]. In particular, the authors described some of the possible effects that hidden symmetries can have in hemispherical domains with Neumann boundary conditions on the equator. Another important generalization was established in [15, 18], where it was shown that the effects of hidden symmetries are retained by discretizations of reaction-diffusion equations in domains which are of a rectangular shape (in particular interval and square). Despite all the possible generalizations, the bulk of spatial hidden symmetry research was developed for PDEs in rectangular domains, and this is what the remainder of this article will elaborate on.

Other types of hidden symmetries (associated with time/space translations, Galilean transformations, etc.) exhibited by Hamiltonian systems have also been discovered (see for example [40]). These symmetries, which correspond to ‘hidden’ conservation laws, are not addressed in this article, and are mentioned here mostly to prevent possible confusion. The geometrical approach used here is directed towards symmetries which are purely spatial.

2. Hidden Translational Symmetries. The subtle effects that the boundary conditions induce on the genericity of bifurcation problems were first noticed by Fujii *et al* [17] and formalized by Armbruster & Dangelmayr [1, 12] for corank two steady-state bifurcations of a reaction-diffusion equation on an interval with Neumann boundary conditions. Crawford *et al* [11] pointed out that such degeneracies, which are associated with hidden translational symmetries, may also be present in the simplest corank one bifurcations. The effects of hidden translational symmetries were explored in considerable detail for steady-state and Hopf bifurcations of elliptic PDEs in n -dimensional rectangles by Gomes & Stewart [21, 22, 23, 24]. These results are a very natural generalization of those referenced above for the interval.

We proceed with a brief description of the method in the context of steady-state bifurcations in the rectangular domain

$$(2.1) \quad \mathcal{D} = [0, d_1] \times [0, d_2] \times \cdots \times [0, d_n].$$

Let $\mathcal{P} : \mathcal{X} \times \mathbb{R} \rightarrow \mathcal{Y}$ denote an elliptic operator (e.g. reaction-diffusion) between function spaces \mathcal{X} and \mathcal{Y} that allow Fourier expansions and the

use of the implicit function theorem. We are interested in steady solutions of the PDE

$$(2.2) \quad u_t + \mathcal{P}(u, \lambda) = 0,$$

which satisfy the Neumann boundary conditions

$$(2.3) \quad \frac{\partial u}{\partial x_j} = 0 \quad \text{for} \quad x_j = 0, d_j$$

for $j = 1, 2, \dots, n$. The quantity $\lambda \in \mathbb{R}$ in equation (2.2) is a bifurcation parameter and, for simplicity, $u : \mathcal{D} \rightarrow \mathbb{R}$ is restricted to being a scalar function (often when considering bifurcations the general situation of $u : \mathcal{D} \rightarrow \mathbb{R}^m$ being vector-valued reduces to this case [31]). We assume that \mathcal{P} is equivariant under an action of the euclidean group $E(n)$ consisting of a semidirect sum of the group of rotations and reflections $O(n)$ with the group of translations \mathbb{R}^n (the method adapts to cases where \mathcal{P} is invariant under a subgroup of the euclidean group, rather than the whole euclidean group [31]). The action of an element $\gamma \in E(n)$ on u is defined as

$$(2.4) \quad (\gamma \cdot u)(x) = u(\gamma^{-1} \cdot x).$$

We assume that there is a trivial solution $u = 0$, undergoing a generic steady-state bifurcation when λ is varied across a critical value λ_c . More formally, the linearization

$$(2.5) \quad L_{\lambda_c} = (d\mathcal{P})_{(0, \lambda_c)}$$

has a nontrivial kernel and the eigenvalues that go through zero do so with nonzero speed.

The reflections across the hyperplanes $x_j = d_j/2$, whose action on the domain \mathcal{D} is

$$(2.6) \quad \mu_j : x_j \rightarrow d_j - x_j.$$

are always symmetries of \mathcal{D} . In the general case of all d_j being different the group generated by the μ_j , is the symmetry group of \mathcal{D} . The boundary conditions (2.3) and equivariance of \mathcal{P} under the euclidean group $E(n)$ allow the use of the subgroup Z_2^n of $E(n)$ generated by the reflections

$$(2.7) \quad \nu_j : x_j \rightarrow -x_j$$

to extend any solution u of $\mathcal{P}(u, \lambda) = 0$ to the larger rectangle

$$(2.8) \quad \mathcal{D}' = [-d_1, d_1] \times [-d_2, d_2] \times \dots \times [-d_n, d_n]$$

by reflection across the boundaries

$$(2.9) \quad u(\nu_j \cdot x) = u(x).$$

Here the n -torus $T^n = \mathbb{R}^n / \mathcal{L}$ represents the effective action of the translations on the spaces $\mathcal{X}_{\mathcal{L}}$ and $\mathcal{Y}_{\mathcal{L}}$. The holohedry H of the lattice \mathcal{L} is the group of rotations and reflections that leave the lattice invariant.

By construction, Z_2^n is a subgroup of H . In fact, almost all rectangular domains lead to

$$(2.14) \quad H = Z_2^n.$$

In the case of equalities between the d_j , the group H will include further rotations.

More subtle is the presence of rotational symmetries not associated with equalities of the d_j . Such degeneracies are due to hidden rotations which we now briefly describe (a more rigorous treatment will be given in the next section). Often one sees patterns that possess more structure than the domain. Such structure may be captured by a refinement of the lattice \mathcal{L} . The specified boundary conditions on \mathcal{D} and \mathcal{D}' are satisfied by functions which are periodic with respect to a finer lattice. Lattice refinements may lead naturally to hidden rotational symmetries as will become clear in the next section. Most rectangular domains do not support such degeneracies, and in this sense we call them *generic rectangular domains*. However, laboratory experiments and numerical simulations are often performed in special rectangles and hidden rotational symmetries are a reality. This issue is the purpose of the next section.

The remaining of this section is concerned with generic rectangular domains. In this case, $H = Z_2^n$, the operator \mathcal{P} in (2.12) is $Z_2^n \times T^n$ -equivariant, and the kernel of the linearization L_{λ_c} in the space of \mathcal{L} -periodic functions is generically an irreducible representation of the same group [20]. Finally, by imposing invariance under the group Z_2^n we obtain the solutions that satisfy Neumann boundary conditions on \mathcal{D} and whose bifurcation is associated with that of the solution u that was originally extended. The resulting subspace, $\text{Fix}(Z_2^n)$, is an abstraction of the kernel

$$(2.15) \quad V = \ker(L_{\lambda_c})$$

in the space of functions satisfying the original Neumann boundary conditions.

The method just described may lead to solutions that do not coincide with the original solution u nor with its images under the symmetries μ_j of the domain. In particular, the simplest periodic solutions satisfying the Neumann boundary conditions (2.3) are of the form

$$(2.16) \quad u_k(x) = \cos((k_1/d_1)x_1\pi) \cos((k_2/d_2)x_2\pi) \cdots \cos((k_n/d_n)x_n\pi)$$

where the k_j are nonnegative integers. The vector

$$(2.17) \quad k = \left(\frac{k_1}{d_1}, \frac{k_2}{d_2}, \dots, \frac{k_n}{d_n} \right)$$

is called *wave vector* and its absolute value $|k|$ is the *wave number*. It can be verified that the reflections μ_j act on the function u_k as

$$(2.18) \quad (\mu_j \cdot u_k)(x) = \begin{cases} -u_k(x) & \text{if } k_j \text{ is odd,} \\ u_k(x) & \text{if } k_j \text{ is even.} \end{cases}$$

If the μ_j and their composites were the only symmetries of the problem, the conjugacy classes would sometimes have one and sometimes two elements. When the symmetry group $Z_2^n + T^n$ is used appropriately it turns out that all conjugacy classes have two elements. The action of this group induces a nontrivial reflection in the eigenspace that acts as a permutation in the conjugacy class $\{u_k, -u_k\}$. An illustration in two dimensions is shown in figure 2.

The eigenspace is typically one dimensional, its elements having the form

$$(2.19) \quad u(x) = Au_k(x),$$

where A is the real amplitude. Using the equivariant Liapunov-Schmidt reduction (see Golubitsky *et al* [20] for a general treatment) the problem $\mathcal{P}(u, \lambda) = 0$ is equivalent to the study of zeros of a smooth function of the form

$$(2.20) \quad f(A^2, \lambda)A = 0$$

where the parameter λ is shifted so the critical value for bifurcation is now $\lambda = 0$ (see [11] for a more detailed deduction of this equation). The bifurcation diagram for equation (2.20) is a pitchfork, and we repeat that this is unexpected for eigenfunctions which are even in all x_j .

By choosing the lengths d_j of the domain appropriately, we may have bifurcations with higher corank. A corank two bifurcation corresponds to a situation where the two solutions u_k and u_{-k} bifurcate simultaneously. These degenerate bifurcations may be unfolded by a family of bifurcation problems parametrized by a path in the d_j parameter space. The parameters d_j are called *unfolding parameters* (see Golubitsky *et al* [19, 20]). In this case, Gomes & Stewart [22, 23] define

$$(2.21) \quad K = \max_j k_j \quad L = \max_j \ell_j$$

and show that the reduced bifurcation equations are of the form

$$(2.22) \quad f_1(A^2, B^2, \lambda)A + f_2(A^2, B^2, \lambda)A^{L-1}B^K = 0$$

$$(2.23) \quad g_1(A^2, B^2, \lambda)B + g_2(A^2, B^2, \lambda)A^L B^{K-1} = 0$$

if all k_j have the same parity and all ℓ_j have the same parity, but they take form

$$(2.24) \quad f(A^2, B^2, \lambda)A = 0$$

$$(2.25) \quad g(A^2, B^2, \lambda)B = 0$$

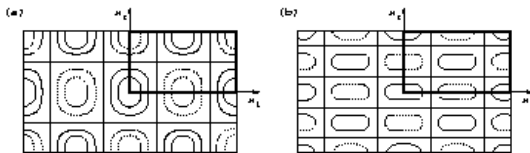


FIG. 2. Contour plots of functions of the form u_2 illustrating a hidden translational symmetry. The domain D is the rectangle limited by the heavy lines. Solid (dotted) lines correspond to positive (negative) values of u_2 . The illustration (a) corresponds to $(k_1, k_2) = (2, 1)$, and it shows no hidden symmetries. All the symmetries of the problem are represented by the symmetries of the domain. A hidden translational symmetry is illustrated in (b) for $(k_1, k_2) = (2, 2)$. Here all the symmetries of the domain leave u_2 invariant. However, by performing a translation of d_1/k_1 on x_1 (or equivalently a translation of d_2/k_2 on x_2), the solution u_2 is mapped to $-u_2$, and the domain does not remain invariant under this transformation.

otherwise. This is a curious outcome of the analysis of Gomes & Stewart [22, 23]: the reduced equations for corank two bifurcations can be characterized in very general terms, independently of the dimension, n , of the domain. Furthermore, Manoel & Stewart [32, 33] established that the singularity theory for these problems can similarly be reduced to the genus defined on a one-dimensional domain by Armbruster & Dangelmayr [1].

In rectangles of dimension n , the parameters d_j provide unfoldings of bifurcations with corank up to $n + 1$. Gomes & Stewart [22, 23] describe how to obtain analogous reduced equations for bifurcations with arbitrary corank. This involves determining the typical form of a map which is equivariant under the 'right' symmetries. The resolution of this problem, which can be achieved analytically for bifurcation of corank one (2.20) and two (2.22, 2.24), relies on algorithmic combinatorial computations for bifurcations of higher corank. This issue will not be pursued further here.

Results in one-dimensional domains were applied to reaction-diffusion models of ecological systems [17]. In two dimensions, these results found a very wide variety of applications: for instance, reaction-diffusion [34, 35, 3, 4], convection [39, 21, 31, 5, 29], solidification of binary fluids [30], Kuramoto-Sivashinsky equation [2], Faraday experiment [7], elastic buckling [28]. The results are equally applicable in higher dimensions like Rayleigh-Bénard convection [22] in a three-dimensional domain with rectangular cross section. The study of three-dimensional patterns is not nearly as developed as in lower dimensions and this is reflected in the demand for this method.

3. Hidden Rotational Symmetries. Ashwin [2] noticed another effect of hidden symmetries in the Kuramoto-Sivashinsky equation in a two-dimensional rectangle with Neumann boundary conditions. He reports

the existence of bifurcations of arbitrarily high corank for some critical domain lengths. This phenomenon is due to number theoretic degeneracies associated with rotational symmetries.

These symmetries are uncovered by applying the extension procedure described in the previous section, which can be summarized in the following six steps:

1. Consider a model consisting of an euclidean invariant elliptic PDE on a rectangular domain \mathcal{D} with Neumann boundary conditions.
2. Extend the problem to the larger rectangle \mathcal{D}' by reflection across the boundaries (these reflections generate the group Z_2^n). Extend to \mathbb{R}^n by periodic replication.
3. Define the lattice \mathcal{L} consisting of the vertices of \mathcal{D}' together with their replicates in \mathbb{R}^n (as in figure 1).
4. Determine the holohedry H of the lattice \mathcal{L} .
5. Construct an irreducible representation of $H \wr T^n$ (for more degenerate bifurcations several of these representations combine nonirreducibly as a direct sum).
6. Restrict the representation to the subspace $\text{Fix}(Z_2^n)$ to obtain an abstraction of the kernel V with the appropriate symmetries.

The previous section was concerned with generic rectangular domains, where no rotational symmetry was present. In those cases, the holohedry group of the lattice \mathcal{L} is $H = Z_2^n$, the group generated by the reflections r_j across $x_j = 0$. This fails when some of d_i are equal, rotational symmetries appearing in the larger holohedry $H \not\subseteq Z_2^n$. There are, however, more subtle cases where $H = Z_2^n$ but rotational symmetries are still detected. For this reason, these are called hidden rotational symmetries.

Consider that when the bifurcation parameter is at the critical value λ_c , the real eigenfunction

$$(3.1) \quad u_{\mathbf{k}}(x) = \cos((k_1/d_1)x_1\pi) \cos((k_2/d_2)x_2\pi) \cdots \cos((k_n/d_n)x_n\pi)$$

is in the kernel V of the linearized Neumann boundary conditions problem. Suppose that we relax the boundary conditions and, as Dionne & Golubitsky [13, 14], define a *plane wave* as the complex-valued function of the form

$$(3.2) \quad w_{\mathbf{k}}(x) = e^{i(\mathbf{k} \cdot \mathbf{x})\pi}$$

where $\mathbf{k} = (k_1/d_1, k_2/d_2, \dots, k_n/d_n)$ is the wave vector as before, and obtain a real eigenfunction of the linearized euclidean invariant problem as

$$(3.3) \quad v_{\mathbf{k}}(x) = zw_{\mathbf{k}}(x) + \bar{z}w_{-\mathbf{k}}(x).$$

Note that by imposing the Neumann boundary conditions (2.3) on $v_{\mathbf{k}}$ we get an eigenfunction of the form $u_{\mathbf{k}}$. The $O(n)$ -equivariance of L_{λ_c} guarantees that if $v_{\mathbf{k}}$ is an eigenfunction, then so is $v_{\mathbf{k}'}$, for every \mathbf{k}' having the same

absolute value as k . These eigenfunctions generate an infinite dimensional function space. Now recall that we have defined a lattice \mathcal{L} . For this lattice the kernel of L_{λ_c} contains at most a finite number of \mathcal{L} -periodic eigenfunctions with wave number $|k|$.

At this point it is convenient to define the dual lattice \mathcal{L}^* as

$$(3.4) \quad \mathcal{L}^* = \{k \in \mathbb{R}^n : (k \cdot \alpha) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{L}\}.$$

Equivalently, \mathcal{L}^* can be defined as the set of all $k \in \mathbb{R}^n$ such that the associated plane wave, v_k , is \mathcal{L} -periodic. It is clear that \mathcal{L}^* is a lattice, and its elements are the wave vectors k that lead to \mathcal{L} -periodic eigenfunctions v_k .

It is also convenient to observe that the action of Z_2^n on v_k can be specified as an action on the wave vector k as

$$(3.5) \quad (r_j \cdot v_k)(x) = v_{(r_j \cdot k)}(x) = v_k(r_j^{-1} \cdot x)$$

where the generators r_j of Z_2^n act on k by reflection across $k_j = 0$ as (2.7).

The construction of the lattice \mathcal{L} implies that if α is an element of \mathcal{L} , then all the vectors in the Z_2^n -orbit of α are also elements of \mathcal{L} . The same is true for any wave vector k in \mathcal{L}^* . Furthermore, since Z_2^n acts on a wave vector k by changing component signs as in (2.7), it is clear that the wave number, $|k|$, is constant along a Z_2^n -orbit. More formally, we define the intersection of \mathcal{L}^* with the circle of radius $|k|$ as

$$(3.6) \quad \mathcal{C}(|k|) = \{k' \in \mathcal{L}^* : |k'| = |k|\}$$

and say that any rectangular domain \mathcal{D} leads to the condition

$$(3.7) \quad \mathcal{C}(|k|) \supset \{(r \cdot k) : r \in Z_2^n\}$$

for every wave vector k in \mathcal{L}^* . Therefore, if v_k is in the kernel of L_{λ_c} , then so is $v_{r \cdot k}$ for every r in the Z_2^n -orbit of k . We say that the dual lattice \mathcal{L}^* is a generic rectangular lattice if and only if given any wave vector k in \mathcal{L}^* , the circle $\mathcal{C}(k)$ coincides with the Z_2^n -orbit of k . More formally, we say that \mathcal{L}^* is a *generic rectangular lattice* if and only if the equality

$$(3.8) \quad \mathcal{C}(|k|) = \{(r \cdot k) : r \in Z_2^n\}$$

is verified for every wave vector k in \mathcal{L}^* . Note that if k and k' are in the same Z_2^n -orbit, then by imposing the Neumann boundary conditions (2.3) on v_k and $v_{k'}$ we obtain the same eigenfunction u_k . This is because functions of the form u_k are Z_2^n -invariant. Consequently, if \mathcal{D} is a generic rectangular domain, then the eigenspace V is generically one dimensional.

Now we make rigorous the generic rectangular domain assumption made in the previous section. Consider a domain \mathcal{D} of the form (2.1) and the associated dual lattice \mathcal{L}^* constructed as above. We say that the

rectangular domain \mathcal{D} is a *generic rectangular domain* if and only if \mathcal{L}^* is a generic rectangular lattice. In other words, the rectangle \mathcal{D} is a generic rectangular domain if and only if given any two wave vectors k and k' in \mathcal{L}^* which have the same wave number

$$(3.9) \quad |k| = |k'|$$

then it is also true that

$$(3.10) \quad \left| \frac{k_x}{d_x} \right| = \left| \frac{k'_x}{d_x} \right|$$

for all $j = 1, 2, \dots, n$. This is the same as saying that if k and k' are on the same circle $\mathcal{C}(k)$, then they are in the same \mathbb{Z}_2^* -orbit.

In two dimensions, it is immediately clear that a square

$$(3.11) \quad \mathcal{D} = [0, 1] \times [0, 1]$$

is not a generic rectangular domain since, for example, the wave vectors $(1, 0)$ and $(0, 1)$ are elements of \mathcal{L}^* (in fact, they generate \mathcal{L}^*) which have the same wave number and do not verify condition (3.10). These two vectors can be mapped onto each other by a suitable rotation of $\pi/2$. Less obvious is the fact that the wave vectors $(5, 0)$ and $(4, 3)$, which also have the same wave number, are connected by a rotation. Crawford [9] calls this a hidden rotational symmetry, which he investigates in considerable detail. See figure 3(a) for an illustration of the symmetries of a square lattice, and figures 4 and 5 for some associated patterns.

More generally, Ashwin [2] proves that the two dimensional rectangular domain

$$(3.12) \quad \mathcal{D} = [0, d_1] \times [0, d_2]$$

supports hidden rotational symmetries if and only if it satisfies the condition $d_1^2/d_2^2 \in \mathbb{Q}$. It is clear that a square belongs to the larger class characterized by Ashwin. It is also clear that this class includes rectangles of aspect ratio $\sqrt{3}$, such as

$$(3.13) \quad \mathcal{D} = [0, 1] \times [0, \sqrt{3}].$$

The effects of hidden rotations in such rectangles were investigated by Pinho [43]. Interestingly, such rectangular domains lead to dual lattices which support hexagonal sublattices. See figure 3(b) for an illustration of the symmetries of a rectangular dual lattice with aspect ratio $1/\sqrt{3}$, and figures 6 and 7 for some associated hexagonal patterns. Note that any sublattice of \mathcal{L}^* is the dual of a particular refinement of the lattice \mathcal{L} . Therefore these rotational symmetries lead to patterns which have more structure than the domain.

in the Faraday experiment. They used a container whose horizontal cross section had the symmetry of a square, but the sides were slightly curved to exclude translational symmetries.

Ashwin & Mei [3, 4] also did an investigation along these lines in the context of Hopf bifurcations of the Brusselator equations in a square domain. These authors observed that this particular set of equations with Neumann boundary conditions admits solutions that do not persist when a homotopy parameter is introduced to interpolate between Neumann and Dirichlet boundary conditions.

This type of interpolation to force symmetry breaking was also applied by Hirschberg & Knobloch [29] to a convection problem in a large aspect ratio rectangular domain. The authors point out that Neumann boundary conditions support a large scale flow generated by interaction of two modes, which prevents the system from approaching the unbounded system in the limit of infinite aspect ratio. This 'unwanted' hidden symmetry is removed with a homotopy parameter interpolating the boundary conditions between Neumann and Dirichlet. Dirichlet boundary conditions support a large scale flow with different properties, which allow the system to approach the results for the unbounded system.

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